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Ultraviolet cutoff and bosonic dominance

Musongela Lubo*
The Abdus Salam International Centre for Theoretical Physics, P.O. Box 586, 34100 Trieste, Italy

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We rederive the thermodynamical properties of a noninteracting gas in the presence of a minimal uncertainty in length. Apart from the phase space measure which is modified due to a change of the Heisenberg uncertainty relations, the presence of an ultraviolet cutoff plays a tremendous role. The theory admits an intrinsic temperature above which the fermion contribution to energy density, pressure and entropy is negligible.

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I. INTRODUCTION

Modern physics is an edifice in which every stone is tightly linked to the others. A slight modification in one area may produce important changes in different fields. Quantum mechanics starts with the commutation relations. Once they have been fixed, using a representation of the operator algebra we can in principle solve the Schrödinger equation whose solutions potentially contain all the physics of the (nonrelativistic) system studied. The correspondence principle which states the link between the commutator of two variables of the phase space and their classical Poisson brackets is one of the basic axioms of quantum mechanics. It is not deduced from another assumption and cannot be judged in an isolated manner: only the whole theory can be confronted to experience through its predictions.

Some authors have investigated the consequences of the alterations of these commutation relations on the observables of some physical systems. In particular, some deformations of the canonical commutators studied by Kempf, Mangano, and Mann (KMM) induce a minimal uncertainty in position (or momentum) in a very simple way, providing a toy model with manifest nonlocality [1]. The implications of some of these quantum structures have been studied for the harmonic oscillator [1–3] and the hydrogen atom [4]. The trans-Planckian problem occurring in the usual description of the Hawking mechanism of black hole evaporation has also been addressed in this framework, for the Schwarzschild and the Bañados-Teitelboim-Zanelli (BTZ) solutions [5,6]. These studies have extended the work on the entropy of the black hole [7] and Hawking radiation [8] in theories with modified dispersion relations [9–13].

One of the purposes of this article is a reinvestigation of the modifications these models induce, not in the characteristics of a single particle, but in the behavior of a macroscopic system. The modern presentation of thermodynamics basically relies on statistical physics. According to the system under study (isolated, closed or opened), one uses an ensemble (microcanonical, canonical or grand canonical) and the corresponding potential (entropy, free energy, grand potential) to derive thermodynamical quantities (pressure, specific heat, chemical potential, etc.).

The thermodynamic potentials are related to the derivatives of the partition function which itself is an average (of a quantity which depends on the ensemble used) on the phase space. To define a measure on the phase space, one needs to know the extension of the fundamentals cells. In the "classical" statistical physics, the Heisenberg uncertainty relations are used to show that this volume is the cube of the Planck constant and the undiscernability of particles justifies the Gibbs factor. This can be inferred from quantum statistical physics in which the sum giving the partition function is made on a discrete set of states.

If one modifies the commutators, one changes the Heisenberg uncertainty relations. The measure on the phase space is no more the same; this results in new partition functions and consequently different thermodynamical behaviors. From the quantum point of view, the energy spectrum of systems are modified by the change in the commutation relations.

The thermodynamics of some models displaying a minimal uncertainty in length has been analyzed before [14–16]. We shall give here better approximations and correct some sign errors. The implications for the early universe have been at the center of many works among which are Refs. [17,18]. The standard big bang in a universe in which an ultraviolet cutoff appears in a toy model exhibiting a modified dispersion relation has similarly been analyzed [19]. The difference between the two approaches relies on the fact that the dispersion relations are different; in the first models, they come from an assumption on the structure of the quantum phase space.

In these works, the equation of state used for radiation was obtained considering bosons. In the usual case the contributions of fermions to energy, pressure and entropy are simply the seven eights of the ones corresponding to bosons. This renders the equation of state insensitive to the ratio between fermionic and bosonic degrees of freedom. We show this to be drastically changed in the new framework.

Many studies have been devoted to cosmological perturbations in trans-Planckian physics [20–31]. The considerations we develop here may, in this context, be seen as relevant only in the pre-inflationary era. However, the phenomenological bounds [4] are much lower than the Planck scale. If one adopts a less restrictive point of view, then the new scale may generate some sizeable effects.

The article is organized as follows. In the second section we give a very brief survey of the two models we will study. They possess a minimal length uncertainty and so the quasiposition representation plays a crucial role. The third section

*Email address: muso@ictp.trieste.it

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is devoted to the study of the noninteracting gas. We obtain its equation of state, entropy, etc. We use its specific heat to set a bound on the deformation parameter and find it to be in agreement with previous ones.

II. DEFORMED THEORIES

As shown by KMM [1], the quantum mechanical theory defined by the commutation relation

$$[\hat{x},\hat{p}]=i\hbar(1+\beta\hat{p}^2) \tag{1}$$

is endowed with a minimal length uncertainty $\Delta x_{min}=\hbar\sqrt{\beta}$. The presence of this minimal length uncertainty implies that no position representation exists. The concept which proves to be the closest to it is the quasiposition representation in which the operators are nonlocal:

$$\hat{p}=\frac{1}{\sqrt{\beta}}\tan(-i\hbar\sqrt{\beta}\partial_\xi),\quad \hat{x}=\xi+\hbar\sqrt{\beta}\tan(i\hbar\sqrt{\beta}\partial_\xi). \tag{2}$$

It should be emphasized that the second part of the position operator plays an important role. If it was not present, the theory would admit position eigenstates which would have finite energy. This is of course forbidden by the existence of a minimal uncertainty in length. This representation is found by projecting on states of maximal localization. For details, see Ref. [1].

