Highly damped quasinormal modes of Kerr black holes

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Motivated by recent suggestions that highly damped black hole quasinormal modes (QNM's) may provide a link between classical general relativity and quantum gravity, we present an extensive computation of highly damped QNM's of Kerr black holes. We perform the computation using two independent numerical codes based on Leaver's continued fraction method. We do not limit our attention to gravitational modes, thus filling some gaps in the existing literature. As already observed by Berti and Kokkotas, the frequency of gravitational modes with $l=m=2$ tends to $\omega_R=2\Omega$, Ω being the angular velocity of the black hole horizon. We show that, if Hod's conjecture is valid, this asymptotic behavior is related to reversible black hole transformations. Other highly damped modes with $m>0$ that we computed do *not* show a similar behavior. The real part of modes with $l=2$ and $m<0$ seems to asymptotically approach a constant value $\omega_R \simeq -m\varpi$, $\varpi \simeq 0.12$ being (almost) independent of a. For any perturbing field, trajectories in the complex plane of QNM's with $m=0$ show a spiraling behavior, similar to the one observed for Reissner-Nordström black holes. Finally, for any perturbing field, the asymptotic separation in the imaginary part of consecutive modes with $m>0$ is given by $2\pi T_H$ (T_H being the black hole temperature). We conjecture that for all values of l and $m>0$ there is an infinity of modes tending to the critical frequency for superradiance ($\omega_R=m$) in the extremal limit. Finally, we study in some detail modes branching off the so-called "algebraically special frequency" of Schwarzschild black holes. For the first time we find numerically that QNM *multiplets* emerge from the algebraically special Schwarzschild modes, confirming a recent speculation.

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I. INTRODUCTION

The study of linearized perturbations of black hole solutions in general relativity has a long history [1]. The development of the relevant formalism, initially motivated by the need for a formal proof of black hole stability, gave birth to a whole new research field. A major role in this field has been played by the concept of quasinormal modes (QNM's): oscillations having purely ingoing wave conditions at the black hole horizon and purely outgoing wave conditions at infinity. These modes determine the late-time evolution of perturbing fields in the black hole exterior. Numerical simulations of stellar collapse and black hole collisions in the "full" (nonlinearized) theory have shown that in the final stage of such processes ("ringdown") QNM's dominate the black hole response to any kind of perturbation. Since their frequencies are uniquely determined by the black hole parameters (mass, charge and angular momentum), QNM's are likely to play a major role in the nascent field of gravitational wave astronomy, providing unique means to "identify" black holes [2].

An early attempt at relating QNM's to the Hawking radiation was carried out by York [3]. More recently Hod made an interesting proposal to infer quantum properties of black holes from their classical oscillation spectrum [4]. It was suggested many years ago by Bekenstein [5] that in a quantum theory of gravity the surface area of a black hole (which by the Bekenstein-Hawking formula is nothing but its entropy) should have a discrete spectrum. The eigenvalues of this spectrum are likely to be uniformly spaced. Hod observed that the real parts of the asymptotic (highly damped) quasinormal frequencies of a Schwarzschild black hole of mass M, as numerically computed by Nollert [6] and later by Andersson [7], can be written as

$$\omega_R = T_H \ln 3, \tag{1}$$

where we have used units such that $c=G=1$ and T_H is the black hole's Hawking temperature. He then exploited Bohr's correspondence principle, requiring that "transition frequencies at large quantum numbers should equal classical oscillation frequencies," to infer that variations in the black hole mass induced by quantum processes should be given by

$$\Delta M = \hbar \omega_R . \tag{2}$$

Finally, he used the first law of black hole thermodynamics to deduce the spacing in the area spectrum for a Schwarzschild black hole. Remarkably, in this quantum gravity context relevant modes are those which damp infinitely fast, do not significantly contribute to the gravitational wave signal, and are therefore typically ignored in studies of gravitational radiation. Following Hod's suggestion, Dreyer recently used a similar argument to fix a free parameter (the so-called

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Barbero-Immirzi parameter) appearing in loop quantum gravity [8]. Supposing that transitions of a quantum black hole are characterized by the appearance or disappearance of a puncture with lowest possible spin j_{min}, Dreyer found that loop quantum gravity gives a correct prediction for the Bekenstein-Hawking entropy if $j_{min}=1$, consequently fixing the Barbero-Immirzi parameter.

When Hod made his original proposal, formula (1) was merely a curious numerical coincidence. Kunstatter [9] suggested that a similar relation may hold also for multidimensional black holes. Since these early speculations, a full formalism for nonrotating black hole perturbations in higher dimensions has been developed [10], and different calculations have now shown that formula (1) holds *exactly* for scalar and gravitational perturbations of nonrotating black holes in any dimension [11–16]. Furthermore, Birmingham *et al.* have recently given intriguing hints corroborating the correspondence suggested by Hod [17], focusing attention on (2+1)-dimensional Bañados-Teitelboim-Zanelli (BTZ) black holes [18]. In this case the QNM frequencies (which belong to two ''families'') can be obtained analytically, and their real parts are independent of the mode damping. They showed that the identification of the fundamental quanta of black hole mass and angular momentum with the real part of the QNM frequencies leads to the correct quantum behavior of the asymptotic symmetry algebra, and thus of the dual conformal field theory.

In light of these exciting new results, Hod's conjecture seems to be a very promising candidate to shed light on quantum properties of black holes. However, it is natural to ask whether the conjecture applies to more general (charged and/or rotating) black holes. If asymptotic frequencies for ''generic'' black holes depend (as they do) on the hole's charge, angular momentum, or on the presence of a cosmological constant, should Hod's proposal be modified in some way? And how does the correct modification look like? The hint for an answer necessarily comes from analytical or numerical calculations of highly damped QNM's for charged and rotating black holes, or for black holes in nonasymptotically flat spacetimes. Some calculations in this direction have now been performed, revealing unexpected and puzzling features [12,19–27].

In particular, the technique originally developed by Nollert to study highly damped modes of Schwarzschild black holes has recently been extended to the RN case [19], showing that highly damped RN QNM's show a peculiar spiraling behavior in the complex-ω plane as the black hole charge is increased. Independently, Motl and Neitzke obtained an analytic formula for the asymptotic frequencies of scalar and electromagnetic-gravitational perturbations of a RN black hole whose predictions show an excellent agreement (at least for large values of the charge) with the numerical results [12]. For computational convenience they fixed their units in a somewhat unconventional way: they introduced a parameter k related to the black hole charge and mass by $Q/M=2\sqrt{k}/(1+k)$, so that $\beta=4\pi/(1-k)=1/T_H$ is the inverse black hole Hawking temperature and $\beta_I=-k^2\beta$ is the inverse Hawking temperature of the inner horizon. Their result is an implicit formula for the asymptotic QNM frequencies,

$$e^{\beta\omega}+2+3e^{-\beta_I\omega}=0, \tag{3}$$

which has recently been confirmed by independent calculations [24]. However, its interpretation in terms of the suggested correspondence is still unclear. Asymptotic quasinormal frequencies of a charged black hole, according to formula (3), depend not only on the black hole's Hawking temperature, but also on the Hawking temperature of the (causally disconnected) inner horizon. Perhaps more worrying is the fact that the asymptotic formula does not yield the correct Schwarzschild limit as the black hole charge Q tends to zero. The mathematical reason for this behavior has been discussed in [12,24]. A calculation of higher-order corrections in $\omega_I^{-1/2}$ may explain the observed disagreement: indeed, as we shall see, the numerical study of Kerr modes we present in this paper seems to support this expectation. Finally and most importantly, it is not at all clear which are the implications of the generally nonperiodic behavior of asymptotic RN modes for the Hod conjecture. Maybe the complicated behavior we observe is an effect of the electromagnetic-gravitational coupling, and we should only consider *pure gravitational perturbations* for a first understanding of black hole quantization based on Hod's conjecture. The latter suggestion may possibly be ruled out on the basis of two simple observations: first of all, in the large damping limit ''electromagnetic'' and ''gravitational'' perturbations seem to be isospectral to each other, and isospectral to scalar perturbations as well [12]; secondly, Kerr modes with $m=0$ show a very similar spiraling behavior, which is clearly *not* due to any form of electromagnetic-gravitational coupling.

The available numerical calculations for highly damped modes of black holes in non-asymptotically flat spacetimes are as puzzling as those for RN black holes in flat spacetime. Cardoso and Lemos [20] have studied the asymptotic spectrum of Schwarzschild black holes in a de Sitter background. They found that, when the black hole radius is comparable to the cosmological radius, the asymptotic spectrum depends not only on the hole's parameters, but also on the angular separation index l. Formula (1) does not depend on dimensionality and gives the same limit for ''scalar'' and ''gravitational'' modes (loosely using the standard four dimensional terminology; see [10,14] for a more precise formulation in higher dimensions). This ''universality'' seems to be lost when the cosmological constant is nonzero. The study carried out in [20] has recently been generalized to higher dimensional Schwarzschild–de Sitter black holes [26] and to take into account higher-order corrections to the predicted behavior [21]. However the issue is not settled yet, and the asymptotics may be different from what was predicted in [20]. Indeed, recent numerical and analytical calculations [25,27] seem to suggest that the result presented in [20] is only correct when the overtone index n satisfies $nk\ll 1$, where k is the surface gravity at the Schwarzschild–de Sitter black hole horizon. For higher overtones, the behavior seems to be different. The problem is not completely solved yet. Numerically, it seems difficult to compute QNM frequencies for $nk>1$ [25]. Furthermore, at present, numerical and analytical results show only a qualitative (but not quantitative) agreement [27].

Calculations of QNM's for Schwarzschild–anti–de Sitter black holes were performed in various papers [28], showing that the nature of the QNM spectrum in this case is remarkably different (basically due to the "potential barrier" arising because of the cosmological constant, and to the changing QNM boundary conditions at infinity). Those calculations were recently extended to encompass asymptotic modes [23]. The basic result is that consecutive highly damped modes (whose real part goes to infinity as the imaginary part increases) have a uniform *spacing* in both the real and the imaginary part; this spacing is apparently independent of the kind of perturbation considered and of the angular separation index l.

