Generalized Friedmann branes

László Á. Gergely*
Institute of Cosmology and Gravitation, University of Portsmouth, Portsmouth PO1 2EG, United Kingdom and Astronomical Observatory and Department of Experimental Physics, University of Szeged, Szeged 6720, Dóm tér 9, Hungary
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We prove that for a large class of generalized Randall-Sundrum type II models the characterization of the brane-gravity sector by the effective Einstein equation, Codazzi equation and the twice-contracted Gauss equation is equivalent to the bulk Einstein equation. We give the complete set of equations in the generic case of non-Z_2-symmetric bulk and arbitrary energy-momentum tensors both on the brane and in the bulk. Among these, the effective Einstein equation contains a varying cosmological ''constant'' and two new source terms. The first of these represents the deviation from Z_2 symmetry, while the second arises from the bulk energy-momentum tensor. We apply the formalism for the case of a perfect fluid on a Friedmann brane embedded in a generic bulk. The generalized Friedmann and Raychaudhuri equations are given in a form independent of both the embedding and the bulk matter. They contain two new functions obeying a first order differential system, both depending on the bulk matter and the embedding. Then we focus on Friedmann branes separating two nonidentical (inner or outer) regions of Reissner–Nordström–anti-de Sitter bulk space-times, generalizing previous non-Z_2-symmetric treatments. Finally the analysis is repeated for the Vaidya–anti-de Sitter bulk space-time, allowing for both ingoing and outgoing radiation in each region.

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I. INTRODUCTION

Since the pioneering idea of Randall and Sundrum [1] of enriching four dimensional space-time with one noncompact spatial dimension, cosmology has advanced towards new types of extensions. Generalized Randall-Sundrum type II models have in common a five dimensional space-time (bulk), governed by the Einstein equations, and a four dimensional brane, representing our physical world, on which ordinary matter fields are confined. At low energies gravity is also localized at the brane [1]; however this feature does not always hold [2]. Generalizations of the original Randall-Sundrum scenario are various and multiple, all allowing for matter with cosmological symmetry on the brane (Friedmann branes) [3,4] (in this case the bulk is Schwarzschild–anti-de Sitter space-time [5,6]). The assumption of Z_2 symmetric embedding was also lifted in a series of papers [7,13], and nonempty bulks have also been considered, with physically reasonable matter content, like null dust [14,15], which can be interpreted as the high frequency (geometrical optics) approximation of unpolarized radiation (even gravitational), whenever the wavelength of the radiation is negligible compared to the curvature radius of the background. In the present paper we present a formalism generic enough to allow for all such types of extensions. Models allowing a *dilatonic* type scalar field in the bulk were also discussed [16], but will not be dealt with in the context of this paper, neither will the possibility of having different coupling constants on the two sides of the brane [13]. Further generalization of our formalism however, is straightforward.

In Sec. II we present the decomposition of the Einstein tensor in an arbitrary $(d+1)$-dimensional space-time with respect to some generic (timelike or spacelike foliation). We carefully monitor the relationship of the tensor, vector and scalar projections of the Einstein equation with the system of effective Einstein and Codazzi equations, widely employed in brane-world scenarios. We show that the latter system should be supplemented by the twice contracted Gauss equation in order to assure the full equivalence. (In this context we mention a recent analysis [17], which also underlines the unsatisfactory feature of ''truncating'' the system of bulk Einstein equations to brane equations.)

Beginning with Sec. III we have in mind the brane-world scenario. By use of the Lanczos-Sen-Darmois-Israel junction conditions [18–21] we derive the generalized effective Einstein equation in a form closely resembling previous works [22]:

$$G_{ab}=-\Lambda g_{ab}+\kappa^2 T_{ab}+\widetilde{\kappa}^4 S_{ab}-\overline{\mathcal{E}_{ab}}+\overline{L_{ab}^{TF}}+\overline{\mathcal{P}_{ab}}\,. \quad (1)$$

Among the source terms on the rhs we find the brane energy-momentum tensor T_{ab}, the term S_{ab} quadratic in the energy-momentum tensor (relevant at high energies) and $\overline{\mathcal{E}_{ab}}$, the electric part of bulk Weyl tensor. Our generic treatment *does not* require the Z_2 symmetry of the bulk across the brane and this leads to three important modifications. First $\overline{\mathcal{E}_{ab}}$ represents an average taken over the two sides of the brane. Second, a new source term $\overline{L_{ab}^{TF}}$ appears. Third, there is a contribution included in Λ, which transforms the cosmological ''constant'' into a function. Bulk energy-momentum is also allowed, resulting in the $\overline{\mathcal{P}_{ab}}$ source term and a second nonconstant contribution to Λ. When allowing bulk matter, we have in mind reasonable sources, like null dust or multi-component null dust, which can model for example the cross-flow of gravitational radiation escaping the brane and

*Email address: Laszlo.Gergely@port.ac.uk; gergely@physx.u-szeged.hu

Hawking radiation leaving the bulk black hole(s). At the end of the section we give the generic form of the brane Bianchi identities.

The most interesting applications of the developed formalism would be for branes containing black holes and for branes containing perfect fluid and obeying cosmological symmetries. Anisotropic cosmological brane-world models can also be considered [23,24].

Among these in Sec. IV we discuss the case of maximally symmetric branes with perfect fluid. By employing the brane Bianchi identities, we derive the generalized Raychaudhuri and Friedmann equations *in a form insensitive to both the choice of the bulk matter and of the left and right embeddings of the brane in the bulk*. The price to pay is a cosmological function instead of a constant, and that the source term usually quoted as dark radiation acquires a much broader interpretation.

An algorithm to solve in a hierarchical way the relevant system of equations for a bulk containing a Friedmann brane with a given perfect fluid is presented in the Appendix. The equations refer to $d=4$. The algorithm is suited for the cases when no *a priori* choice of the bulk is performed, instead the matter content of the bulk and the details of the embedding are specified. There are constraints on both of these choices, as detailed in the Appendix.