The deformations studied here produce a theory which is endowed with a minimal uncertainty in length. This means one cannot measure distances with arbitrary precision, even by increasing the momenta of the particles involved. It has been argued that a similar situation is likely to occur when quantum gravity sets in. This can be taken as a motivation for studying some aspects of this toy model.

The most obvious extension to a three dimensional space is obtained by taking the tensorial product of three such copies. It will be referred to as the A_1 model. It has translational invariance but lacks the symmetry under rotations. Another extension preserves rotational and translational symmetry; it will be referred to as the A_2 model. Its commutation relations are

$$[\hat{x}_j,\hat{p}_k]=i\hbar(f(\hat{p}^2)\delta_{j,k}+g(\hat{p}^2)\hat{p}_j\hat{p}_k),\quad g(p^2)=\beta,$$

$$f(p^2)=\frac{\beta p^2}{-1+\sqrt{1+2\beta p^2}}. \tag{3}$$

We will need the form of the momentum operators in this model:

$$p_k=-i\hbar\sum_{r=0}^{\infty}\left(\frac{\hbar^2\beta}{2}\Delta\right)^r\frac{\partial}{\partial\xi_k},\quad \text{where}\quad \Delta=\sum_{l=1}^{3}\frac{\partial^2}{\partial\xi_l^2}. \tag{4}$$

Although the model A_2 is physically better motivated, we shall analyze the thermodynamics of both extensions. The interest of such a study is that it reveals some generic properties which are shared by these kinds of deformations.

III. NONINTERACTING GAS IN DEFORMED THEORIES

Introducing the momentum scale β, it is straightforward that with the Boltzmann constant k, the light velocity c and the mass m of any particle, one can construct on purely dimensional grounds the characteristic temperatures

$$T_c=\frac{1}{\beta mk},\quad T_{cr}=\frac{c}{k\sqrt{\beta}}. \tag{5}$$

The first is ''nonrelativistic,'' particle dependent while the second is ''relativistic'' (c dependent) and universal. We are interested in what happens at and above these temperatures.

The constant β which appear in the KMM algebra is a free parameter. What numerical value can it assume? It was suggested [32] that the minimal length uncertainty $\hbar\sqrt{\beta}$ should be of the order of the Planck scale. We shall adopt a less restrictive point of view here. The only constraint is that the deformed commutator should not lead to contradictions with the predictions of the orthodox theory which have been observed experimentally. Our attitude is inspired by recent works which have shown that a new physics may take place well before the Planck scale [33]. Taking the most stringent phenomenological constraint $\hbar\sqrt{\beta}\leqslant 10^{-16}m$ [4], one finds a lower bound to the characteristic relativistic temperature: $T_{cr}\geqslant 10^{13}$ K. If one assumes the minimal length uncertainty to be of the order of the Planck scale, then $T_{cr}=10^{16}$ K.

We now briefly summarize the formula of statistical physics we will need. In the usual theory, a system which is in contact with a large heat reservoir and does not exchange particles with the surroundings has to be studied in the canonical ensemble [34,35]. Its equilibrium state will be described by a fixed temperature and a fixed particle number while its energy will fluctuate around a mean value. Strictly speaking, for such a system the particle number N is fixed once and for all. But, one knows that when phase transitions are not present, the descriptions given by the canonical and the grand canonical ensembles are very close. This will be used to compute the chemical potential in the canonical ensemble where the calculations are easier.

The most important quantity will be the canonical partition function $Z(T,V,N)$ which is defined in terms of the Hamiltonian operator $\hat{H}$ by the equation

$$Z(T,V,N)=Tr\exp\left(-\frac{\hat{H}}{kT}\right). \tag{6}$$

The free energy is related to the partition function by

$$F(T,V,N)=-kT\ln Z. \tag{7}$$

In these variables the pressure P, the entropy S, the chemical potential μ, the internal energy U and the constant volume specific heat C_V read

$$P=-\frac{\partial F}{\partial V},\quad S=-\frac{\partial F}{\partial T},\quad \mu=\frac{\partial F}{\partial N},$$

$$U=F+TS,\quad \text{and}\quad C_V=\frac{\partial U}{\partial T}. \tag{8}$$

A. A_1 model

1. Low temperatures

Low temperatures correspond to nonrelativistic behaviors. Let us study a noninteracting nonrelativistic gas. One has to solve the Schrödinger equation for a particle in a cubic box of length L. This will be done in the quasiposition representation because of the lack of a position one. For simplicity, let us first consider a one dimensional system; one obtains the solution

$$\psi(t,\xi)\propto e^{-iEt}\exp\left(\pm\frac{i\xi}{\hbar\sqrt{\beta}}\arctan\sqrt{2m\beta\hbar E}\right). \tag{9}$$

When the periodic boundary condition $\psi(t,\xi=0)=\psi(t,\xi=L)$ is imposed, one finds the quantization of energy

$$E_n=\frac{1}{2m\beta}\tan^2\left(\frac{2\pi\hbar\sqrt{\beta}n}{L}\right), \tag{10}$$

(n being an integer) already obtained in Refs. [1,5]. This leads to a cutoff in order to avoid a nonmonotonic dispersion relation:

$$n_{sup}=E\left[\frac{L}{4\hbar\sqrt{\beta}}\right], \tag{11}$$

E being the integer part function (not to be confused with the energy).