The aim of this paper is to study in depth the behavior of highly damped Kerr QNM's, complementing and clarifying results that were presented in previous works [19,29]. The plan of the paper is as follows. In Sec. II we briefly introduce our numerical method. In Sec. III we discuss some results presented in [19] and show a more comprehensive calculation of gravitational QNM's, considering generic values of m and higher multipoles (namely, $l=3$). In Sec. IV we display some results for scalar and electromagnetic perturbations. If our numerics for nongravitational modes are indicative of the true asymptotic behavior, the asymptotic formula which is valid for $l=m=2$ gravitational perturbations may be very special. In Sec. V we briefly summarize our results and we discuss the asymptotic behavior of the modes' imaginary part. Finally, in Sec. VI we turn our attention to a different open problem concerning Kerr perturbations. Motivated by some recent, surprising developments arising from the study of the branch cut in the Schwarzschild problem [30] and by older conjectures derived from analytical calculations of the properties of algebraically special modes [31], we turn our attention to Kerr QNM's in the vicinity of the Schwarzschild algebraically special frequencies. As the black hole is set into rotation, we find for the first time that a QNM multiplet appears close to the algebraically special Schwarzschild modes. A summary, conclusions and an outlook on possible future research directions follow.

II. NUMERICAL METHOD

A first numerical study of Kerr QNM's was carried out many years ago by Detweiler [32]. Finding highly damped modes through a straightforward integration of the perturbation equations is particularly difficult even for nonrotating black holes [2]. In the Kerr case the situation is even worse, because, due to the nonspherical symmetry of the background, the perturbation problem does not reduce to a single ordinary differential equation for the radial part of the perturbations, but rather to a system of differential equations (one equation for the angular part of the perturbations, and a second equation for the radial part).

A method to find the eigenfrequencies without resorting to integrations of this system was developed by Leaver, and has been extensively discussed in the literature [19,29,33]. In this paper we will apply exactly the same method. Following Leaver, we will choose units such that $2M=1$. Then the perturbation equations depend on a parameter s denoting the spin of the perturbing field ($s=0,-1,-2$ for scalar, electromagnetic and gravitational perturbations respectively), on the Kerr rotation parameter a ($0<a<1/2$), and on an angular separation constant A_{lm}. In the Schwarzschild limit the angular separation constant can be determined analytically, and is given by the relation $A_{lm}=l(l+1)-s(s+1)$.

The basic idea in Leaver's method is the following. Boundary conditions for the radial and angular equations translate into convergence conditions for the series expansions of the corresponding eigenfunctions. In turn, these convergence conditions can be expressed as two equations involving continued fractions. Finding QNM frequencies is a two-step procedure: for assigned values of a, ℓ, m and ω, first find the angular separation constant $A_{lm}(\omega)$ looking for zeros of the *angular* continued fraction equation; then replace the corresponding eigenvalue into the *radial* continued fraction equation, and look for its zeros as a function of ω. Leaver's method is relatively well convergent and numerically stable for highly damped modes, when compared to other techniques [34]. We mention that an alternative, approximate method for finding Kerr quasinormal frequencies has recently been presented [35], which has the advantage of highlighting some physical features of the problem.

In the next sections we will use Leaver's technique to complement numerical studies of Kerr quasinormal overtones carried out by some of us in the past [19,29]. The method we use for our analysis is the one described in those papers. Exploring the high-damping regime necessarily requires pushing our numerics to their limits. Therefore we have systematically cross-checked the reliability of our results using two independent codes. As we shall see, our study will uncover a plethora of interesting new features.

III. GRAVITATIONAL PERTURBATIONS

A. $l=m=2$ modes: A more extensive discussion

Let us consider rotating black holes, having angular momentum per unit mass $a=J/M$. The black hole's (event and inner) horizons are given in terms of the black hole parameters by $r_{\pm}=M\pm\sqrt{M^2-a^2}$. The hole's temperature $T_H=(r_+-r_-)/A$ where $A=8\pi M r_+$ is the hole's surface area, related to its entropy S by the relation $S=A/4$. Introducing the angular velocity of the horizon $\Omega=4\pi a/A$, applying the first law of black hole thermodynamics,

$$\Delta M=T_H\Delta S+\Omega\Delta J, \tag{4}$$

and *assuming that the formula for the area spectrum derived for a Schwarzschild black hole still holds in this case*, Hod conjectured that the real parts of the asymptotic frequencies for rotating black holes are given by

$$\omega_R=\tilde{\omega}_R=T_H\ln 3+m\Omega, \tag{5}$$

where m is the azimuthal eigenvalue of the field [4]. Hod later used a systematic exploration of moderately damped Kerr black hole QNM's carried out a few years ago by one of us [29] to lend support to formula (5), at least for modes with

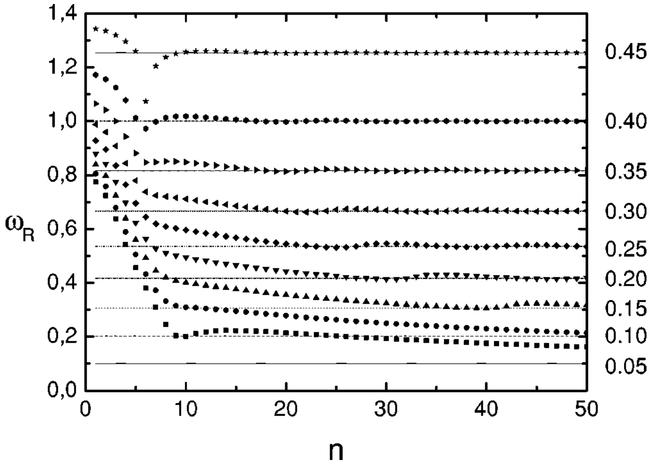

FIG. 1. Each different symbol corresponds to the (numerically computed) value of ω_R as a function of the mode index n, at different selected values of the rotation parameter a. The selected values of a are indicated on the right of the plot. Horizontal lines correspond to the predicted asymptotic frequencies 2Ω at the given values of a. Convergence to the asymptotic value is clearly faster for larger a. In the range of n allowed by our numerical method ($n \lesssim 50$) convergence is not yet achieved for $a \lesssim 0.1$.

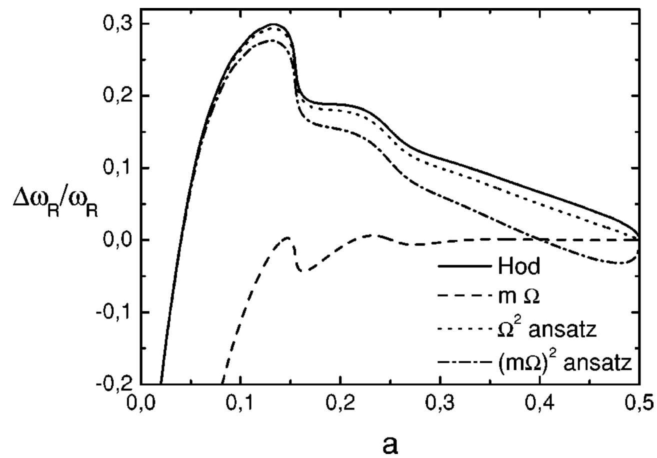

FIG. 2. Relative difference between various fit functions and numerical results for the mode with overtone index $n=40$. From top to bottom in the legend, the lines correspond to the relative errors for formulas (5), (6), (7) and (8).

$l=m$ [36]. His conclusions were shown to be in contrast with the observed behavior of modes having stronger damping in [19]: the deviations between the numerics and formula (5) were indeed shown to *grow* as the mode order grows (see Fig. 7 in [19]). Hod even used Eq. (4), *without including the term due to variations of the black hole charge* ΔQ, to conjecture that Eq. (5) holds for Kerr-Newman black holes as well [4]. This second step now definitely appears to be a bold extrapolation. Not only does formula (5) disagree with the observed numerical behavior for perturbations of Kerr black holes having $l=m=2$ [19] (not to mention other values of m, as we shall see below); by now, analytic and numerical calculations have shown that RN QNM's have a much more rich and complicated behavior [12,13,19].

In summary, there is now compelling evidence that the conjectured formula (5) must be wrong. However it turns out [19], quite surprisingly, that an extremely good fit to the numerical data for $l=m=2$ is provided by an even simpler relation, not involving the black hole temperature:

$$\omega_R = m\Omega. \tag{6}$$

At first sight, the good fitting properties of this formula may be regarded as a coincidence. After all, this formula does not yield the correct Schwarzschild limit. Why should we trust it when it is only based on numerical evidence? A convincing argument in favor of formula (6) is given in Fig. 1. There we show the real part of modes having $l=m=2$ as a function of n for some selected values of a (namely, $a=0.05,0.10,\dots,0.45$). The convergence towards the limiting value $\omega_R=2\Omega$ (horizontal lines in the plot) is evident. Furthermore, the convergence is much faster for holes spinning closer to the extremal limit, and becomes slower for black holes which are slowly rotating. The behavior we observe presents interesting analogies with the asymptotic formula (3). The Schwarzschild limit may not be recovered straightforwardly as $a\to 0$. Some order-of-limits issues may be at work, as recently claimed in [13] to justify the incorrect behavior of formula (3) as the black hole charge $Q\to 0$.

Is formula (6) merely an approximation to the "true" asymptotic behavior, for example a lowest-order expansion in powers of Ω? To answer this questions we can try and replace Eq. (6) by some alternative relation. Since in the Schwarzschild limit equation (6) does not give the desired "ln 3" behavior, we would like a higher-order correction which *does* reproduce the nonrotating limit, while giving a good fit to the numerical data. Therefore, in addition to Eqs. (5) and (6), we considered the following fitting relations:

$$\omega_R = 4\pi T_H^2 \ln 3 + m\Omega = T_H \ln 3 (1-\Omega^2) + m\Omega, \tag{7}$$

$$\omega_R = T_H \ln 3 (1-m^2\Omega^2) + m\Omega. \tag{8}$$

Formula (7) enforces the correct asymptotic limit at $a=0$, and can be considered as an Ω^2 correction to Hod's conjectured formula (5). Since numerical results suggest a dependence on $m\Omega$ we also used the slight modification given by formula (8), hoping for a better fit to our numerical data. The relative errors of the various fitting formulas with respect to the numerical computation for the $n=40$ QNM are given in Fig. 2. Equation (6) is clearly the one which performs better. All relations are seen to fail quite badly for small rotation rate, but this apparent failure is only due to the onset of the asymptotic behavior occurring *later* (that is, when $n>40$) for small values of a.