Section V deals with Friedmann branes embedded in the Reissner–Nordström–anti-de Sitter bulk. In the case of a cosmological bulk with maximally symmetric spatial 3 sections (case without the charge), with the exception of the static case, where exotic solutions are equally possible [25], a generalized Birkhoff theorem holds [6], which states that such a bulk is the 5D Schwarzschild–anti-de Sitter space-time. We develop a formalism which is suitable for matching inner and outer regions of the Reissner–Nordström–anti-de Sitter space-time, thus allowing for two, one or no charged black hole. We give the energy-momentum tensor leading to the solution already employed in [26] in the study of the Z_2 symmetric embedding. Then by straightforward algebra we find the generalized Friedmann and Raychaudhuri equations. These were checked to reproduce all previous non-Z_2 symmetric results derived in the particular case of pure cosmological bulk: different black hole masses on the two sides on the brane [7–9], zero black hole mass and different cosmological constants on the two sides [10,11] and allowing for both types of generalizations [13,6,12].

In Sec. VI we study a generic *asymptotically* anti-de Sitter bulk compatible with type II fluid. Such a bulk is a generalization of the charged Vaidya solution in the presence of a cosmological constant. In [27] the charged Vaidya solution was employed to model evaporating charged black holes. Here first we derive the 5 dimensional solution, which agrees with [14], generated by null dust and an electromagnetic field on a cosmological background. The details of these source terms were not given in the literature before. We then follow the generic prescription described in the Appendix and we write down the generalized Friedmann and Raychaudhuri equations for this case. The generic results derived here are also new. In the particular case of Z_2 symmetric brane embeddings, the Friedmann equation for Vaidya–anti-de Sitter bulk [14] is recovered.

Throughout the paper a tilde distinguishes the quantities defined on the $(d+1)$-dimensional space-time. The only exception is the normal n to the leaves of the foliation. Its norm is $n^c n_c=\epsilon=\pm 1$ ($\epsilon=1$ stands for timelike and $\epsilon=-1$ for spacelike foliations). Latin indices represent abstract indices running from 0 to d. Vector fields in Lie derivatives are represented by boldface characters. For example $\widetilde{\mathcal{L}}_{\widetilde{\mathbf{V}}}T$ denotes the Lie derivative along the integral lines of the vector field $\widetilde{V}^a$. From section to section, as the paper converges toward its conclusion and the results derived apply to more specific situations, the degree of generality decreases accordingly. Up to Sec. III B everything holds for arbitrary ϵ. Up to and including Sec. IV, the results are dimension-independent, $d=4$ being imposed only in the application described in Secs. V and VI and in the Appendix. In Sec. VI, ϵ distinguishes between the outgoing or ingoing character of the radiation.

II. THE EQUIVALENCE OF TWO $(d+1)$-DIMENSIONAL DECOMPOSITIONS

The $(d+1)$-metric $\widetilde{g}_{ab}$ induces a metric g_{ab} on the leaves,

$$\widetilde{g}_{ab}=g_{ab}+\epsilon n_a n_b . \tag{2}$$

By introducing the projectors $g^{a_1 \dots a_r d_1 \dots d_s}_{c_1 \dots c_r b_1 \dots b_s} = g^{a_1}_{c_1} \dots g^{a_r}_{c_r} g^{d_1}_{b_1} \dots g^{d_s}_{b_s}$, one can define the projected covariant derivative and the projected Lie derivative of any tensor $\widetilde{T}^{a_1 \dots a_r}_{b_1 \dots b_s}$ as

$$\nabla_a \widetilde{T}^{a_1 \dots a_r}_{b_1 \dots b_s} = g^{c a_1 \dots a_r d_1 \dots d_s}_{a c_1 \dots c_r b_1 \dots b_s} \widetilde{\nabla}_c \widetilde{T}^{c_1 \dots c_r}_{d_1 \dots d_s}, \tag{3}$$

$$\mathcal{L}_{\widetilde{\mathbf{V}}} \widetilde{T}^{a_1 \dots a_r}_{b_1 \dots b_s} = g^{a_1 \dots a_r d_1 \dots d_s}_{c_1 \dots c_r b_1 \dots b_s} \widetilde{\mathcal{L}}_{\widetilde{\mathbf{V}}} \widetilde{T}^{c_1 \dots c_r}_{d_1 \dots d_s}. \tag{4}$$

If both the tensor $\widetilde{T}^{a_1 \dots a_r}_{b_1 \dots b_s}$ and the vector $\widetilde{V}^a$ are defined on the leaves, the above equations are the g_{ab}-compatible covariant derivative and Lie derivative on the lower-dimensional space, respectively. If $\widetilde{V}^a$ however is transverse to the leaves, the projected Lie derivative describes transverse evolution. The embedding of the leaves in the $(d+1)$-dimensional space-time is characterized by the extrinsic curvature $K_{ab}=\nabla_a n_b$. Its trace will be denoted by K. It is immediate to see that K_{ab} is symmetric by noting that $n_b=\beta \widetilde{\nabla}_b \chi$ (β is an arbitrary function; as the condition $\chi=$const defines the leaves, χ is time for $\epsilon=-1$ and any coordinate transverse to the leaves for $\epsilon=1$). The extrinsic curvature obeys

$$2K_{ab}=\mathcal{L}_{\mathbf{n}} g_{ab} . \tag{5}$$

The congruence n^a has its own curvature [28] $\alpha^b=n^c \widetilde{\nabla}_c n^b = g^b_d \alpha^d$. For spacelike foliations this is the nongravitational acceleration of observers with velocity n^a. With this we find

$$\widetilde{\nabla}_a n_b = K_{ab}+\epsilon n_a \alpha_b . \tag{6}$$

For notational convenience we also introduce the tensors

$$E_{ab}=K_{ac}K_b^c-\mathcal{L}_{\mathbf{n}}K_{ab}+\nabla_b\alpha_a-\epsilon\alpha_b\alpha_a\,, \tag{7}$$

$$F_{ab}=KK_{ab}-K_{ac}K_b^c\,, \tag{8}$$

together with their traces, E and F. Note that E_{ab} carries the information about the transverse evolution of K_{ab}.