This cutoff n_{sup} is necessary in order to prevent a divergence of the partition function which would take place otherwise, due to the periodic nature of the energy [Eq. (10)]. But this still allows an infinite energy provided that the length of the box is fine tuned in such a way that the result of its division by the minimal length uncertainty is an integer. This would not be problematic since such an energy would have a vanishing contribution to the partition function. This conclusion will remain untouched at high temperatures.

The one particle partition function is given by the formula

$$Z(T,V,1)=\sum_{n=0}^{n_{sup}} e^{-E_n/kT}$$
$$=\sum_{n=0}^{n_{sup}}\exp\left[-\frac{1}{2\beta mkT}\tan^2\left(\hbar\sqrt{\beta}\,\frac{2\pi n}{L}\right)\right]. \tag{12}$$

The sum on n can be approximated by an integral on dn if the size of the box L is big enough. Introducing the integration variable p by

$$n=\frac{L}{2\pi\hbar\sqrt{\beta}}\arctan(\sqrt{\beta}p) \tag{13}$$

and replacing, as usual in statistical physics, the length L by an integral on position, we find

$$Z(T,V,1)=\frac{1}{h}\int dx\,dp\,\frac{1}{1+\beta p^2}e^{-p^2/2mkT}. \tag{14}$$

The formula given in Eq. (14) admits a semiclassical interpretation. Let us first consider its limiting case $\beta=0$. Classically, the system can be seen as a point evolving in phase space. The probability for the system to be in a configuration in which the first particle is in the region $\vec{q}_1\pm\vec{dq}_1,\vec{p}_1\pm\vec{dp}_1$, the second particle in the region $\vec{q}_2\pm\vec{dq}_2,\vec{p}_2\pm\vec{dp}_2,\ldots$ is proportional to $e^{-E(\vec{q},\vec{p})}$ and proportional to the number of elementary cells contained in the volume of the aforementioned region. At the quantum level, the Heisenberg uncertainty relation of the usual theory (written in the one dimensional case) $\Delta p_i\Delta q_i\geqslant\hbar/2$ assigns to each elementary cell a volume h. The number of such cells contained in the region under consideration is therefore

$$\prod_{i=1}^{N}\frac{d^3\vec{p}_i d^3\vec{q}_i}{h^3}. \tag{15}$$

One sees that in the new theory, one can simply keep the usual dispersion relation and modify the elementary cell volume. At the quantum level, this appears as a Jacobian linked to the change of variables ($\vec{n}\rightarrow\vec{p}$). This could be anticipated with a semiclassical reasoning. The new Heisenberg uncertainty relation implies

$$\Delta x\Delta p\geqslant\frac{\hbar}{2}(1+\beta\langle p^2\rangle). \tag{16}$$

It assigns to the elementary cells of the phase space of the new theory a volume $h(1+\beta p^2)$ which replaces the usual factor h. From this we could conjecture a simple recipe when dealing with the semiclassical approximation: it is obtained by keeping the classical dispersion relation but modifying the measure in a way consistent with the new Heisenberg uncertainty relation.

However, it should be noted that, in the new theory, the range over which one integrates is finite, due to the presence of the cutoff which depends on the volume but not the temperature:

$$p_{sup}=\frac{t}{\sqrt{\beta}},\quad \text{with}\quad t=\tan\left[\frac{\pi}{2}\left(\frac{L}{4\hbar\sqrt{\beta}}\right)^{-1}\left\lfloor\frac{L}{4\hbar\sqrt{\beta}}\right\rfloor\right]. \tag{17}$$

In this formula $\lfloor x\rfloor$ represents the integer part of the number x. Due to the equality

$$\lim_{x\rightarrow\infty}\frac{\lfloor x\rfloor}{x}=1, \tag{18}$$

the upper bound goes to infinity and the volume of the elementary cell tends to the usual one as the deformation parameter is sent to zero. It can be anticipated that, because the integrand of Eq. (14) is a rapidly decreasing function, taking the upper bound to be infinite will not introduce an appreciable error in most cases.