We believe that the excellent fitting properties and the convergence plot, when combined together, are very good evidence in favor of Eq. (6). Maybe the impressive visual agreement between the numerics and the conjectured asymptotic formula (6), displayed in the left panel of Fig. 3, is even more convincing. Therefore, let us assume as a working hypothesis that Eq. (6) *is* the correct asymptotic formula (at least for $l=m=2$, and maybe for large enough a), and

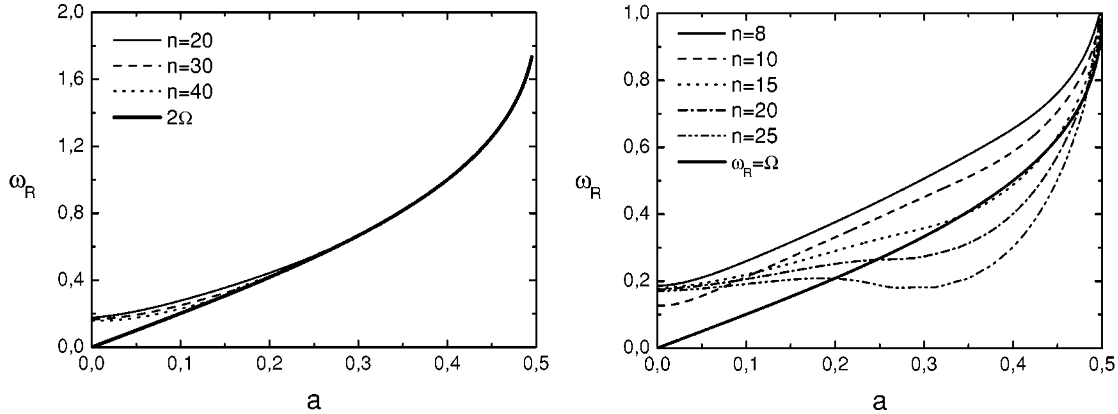

FIG. 3. Real part of the frequency for different modes with $l=2$ and $m>0$. In both panels we overplot (bold solid line) the prediction of formula (6). The left panel shows the excellent agreement between modes with $l=m=2$ and the asymptotic formula. The right panel shows the different behavior of modes with $m=1$; these modes have a frequency that ''bends'' downwards as n increases, showing a local minimum as a function of a. In both cases, $\omega_R \to m$ in the extremal limit $a \to 1/2$.

let us consider the consequences of such an assumption in computing the area spectrum for Kerr black holes. Modes having $l=m$ may indeed be the relevant ones to make a connection with quantum gravity, as recently claimed in [36]. Furthermore, the proportionality of these modes to the black hole's angular velocity Ω seems to suggest that something ''deep'' is at work in this particular case.

In the following, we will essentially repeat the calculation carried out by Abdalla *et al.* [22] for near-extremal ($a \to M$) Kerr black holes. We will argue that the conclusion of their calculation is in fact wrong, since those authors did not take into account the functional behavior of $\omega_R(a)$ (which was unknown when they wrote the paper), but rather assumed that $\omega_R = m/2M$ is *constant* in the vicinity of the extremal limit. In following the steps traced out in [22] we will restore for clarity all factors of M. This means, for example, that the asymptotic frequency for $m>0$ in the extremal limit is $\omega_R = m/2M$. Let us also define $x=a/M$. The black hole inner and outer horizons are $r_{\pm} = M[1 \pm (1-x^2)^{1/2}]$. The black hole temperature is

$$T = \frac{r_+ - r_-}{A} = \frac{1}{4\pi M}\frac{\sqrt{1-x^2}}{1+\sqrt{1-x^2}}, \tag{9}$$

and we recall that the black hole surface area $A = 8\pi M^2[1+(1-x^2)^{1/2}]$ is related to its entropy S by the relation $S = A/4$. The hole's rotational frequency is

$$\Omega = \frac{4\pi a}{A} = \frac{a}{2Mr_+} = \frac{1}{2M}\frac{x}{1+\sqrt{1-x^2}}. \tag{10}$$

Let us now apply the first law of black hole thermodynamics and the area-entropy relation to find

$$\Delta A = \frac{4}{T}(\Delta M - \Omega \Delta J). \tag{11}$$

The authors of [22] focused on the extremal limit. They used $\Delta J = \hbar m$ and $\Delta M = \hbar \omega_R(x=1) = \hbar m/2M$ to deduce that

$$\Delta A = 4\hbar m\left[\frac{1/2M - \Omega}{T}\right] = \hbar m \mathcal{A}, \tag{12}$$

where $\mathcal{A}$ is the area quantum. Now, the square parenthesis is undefined, since $\Omega \to 1/2M$ when $x \to 1$. Taking the limit $x \to 1$ *and keeping* $\Delta M = \hbar m/2M$ *constant* leads to

$$\mathcal{A} = 8\pi\left(1+\sqrt{\frac{1-x}{2}}\right) \simeq 8\pi, \tag{13}$$

which is the final result in [22]. The fundamental assumption in this argument is that the asymptotic frequency is $\omega_R = m/2M$, which is strictly true only for $x=1$. However, one has to consider how the QNM frequency changes with x. What is the effect of assuming $\omega_R = m\Omega$ on the area spectrum? The calculation is exactly the same, but the equation $\Delta M = \hbar m/2M$ is replaced by $\Delta M = \hbar m \Omega$, and we conclude that

$$\Delta A = 0. \tag{14}$$

The area variation is *zero* at any black hole rotation rate $a < M$. At first sight, this result may look surprising. It is not, and it follows from fundamental properties of black holes. Indeed, we are looking at reversible black hole transformations. It is well known that the gain in energy ΔE and the gain in angular momentum ΔJ resulting from a particle with negative energy $-E$ and angular momentum $-L_z$ arriving at the event horizon of a Kerr black hole is subject to the inequality

$$\Delta M \geqslant \Omega \Delta J. \tag{15}$$

See, for example, Eq. (352) on p. 373 in [1] and the related discussion. This inequality is equivalent to the statement that the irreducible mass $M_{irr} \equiv (Mr_+/2)^{1/2}$ of the black hole can only increase [37]. In other words, by no continuous infinitesimal process involving a single Kerr black hole can the surface area of the black hole be decreased (Hawking's area theorem). Assuming the validity of Hod's conjecture (2), and

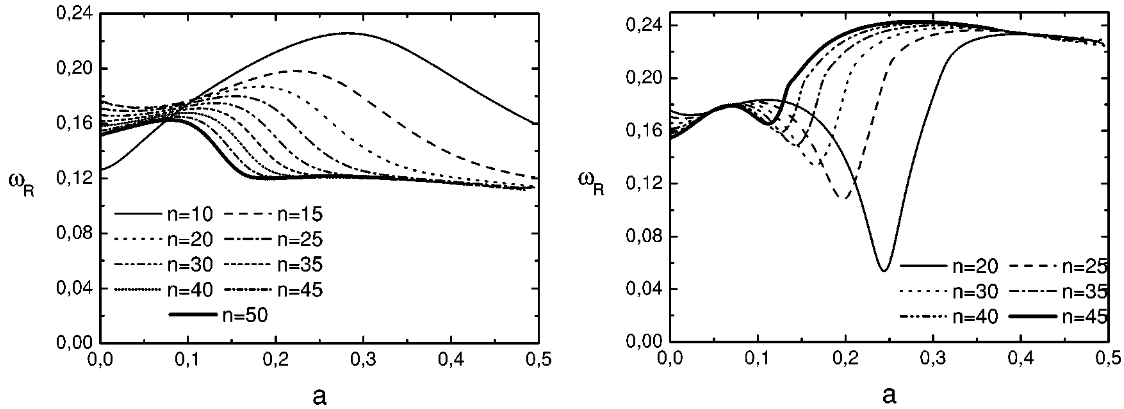

FIG. 4. Real part of the first few modes with $l=2$ and $m<0$. Modes with $m=-1$ are shown in the left panel, modes with $m=-2$ in the right panel. As the mode order n increases, ω_R seems to approach a (roughly) constant value $\omega_R=-m\varpi$, where $\varpi\simeq 0.12$. Convergence to this limiting value is faster for large values of the rotation parameter a (compare Fig. 1).

using the result (6) for asymptotic QNM's, we are saturating the inequality (15): we are considering a *reversible* process, in which the area (or, equivalently, the irreducible mass) is conserved. Classically, this result makes sense. Perturbations of Kerr black holes dying out on a vanishingly small timescale are likely to be a process for which the horizon area is an *adiabatic invariant*. Some physical processes exhibiting this feature were considered in detail in [38].

What does the result (14) mean from the point of view of area quantization? It could mean that using modes having $l=m$ in Hod's conjecture is wrong, or that we cannot use Bohr's correspondence principle to deduce the area spectrum for Kerr. A speculative suggestion may be to *modify Bohr's correspondence principle as introduced by Hod*. Suppose for example that we do not interpret the asymptotic frequencies as a change in *mass* ($\Delta M=\hbar\omega_R$), but rather impose $T\Delta S=\hbar\omega_R$. This is of course equivalent to Hod's original proposal when $a=0$. The asymptotic formula would then imply, using the first law of black hole thermodynamics, that the minimum possible variation in mass is $\Delta M=2m\hbar\Omega$.

We notice that the above arguments do not apply to strictly extremal Kerr black holes, for which $a=M$. In the extremal case the horizon area is *not* an adiabatic invariant [39], and its quantization probably requires some special treatment.