The $(d+1)$-dimensional Einstein tensor is equivalent with the following set of projections:

$$g_a^c g_b^d \widetilde{G}_{cd}=G_{ab}-\epsilon\left[F_{ab}-E_{ab}-\frac{1}{2}g_{ab}(F-2E)\right], \tag{9a}$$

$$g_a^c n^d \widetilde{G}_{cd}=g_a^c n^d \widetilde{R}_{cd}=\nabla_c K_a^c-\nabla_a K, \tag{9b}$$

$$2n^a n^b \widetilde{G}_{ab}=-\epsilon R+F. \tag{9c}$$

These equations have the following meaning, provided $\widetilde{G}_{ab}$ is determined by a $(d+1)$-dimensional Einstein equation. The tensor equation (9a) determines (through E_{ab}) the evolution of K_{ab} normal to the foliation. Together with Eq. (5) they give the off-leave evolutions of the variables (g_{ab},K_{ab}) defined on the leaves. The vector equation (9b) is the Codazzi equation and represents a constraint on these variables. Similarly does the scalar equation (9c). For spacelike foliations the vector and scalar equations are the diffeomorphism and Hamiltonian constraints, respectively. These ‘‘instantaneous constraints’’ become dynamical for timelike foliations and the evolution equations form an elliptic, rather than hyperbolic system.

In what follows we would like to set up an equivalent set of equations, most commonly employed in brane-world scenarios, e.g., suitable for timelike foliations. We would like to keep, however a strict account of the sets of equations which are equivalent to each other in the two pictures. For this purpose first we decompose the tensor equation (9a) into its trace

$$2\epsilon g^{ab}\widetilde{G}_{ab}=(d-2)(-\epsilon R+F)-2(d-1)E, \tag{10}$$

and trace-free parts

$$-\epsilon(g_a^c g_b^d \widetilde{G}_{cd})^{TF}=(-\epsilon R_{ab}+F_{ab}-E_{ab})^{TF}, \tag{11}$$

where ^{TF} denotes tracefree, e.g., $f_{ab}^{TF}=f_{ab}-fg_{ab}/d$ for any tensor f_{ab} defined on the leaves.

The trace equation (10), properly combined with the scalar equation (9c) gives the twice contracted Gauss equation:

$$-\epsilon\widetilde{R}=-\epsilon R+F-2E. \tag{12}$$

Eliminating R from Eqs. (9c) and (12) gives E solely in terms of bulk tensors:

$$E=n^a n^b \widetilde{G}_{ab}+\frac{\epsilon}{2}\widetilde{R}. \tag{13}$$

Concerning the tracefree part E_{ab}^{TF}, it is commonly expressed in terms of the Weyl tensor (the purely radiative contribution to gravity)

$$\widetilde{C}_{abcd}=\widetilde{R}_{abcd}+\frac{2}{d(d-1)}\widetilde{g}_{a[c}\widetilde{g}_{d]b}\widetilde{R}-\frac{2}{d-1}(\widetilde{g}_{a[c}\widetilde{R}_{d]b}-\widetilde{g}_{b[c}\widetilde{R}_{d]a}). \tag{14}$$

Its ‘‘electric’’ part with respect to n^a is defined as

$$\mathcal{E}_{ac}=\widetilde{C}_{abcd}n^b n^d=g_a^i n^j g_c^k n^l \widetilde{R}_{ijkl}+\frac{\epsilon\widetilde{R}}{d(d-1)}g_{ac}-\frac{1}{d-1}(\epsilon g_a^i g_c^k \widetilde{R}_{ik}+g_{ac}n^i n^k \widetilde{R}_{ik}). \tag{15}$$

Inserting the projections

$$g_a^i n^j g_c^k n^l \widetilde{R}_{ijkl}=E_{ac}\,,$$

$$-\epsilon g_a^i g_c^k \widetilde{R}_{ik}=-\epsilon R_{ac}+F_{ac}-E_{ac}\,,$$

$$n^i n^k \widetilde{R}_{ik}=-K_{bd}K^{bd}-E, \tag{16}$$

we find

$$(d-1)\mathcal{E}_{ab}=[-\epsilon R_{ab}+F_{ab}+(d-2)E_{ab}]^{TF}. \tag{17}$$

Eliminating R_{ab} from Eqs. (11) and (17) leads to

$$\mathcal{E}_{ab}+\epsilon\frac{1}{d-1}(g_a^c g_b^d \widetilde{G}_{cd})^{TF}=E_{ab}^{TF}\,. \tag{18}$$

In what follows, this equation containing the tracefree part of the off-leave evolution $\mathcal{L}_{\mathbf{n}}K_{ab}$ will be regarded as the definition of $\mathcal{E}_{ab}$. Eliminating the off-leave derivative term from Eqs. (11) and (17) results in

$$\mathcal{E}_{ab}-\epsilon\frac{d-2}{d-1}(g_a^c g_b^d \widetilde{G}_{cd})^{TF}=(-\epsilon R_{ab}+F_{ab})^{TF}. \tag{19}$$

Combining this trace-free equation with the scalar equation (9c) we obtain the effective Einstein equation on the leaves:

$$G_{ab}=\frac{d-2}{d-1}(g_a^c g_b^d \widetilde{G}_{cd})^{TF}+\frac{d-2}{d}g_{ab}\epsilon n^c n^d \widetilde{G}_{cd}+\epsilon\left(F_{ab}-\frac{g_{ab}}{2}F\right)-\epsilon\mathcal{E}_{ab}\,. \tag{20}$$

Note that the trace of the effective Einstein equation (20) and the scalar equation (9c) coincide by construction. Therefore the second scalar equation is given by the trace of the original tensor equation (9a) which, as we have seen, is equivalent (modulo the trace of the effective Einstein equation) to either the twice-contracted Gauss equation (12) or to Eq. (13).