Going to the three dimensional extension A_1, one has simply to take the product of the elementary cells in the three directions. The integral over the positions is obvious; the change of variable $w=\vec{p}^2/2mkT$ allows one to write the one point partition function as

$$Z(T,V,1)=\frac{V}{\lambda^3}J,$$

with

$$J=\left[\frac{1}{\sqrt{\pi}}\int_0^{\omega_{sup}}\exp(-\omega)\omega^{-1/2}\left(1+2\frac{T}{T_c}\omega\right)^{-1}d\omega\right]^3,$$

$$\lambda=\left(\frac{h^2}{2\pi mkT}\right)^{1/2},\quad \omega_{sup}=\frac{t^2}{2}\frac{T_c}{T}. \tag{19}$$

When β vanishes, the upper bound equals infinity and the integral giving J assumes the value one so that the undeformed theory is recovered. The total partition function Z is found to be related to the usual one (corresponding to $\beta=0$ and now denoted Z_*) by the relation

$$Z=Z_*J^N. \tag{20}$$

The free energy then becomes

$$F=F_*-NkT\ln J. \tag{21}$$

The thermodynamical quantities are affected in the following way:

$$P=P_*+NkT\frac{1}{J}\frac{\partial J}{\partial V},\quad S=S_*+Nk\ln J+NkT\frac{1}{J}\frac{\partial J}{\partial T},$$

$$\mu=\mu_*-kT\ln J-NkT\frac{1}{J}\frac{\partial J}{\partial N},\quad U=U_*+NkT^2\frac{1}{J}\frac{\partial J}{\partial T},$$

$$C_V=C_V^*+2NkT\frac{1}{J}\frac{\partial J}{\partial T}+NkT^2\left[-\frac{1}{J^2}\left(\frac{\partial J}{\partial T}\right)^2+\frac{1}{J}\frac{\partial^2 J}{\partial T^2}\right]. \tag{22}$$

As can be seen from Eq. (19), J depends on the volume only through the cutoff whose influence will be seen to be negligible for the system under study. Thus, the equation of state $PV=NkT$ will remain valid, thanks to Eqs. (22). The presence of the temperature and the absence of the number of particles in the expression of J results in the fact that the entropy receives two contributions while the last term of the chemical potential in Eqs. (22) vanishes. The internal energy is also modified and by way of consequence the specific heat at constant volume too.

The integral giving J cannot be computed analytically. However, the qualitative features of the theory can be obtained quite easily. When the deformation parameter goes to zero, J assumes the value 1. As we shall show in a moment, the following parametrization holds:

$$J=1+\sigma_1\frac{T}{T_c}+\sigma_2\left(\frac{T}{T_c}\right)^2+\cdots. \tag{23}$$

Working to second order, we find

$$\frac{T}{J}\frac{\partial J}{\partial T}=\sigma_1\frac{T}{T_c}+(2\sigma_2-\sigma_1^2)\left(\frac{T}{T_c}\right)^2$$
$$=\log\left[1+\sigma_1\frac{T}{T_c}+\left(2\sigma_2-\frac{1}{2}\sigma_1^2\right)\left(\frac{T}{T_c}\right)^2\right]. \tag{24}$$

The last formula's interest lies in the fact that it allows a more compact expression of the entropy:

$$S=Nk\left(\frac{5}{2}+\log\sigma_0\right)+Nk\log\left\{\frac{V}{N}\left(\frac{2\pi mkT}{h^2}\right)^{3/2}\right.$$
$$\left.\times\left[1+2\sigma_1\frac{T}{T_c}+\left(3\sigma_2+\frac{1}{2}\sigma_1^2\right)\left(\frac{T}{T_c}\right)^2\right]\right\}, \tag{25}$$

so that an adiabatic process takes the form:

$$V\sim T^{-3/2}\left[1-2\sigma_1\frac{T}{T_c}+\left(-3\sigma_2+\frac{7}{2}\sigma_1^2\right)\left(\frac{T}{T_c}\right)^2\right]. \tag{26}$$

The formulas displayed in Eqs. (22) are used to recast the equation of state and the specific heat as

$$\rho=P\left[\frac{3}{2}+\sigma_1\frac{T}{T_c}+(2\sigma_2-\sigma_1^2)\left(\frac{T}{T_c}\right)^2\right],$$

$$C_V=Nk\left[\frac{3}{2}+2\sigma_1\frac{T}{T_c}+(6\sigma_2-3\sigma_1^2)\left(\frac{T}{T_c}\right)^2\right]. \tag{27}$$

Considering the reaction

$$aA+bB\rightleftharpoons dD+eE, \tag{28}$$

the expression of the chemical potential μ shows that the densities at equilibrium $X_i=N_i/V$ obey the modified law of action of masses

$$\frac{X_D^dX_E^e}{X_A^aX_B^b}=c^{te}T^{3/2(-a-b+d+e)}\left(1+2d\sigma_{1,D}\frac{T}{T_{c,D}}+2e\sigma_{1,E}\frac{T}{T_{c,E}}\right.$$
$$\left.-2a\sigma_{1,A}\frac{T}{T_{c,A}}-2b\sigma_{1,B}\frac{T}{T_{c,B}}\right), \tag{29}$$

since each particle, having its own mass, possesses a specific critical temperature.