B. Modes with $l=2$, $l\neq m$

As discussed in the previous paragraph, we feel quite confident that the real part of modes with $l=m=2$ approaches the limit $\omega_R=m\Omega$ as the mode damping tends to infinity. What about modes having $l\neq m$? In [19] it was shown that modes with $m=0$ show a drastically different behavior. As the damping increases, modes show more and more loops. Pushing the calculation to very large imaginary parts is not easy, but the trend strongly suggests a spiraling asymptotic behavior, reminiscent of RN modes. In this section we present results for the cases not considered in [19], concentrating on the real parts of modes with $l=2$ and $m=1$, -1, -2.

Modes for which $l=2$, $m=1$ are displayed in the right panel of Fig. 3. They do not seem to approach the limit one could naively expect, that is, $\omega_R=\Omega$. Instead, the real part of the frequency shows a minimum as a function of a, and approaches the limit $\omega_R=m$ as $a\to 1/2$. To our knowledge, the fact that the real part of modes with $l=2$ and $m=1$ approaches $\omega_R=m=1$ as $a\to 1/2$ has not been observed before. In the following we will see that this behavior is characteristic of QNM's due to perturbation fields having arbitrary spin, as long as $m>0$.

The real parts of modes with $l=2$, $m<0$ as functions of a (for some selected values of n) are displayed in Fig. 4. From the left panel, displaying the real part of modes with $m=-1$, we infer an interesting conclusion: the frequencies tend to approach a constant (presumably a-independent) limiting value, with a convergence rate which is faster, as in the $l=m=2$ case, for large a. The limiting value is approximately given by 0.12. A similar result holds for modes with $l=2$, $m=-2$ (right panel). Once again the frequencies asymptotically approach a (roughly) constant value, with a convergence rate which is faster for large a. The limiting value is now approximately given by $\omega_R=0.24$, about twice the value we got for $m=-1$. In summary, the real part of modes with $m<0$ seems to asymptotically approach the limit

$$\omega_R=-m\varpi, \tag{16}$$

where $\varpi\simeq 0.12$ is (to a good approximation) independent of a, at least in the extremal limit $a\to 1/2$.

We will see below that this surprising result is quite general. It is supported by calculations of gravitational QNM's for different values of l, and it also holds for electromagnetic and scalar perturbations, as long as $m<0$. An analytical derivation of this result is definitely needed. It may offer some insight on the physical interpretation of the result, and help explain the surprising qualitative difference in the asymptotic behavior of modes having different values of m.

C. Modes with $l=3$

Results for a few highly-damped QNM's with $l=3$, $m=0$ were shown in [19]. Those modes exhibit the usual "spiraling" behavior in the complex plane as the imaginary part increases. In this paragraph we present a more complete cal-

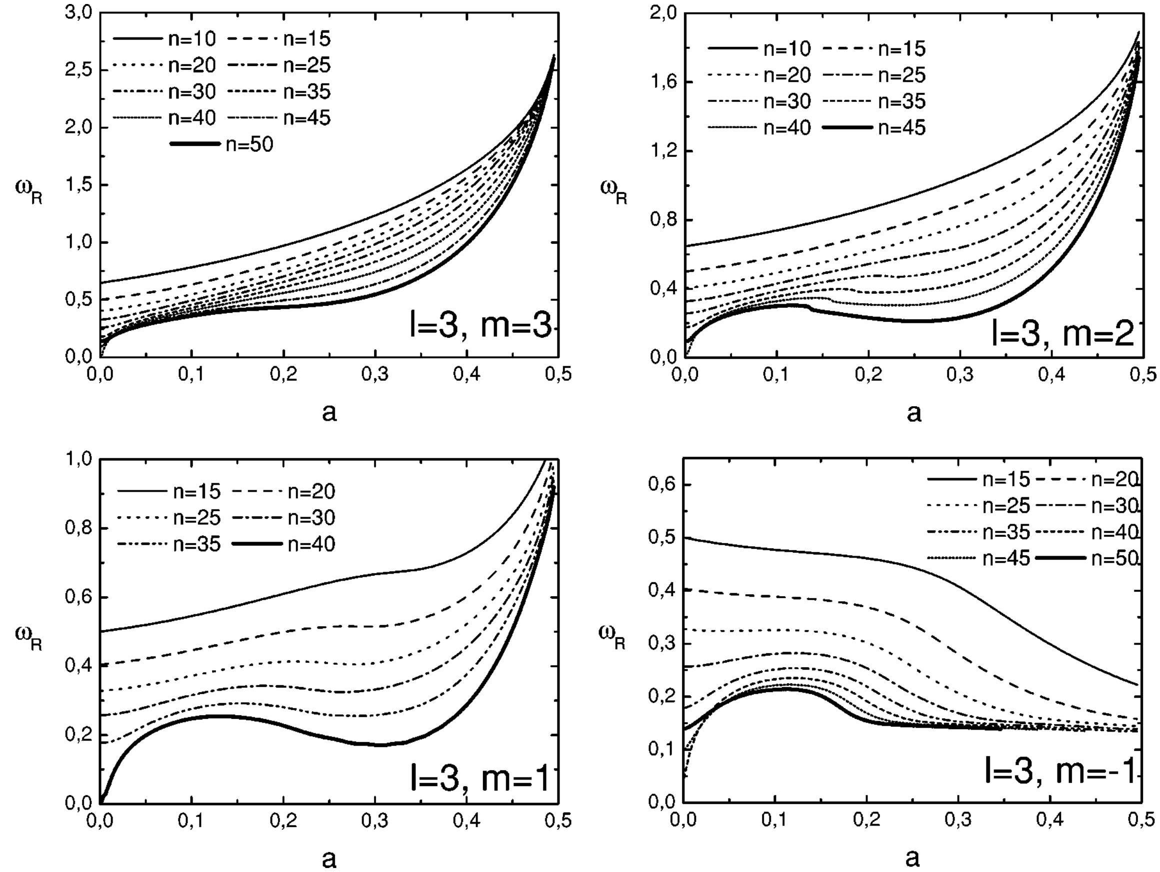

FIG. 5. Real parts of some modes with $l=3$ and different values of m (indicated in the plots). When $m>0$, the observed behavior is reminiscent of modes with $l=2$, $m=1$ (see Fig. 3). Modes with $m<0$ approach a (roughly) constant value $\omega_R=-m\varpi$ (we only show modes with $m=-1$), as they do for $l=2$ (see Fig. 4).

culation of modes with $l=3$. Some care is required in considering the results of this section as representative of the asymptotic behavior. In fact the pure imaginary Schwarzschild algebraically special mode (separating the lower QNM branch from the upper branch) is located at

$$\tilde{\Omega}_l=-i\,\frac{(l-1)l(l+1)(l+2)}{6}, \tag{17}$$

and can be taken as (roughly) marking the onset of the asymptotic regime. The algebraically special mode quickly moves downwards in the complex plane as l increases, and corresponds to an overtone index $n=41$ when $l=3$. Unfortunately we did not manage to push our numerical calculations for $l=3$ to values of n larger than about 50. Therefore we cannot be completely sure that our results are indicative of the ''true'' QNM asymptotics.

In any event, some prominent features emerge from the general behavior of the real part of the modes, as displayed in the different panels of Fig. 5. First of all, contrary to our expectations, neither the branch of modes with $m=3$ nor the branch with $m=2$ seem to approach the limit we would expect, $\omega_R=m\Omega$. These modes show a behavior which is more closely reminiscent of modes having $l=2$, $m=1$: the modes' real part ''bends'' towards the zero-frequency axis, shows a minimum as a function of a, and tends to $\omega_R=m$ as $a\to 1/2$. If the qualitative behavior of QNM's does not drastically change at larger overtone indices, we would be facing a puzzling situation. Indeed, gravitational modes with $l=m=2$ would have a rather unique asymptotic behavior, that would require more physical understanding to be motivated.

Another prominent feature is that, whenever $m>0$, there seems to be an infinity of modes approaching the limit $\omega_R=m$ as $a\to 1/2$. This behavior confirms the general trend we observed for $l=2$, $m>0$.

Finally, our calculations of modes with $m<0$ show, once again, that these modes tend to approach $\omega_R=-m\varpi$, where $\varpi\simeq 0.12$. We display, as an example, modes with $l=3$ and $m=-1$ in the bottom right panel of Fig. 5.

IV. SCALAR AND ELECTROMAGNETIC PERTURBATIONS

The calculations we have performed for $l=3$ hint at the possibility that modes with $l=m=2$ are the only ones approaching the limit $\omega_R=m\Omega$. However, for reasons explained in the previous paragraph, carrying out numerical calculations in the asymptotic regime when $l>2$ is very difficult.

This technical difficulty is a hindrance if we want to test the ''uniqueness'' of gravitational modes with $l=m=2$ by looking at gravitational modes having $l>2$. An alternative