For spacelike foliations the usual way to think of the above system of equations is to choose variables g_{ab} and K_{ab} satisfying the constraints (9b) and (9c) on the leaves and let them evolve via Eqs. (5) and (9a). When the foliation is timelike, another viewpoint is common. In the brane-world scenario the central role is played by the effective Einstein equation (20), in which the bulk matter (via the bulk Einstein equation), the extrinsic curvature of the brane (the F terms) and the electric part of the bulk Weyl tensor are all considered sources for the brane gravity sector. While the extrinsic curvature of the brane is determined by brane matter and brane tension through the junction mechanism, and (in the Z_2 symmetric cosmological bulk) the longitudinal part of $\mathcal{E}_{ab}$ is fixed by the vector equation [22], nothing constraints the behavior of the transverse part of $\mathcal{E}_{ab}$, which remains arbitrary from a brane point of view. This feature is the source of several difficulties, frequently formulated as the lack of a *temporal* evolution equation for $\mathcal{E}_{ab}$.

The *off-brane* evolution of $\mathcal{E}_{ab}$ was deduced from the bulk Bianchi identities in Ref. [22] (in the case of a Z_2 symmetric cosmological bulk). It also follows from the above equations. As $\mathcal{E}_{ab}$ is completely determined by the bulk matter, the induced metric and the extrinsic curvature via Eq. (19), the evolution of $\mathcal{E}_{ab}$ follows from the metric evolution (5) and the evolution of the extrinsic curvature. Let us recall that the latter was given by the tensor equation (9a). Should one choose a brane-world viewpoint, the situation is different: the effective Einstein equation (20) together with Eq. (18) gives only the traceless part of $\mathcal{L}_{\mathbf{n}}K_{ab}$. The complementary equation is either the twice contracted Gauss equation (12) or Eq. (13), which both contain $g^{ab}\mathcal{L}_{\mathbf{n}}K_{ab}$. With this equation, the bulk Bianchi identities emerge as a consequence.

It is clear now that in the brane-world scenario the effective Einstein equation and the Codazzi equations do not provide a complete characterization of gravity, but they should be supplemented by the twice-contracted Gauss equation [or the expression (13) for E]. This set of equations is equivalent with the Einstein equation in the $(d+1)$-dimensional spacetime.

III. THE EFFECTIVE EINSTEIN EQUATION FOR NON-Z_2-SYMMETRIC BULK

A. The junction conditions

Both in general relativity and in the brane world scenarios the possibility of a distributional matter source on a hypersurface is of interest. Such a hypersurface divides the spacetime into two distinct regions. In both of these regions one of the systems (9) or (9b), (13) and (20) should be imposed separately. Quantities defined on these domains will be distinguished by $+$ or $-$ symbols. The passage from one zone to the other is described in a coordinate-independent manner by the junction conditions [21] (see also [6]). These conditions include the continuity of the induced metric across the hypersurface, $g^{+}_{ab}=g^{-}_{ab}$, and the Lanczos equation [18], a condition on the jump of the extrinsic curvature. It is straightforward to deduce the latter equation from the equations derived in the preceding section.

From Eqs. (7), (8) and the second relation (16) we find

$$\mathcal{L}_{\mathbf{n}}K_{ab}=-\epsilon g^{i}_{a}g^{k}_{b}\widetilde{R}_{ik}+Z_{ab}\,, \tag{21}$$

$$Z_{ab}=\epsilon R_{ab}+2K_{ac}K^{c}_{b}-KK_{ab}+\nabla_{b}\alpha_{a}-\epsilon\alpha_{b}\alpha_{a}\,. \tag{22}$$

The Einstein equation gives

$$g^{i}_{a}g^{k}_{b}\widetilde{R}_{ik}=\widetilde{\kappa}^{2}\left(g^{i}_{a}g^{k}_{b}\widetilde{T}_{ik}-\frac{1}{d-1}g_{ab}\widetilde{T}\right). \tag{23}$$

If l is the coordinate adapted to the normal, $\mathbf{n}=\partial/\partial l$, the energy-momentum tensor can be written as $\widetilde{T}_{ik}=\widetilde{\Pi}_{ik}+\tau_{ik}\delta(l)$, with $\widetilde{\Pi}_{ik}$ the regular part and τ_{ik} the distributional part on the layer, obeying $\tau_{ik}n^{i}=0$. Thus Eq. (21) becomes

$$\frac{\partial}{\partial l}K_{ab}=-\epsilon\widetilde{\kappa}^{2}\left(\tau_{ab}-\frac{1}{d-1}g_{ab}\tau\right)\delta(l)+W_{ab}+Z_{ab}\,, \tag{24}$$

$$W_{ab}=-\epsilon\widetilde{\kappa}^{2}\left(g^{i}_{a}g^{k}_{b}\widetilde{\Pi}_{ik}-\frac{1}{d-1}g_{ab}\widetilde{\Pi}\right). \tag{25}$$

As both Z_{ab} and W_{ab} are finite, integration across the layer on an infinitesimal integration range gives the Lanczos equation:

$$\Delta K_{ab}=-\epsilon\widetilde{\kappa}^{2}\left(\tau_{ab}-\frac{1}{d-1}g_{ab}\tau\right), \tag{26}$$

or equivalently

$$-\epsilon\widetilde{\kappa}^{2}\tau_{ab}=\Delta K_{ab}-g_{ab}\Delta K. \tag{27}$$

Here we have introduced the notation $\Delta f_{ab}=f^{+}_{ab}-f^{-}_{ab}$ for the jump of any tensor f_{ab} and Δf for its trace. (By construction, $+$ means the region towards which n is pointing. We emphasize, that the Lanczos equation is not affected by the choice of the orientation of the normal n, because the change in the orientation implies that both the $+$ and $-$ regions and the sign of the extrinsic curvature are reversed.) We also introduce the mean value $\overline{f}_{ab}=(f^{+}_{ab}+f^{-}_{ab})/2$. Obviously $\Delta g_{ab}=0$ and $\overline{g}_{ab}=g_{ab}$. Straightforward algebra then shows

$$\begin{aligned}\overline{F}_{ab}&=\overline{K}_{ab}\overline{K}-\overline{K}_{ac}\overline{K}^{c}_{b}+\delta F_{ab}\,,\\ \overline{F}&=\overline{K}^{2}-\overline{K}_{ab}\overline{K}^{ab}+\delta F,\\ \Delta F_{ab}&=-\epsilon\widetilde{\kappa}^{2}\left[\overline{K}\left(\tau_{ab}-\frac{1}{d-1}g_{ab}\tau\right)+\overline{K}_{ab}\frac{\tau}{d-1}-2\overline{K}_{c(a}\tau^{c}_{b)}\right],\\ \Delta F&=2\epsilon\widetilde{\kappa}^{2}\overline{K}_{ab}\tau^{ab},\end{aligned} \tag{28}$$

where we have denoted by

$$\delta F_{ab}=-\frac{\tilde{\kappa}^4}{4}\left(\tau_{ac}\tau_b^c-\frac{1}{d-1}\tau\tau_{ab}\right) \tag{29}$$

the contribution which distinguishes the functional form of $\overline{F}_{ab}$ from the one of F_{ab}.