Let us now evaluate the integral J in order to have numerical estimates of the constants σ_i. We shall use the MacLaurin expansion with a remainder. Let us separate the integral of Eq. (19) in two parts:

$$J^{1/3}=I_1+I_2=\frac{1}{\sqrt{\pi}}\left[\int_0^{T_c/2T}\cdots+\int_{T_c/2T}^{\omega_{sup}}\cdots\right]. \tag{30}$$

On the second interval, the inequality

$$\frac{1}{1+\frac{2T}{T_c}\omega}\leqslant\frac{1}{2} \tag{31}$$

can be used to obtain the majorization

$$I_2\leqslant\frac{1}{2\sqrt{\pi}}\int_{T_c/2T}^{\omega_{sup}}e^{-\omega}\omega^{-1/2}=\frac{1}{2\sqrt{\pi}}\left[\Gamma\left(\frac{1}{2},\frac{T_c}{2T}\right)-\Gamma\left(\frac{1}{2},\omega_{sup}\right)\right]. \tag{32}$$

The asymptotic formula

$$\Gamma(a,z)\sim z^{a-1}e^{-z}\quad\text{for}\quad z\to\infty \tag{33}$$

shows that

$$I_2\leqslant c^{st}\left(\frac{T_c}{2T}\right)^{-1/2}\exp\left(-\frac{T_c}{2T}\right)+c^{st}\omega_{sup}^{-1/2}\exp(-\omega_{sup}). \tag{34}$$

Since, for a reasonable β, the characteristic temperature T_c is reasonably expected to be very high, the integral I_2 which contains the influence of the upper bound can therefore be neglected.

To evaluate I_1, we shall use the MacLaurin theorem which states that for any sufficiently regular function f defined on an interval $[0,a]$ and for any point ω on that interval, there exists another point $\theta(\omega)$ on the same interval such that

$$f(\omega)=f(0)+f'(0)\omega+\frac{1}{2}f''(0)\omega^2+\frac{1}{6}f'''[\theta(\omega)]\omega^3. \tag{35}$$

This gives

$$\frac{1}{1+2\frac{T}{T_c}\omega}=1-\frac{2T}{T_c}\omega+\left(\frac{2T}{T_c}\right)^2\omega^2+\frac{1}{6}f'''[\theta(\omega)]\omega^3. \tag{36}$$

This enables us to find

$$\left|I_1-\frac{1}{\sqrt{\pi}}\left\{\int_0^{T_c/2T}d\omega e^{-\omega}\omega^{-1/2}\left[1-\frac{2T}{T_c}\omega+\left(\frac{2T}{T_c}\right)^2\omega^2\right]\right\}\right|$$
$$\leqslant\max|f'''(x)|\frac{1}{6\sqrt{\pi}}\int_0^{T_c/2T}d\omega e^{-\omega}\omega^{5/2}. \tag{37}$$

On the interval involved, the following inequality is verified:

$$|\max f'''(x)|\leqslant c^{st}\left(\frac{2T}{T_c}\right)^3. \tag{38}$$

The integrals remaining in Eq. (37) can be expressed in terms of complete and incomplete Γ functions. The final result reads:

$$I_1=\frac{1}{\sqrt{\pi}}\left[\Gamma\left(\frac{1}{2}\right)-\Gamma\left(\frac{3}{2}\right)\frac{2T}{T_c}+\Gamma\left(\frac{5}{2}\right)\left(\frac{2T}{T_c}\right)^2\right]. \tag{39}$$

From this one reads the values of the coefficients σ_i.

The predictions of usual statistical physics are known to be quite accurate in ordinary conditions. We shall use this to give an estimate of the bound thermodynamics imposes on the deformation parameters. Consider the helium whose specific heat at constant volume assumes an experimental value comprised between 12.39 and 12.41 $\mathrm{J\,K^{-1}\,mole^{-1}}$. The undeformed theory assigns the value 12.47 $\mathrm{J\,K^{-1}\,mole^{-1}}$ to any nonrelativistic gas. Thanks to Eqs. (27), we can write the specific heat in the new theory as

$$C_V=12.47\left(1+\sigma_1\frac{T}{T_c}\right). \tag{40}$$

The measured value tells us that $12.39<C_V<12.41$. Assuming $T=300$ K, the prediction of the model does not get out of the experimental bounds provided that $T_c\geqslant10^6$ K which induces $\beta\leqslant10^{-45}$. One then finds a minimal length uncertainty $\gamma\leqslant10^{-12}$ m which does not disagree with the bound derived from atomic physics considerations $\gamma\leqslant10^{-16}$ m [4] but is less precise. It should be stressed that this only gives an idea of the order of magnitude since we did not include interactions between the atoms. The estimated temperature T_c at which something new should happen is too high for the nonrelativistic approach to be reliable. Thus the interest of the next subsection.

