Action principle formulation for the motion of extended bodies in general relativity

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We present an action principle formulation for the study of the motion of an extended body in general relativity in the limit of a weak gravitational field. This gives the classical equations of motion for multipole moments of arbitrary order coupling to the gravitational field. In particular, a new force due to the octupole moment is obtained. The action also yields the gravitationally induced phase shifts in quantum interference experiments due to the coupling of all multipole moments.

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The study of the motion of extended bodies possessing multipole moments in a gravitational field has a long history [1]. The starting point was the Einstein-Infeld-Hoffman [2] derivation of the geodesic equation for a point test particle from the gravitational field equation and the conservation law for a stress energy tensor. The test particle approximation breaks down if the body's extension in space is non-negligible compared to the radius of curvature of the background field and also when the back reaction due to the body on the background field cannot be ignored. In this article, we are concerned with the former aspect. This is particularly motivated by the fact that astrophysical bodies such as planets and stars are extended and should in a realistic analysis be treated as such. The interaction of covariant generalization of Newtonian multipole moments with the gravitational field will be given by their coupling to Riemann curvature and its derivatives. This would appear as a modification to the geodesic equation.

The modification to the geodesic for a spinning body is given by the Mathisson–Papapetrou equation [3,4], which may be extended to a particle with intrinsic spin [5]. Subsequent to the treatment of spinning bodies, various authors have obtained the corrections up to the covariant generalization of Newtonian quadrupole moment [6–8]. A comprehensive study of the problem including comparison of various approaches and results is carried out by Dixon [8], but, to our knowledge, no one has obtained corrections to the geodesic equation arising due to the coupling of covariantized higher-order Newtonian multipole moments with gravitational field. More importantly a procedure to derive equations of motion of extended bodies, with arbitrary multipole moments, through an action principle has not been obtained during the past 65 years in which the equations of motion in a gravitational field have been studied [9]. In the absence of a general principle to obtain the action made up of terms uniquely attributed to couplings of all multipole moments, such a task is very difficult. This is precisely what we wish to do in the following.

While the action is used in classical physics only as a tool to obtain the equations of motion, the action is directly observable in quantum physics as the phase of the wave function. Therefore, the phase shift produced by the coupling of multipole moments with the gravitational field can in principle be measured (the action giving an algorithm to calculate various multipole phase shifts). Neutron interferometry provided the first instance where the effect of the earth's gravitational field on the phase of the neutron wave function was observed [10]. Interesting gravitational analogs [11] of the topological Aharanov-Bohm [12] and Aharanov-Casher [13,14] phase shifts have also been proposed.

In this paper we present a formalism which yields in a simple and elegant way the corrections to the geodesic equation up to all multipoles of the extended body. The equations of motion for multipoles simply follow from variation of our action. As a demonstration of our formalism we obtain for the first time corrections to the geodesic equation up to the covariant generalization of Newtonian octupole moment. Moreover, our action gives the quantum phase shifts in interferometry due to the coupling of all multiple moments to the gravitational field. Our formulation of action principle for an extended body may facilitate and prompt further investigations in the important and emerging area of interface between quantum and gravitational realms [15]. Particularly, the ongoing experiments in atomic [16], molecular [17], and Bose–Einstein condensate interferometry [18] hold promise for experimentally testing the new gravitational phase shifts that will be obtained in the present article.

We envision an extended rigid body as a thin world tube in space-time and its thickness is small compared to the scale over which curvature varies. We would further assume that there are no external or internal forces acting on this body apart from the gravitational field in which it is propagating. In the thin world tube we choose a reference world line (z^μ) having the 4-velocity $u^\mu = dz^\mu/ds = (1,0,0,0)$ and define multipole moments with respect to it on a spacelike hypersurface. The multipole moments of order $2n$ are defined as [8]

$$t^{\kappa_1 \cdots \kappa_n \mu\nu} = \int \delta x^{\kappa_1} \ldots \delta x^{\kappa_n} \sqrt{-g}\, T^{\mu\nu} w^\alpha d\Sigma_\alpha , \qquad (1)$$

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where $\delta x^\mu = x^\mu - z^\mu$, $T^{\mu\nu}$ is the energy-momentum tensor, and the integration is over the spacelike hypersurface identified by the unit normal vector field w^α. The above multipole moments are defined in a class of coordinate systems that are related by linear transformations in order that the expression (1) is covariant. But once they are defined this way, $t^{\kappa_1 \cdots \kappa_n \mu\nu}$ can be transformed to any arbitrary coordinate system as a tensor. All the relativistic equations in the present letter are covariant with respect to the above linear transformations, if not with respect to general coordinate transformations.