Let us now consider a region of space-time of a finite thickness 2η, which contains this temporal hypersurface. The set of equations (9b) and (20) holds in any of the two regions even in the limit $\eta\to 0$. Their sum and difference give

$$\tilde{\kappa}^2\overline{(g_a^c n^d \tilde{\Pi}_{cd})}=\nabla_c\overline{K}_a^c-\nabla_a\overline{K}, \tag{30a}$$

$$\Delta(g_a^c n^d \tilde{\Pi}_{cd})=-\epsilon\nabla_c\tau_a^c, \tag{30b}$$

$$2\tilde{\kappa}^2\overline{(n^a n^b \tilde{\Pi}_{ab})}=-\epsilon R+\overline{F}, \tag{30c}$$

$$\Delta(n^a n^b \tilde{\Pi}_{ab})=\epsilon\overline{K}_{ab}\tau^{ab}, \tag{30d}$$

$$\tilde{\kappa}^2\frac{d-2}{d-1}\overline{(g_a^c g_b^d \tilde{\Pi}_{cd})^{TF}}=R_{ab}^{TF}-\epsilon\overline{F}_{ab}^{TF}+\epsilon\overline{\mathcal{E}}_{ab}, \tag{30e}$$

$$\tilde{\kappa}^2\frac{d-2}{d-1}\Delta(g_a^c g_b^d \tilde{\Pi}_{cd})^{TF}=-\epsilon\Delta F_{ab}^{TF}+\epsilon\Delta\mathcal{E}_{ab}. \tag{30f}$$

The last four equations are the trace and trace-free parts of the sum and difference of the effective Einstein equations in the two regions, respectively. From among them the last two equations define the mean value and the jump of $\mathcal{E}_{ab}$ (the trace-free part of E_{ab}). Let us recall that the trace E is also determined in both regions by Eq. (13), which in terms of the bulk energy-momentum tensor reads

$$E^{\pm}=\tilde{\kappa}^2\left(n^a n^b\tilde{\Pi}_{ab}^{\pm}-\frac{\epsilon}{d-1}\tilde{\Pi}^{\pm}\right). \tag{31}$$

As it will be employed in the next subsection, we also give the undecomposed form of the equation obtained by the sum of the effective Einstein equations on each side [equivalent to Eqs. (30c) and (30e)]:

$$G_{ab}=\tilde{\kappa}^2\left[\frac{d-2}{d-1}\overline{(g_a^c g_b^d\tilde{\Pi}_{cd})^{TF}}+\epsilon\frac{d-2}{d}g_{ab}\overline{(n^c n^d\tilde{\Pi}_{cd})}\right]+\epsilon\left(\overline{F}_{ab}-\frac{g_{ab}}{2}\overline{F}-\overline{\mathcal{E}}_{ab}\right). \tag{32}$$

B. The effective Einstein equation

From now on we specialize to the brane-world scenarios, where a $(d-1)$-dimensional distributional source evolves in time and in consequence the hypersurface is temporal. Thus we apply the above formulas for $\epsilon=1$. For the generic brane energy-momentum tensor $\tau_{ab}=-\lambda g_{ab}+T_{ab}$ (where λ is the brane tension and T_{ab} represents ordinary matter on the brane) we have

$$\delta F_{ab}-\frac{g_{ab}}{2}\delta F=\tilde{\kappa}^4\left[S_{ab}+\lambda\frac{d-2}{4(d-1)}T_{ab}-\frac{d-2}{8(d-1)}g_{ab}\lambda^2\right]. \tag{33}$$

Here S_{ab} denotes a quadratic expression in T_{ab}:

$$S_{ab}=\frac{1}{4}\left[-T_{ac}T_b^c+\frac{1}{d-1}TT_{ab}-\frac{g_{ab}}{2}\left(-T_{cd}T^{cd}+\frac{1}{d-1}T^2\right)\right]. \tag{34}$$

By defining the brane gravitational constant and the brane cosmological "constant" through

$$\kappa^2=\frac{d-2}{4(d-1)}\tilde{\kappa}^4\lambda, \tag{35}$$

$$\Lambda=\frac{\kappa^2\lambda}{2}-\frac{\overline{L}}{d}-\tilde{\kappa}^2\frac{d-2}{d}\overline{(n^c n^d\tilde{\Pi}_{cd})}, \tag{36}$$

we obtain the effective Einstein equation (1). Among the source terms we find $\overline{L}_{ab}$, which is defined as

$$\overline{L}_{ab}=\overline{K}_{ab}\overline{K}-\overline{K}_{ac}\overline{K}_b^c-\frac{g_{ab}}{2}(\overline{K}^2-\overline{K}_{ab}\overline{K}^{ab}), \tag{37}$$

with $\overline{L}_{ab}^{TF}$ and $\overline{L}$ its tracefree part and trace. Finally $\overline{\mathcal{P}}_{ab}$ is given by the pull-back of the bulk energy-momentum tensor to the brane:

$$\overline{\mathcal{P}}_{ab}=\tilde{\kappa}^2\frac{d-2}{d-1}\overline{(g_a^c g_b^d\tilde{\Pi}_{cd})^{TF}}. \tag{38}$$

The first four terms of the right-hand side (RHS) of the effective Einstein equation are well known [22]. They are the cosmological term, the ordinary brane matter source term (dominant at low energies), a quadratic term in the brane energy-momentum (relevant at high energies), and the bulk electric Weyl-curvature contribution. The only modification up to here is the possibility of a varying cosmological "constant" (it depends both on the projection $\overline{(n^c n^d\tilde{\Pi}_{cd})}$ of the bulk energy-momentum tensor and on the embedding of the brane). In addition to these there are two new terms. The first of them, $\overline{L}_{ab}^{TF}$ represents the imprint of the particular way the time-evolving brane is bent into the bulk from both sides. This contribution disappears in the Z_2-symmetric case (as well as the contribution $\overline{L}$ to Λ). The last term, $\overline{\mathcal{P}}_{ab}$ arises from the projection of the bulk energy-momentum tensor on the brane, and is traceless by definition.