2. High temperatures

The deformed Klein-Gordon wave equation for a massless particle in a box leads to the spectrum

$$E_{\vec{n}}^2=\frac{c^2}{\beta}\left[\tan^2\left(\hbar\sqrt{\beta}\,\frac{2\pi n_x}{L}\right)+\tan^2\left(\hbar\sqrt{\beta}\,\frac{2\pi n_y}{L}\right)+\tan^2\left(\hbar\sqrt{\beta}\,\frac{2\pi n_z}{L}\right)\right]. \tag{41}$$

Replacing the sum by an integral, one obtains for the factor J appearing in the partition function the expression

$$J=\int_0^{\omega_{sup}}dx\int_0^{\omega_{sup}}dy\int_0^{\omega_{sup}}dz\quad\varphi(x,y,z), \tag{42}$$

where

$$\varphi(x,y,z)=\frac{1}{\pi}\left[\left(1+\frac{T^2}{T_{cr}^2}x^2\right)\left(1+\frac{T^2}{T_{cr}^2}y^2\right)\left(1+\frac{T^2}{T_{cr}^2}z^2\right)\times[\exp(\sqrt{x^2+y^2+z^2})]\right]^{-1},\quad \omega_{sup}=\frac{T_{cr}}{T}t. \tag{43}$$

Let us first consider a temperature verifying $T/T_{cr}<1$, with an upper bound which is practically infinite $[T_{cr}t/T>1$, t being given by Eq. (17)]. In this case, replacing the domain of integration, a cube, by a sphere should not introduce an important error. The result reads

$$J=1-12\frac{\zeta(5)}{\zeta(3)}\left(\frac{T}{T_{cr}}\right)^2+288\frac{\zeta(7)}{\zeta(3)}\left(\frac{T}{T_{cr}}\right)^4. \tag{44}$$

From Eqs. (22), one finds the equation of state and the expression of the entropy

$$\frac{\rho}{p}=3-24\frac{\zeta(5)}{\zeta(3)}\left(\frac{T}{T_{cr}}\right)^2+288\left[4\frac{\zeta(7)}{\zeta(3)}-\left(\frac{\zeta(5)}{\zeta(3)}\right)\right]\left(\frac{T}{T_{cr}}\right)^4,$$

$$S=4Nk+Nk\log\left(8\pi\frac{V}{N}\left(\frac{kT}{hc}\right)^3\left\{1-36\frac{\zeta(5)}{\zeta(3)}\left(\frac{T}{T_{cr}}\right)^2+288\left[5\frac{\zeta(7)}{\zeta(3)}+\left(\frac{\zeta(5)}{\zeta(3)}\right)\right]\left(\frac{T}{T_{cr}}\right)^4\right\}\right). \tag{45}$$

Like in Eqs. (25) and (27), one obtains small departures from the unmodified theory.

3. Very high temperatures

What happens at very high temperatures? Like in the preceding subsection, the most salient features can be captured from the behavior of J. As the temperature is increased, the form of the its integrand and its upper bound [Eqs. (43)] show that J goes to zero. An approximation of the form

$$J=\sigma_n\left(\frac{T_{cr}}{T}\right)^n+\sigma_{n+1}\left(\frac{T_{cr}}{T}\right)^{n+1}+\sigma_{n+2}\left(\frac{T_{cr}}{T}\right)^{n+2}+\cdots \tag{46}$$

with n a positive integer will hold. Keeping the first two corrections one shows, by computations similar to those of the preceding subsection, that the entropy takes the form

$$S=4Nk+Nk\log\left\{8\pi e^{-n}\sigma_n\frac{k^3}{(hc)^3}T_{cr}^n\frac{V}{N}T^{3-n}\times\left[1+\left(\frac{1}{2}\frac{\sigma_{n+1}^2}{\sigma_n^2}-\frac{\sigma_{n+2}}{\sigma_n}\right)\left(\frac{T_{cr}}{T}\right)^2\right]\right\} \tag{47}$$

(from which the equation of an adiabatic process can be deduced) while the equation of state reads

$$\rho=P\left[(3-n)-\frac{\sigma_{n+1}}{\sigma_n}\frac{T_{cr}}{T}+\left(\frac{\sigma_{n+1}^2}{\sigma_n^2}-2\frac{\sigma_{n+2}}{\sigma_n}\right)\left(\frac{T_{cr}}{T}\right)^2\right]. \tag{48}$$

The law of action of masses now assumes the form:

$$\frac{X_D^dX_E^e}{X_A^aX_B^b}=c^{st}T^{(3-n)(d+e-a-b)}\left[1+(d+e-a-b)\times\left(\frac{1}{2}\frac{\sigma_{n+1}^2}{\sigma_n^2}-\frac{\sigma_{n+2}}{\sigma_n}\right)\left(\frac{T_{cr}}{T}\right)^2\right]. \tag{49}$$

Let us find the integer n. One has $T/T_{cr}<1$ and $T_{cr}t/T<1$. Now, the domain of integration is small and so the exponential appearing in the function φ can be expanded in polynomials. The function $1+T_{cr}^2/T^2x^2$ admits two Taylor expansions; the region in which $x<T/T_{cr}$ will be denoted A; the other one will be denoted B. The same situation occurs for y and z; this leads to a partition of the domain of integration. The most important contribution reads

$$J_{AAA}=\frac{T_{cr}^3}{T^3}. \tag{50}$$

The dependence on the volume is negligible. Comparing with Eq. (46), one has $n=3$ so that the equation of state is, to first order, $\rho\sim 0$.