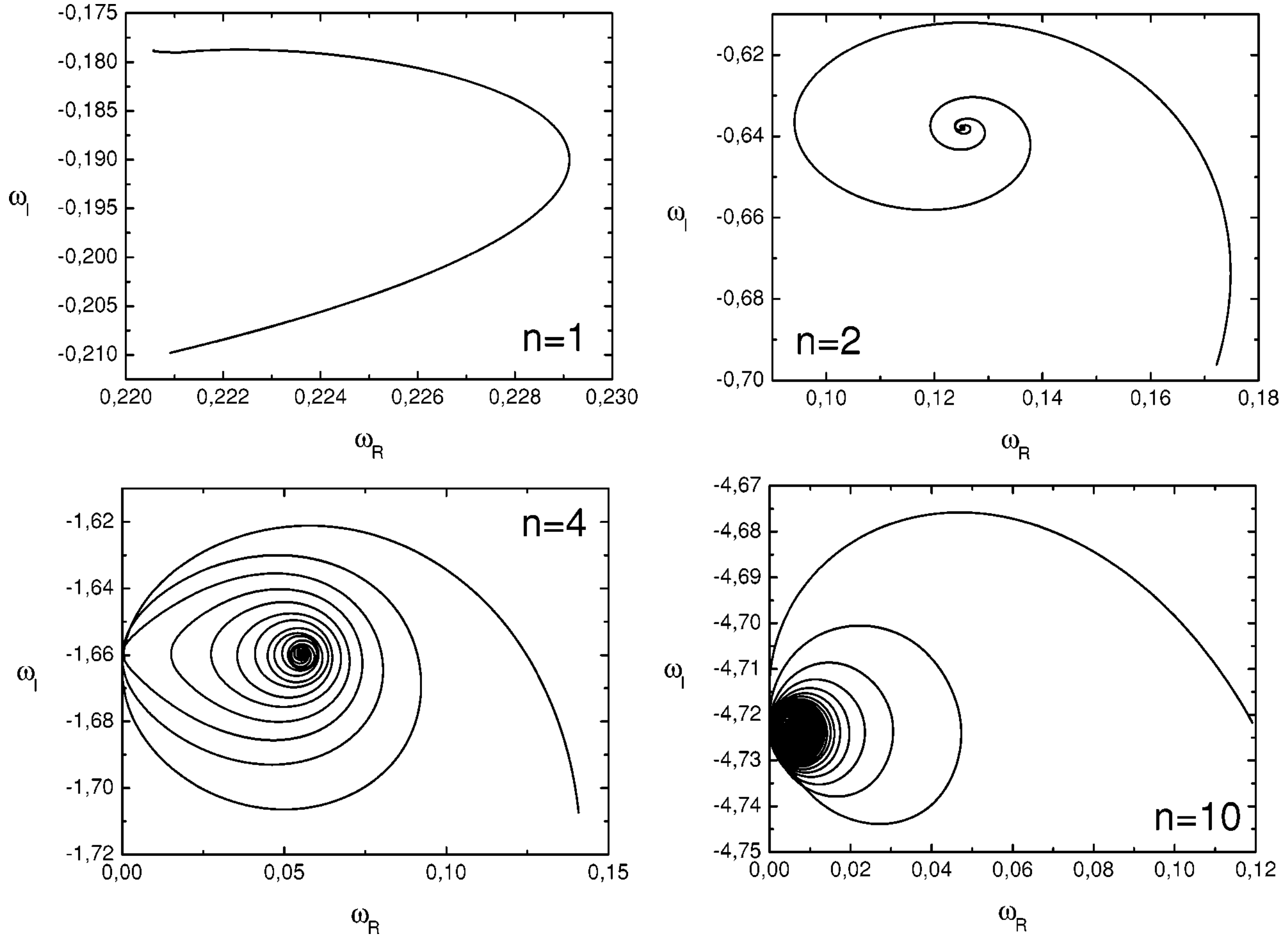

FIG. 6. Trajectories of a few scalar modes with $l=m=0$. The different panels correspond to the fundamental mode (top left), which does not show a spiraling behavior, and to modes with overtone indices $n=2$, 4, 10.

idea to check this "uniqueness" is to look instead at perturbations due to fields having *different spin* and $l\leqslant 2$. In particular, here we show some results we obtained extending our calculation to scalar $(s=0)$ and electromagnetic $(s=-1)$ modes. To our knowledge, results for Kerr scalar modes have only been published in [35]. Some highly damped electromagnetic modes were previously computed in [29].

A. Scalar modes

In Fig. 6 we show a few scalar modes with $l=m=0$. As we could expect from existing calculations [19,35] the modes show the typical spiraling behavior; the surprise here is that this spiraling behavior sets in very quickly, and is particularly pronounced even if we look at the first overtone $(n=2)$. As the mode order grows, the number of spirals grows, and the center of the spiral (corresponding to extremal Kerr holes) moves towards the pure imaginary axis (at least for $n\lesssim 10$).

In Fig. 7 we show the trajectories of some scalar modes for $l=2$. As can be seen in the top left panel, rotation removes the degeneracy of modes with different values of *m*. If we follow modes with $m=0$ we see the usual spiraling behavior, essentially confirming results obtained in [35] using the Prüfer method. However our numerical technique seems to be more accurate than the (approximate) Prüfer method, and we are able to follow the modes up to larger values of the rotation parameter: compare the bottom right panel in our Fig. 7 to Fig. 6 in [35], and remember that their numerical values must be multiplied by a factor 2 (due to the different choice of units). On the basis of our numerical results, it is quite likely that the asymptotic behavior of scalar modes with $l=m=0$ is described by a relation similar to Eq. (3). However, at present, no such relation has been derived analytically.

In Fig. 8 we show the real part of scalar modes with $l=m=1$ and $l=m=2$ as a function of *a*, for increasing values of the overtone index *n*. In both cases modes do not show a tendency to approach the $\omega_R=m\Omega$ limit suggested by gravitational modes with $l=m=2$. As we observed for modes with $l=3$ and $m>0$, their behavior is rather similar to that of gravitational modes with $l=2$ and $m=1$. This may be considered further evidence that gravitational perturbations with $l=m=2$ are, indeed, very special.

B. Electromagnetic modes

The calculation of highly damped electromagnetic QNM's basically confirms the picture we obtained from the computation of scalar QNM's presented in the previous section. We show some selected results in Fig. 9. The top left panel

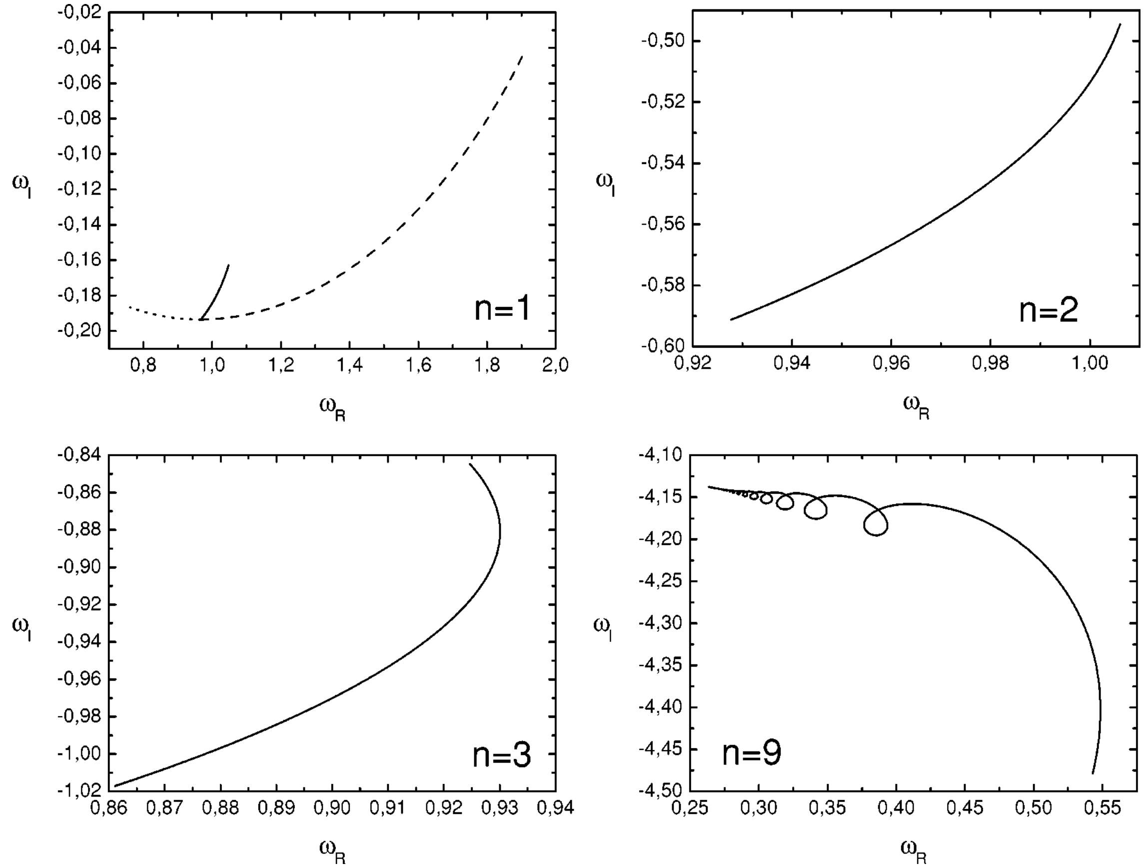

FIG. 7. Trajectories of a few scalar modes for $l=2$. In the top left panel we show how rotation removes the degeneracy of modes with different m's, displaying three branches (corresponding to $m=2,\ 0,\ -2$) "coming out of the Schwarzschild limit" for the fundamental mode ($n=1$). In the top right and bottom left panel we show the progressive "bending" of the trajectory of the $m=0$ branch for the first two overtones ($n=2,\ 3$). Finally, in the bottom right panel we show the typical spiraling behavior for a mode with $m=0$ and $n=9$. This plot can be compared to Fig. 6 in [35] (notice that their scales have to be multiplied by two to switch to our units). The continued fraction method allows us to compute modes for larger values of a (and is presumably more accurate) than the Prüfer method.

shows that, for large damping, the real part of electromagnetic QNM's with $l=1$ and $m>0$ shows a local minimum, approaching the limit $\omega_R=m$ as $a\to 1/2$. The top right panel shows that the real parts of modes with $l=1$ and $m=0$ quickly start oscillating (that is, QNM's display spirals in the complex-ω plane). Finally, the bottom plots show the behavior of modes with $l=1,\ m=-1$ (left) and $l=2,\ m=-2$ (right). Once again, if our calculations are indicative of the asymptotic behavior, modes seem to approach a roughly constant value $\omega \simeq -m\varpi$.