In terms of λ and T_{ab}, Eqs. (30d), (30b) and (30f) can be written as

$$\Delta(n^a n^b\tilde{\Pi}_{ab})=-\lambda\overline{K}+T_{ab}\overline{K}^{ab}, \tag{39a}$$

$$\Delta(g_a^c n^d\tilde{\Pi}_{cd})=-\nabla_c T_a^c, \tag{39b}$$

$$\tilde{\kappa}^2\frac{d-2}{d-1}\Delta(g_a^c g_b^d \tilde{\Pi}_{cd})^{TF}=\Delta\mathcal{E}_{ab}+\tilde{\kappa}^2\bigg[\overline{K}T_{ab}+\frac{T}{d-1}\overline{K}_{ab}+\frac{d-2}{d-1}\lambda\overline{K}_{ab}-2\overline{K}^c_{(a}T_{b)c}\bigg]^{TF}. \tag{39c}$$

Thus Eqs. (30a), (31), (1) and (39) are the complete set of equations in the generic case of non-Z_2-symmetric bulk and arbitrary energy-momentum tensors both on the brane and in the bulk.

The Bianchi identity in d dimensions allows for the expression of the longitudinal part of $(\overline{\mathcal{E}}_{ab}-\overline{L}_{ab}^{TF}-\overline{\mathcal{P}}_{ab})$:

$$\begin{aligned}\nabla^a(\overline{\mathcal{E}}_{ab}-\overline{L}_{ab}^{TF}-\overline{\mathcal{P}}_{ab})&=\frac{\nabla_b\overline{L}}{d}+\tilde{\kappa}^2\frac{d-2}{d}\nabla_b\overline{(n^c n^d\tilde{\Pi}_{cd})}-\kappa^2\Delta(g_b^c n^d\tilde{\Pi}_{cd})\\&\quad+\frac{\tilde{\kappa}^4}{4}\left(T_b^c-\frac{T}{d-1}g_b^c\right)\Delta(g_c^a n^d\tilde{\Pi}_{ad})+\frac{\tilde{\kappa}^4}{4}\bigg[2T^{ac}\nabla_{[b}T_{a]c}\\&\quad+\frac{1}{d-1}(T_{ab}\nabla^a T-T\nabla_b T)\bigg].\end{aligned} \tag{40}$$

This equation has important cosmological implications, as will be discussed in the next section.

IV. PERFECT FLUID ON FRIEDMANN BRANE

Friedmann branes, obeying cosmological symmetries are characterized by the metric

$$g_{ab}=-u_a u_b+a^2(\tau)h_{ab}, \tag{41}$$

where $a(\tau)$ is the scale factor and h_{ab} is a $(d-1)$ metric with *constant* curvature (characterized by the curvature index $k=1,0,-1$) of the maximally symmetric spatial slices with constant τ. The timelike congruence $u^a=(\partial/\partial\tau)^a$ obeys $u^a u_a=-1$ and $h_{ab}u^a=0$. It is not difficult to prove $u_b\nabla_a u^b=u^b\nabla_b u^a=0$. We denote by a dot the time derivative with respect to τ, which in the generic case is defined as the Lie derivative in the u^a direction, projected into the hypersurface perpendicular to u^a (of constant τ). Then, from the condition $\dot{h}_{ab}=0$ we find

$$u^c\nabla_c h_{ab}=-\frac{1}{a^2}(\nabla_a u_b+\nabla_b u_a), \tag{42}$$

the trace of which implies $\nabla_a u^a=(d-1)\dot{a}/a$.

When there is a perfect fluid on the brane, it has the energy-momentum tensor

$$T_{ab}=\rho(\tau)u_a u_b+p(\tau)a^2 h_{ab}, \tag{43}$$

with u^a representing its d-velocity. Spatial isotropy and homogeneity implies $h_{ab}\nabla^b a=h_{ab}\nabla^b\rho=h_{ab}\nabla^b p=0$.

The quadratic term (34) then becomes

$$\tilde{\kappa}^4 S_{ab}=\kappa^2\frac{\rho}{\lambda}\left[\frac{\rho}{2}u_a u_b+\left(\frac{\rho}{2}+p\right)a^2 h_{ab}\right]. \tag{44}$$

We complete the bookkeeping of the source terms by introducing the effective nonlocal energy density U arising from the totality of nonlocal *tracefree* terms in the effective Einstein equation (1):

$$-\overline{\mathcal{E}}_{ab}+\overline{L}_{ab}^{TF}+\overline{\mathcal{P}}_{ab}=\kappa^2 U\left(u_a u_b+\frac{a^2}{d-1}h_{ab}\right). \tag{45}$$

U is a generalization of the effective nonlocal energy density introduced in the case of Z_2 symmetric, cosmological bulk [29].