Since what we do must be consistent with the Newtonian theory in the appropriate limit, we shall now establish the relation between the covariant multipole moments of order $2n$ and the covariant generalizations of the antisymmetric spin tensor S^{ij}, the symmetric quadrupole moment tensor I^{ij}, and the symmetric octupole moment tensor O^{ijk} in Newtonian gravity. We write the Newtonian potential energy U of the body with mass density $\rho(x)$ in terms of the potential $\phi(x)$ expanded in a Taylor series around the central world line,

$$U = \int \rho(x)\phi(x)d^3x = m\phi(z) + d^i\partial_i\phi|_z + \frac{1}{2}I^{ij}\partial_i\partial_j\phi|_z + \frac{1}{6}O^{ijk}\partial_i\partial_j\partial_k\phi|_z + \cdots, \tag{2}$$

where the mass $m = \int \rho(x)d^3x$, the dipole moment $d^i = \int \rho(x)\delta x^i d^3x$, the quadrupole moment $I^{ij} = \int \rho(x)\delta x^i \delta x^j d^3x$, and the octupole moment $O^{ijk} = \int \rho(x)\delta x^i \delta x^j \delta x^k d^3x$, with $\delta x^i = x^i - z^i$. In view of Eq. (1), we identify $t^{i00} = d^i$, $t^{ij00} = I^{ij}$, $t^{ijk00} = O^{ijk}$. The spin tensor (orbital angular momentum) in the Newtonian limit is defined as $S^{ij} = 2\int \rho \delta x^{[i}v^{j]}d^3x$, where $v^i = dx^i/dt$. The spin tensor then satisfies $dS^{ij}/dt = 2p^{[i}u^{j]} - 2\int \rho \delta x^{[i}\partial^{j]}\phi d^3x$, where $u^i = dz^i/dt$ and the momentum $p^i = \int \rho(x)v^i d^3x$. Using a Taylor expansion of the potential, and choosing z^i to be the center of mass so that $d^i = 0$, the spin propagation equation up to the octupole term becomes

$$\frac{d}{dt}S^{ij} = 2p^{[i}u^{j]} - 2I^{k[i}\partial^{j]}\partial_k\phi|_z - O^{kr[i}\partial^{j]}\partial_k\partial_r\phi|_z . \tag{3}$$

Converting this spin tensor into a covariant leads to

$$S^{\mu\nu} = t^{\mu\nu 0} - t^{\nu\mu 0}. \tag{4}$$

In the weak field limit, the metric is $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $h_{\mu\nu} \ll 1$. In the Newtonian limit $g_{00} = 1 + 2\phi$, which implies $\phi = h_{00}/2$. Thus the dipole term in Eq. (2) leads to the covariant form $t^{\mu\alpha\beta}h_{\alpha\beta,\mu}|_z$, where $t^{\mu\alpha\beta}$ includes the spin tensor. We shall now choose the reference world line to be the center of mass so that $d^i = 0$. We consider only matter distributions for which [19]

$$t^{\mu\nu\alpha} = S^{\mu(\nu}u^{\alpha)}, \tag{5}$$

where $S^{\mu\nu}$ satisfies $S^{\mu\nu}u_\nu = 0$.

The quadrupole term leads to $\partial_i\partial_j\phi|_z I^{ij} = \frac{1}{2}\partial_i\partial_j h_{00}|_z I^{ij} = -R_{0i0j}|_z I^{ij}$. We thus have $\partial_i\partial_j h_{00}|_z t^{ij00} = -2R_{0i0j}|_z I^{ij}$, which through conversion into a covariant leads to

$$h_{\alpha\beta,\mu\nu}|_z t^{\mu\nu\alpha\beta} = -2R_{\alpha\mu\beta\nu}|_z I^{\mu\nu}u^\alpha u^\beta, \tag{6}$$

where the covariant quadrupole tensor $I^{\mu\nu} = I^{\nu\mu}$. Similarly, the octupole term leads to

$$h_{\alpha\beta,\mu\nu\sigma}|_z t^{\mu\nu\sigma\alpha\beta} = -2R_{\alpha\mu\beta\nu,\sigma}|_z O^{\mu\nu\sigma}u^\alpha u^\beta, \tag{7}$$

where $O^{\mu\nu\sigma}$ is the fully symmetric covariant octupole tensor. Equations (5)–(7) are the key relations which would unambiguously provide the connection between covariant multipole moments with their Newtonian analogs.