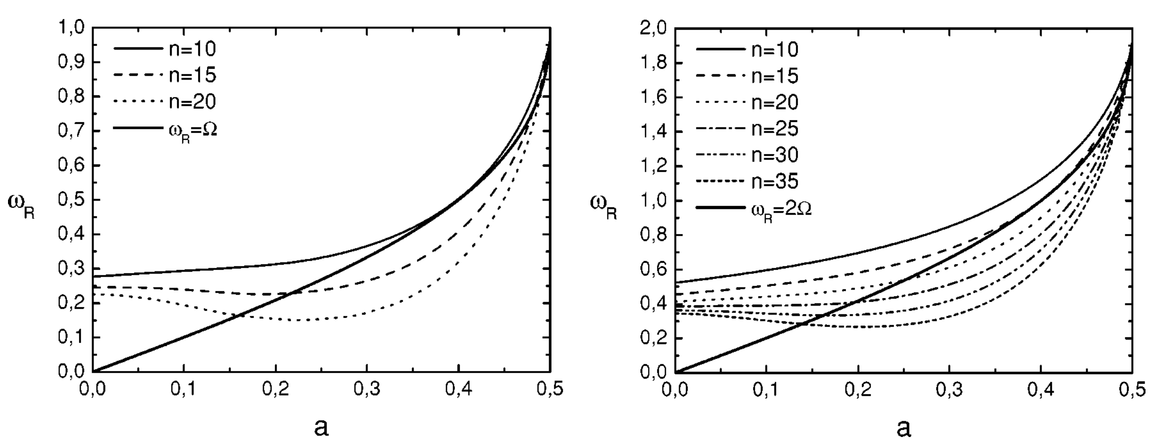

FIG. 8. Real parts of the scalar modes with $l=m=1$ (left) and $l=m=2$ (right). The observed behavior is reminiscent of Figs. 3 and 5.

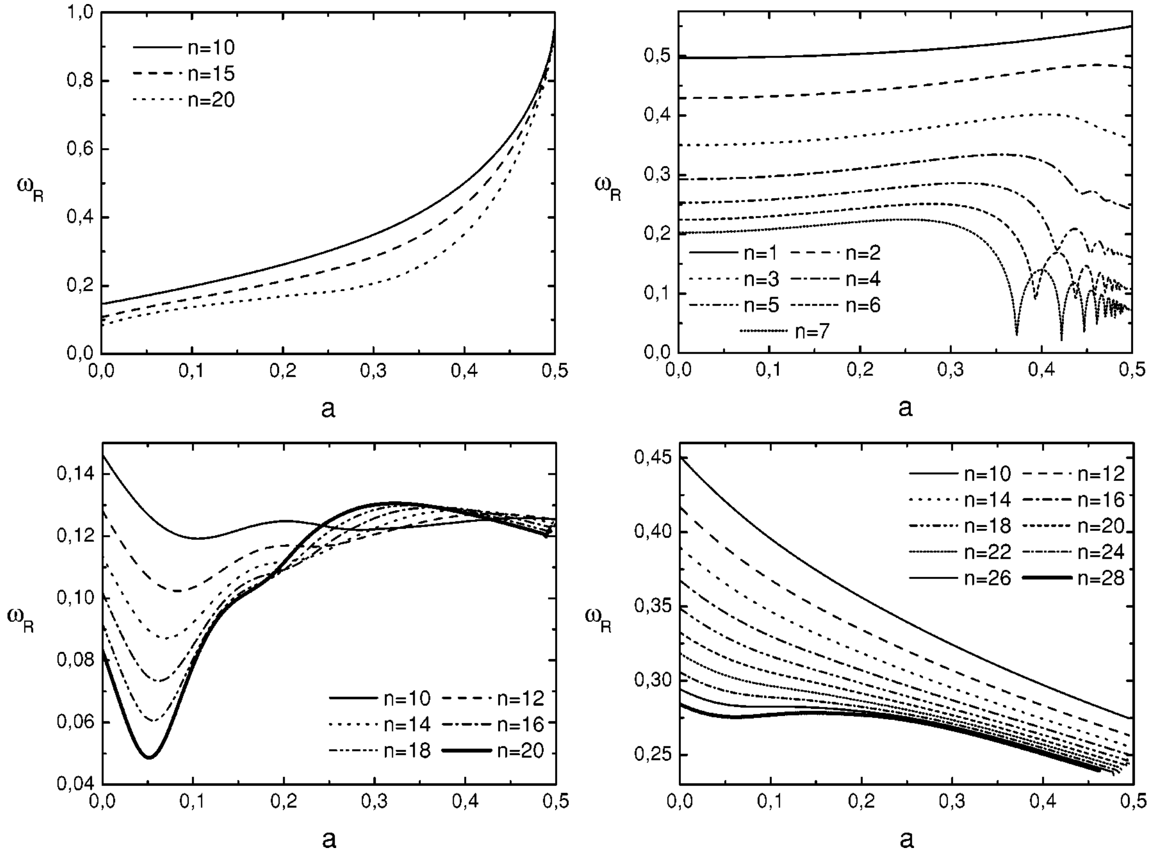

FIG. 9. Real part of electromagnetic modes with $l=m=1$ (top left), $l=1$, $m=0$ (top right), $l=1$, $m=-1$ (bottom left) and $l=2$, $m=-2$ (bottom right) as a function of the rotation parameter a, for increasing values of the mode index.

V. THE ASYMPTOTIC BEHAVIOR OF THE MODES' IMAGINARY PART

The evidence for a universal behavior emerging from the calculations we have presented is suggestive. For reasons we explained in the previous sections, in some instances we may not have reached the asymptotic regime when our numerical codes become unreliable. With this caution, we can still try and draw some conclusions. Our results suggest that, whatever the kind of perturbation (scalar, electromagnetic or gravitational) that we consider, asymptotic modes belong to one of three classes:

(1) Modes with $m>0$: their real part probably approaches the limit $\omega_R = m\Omega$ only for gravitational modes with $l=m$. Our calculation for $l=m=3$ cannot be considered as a trustworthy counterexample to this prediction, since it is not really representative of the asymptotic regime. For other kinds of perturbations (and for $m\neq l$) ω_R apparently shows a minimum as a function of a. This may be a real feature of asymptotic modes, but it may as well be due to the asymptotic behavior emerging only for larger values of n. To choose between the two alternatives we would either require better numerical methods or the development of analytical techniques. A "universal" feature is that, whatever the spin of the perturbing field, QNM frequencies approach the limiting value $\omega_R = m$ as $a\to 1/2$.

(2) Modes with $m=0$: these modes show a spiraling behavior in the complex plane, reminiscent of RN QNM's.

(3) Modes with $m<0$: their real part seems to asymptotically approach a constant (or weakly a-dependent) limit $\omega_R \simeq -m\varpi$, where $\varpi\simeq 0.12$, whatever the value of l and the spin of the perturbing field. Maybe this limit is not exactly independent of a, but on the basis of our numerical data we are quite confident that highly damped modes with $m<0$ tend to a universal limit $\omega_R \simeq -m\varpi_{ext}$, where ϖ_{ext} has some value between 0.11 and 0.12, as $a\to 1/2$.

Another interesting result concerns the modes' imaginary part. In [19] we observed that the following formula holds for gravitational modes with $l=m=2$:

$$\omega^{\rm Kerr}_{l=m=2} = 2\Omega + i2\pi T_H n. \tag{18}$$

Our numerical data show that, in general, all modes with $m>0$ have an asymptotic separation equal to $2\pi T_H$. This result holds *for all kinds of perturbations* (scalar, electromagnetic or gravitational) we considered, as long as $m>0$. For $m=0$ the imaginary part oscillates, and this beautiful, general result does not hold. It turns out that it does not hold as well for modes with $m<0$. So far the analysis of our numerical data did not lead us to any conclusion on the asymptotic separation of modes with $m<0$. This may hint at the fact that for $m<0$ our calculations are not yet indicative of the asymptotic regime. Therefore, some care is required in drawing conclusions on asymptotic modes from our results for $m<0$.

VI. ALGEBRAICALLY SPECIAL MODES

A. An introduction to the problem

Algebraically special modes of Schwarzschild black holes have been studied for a long time, but only recently their understanding has reached a satisfactory level. Among the early studies rank those of Wald [40] and of Chandrasekhar [41], who gave the exact solution of the Regge-Wheeler, Zerilli and Teukolsky equations at the algebraically special frequency. The nature of the QNM boundary conditions at the Schwarzschild algebraically special frequency is extremely subtle, and has been studied in detail by Maassen van den Brink [31]. Black hole oscillation modes belong to three categories:

(1) ''standard'' QNM's, which have outgoing wave boundary conditions at both sides (that is, they are outgoing at infinity and ''outgoing into the horizon,'' using Maassen van den Brink's ''observer-centered definition'' of the boundary conditions);

(2) total transmission modes from the left (TTM_L's) are modes incoming from the left (the black hole horizon) and outgoing to the other side (spatial infinity);

(3) total transmission modes from the right (TTM_R's) are modes incoming from the right and outgoing to the other side.

In our units, the Schwarzschild ''algebraically special'' frequency is given by formula (17), and has been traditionally associated with TTM's. However, when Chandrasekhar found the exact solution of the perturbation equations at the algebraically special frequency he did not check that these solutions satisfy TTM boundary conditions. In [31] it was shown that, in general, they do not. An important conclusion reached in [31] is that the Regge-Wheeler equation and the Zerilli equation (which are known to yield the same QNM spectrum, being related by a supersymmetry transformation) have to be treated on different footing at $\tilde{\Omega}_l$, since the supersymmetry transformation leading to the proof of isospectrality is singular there. In particular, the Regge-Wheeler equation has *no modes at all* at $\tilde{\Omega}_l$, while the Zerilli equation has *both a QNM and a* TTM_L.

Numerical calculations of algebraically special modes have yielded some puzzling results. Studying the Regge-Wheeler equation (that should have no QNM's at all according to Maassen van den Brink's analysis) and not the Zerilli equation, Leaver [33] found a QNM which is very close, but not exactly located *at*, the algebraically special frequency. Namely, he found QNM's at frequencies $\tilde{\Omega}_l'$ such that

$$\tilde{\Omega}_2' = 0.000000 - 3.998000i,$$

$$\tilde{\Omega}_3' = -0.000259 - 20.015653i. \tag{19}$$

Notice that the ''special'' QNM's $\tilde{\Omega}_l'$ are such that $\Re i\tilde{\Omega}_2' < |\tilde{\Omega}_2|$, $\Re i\tilde{\Omega}_3' > |\tilde{\Omega}_3|$, and that the real part of $\tilde{\Omega}_3'$ is not zero. Maassen van den Brink [31] speculated that the numerical calculation may be inaccurate and the last three digits may not be significant, so that no conclusion can be drawn on the coincidence of $\tilde{\Omega}_l$ and $\tilde{\Omega}_l'$, *''if the latter does exist at all.''*

An independent calculation was carried out by Andersson [42]. Using a phase-integral method, he found that the Regge-Wheeler equation has pure imaginary TTM_R's which are very close to $\tilde{\Omega}_l$ for $2 \leqslant l \leqslant 6$. He therefore suggested that the modes he found coincide with $\tilde{\Omega}_l$, which would then be a TTM. Maassen van den Brink [31] observed that, if all figures in the computed modes are significant, the coincidence of TTM's and QNM's is not confirmed by this calculation, since $\tilde{\Omega}_l'$ and $\tilde{\Omega}_l$ are numerically (slightly) different.