Next we specify the system (39) for the perfect fluid energy-momentum tensor. Equations (39a) and (39c) give

$$\Delta(n^a n^b\tilde{\Pi}_{ab})=(p-\lambda)\overline{K}+(\rho+p)u_a u_b\overline{K}^{ab}, \tag{46}$$

$$\begin{aligned}\tilde{\kappa}^2\frac{d-2}{d-1}\Delta(g_a^c g_b^d\tilde{\Pi}_{cd})^{TF}&=\Delta\mathcal{E}_{ab}+\tilde{\kappa}^2\bigg[(\rho+p)\overline{K}u_a u_b-2(\rho+p)u_{(a}\overline{K}^c_{b)}u_c\\&\quad-\frac{\rho+(d-1)p-(d-2)\lambda}{d-1}\overline{K}_{ab}\bigg]^{TF},\end{aligned} \tag{47}$$

while Eq. (39b) decouples into the following time and space components:

$$\Delta(u^c n^d\tilde{\Pi}_{cd})=\dot{\rho}+(d-1)\frac{\dot{a}}{a}(\rho+p), \tag{48}$$

$$h_{ab}\Delta(g^{ac}n^d\tilde{\Pi}_{cd})=0. \tag{49}$$

By virtue of Eqs. (45), (48) and (49) the space projection of the Bianchi identity (40) trivially vanishes. [We also use in the proof that in order to satisfy the cosmological symmetries, U as well as $\overline{L}$ and $\overline{(n^c n^d\tilde{\Pi}_{cd})}$, (both contributing to Λ) are pure time-functions, i.e., they have vanishing spatial derivatives.] The time projection of the Bianchi identity (40) gives

$$\kappa^2\left(\dot{U}+d\frac{\dot{a}}{a}U\right)=\frac{1}{d}[\tilde{\kappa}^2(d-2)\overline{(n^c n^d\tilde{\Pi}_{cd})}+\overline{L}]^{\bullet}-\kappa^2\left(1+\frac{\rho}{\lambda}\right)\Delta(u^c n^d\tilde{\Pi}_{cd}). \tag{50}$$

The homogeneous part of the above equation integrates to

$$U=U_0\left(\frac{a_0}{a}\right)^d, \tag{51}$$

where $U_0 a_0^d$ is an integration constant. Variation of the constant U_0 gives a first order ordinary differential equation:

$$\kappa^2\left(\frac{a_0}{a}\right)^d \dot{U}_0+\dot{\Lambda}+\kappa^2\left(1+\frac{\rho}{\lambda}\right)\Delta(u^c n^d \widetilde{\Pi}_{cd})=0. \quad (52)$$

[We have employed the definition of Λ given in Eq. (36).] The solution of Eq. (52) depends both on the bulk matter and on the details of the embedding of the brane in the bulk.

Finally we compute the Einstein tensor:

$$G_{ab}=\frac{(d-2)}{2}\left\{(d-1)\frac{\dot{a}^2+k}{a^2}u_a u_b -[2a\ddot{a}+(d-3)(\dot{a}^2+k)]h_{ab}\right\}, \quad (53)$$

which is the last piece of information required in order to write the effective Einstein equation (1). Its nontrivial projections combine to give the generalized Friedmann and generalized Raychaudhuri equations:

$$(d-1)(d-2)\frac{\dot{a}^2+k}{a^2}=2\Lambda+2\kappa^2\rho\left(1+\frac{\rho}{2\lambda}\right)+2\kappa^2 U_0\left(\frac{a_0}{a}\right)^d, \quad (54)$$

$$(d-1)(d-2)\frac{\ddot{a}}{a}=2\Lambda-\kappa^2\left\{\left[d-3+(d-2)\frac{\rho}{\lambda}\right]\rho +(d-1)\left(1+\frac{\rho}{\lambda}\right)p\right\}-(d-2)\kappa^2 U_0\left(\frac{a_0}{a}\right)^d. \quad (55)$$

Apart from the dimension-carrying index d, at first glance the above equations are *identical* with the corresponding equations of [4] and [3], obtained for Z_2 symmetric cosmological bulk. Still, important differences arise from the nonconstant character of Λ and U_0, given in the generic case by Eqs. (36) and (52). Another distinctive feature is that the U_0 term cannot be interpreted any more as pure dark radiation. A glance at Eqs. (45) and (51) shows that it carries both radiative degrees of freedom (from the electric bulk Weyl tensor), as well as imprints of the bulk matter and of the particular way the brane is bent into the left and right bulk regions. Equation (52) defining the potential U_0 is an integrability condition, which can be equivalently derived by taking the time derivative of the generalized Friedmann equation, then employing Eq. (48) to eliminate $\dot{\rho}$ and Eqs. (54) and (55) to eliminate all derivatives of a.

Let us recall, that the information on the gravitational field is completed by Eqs. (30a) and (46) [these represent $(d+1)$ constraints on the mean extrinsic curvature and bulk matter], Eqs. (48) and (49) (d constraints on the bulk matter), Eq. (47) determining $\Delta\mathcal{E}_{ab}$ and Eq. (31) giving $E^{\pm}$. Also, the Lanczos equation

$$\Delta K_{ab}=-\widetilde{\kappa}^2\left[\left(\frac{d-2}{d-1}\rho+p-\frac{\lambda}{d-1}\right)u_a u_b+\frac{\rho+\lambda}{d-1}a^2 h_{ab}\right] \quad (56)$$

remains a useful link between dynamical and geometrical quantities defined on the brane.

We summarize the above results in the Appendix for $d=4$. The algorithmic way the equations are grouped is meant to facilitate the search for non-Z_2-symmetric FRW braneworld solutions. The equations governing off-brane evolution are also presented there.

In the following two sections we apply these generic equations for the *five-dimensional* Reissner–Nordström–anti-de Sitter bulk and charged Vaidya–anti-de Sitter bulk, both containing a *four-dimensional* Friedmann brane. On the two sides of the brane the bulk is characterized by different mass and charge functions, also by different cosmological constants. As the generalized Friedmann and Raychaudhuri equations are given in a form invariant with respect to different choices of the bulk and the embeddings, only the complementary set of equations (30a), (31), and (46)–(49) will change from case to case.