In the Newtonian approximation the phase shift in quantum mechanical interference due to the gravitational field is

$$\begin{aligned}\Phi &= -\frac{1}{\hbar}\int U dt \\ &= -\frac{1}{\hbar}\bigg(\int m\phi dt + \int d^i\partial_i\phi dt + \int \frac{1}{2}I^{ij}\partial_i\partial_j\phi dt \\ &\quad + \int \frac{1}{6}O^{ijk}\partial_i\partial_j\partial_k\phi dt + \cdots\bigg),\end{aligned} \tag{8}$$

where U is given by the expansion in using Eq. (2). The first term of Eq. (8) corresponds to the phase shift observed in the Collela–Overhauser–Werner (COW) experiment [10], and the subsequent dipole and quadrupole terms are corrections to it. The higher-order multipole contributions to the phase shift may also be obtained from this expansion. In general relativity this phase shift is obtained by letting the path ordered operator resulting from the covariant generalization of this Newtonian phase shift act on the initial wave function. Using Eqs. (2) and (5)–(7) and noting that in the linear field limit $\omega^a{}_{b\beta}S^b{}_a = h_{\alpha\beta,\mu}S^{\alpha\mu}$, where $\omega^a{}_{b\beta}$ are Ricci rotation coefficients, this path ordered operator is given by

$$\begin{aligned}g = \mathcal{P}\exp\bigg[&-\frac{i}{\hbar}\int\bigg(-m + \frac{1}{2}\omega^a{}_{b\beta}S_a{}^b u^\beta - \frac{1}{2}R_{\alpha\mu\beta\nu}I^{\alpha\beta}u^\mu u^\nu \\ &- \frac{1}{6}\int R_{\alpha\mu\beta\nu;\rho}O^{\alpha\beta\rho}u^\mu u^\nu + \cdots\bigg)ds\bigg].\end{aligned} \tag{9}$$

In the special case where $I^{\alpha\beta}$ and $O^{\alpha\beta\rho}$ are zero, this result is in agreement with the gravitational phase shift for intrinsic spin [5].

We shall now obtain this expression in the weak field limit of general relativity starting from the action principle. Choose a coordinate system such that along the reference world line $g_{\mu\nu} = \eta_{\mu\nu}$. We would here like to recall that the extended body under consideration is rigid and subject to only external gravitational force and no other forces (external or internal). In that case the monopole moment corresponds to the mass (m) of the body with whole matter concentrated on the reference world line, and ignoring the back reaction on the background gravitational field we write $m = p_\alpha u^\alpha$. The higher-order multipole moments are defined by considering the metric perturbations as one moves away from the reference world line z^μ. For simplicity we restrict perturbations of the metric to the first order [20], and later covariantize the equation of motion by changing ordinary derivative to covariant derivative. The action

$$\mathcal{S}=\int \sqrt{-g}\mathcal{L}d^4x=\int \sqrt{-g}\mathcal{L}|_{g_{\mu\nu}=\eta_{\mu\nu}}d^4x+\left(\int \sqrt{-g}\mathcal{L}d^4x - \int \sqrt{-g}\mathcal{L}|_{g_{\mu\nu}=\eta_{\mu\nu}}d^4x\right). \tag{10}$$

The first term is the kinetic energy term,

$$\int \sqrt{-g}\mathcal{L}|_{g_{\mu\nu}=\eta_{\mu\nu}}d^4x=-\int p_\alpha u^\alpha ds.$$

So,

$$\mathcal{S}=-\int p_\alpha u^\alpha ds+\int \delta g_{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}}(\sqrt{-g}\mathcal{L})d^4x+\cdots$$
$$=-\int p_\alpha u^\alpha ds+\frac{1}{2}\int \delta g_{\mu\nu}\sqrt{-g}T^{\mu\nu}d^4x+\cdots. \tag{11}$$