Onozawa [29] showed that the Kerr mode with overtone index $n=9$ tends to $\tilde{\Omega}_l$ as $a \to 0$, but suggested that modes approaching $\tilde{\Omega}_l$ from the left and the right may cancel each other at $a=0$, leaving only the special (TTM) mode. He also calculated this (TTM) special mode for Kerr black holes, solving the relevant condition that the Starobinsky constant should be zero and finding the angular separation constant by a continued fraction method; his results improved upon the accuracy of those previously obtained in [41].

The analytical approach adopted in [31] clarified many aspects of the problem for Schwarzschild black holes, but the situation concerning Kerr modes branching from the algebraically special Schwarzschild mode is still far from clear. In [31] Maassen van den Brink, using slow-rotation expansions of the perturbation equations, drew two basic conclusions on these modes. The first is that, for $a>0$, the so-called Kerr special modes (that is, solutions to the condition that the Starobinsky constant should be zero [29,41]) are all TTM's (left or right, depending on the sign of s). The TTM_R's cannot survive as $a \to 0$, since they do not exist in the Schwarzschild limit; this is related to the limit $a \to 0$ being a very tricky one. In particular, in this limit, the special Kerr mode becomes a TTM_L for $s=-2$; furthermore, the special mode and the TTM_R cancel each other for $s=2$. Studying the limit $a \to 0$ in detail, Maassen van den Brink reached a second important conclusion: the Schwarzschild special frequency $\tilde{\Omega}_l$ is a limit point for a multiplet of ''standard'' Kerr QNM's, which for small a are well approximated by

$$\omega = -4i - \frac{33078176}{700009} ma + \frac{3492608}{41177} ia^2 + \mathcal{O}(ma^2) + \mathcal{O}(a^4) \tag{20}$$

when $l=2$, and by a more complicated formula—his equation (7.33)—when $l>2$. None of the QNM's we numerically found seems to agree with the analytic prediction when the rotation rate a is small.

Maassen van den Brink suggested (see note [46] in [31]) that QNM's corresponding to the algebraically special frequency with $m>0$ may have one of the following three behaviors in the Schwarzschild limit: they may merge with those having $m<0$ at a frequency $\tilde{\Omega}_l'$ such that $|\tilde{\Omega}_l'| < |\tilde{\Omega}_l|$ (but $|\tilde{\Omega}_l'| > |\tilde{\Omega}_l|$ for $l \geqslant 4$) and disappear, as suggested by Onozawa [29]; they may terminate at some (finite) small a; or, finally, they may disappear towards $\omega = -i\infty$. Recently

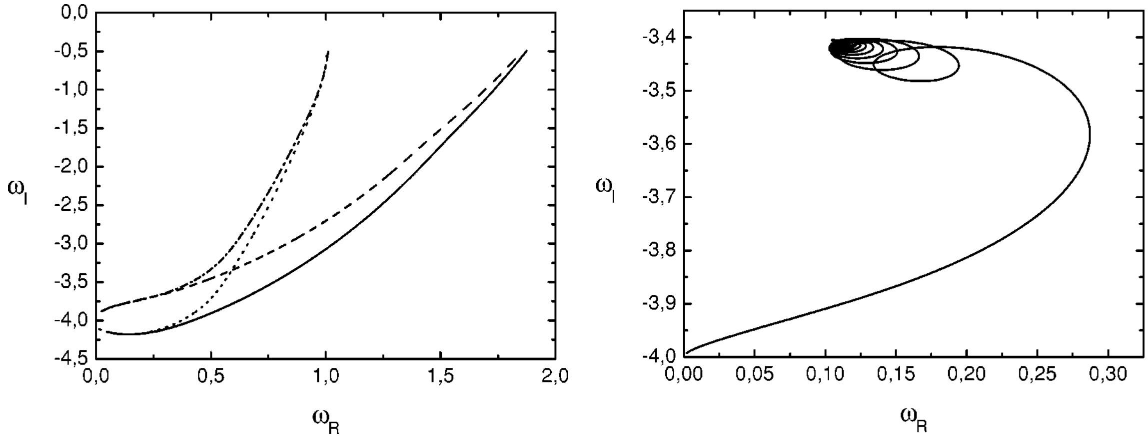

FIG. 10. The left panel shows the trajectories described in the complex-ω plane by the doublets emerging close to the Schwarzschild algebraically special frequency ($\tilde{\Omega}_2=-4i$) when $m>0$ and $l=2$. Notice that the real part of modes with $m>0$ tends to $\omega_R=m$ as $a\to 1/2$. The right panel shows the spiraling trajectory of the mode with $m=0$.

Maassen van den Brink *et al.* [30] put forward another alternative: studying the branch cut on the imaginary axis, they found that in the Schwarzschild case a pair of "unconventional damped modes" should exist. For $l=2$ these modes were identified by a fitting procedure to be located on the unphysical sheet lying behind the branch cut (hence the name "unconventional") at

$$\omega_{\pm}=\mp 0.027+(0.0033-4)i. \tag{21}$$

An approximate analytical calculation confirmed the presence of these modes, yielding

$$\omega_{+}\simeq -0.03248+(0.003436-4)i, \tag{22}$$

in reasonable agreement with Eq. (21). If their prediction is true, an *additional* QNM multiplet should emerge near $\tilde{\Omega}_l$ as a increases. This multiplet "may well be due to $\omega_{\pm}$ splitting (since spherical symmetry is broken) and moving through the negative imaginary axis as a is tuned" [30]. In the following paragraph we will show that a careful numerical search indeed reveals the emergence of such multiplets, but these do not seem to behave exactly as predicted in [30].

B. Numerical search and QNM multiplets

As we have summarized in the previous paragraph, the situation for Kerr modes branching from the algebraically special Schwarzschild mode is still unclear, and there are still many open questions. Is a multiplet of modes emerging from the algebraically special frequency when $a>0$? Can QNM's be matched by the analytical prediction (20) at small values of a? If a doublet does indeed appear, as recently suggested in [30], does it tend to the algebraically special frequency $\tilde{\Omega}_2=-4i$ as $a\to 0$, does it tend to the values predicted by formula (21), or does it go to some other limit?

After carrying out an extensive numerical search with both our numerical codes, we have found some surprises. Our main new result is shown in the left panel of Fig. 10. There we show the trajectories in the complex plane of QNM's with $l=2$ and $m>0$: a *doublet* of modes does indeed appear close to the algebraically special frequency. Both modes in the doublet tend to the usual limit ($\tilde{\Omega}_2=m$) as $a\to 1/2$. We have tried to match these "twin" modes with the predictions of the analytical formula (20). Unfortunately, none of the two branches we find seems to agree with Eq. (20) at small a. Our searches succeeded in finding a mode doublet only when $m>0$. For $m\leqslant 0$ the behavior of the modes is, in a way, more conventional. For example, in the right panel of Fig. 10 we see the $l=2$, $m=0$ mode emerging from the standard algebraically special frequency $\tilde{\Omega}_2$ and finally describing the "usual" spirals as a increases.

In the top left panel of Fig. 11 we see that the real part of all modes having $m\geqslant 0$ does indeed go to zero as $a\to 0$, with an m-dependent slope. However, the top right panel in the same figure shows that the imaginary part of the modes does *not* tend to -4 as $a\to 0$. Qualitatively this behavior agrees rather well with that predicted by Eq. (21). Extrapolating our numerical data to the limit $a\to 0$ yields, however, slightly different numerical values; our extrapolated values for $l=2$ are $\omega=(-4-0.10)i$ and $\omega=(-4+0.09)i$.

At present, we have no explanation for the appearance of the doublet only when $m>0$. A confirmation of this behavior comes from numerical searches we have carried out for $l=3$, close to the algebraically special frequency $\tilde{\Omega}_3$. Once again, a QNM multiplet only appears when $m>0$. In particular, we see the appearance of a doublet that behaves quite similarly to the modes shown in the left panel of Fig. 10. Extrapolating the numerical data for the $l=3$ doublet yields the values $\omega=(-20-0.19)i$ and $\omega=(-20+0.24)i$ as $a\to 0$.

A more careful search near the algebraically special frequency $\tilde{\Omega}_3$ surprisingly revealed the existence of other QNM's. However, the additional modes we find may well be "spurious" modes due to numerical inaccuracies, since we are pushing our method to its limits of validity (very high dampings and very small imaginary parts).