In this theory, statistics will play a role at high temperature. As is evident from Eqs. (42) and (43), J will go to zero as the temperature increases. The Bose-Einstein distribution

$$N_i=\frac{g_i}{\exp\left(\frac{\epsilon_i}{kT}-\nu\right)-1} \tag{51}$$

reduces to the Maxwell-Boltzmann's one only in the limiting case

$$e^{\nu}\ll 1. \tag{52}$$

Thanks to the relation giving the total number of particles $N=\Sigma_iN_i$, one is led to the condition

$$e^{\nu}=\frac{N}{Z(T,V,1)}\ll 1\Rightarrow\frac{N}{V}\ll 8\pi\left(\frac{kT}{hc}\right)^3J. \tag{53}$$

One concludes that neglecting statistics is accurate, at very high temperatures, only for systems whose densities are very small ($J\sim T^{-3}$). If this is not the case, the appropriate integrand in the evaluation of J for bosons for example is

$$\varphi(x,y,z)=\frac{1}{\pi_1}\left\{\left(1+\frac{T^2}{T_{cr}^2}x^2\right)\left(1+\frac{T^2}{T_{cr}^2}y^2\right)\left(1+\frac{T^2}{T_{cr}^2}z^2\right)\times[\exp(\sqrt{x^2+y^2+z^2})-1]\right\}^{-1}. \tag{54}$$

The normalization constant π_1 ensures that $J=1$ in the undeformed theory. One finds the dominant part is given by

$$J_{AAA}=\left(-\frac{\pi}{4}-\log(2\sqrt{2})+3\log(1+\sqrt{3})\right)\left(\frac{T_{cr}}{T}\right)^2. \tag{55}$$

In the formula giving J, one now has $n=2$ and the associate equation of state for a gas of bosons takes the form $\rho=P$.

We treat in more detail the A_2 model in the following section. Our choice is mostly due to the fact that the A_2 model, possessing spherical symmetry, gives J as a one dimensional integral, contrary to the A_1 model [see Eq. (54)]. Apart from leading to simpler formulas, this model seems also more suitable to the treatment of a Robertson-Walker universe because of this rotational symmetry.

B. A_2 model

In this case, the action of the first position operator on the plane wave $\psi_k(t,\vec{\xi})=\exp(-iEt+k_1\xi_1+k_2\xi_2+k_3\xi_3)$ reduces to

$$\hat{p}_1\psi_k(t,\vec{\xi})=-i\hbar\,\psi_k(t,\vec{\xi})\sum_{r=0}^{\infty}\left(\frac{\hbar^2\beta}{2}\vec{k}^2\right)^r. \tag{56}$$

The sum of the left hand side converges only when $\hbar^2k^2<2/\sqrt{\beta}$; this is the cutoff. As shown in the last subsection, we do not learn much from the deformed nonrelativistic theory; we then go directly to high temperatures. The solution to the wave equation gives the dispersion relation

$$E=c\hbar k\left(1+\frac{\hbar^2\beta}{2}k^2\right)^{-1}, \tag{57}$$

from which one infers the quantity controlling the departure from the unmodified theory, for fermions and bosons:

$$J_{bo}=\frac{1}{2\zeta(3)}\int_0^{\sqrt{2}(T_{cr}/T)}dx\,x^2\left[\exp\left(\frac{x}{1+\frac{1}{2}\frac{T^2}{T_{cr}^2}x^2}\right)-1\right]^{-1},$$

$$J_{fe}=\frac{2}{3\zeta(3)}\int_0^{\sqrt{2}(T_{cr}/T)}dx\,x^2\left[\exp\left(\frac{x}{1+\frac{1}{2}\frac{T^2}{T_{cr}^2}x^2}\right)+1\right]^{-1}. \tag{58}$$

For temperatures smaller than T_{cr}, a Taylor expansion of the term in parentheses and an approximation of the upper bound of the integral by infinity holds. This leads to the following expressions for the equation of state and the entropy in the bosonic case:

$$\left(\frac{\rho}{p}\right)_{bo}=3+60\frac{\zeta(5)}{\zeta(3)}\left(\frac{T}{T_{cr}}\right)^2+360\left[21\frac{\zeta(7)}{\zeta(3)}-5\left(\frac{\zeta(5)}{\zeta(3)}\right)^2\right]\left(\frac{T}{T_{cr}}\right)^4,$$

$$S_{bo}=4Nk+Nk\log\left(8\pi\frac{V}{N}\left(\frac{kT}{\hbar c}\right)^3\left\{1+90\frac{\zeta(5)}{\zeta(3)}\left(\frac{T}{T_{cr}}\right)^2+450\left[21\frac{\zeta(7)}{\zeta(3)}+4\left(\frac{\zeta(5)}{\zeta(3)}\right)^2\right]\left(\frac{T}{T_{cr}}\right)^4\right\}\right). \tag{59}$$

For fermions, the behavior is roughly similar but the details are different:

$$\left(\frac{\rho}{p}\right)_{fe}=3+75\frac{\zeta(5)}{\zeta(3)}\left(\frac{T}{T_{cr}}\right)^2+\frac{45}{2}\left[441\frac{\zeta(7)}{\zeta(3)}-125\left(\frac{\zeta(5)}{\zeta(3)}\right)^2\right]\left(\frac{T}{T_{cr}}\right)^4,$$

$$S_{fe}=4Nk+Nk\log\left(8\pi\frac{V}{N}\left(\frac{kT}{\hbar c}\right)^3\left\{1+\frac{225}{2}\frac{\zeta(5)}{\zeta(3)}\left(\frac{T}{T_{cr}}\right)^2+\frac{225}{8}\left[441\frac{\zeta(7)}{\zeta(3)}+100\left(\frac{\zeta(5)}{\zeta(3)}\right)^2\right]\left(\frac{T}{T_{cr}}\right)^4\right\}\right). \tag{60}$$