Note $\delta g_{\mu\nu}=g_{\mu\nu}-\eta_{\mu\nu}=h_{\mu\nu}$ and hence it can be written as $\delta g_{\mu\nu}=h_{\mu\nu,\sigma}|_z\delta x^\sigma+\frac{1}{2}h_{\mu\nu,\sigma\rho}|_z\delta x^\sigma\delta x^\rho+\cdots$ where we have $h_{\mu\nu}(z)=0$. In the present weak field limit, we neglect all terms that are quadratic or higher order in metric perturbations. Substituting for $\delta g_{\mu\nu}$ in Eq. (11) and using Eqs. (1) and (5)–(7), we finally obtain

$$\mathcal{S}=-\int p_\alpha u^\alpha ds+\frac{1}{2}\int h_{\alpha\beta,\mu}S^{\mu\alpha}u^\beta ds - \frac{1}{2}\int R_{\alpha\mu\beta\nu}I^{\alpha\beta}u^\mu u^\nu ds-\frac{1}{6}\int R_{\alpha\mu\beta\nu,\rho}O^{\alpha\beta\rho}u^\mu u^\nu ds \tag{12}$$

up to octupole (all derivatives are evaluated on z^μ). In the linear field limit, since $h_{\alpha\beta,\mu}S^{\mu\alpha}=\omega^a{}_{b\beta}S_a{}^b$, hence the accumulation of infinitesimal phases arising from Eq. (12) is the same as that obtained from the path ordered operator, Eq. (9), in this limit.

We now obtain the equation of motion by extremizing the action ($\delta S=0$) and requiring that coordinate variations vanish at end points of the path. This leads to the equation of motion

$$\frac{dp_\sigma}{ds}=R_{\sigma\lambda\mu\nu}\left(\frac{1}{2}u^\lambda S^{\mu\nu}+\frac{d}{ds}(I^{\lambda\mu}u^\nu)\right)+\frac{1}{2}h_{\nu\sigma,\mu}\frac{d}{ds}S^{\mu\nu} - R_{\sigma\lambda\mu\nu,\rho}u^\mu I^{\nu(\rho}u^{\lambda)}+\frac{1}{3}R_{\alpha\mu\beta\sigma,\nu\rho}u^\alpha O^{\mu\rho(\nu}u^{\beta)} - \frac{1}{3}R_{\alpha\mu\sigma\nu,\rho}\frac{d}{ds}(O^{\mu\nu\rho}u^\alpha). \tag{13}$$

In a coordinate system in which $h_{\alpha\beta,\mu}=0$, Eq. (13) gives

$$\frac{Dp_\sigma}{Ds}=R_{\sigma\lambda\mu\nu}\left(\frac{1}{2}u^\lambda S^{\mu\nu}+\frac{D}{Ds}(I^{\lambda\mu}u^\nu)\right)-u^\mu I^{\nu(\rho}u^{\lambda)}\nabla_\rho R_{\sigma\lambda\mu\nu} + \frac{1}{3}\nabla_\nu\nabla_\rho R_{\alpha\mu\beta\sigma}u^\alpha O^{\mu\rho(\nu}u^{\beta)} - \frac{1}{3}\nabla_\rho R_{\alpha\mu\sigma\nu}\frac{D}{Ds}(O^{\mu\nu\rho}u^\alpha). \tag{14}$$

Equation (14) is generally covariant and thus is valid in every coordinate system. Since the dipole moment couples to Riemann curvature, it is expected that the quadrupole should couple to its first derivative and the octupole to its second derivative. However, there is a coupling of the quadrupole with Riemann curvature in the second term and a coupling of the octupole with the first derivative of Riemann curvature in the last term, which suggests that p_α should be suitably redefined [21], as $p^*_\sigma=p_\sigma-R_{\sigma\lambda\mu\nu}I^{\lambda\mu}u^\nu-\frac{1}{3}\nabla_\rho R_{\sigma\mu\nu\alpha}O^{\mu\nu\rho}u^\alpha$, which would then yield the expected form

$$\frac{Dp^*_\sigma}{Ds}=\frac{1}{2}R_{\sigma\lambda\mu\nu}u^\lambda S^{\mu\nu}+\frac{1}{2}\nabla_\sigma R_{\alpha\mu\beta\nu}u^{[\mu}I^{\alpha][\beta}u^{\nu]} + \frac{1}{6}\nabla_\rho\nabla_\sigma R_{\alpha\mu\beta\nu}u^{[\alpha}O^{\mu]\rho[\nu}u^{\beta]}. \tag{15}$$