VII. CONCLUSIONS

In this paper we have numerically investigated the behavior of highly damped QNM's for Kerr black holes, using two

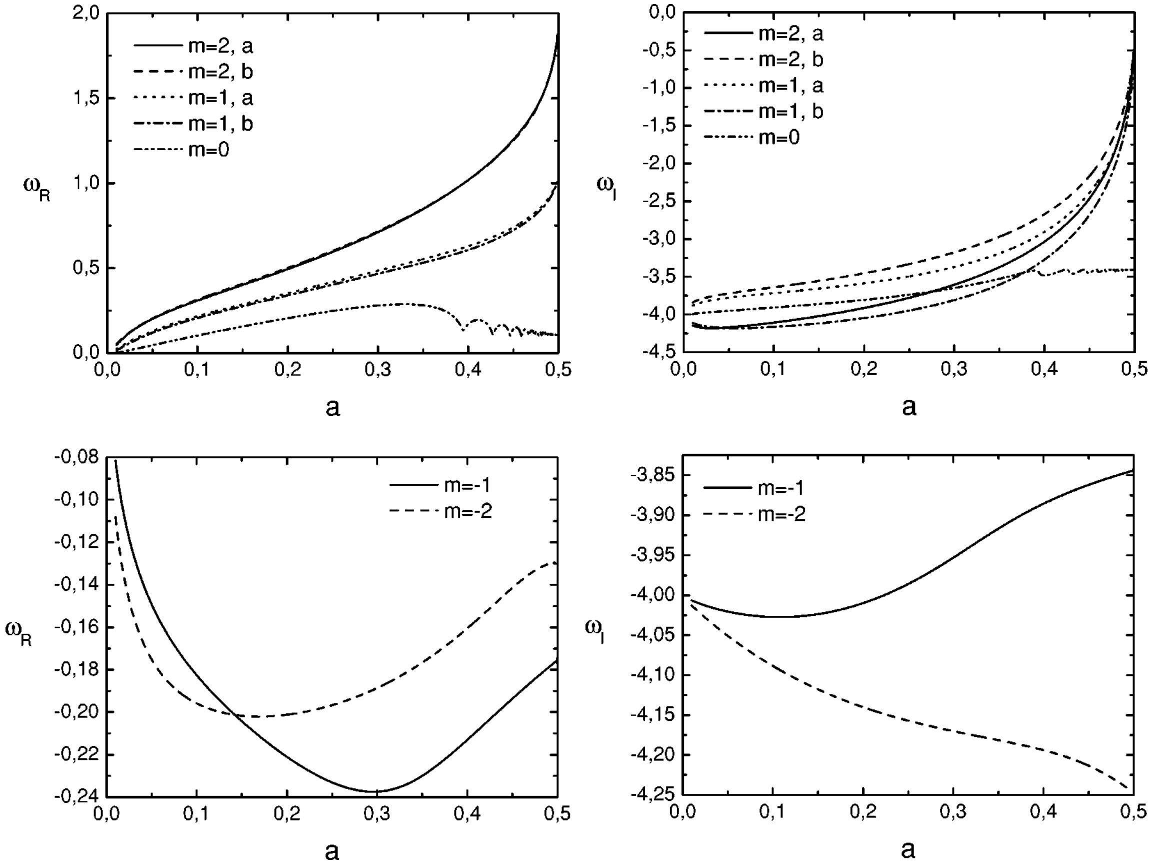

FIG. 11. The top row shows the real and imaginary parts (left and right, respectively) of the "doublet" of QNMs emerging from the algebraically special frequency as functions of a. The doublets only appear when $m>0$. We also overplot the real and imaginary parts of the mode with $l=2$, $m=0$ (showing the usual oscillatory behavior). The bottom row shows, for completeness, the real and imaginary parts (left and right, respectively) of modes with negative m and branching from the algebraically special frequency.

independent numerical codes to check the reliability of our results. Our findings do not agree with the simple behavior conjectured by Hod for the real part of the frequency [4,36] as given in Eq. (5). We did not limit our attention to gravitational modes, thus filling some gaps in the existing literature.

Our main results concerning highly damped modes can be summarized as follows. Scalar, electromagnetic and gravitational modes show a remarkable universality of behavior in the high damping limit. The asymptotic behavior crucially depends, for any kind of perturbation, on whether $m>0$, $m=0$ or $m<0$. As already observed in [19], the frequency of gravitational modes with $l=m=2$ tends to $\omega_R=2\Omega$, Ω being the angular velocity of the black hole horizon. We showed that, if Hod's conjecture is valid, this asymptotic behavior is related to *reversible black hole transformations*, that is, transformations for which the black hole irreducible mass (and its surface area) does not change.

Other (gravitational and nongravitational) modes with $m>0$ do *not* show a similar asymptotic behavior in the range of n allowed by our numerical method. In particular, in the high-damping limit, the real part of (gravitational and nongravitational) modes with $m>0$ typically shows a minimum as a function of the rotation parameter a, and then approaches the limit $\omega_R=m$ as the black hole becomes extremal. At present we cannot exclude the possibility that our calculations actually break down *before* we reach the asymptotic regime. Better numerical methods or analytical techniques are needed to give a final answer concerning the asymptotic behavior of modes with $m>0$.

Hod [43] recently used a continued-fraction argument modelled on that used in [11] and claimed that the asymptotic Kerr QNM frequency is given (for any m) by

$$\omega^{\mathrm{Kerr}}=m\Omega+i2\pi T_H n. \tag{23}$$

This result is obviously compatible with our calculations only for $m>0$, so there is some reason to be cautious about Hod's derivation. Essential in his argument is a comparison of the order of magnitude of the recursion coefficients α_n, β_n and γ_n defined in Eqs. (6), (7) and (8) of [43]. Looking at his formulas (14) and (15), it is apparent that the magnitudes of the α_n and γ_n recursion coefficients as $n\to\infty$ are of the same order. However, in [43] the α_n terms are treated as negligible with respect to the β_n and γ_n terms. Equation (23) comes from imposing $\gamma_n=0$ for $n>N$, where N is some (large) integer. However, if $\gamma_n=0$ for all $n>N$, then it is not legitimate to say that the γ_n-terms are much larger than the α_n terms. Neglecting the α_n terms is not correct: after imposing $\gamma_n=0$, the expansion coefficients d_n for large n are computed by comparing the α_n and β_n terms, not the β_n and γ_n terms. Furthermore, if Hod's argument were correct, it would allow the calculation of the real part of the frequency for Schwarzschild gravitational perturbations. However, applying his argument to the Schwarzschild case, Hod could only derive the asymptotic behavior of electromagnetic perturbations, for which the QNM frequency vanishes. Finally, a contradiction with our numerics would result if the asymptotic limit were reached for $n \gg (ab)^{-2}$, where $b \equiv (1-4a^2)^{1/2}$, as stated in [43]. Our numerical results show that this is only valid for $l=m=2$, otherwise we would clearly see convergence to the asymptotic behavior already for $n=30$–50 (at least for intermediate values of a and b).

Recently Musiri and Siopsis showed that Eq. (5) holds in an intermediate regime, when $|\omega|$ is large but $|\omega a| \lesssim 1$ [44]. Their result is compatible with our calculations, and (unfortunately) it does not provide a final answer on the asymptotic behavior. Concluding, despite these recent efforts, a more careful analytical analysis is needed before drawing any final conclusion on asymptotic Kerr QNM frequencies.

An interesting new finding of this paper is that for all values of $m>0$, and for any kind of perturbing field, there seems to be an infinity of modes tending to the critical frequency for superradiance, $\omega_R=m$, in the extremal limit. This finding generalizes a well-known analytical result by Detweiler for QNM's with $l=m$ [32,35]. It would be interesting to generalize Detweiler's proof, which only holds for $l=m$, to confirm our conjecture that for *any* $m>0$ there is an infinity of QNM's tending to $\omega_R=m$ as $a\to 1/2$.

The real part of modes with $l=2$ and *negative m* asymptotically approaches a value $\omega_R \simeq -m\varpi$, $\varpi \simeq 0.12$ being (almost) independent of a. Maybe this limit is not exactly independent of a, but on the basis of our numerical data we feel confident that highly damped modes with $m<0$ do tend to a universal limit $\omega_R \simeq -m\varpi_{ext}$ (where ϖ_{ext} has some value between 0.11 and 0.12) as $a\to 1/2$. This is an interesting prediction, and it would again be extremely useful to confirm it using analytic techniques. So far we have not been able to find any simple physical explanation for this limiting value. For example, we have tentatively explored a possible connection between ϖ and the frequencies of marginally stable counterrotating photon orbits, but we could not find any obvious correlation between the two.

Both for gravitational and for nongravitational perturbations, the trajectories in the complex plane of modes with $m=0$ show a spiraling behavior, strongly reminiscent of the one observed for Reissner-Nordström (RN) black holes, and probably well approximated in the high damping limit by an equation similar to Eq. (3).

Last but not least, an important result concerning highly damped modes is that, for any perturbing field, the asymptotic separation in the imaginary part of consecutive modes with $m>0$ is given by $2\pi T_H$ (T_H being the black hole temperature). An heuristic explanation for this fact was put forward for the Schwarzschild case in [12]. The idea is as follows. Since QNM's determine the position of the poles of a Green's function on the black hole background, and the Euclidean black hole solution converges to a thermal circle at infinity having temperature T_H, it may not be too surprising that the spacing in asymptotic QNM's coincides with the spacing $2\pi i T_H$ expected for a thermal Green's function. However, this simple relation concerning the mode spacing does not seem to hold when $m\leq 0$. Analytic derivations for the spacing in the QNM imaginary parts have been provided in [45] and [46]. These calculations use the fact that QNM's are poles in the scattering amplitude of the relevant wave equation. They are based on the Born approximation, and they only apply to static spacetimes. A generalization to stationary spacetimes, if possible, might provide an analytical confirmation of our numerical result.

Finally, we studied in some detail modes branching from the so-called ''algebraically special frequency'' of Schwarzschild black holes. We found numerically for the first time that QNM *multiplets* emerge from the algebraically special modes as the black hole rotation increases, confirming a recent speculation [30]. However, we found some quantitative disagreement with the analytical predictions in [30,31]. The problem deserves further investigation.

Hopefully our numerical results will serve as a guide in the analytical search for asymptotic QNM's of Kerr black holes. Although one can in principle apply Motl and Neitzke's [12] method in the present case, the Kerr geometry has some special features that complicate the analysis. The Teukolsky equation describing the field's evolution no longer has the Regge-Wheeler-Zerilli (Schrödinger-like) form; however, it can be reduced to that form by a suitable transformation of the radial coordinate. The main technical difficulty concerns the fact that the angular separation constant A_{lm} is not given analytically in terms of l, as it is in the Schwarzschild or RN geometry; even worse, it depends on the frequency ω in a nonlinear way. Therefore, an analytical understanding of the problem must also encompass an understanding of the asymptotic properties of the separation constant. The scalar case is well studied, both analytically and numerically [47], but a similar investigation for the electromagnetic and gravitational perturbations is still lacking. An idea we plan to exploit in the future is to use a numerical analysis of the angular equation as a guideline to find the asymptotic behavior of A_{lm}. Once the asymptotic behavior of A_{lm} is determined, the analysis of the radial equation may proceed along the lines traced in [12].

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