When the temperature becomes of the order of the critical temperature, an important change takes place. The domain of integration for J in Eq. (58) becomes small and so one can approximate the integrand by its Taylor expansion near the origin. The difference between bosons and fermions enters into play through the difference of signs $\pm$ which leads to different powers in terms of the temperature. Developing the full integrand in Eq. (58) to the fourth order in x, one finds

$$J_{bo}=\frac{1}{2\zeta(3)}\left[\frac{3}{2}\left(\frac{T_{cr}}{T}\right)^2-\frac{1}{3}\sqrt{2}\left(\frac{T_{cr}}{T}\right)^3\right],$$

$$J_{fe}=\frac{2}{3\zeta(3)}\left[\frac{1}{6}\left(\frac{T_{cr}}{T}\right)^3-\frac{1}{16}\left(\frac{T_{cr}}{T}\right)^4\right]. \tag{61}$$

A computation to an order greater than the one to which we have limited ourselves brings in small corrections to the coefficients 13/16, etc. Using Eqs. (48), one then finds the expression of the entropy and the equation of state. The difference is significant between the two statistics as can be seen from the following equations:

$$\left(\frac{\rho}{p}\right)_{bo}=1+\frac{2}{9}\sqrt{2}\,\frac{T_{cr}}{T}+\frac{8}{81}\left(\frac{T_{cr}}{T}\right)^2,$$

$$\left(\frac{\rho}{p}\right)_{fe}=\frac{3}{8}\frac{T_{cr}}{T}+\frac{9}{64}\left(\frac{T_{cr}}{T}\right)^2. \tag{62}$$

A numerical computation supports our approximation scheme.

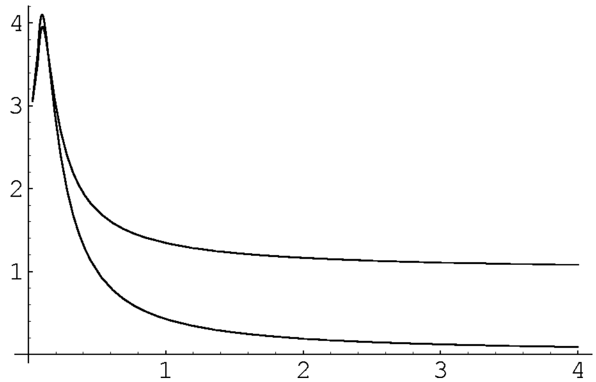

FIG. 1. ρ/p plotted as a function of T/T_{cr} for bosons and fermions.

In Fig. 1 we plot the ratios between the density and the pressure for a gas of bosons (fermions). For temperatures smaller than the critical one, these ratios are close to their known value 3 in the undeformed theory. The two functions then rise above this value as predicted by Eqs. (59) and (60). They finally tend to the asymptotic values 1 and 0 as obtained in Eqs. (62).

Let us note that the behavior of the two models, although similar in the limiting case of very high temperatures, display some qualitative differences. As can be seen by comparing Eqs. (45) and (60), in the A_1 model the density-pressure ratio does not rise above the usual value 3, contrary to what happens in A_2.

We have here a manifestation of the domination of bosons in this context: their density-pressure ratio goes like a constant while the one corresponding to fermions vanishes. Their entropy is also the only one to be considered at scales much higher than T_{cr} as they correspond to $n=2$ in Eq. (47) while for fermions one has $n=3$ [see Eqs. (61)]. Although the ratio we considered is much higher for bosons, one still cannot infer this to be the case for each of its parts. This will be established for black body radiation elsewhere.

IV. CONCLUSIONS

We have studied the thermodynamics induced by a nonlocal theory which exhibits a minimal uncertainty in length. We have obtained that a new behavior sets in at very high temperatures. The difference between fermions and bosons is more important than in the usual case.

The fact that bosons dominate over fermions may mean that as the temperature is increased, fermion modes start crowding close to the cutoff while bosons can keep packing more energy into modes of high energies.

It is worth mentioning some aspects which have not been raised in this work. At the fundamental level, one can ask if the concept of spin is relevant in these theories and, in the case the answer is positive, one still has to study the relation between spin and statistics in the new context. As the spin of a particle is defined, in the modern approach, through the behavior of its wave function under the Lorentz group, one has to find its generalization in the new context. For example, taking the ultimate structure of space-time to be given by a particular noncommutative geometry, the relevant algebra is not the Poincaré algebra but its q deformation. A notion of spin has been defined in these theories and the wave equations for particles of spin 0, 1/2 and 1 have been found [36]. Although the question has not been addressed in KMM theory, we hope for a similar situation to occur. We expect a generalization of the notion of spin which conserves the spin-statistic theorem.

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