This is the equation of motion for a body possessing dipole, quadrupole, and octupole moments in a gravitational field. The spin propagation equation (3) can also be covariantly generalized to

$$\frac{D}{Ds}S^{\alpha\beta}=2p^{[\alpha}u^{\beta]}-2R^{[\alpha}{}_{\mu\nu\sigma}u^{[\mu}I^{\beta]][\nu}u^{\sigma]} - R^{[\alpha}{}_{\mu\nu\sigma;\rho}u^{[\mu}O^{\beta]]\rho[\nu}u^{\sigma]}, \tag{16}$$

which on the redefinition of momentum vector form p_α to p^*_α modifies as follows:

$$\frac{D}{Ds}S^{\alpha\beta}=2p^{*[\alpha}u^{\beta]}+2(R^\alpha{}_{\mu\nu\sigma}u^{[\beta}I^{\mu][\nu}u^{\sigma]} - R^\beta{}_{\mu\nu\sigma}u^{[\alpha}I^{\mu][\nu}u^{\sigma]})-2R^{[\alpha}{}_{\mu\nu\sigma}u^{[\mu}I^{\beta]][\nu}u^{\sigma]} - \frac{5}{3}R^{[\alpha}{}_{\mu\nu\sigma;\rho}u^{[\mu}O^{\beta]]\rho[\nu}u^{\sigma]}. \tag{17}$$

To simplify the notation we would further define $J^{\mu\alpha\beta\nu}:=-3u^{[\mu}I^{\alpha][\beta}u^{\nu]}$ and $G^{\mu\alpha\beta\sigma\nu}:=-u^{[\mu}O^{\alpha]\beta[\sigma}u^{\nu]}$ and then finally obtain

$$\frac{Dp^*_\sigma}{Ds}=\frac{1}{2}R_{\sigma\lambda\mu\nu}u^\lambda S^{\mu\nu}+\frac{1}{6}J^{\mu\alpha\beta\nu}\nabla_\sigma R_{\mu\alpha\beta\nu} + \frac{1}{6}\nabla_\rho\nabla_\sigma R_{\alpha\mu\beta\nu}G^{\alpha\mu\rho\nu\beta}, \tag{18}$$

$$\frac{D}{Ds}S^{\alpha\beta}=2p^{*[\alpha}u^{\beta]}-\frac{4}{3}R^{[\alpha}{}_{\mu\nu\sigma}J^{\beta]\mu\nu\sigma}-\frac{5}{3}R^{[\alpha}{}_{\mu\nu\sigma;\rho}G^{\beta]\mu\rho\nu\sigma}. \tag{19}$$

Thus we have obtained for the first time the correction to propagation equations upto the coupling of covariant generalization of Newtonian octupole moment with the background gravitational field. These equations agree with the earlier results obtained up to quadrupole [8] but the force due to the octupole moment is a new result. Our procedure can be easily generalized to obtain further corrections from due to all higher multipole moments.

This is a very simple and elegant method of deriving the equation of motion for an extended body incorporating the coupling of multipole moments of arbitrary order with the gravitational field. And above all we have found the action for such a body which we have also used to compute the gravitational phase shifts in its quantum wave function due to the multipoles. We hope that the new gravitationally induced phase shifts would be measured in future interferometry experiments based on atomic, molecular, and Bose-Einstein condensates [16–18]. Our novel algorithm also yields modifications to the geodesic equation, which, with present day high precision astronomical observations yielding multipole moments of planetary or stellar bodies, can be applied to obtain aberrations in their orbits. It would be interesting to relax the other aspect of test body character, i.e., no longer ignore the back reaction, and then study the motion in full generality.

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[20] If the size of the body is L and the radius of curvature of the background space-time is R, restricting to the first order would imply that we can go up to octupole. This is because the metric perturbation varies as L^2/R^2 while the expansion in multipole coupling varies as L/R. That means octupole would vary as L^3/R^3, which would be fine. However, if one wishes to go to higher multipole expansion, one would have to include the second-order term in the metric perturbation.

[21] To compare his results with earlier work dealing up to the quadrupole correction, Dixon defined $p^*_\sigma = p_\sigma - R_{\sigma\lambda\mu\nu} I^{\lambda\mu} u^\nu$.