V. REISSNER–NORDSTRÖM–ANTI-de SITTER BULK

The generalization of the 4-dimensional Reissner-Nordström solution to a cosmological context in 5 dimensions was discussed in [26]. The metric

$$d\widetilde{s}^2=-f(r;k)dt^2+\frac{dr^2}{f(r;k)}+r^2[d\chi^2+\mathcal{H}^2(\chi;k)(d\theta^2 +\sin^2\theta d\phi^2)], \quad (57)$$

with

$$\mathcal{H}(\chi;k)=\begin{cases}\sin\chi, & k=1,\\ \chi, & k=0,\\ \sinh\chi, & k=-1\end{cases} \quad (58)$$

can be also written as

$$\widetilde{g}_{ab}=-u_a u_b+n_a n_b+r^2 h_{ab}, \quad (59)$$

where a time coordinate τ was introduced through the timelike vector

$$u=\frac{\partial}{\partial\tau}=\dot{t}\frac{\partial}{\partial t}+\dot{r}\frac{\partial}{\partial r}, \quad (60)$$

with unit negative norm, implying

$$f\dot{t}=(\dot{r}^2+f)^{1/2}. \quad (61)$$

We have chosen the $\dot{t}>0$ root and a dot denotes derivatives with respect to τ. Then the unit normal 1-form to both h_{ab} and u_a is determined up to a sign:

$$n=\pm(-1)^\sigma(-\dot{r}dt+\dot{v}dr). \quad (62)$$

The $+$ sign refers to right-pointing normal. The final results will not depend on this choice of the orientation. We have inserted an additional sign $(-1)^\sigma$ to allow the *outgoing* coordinate y, defined by

$$n=\pm(-1)^{\sigma}dy, \tag{63}$$

to increase either in the right or left directions. We will specify later the meaning of the exponent σ. Finally the 1-form field u_a is

$$u=-(\dot{r}^2+f)^{1/2}dt+\dot{a}dr=-d\tau. \tag{64}$$

If the bulk contains an electromagnetic field characterized by the potential 1-form

$$A=\frac{q}{r^2}dt, \tag{65}$$

its energy-momentum tensor will be

$$\widetilde{T}_{ab}^{EM}=\frac{3q^2}{r^6}(u_a u_b-n_a n_b+r^2 h_{ab}). \tag{66}$$

Then the bulk Einstein equation for the metric ansatz (57) with the source term

$$\widetilde{\Pi}_{cd}=-\widetilde{\Lambda}\widetilde{g}_{ab}+\widetilde{T}_{ab}^{EM} \tag{67}$$

is satisfied for

$$f(r;k)=k-\frac{2m}{r^2}-\frac{\widetilde{\kappa}^2\widetilde{\Lambda}}{6}r^2+\frac{q^2}{r^4}, \tag{68}$$

m and q being the mass and charge of the central black hole (or "stellar object," in the absence of a horizon) and $\widetilde{\Lambda}<0$ the bulk cosmological constant. In principle, all of these constants can take different values on the two sides of the brane. Thus we will drop the assumption of Z_2 symmetry and obtain more generic results than in [26].

If one passes from the coordinates (t,r) to (τ,y), the position of the brane can be simply specified as $y=\text{const}$. This choice is equivalent to the embedding relations $t=t(\tau)$ and $r=a(\tau)$. Therefore we replace $(r,\dot{r})$ with $(a,\dot{a})$. As for the embedding relation $t=t(\tau)$, we know only Eq. (61). By construction, n is the normal and u the tangent vector to the brane. Thus we take for the induced metric the expression (41). By this we have assumed that the bulk has the spatial symmetries of the brane.[1]

The extrinsic curvature for such hypersurfaces is

$$K_{ab}=\mp(-1)^{\sigma}\left[\frac{\ddot{a}+\frac{1}{2}\frac{\partial f}{\partial a}}{(\dot{a}^2+f)^{1/2}}u_a u_b-(\dot{a}^2+f)^{1/2}a h_{ab}\right]. \tag{69}$$

K_{ab} on the two sides depends on the actual value of the function f and the sign ambiguity arises from the ambiguity in the choice of the normal.

As we have not imposed the Z_2 symmetry, at this point we have to raise the question of whether inner or outer regions of the Reissner–Nordström–anti-de Sitter space-time will be glued together. We introduce a pair of indices $\eta_{R,L}$ which take the value 1 for inner regions and 0 for outer regions. Then, according to our definitions

$$\sigma=\begin{cases}\eta_R, & \text{right region,}\\ \eta_L+1, & \text{left region.}\end{cases} \tag{70}$$

Then the extrinsic curvatures on the two sides of the brane are

$$K_{ab}^R=\mp\left(\frac{A_R}{B_R}u_a u_b-B_R a h_{ab}\right),$$

$$K_{ab}^L=\pm\left(\frac{A_L}{B_L}u_a u_b-B_L a h_{ab}\right), \tag{71}$$

where R,L refer to right and left regions, respectively, and the following notations were introduced for convenience ($I=R,L$):

$$A_I=\ddot{a}+\frac{1}{2}\frac{\partial f_I}{\partial a}, \tag{72}$$

$$B_I=(-1)^{\eta_I}(\dot{a}^2+f_I)^{1/2}. \tag{73}$$

Then, according to the definition of the jump and mean value of the extrinsic curvature:

$$\Delta K_{ab}=-\left[\left(\frac{A_R}{B_R}+\frac{A_L}{B_L}\right)u_a u_b-(B_R+B_L)a h_{ab}\right],$$

$$2\overline{K}_{ab}=\mp\left[\left(\frac{A_R}{B_R}-\frac{A_L}{B_L}\right)u_a u_b-(B_R-B_L)a h_{ab}\right]. \tag{74}$$

We also have

$$\frac{2\overline{L}}{3}=-\frac{B_R-B_L}{a}\left[\frac{A_R}{B_R}-\frac{A_L}{B_L}+\frac{B_R-B_L}{a}\right]. \tag{75}$$

Equations (A1) and (A3) are then satisfied, while Eq. (A2) gives

$$\dot{\rho}+3\frac{\dot{a}}{a}(\rho+p)=0. \tag{76}$$

This first order differential relation among the scale factor, pressure and density of the perfect fluid (being independent

[1]For such a brane and pure cosmological bulk a "generalized Birkhoff theorem" holds [6], stating that the bulk is the 5 dimensional Schwarzschild–anti-de Sitter space-time, a particular case in our treatment. However when the brane has the additional static symmetry (Einstein brane), the derivation of the above mentioned "generalized Birkhoff theorem" is obstructed, and other bulk solutions are possible, as shown in [25].

both of Λ and U_0) guarantees that for a given equation of state $p(\rho)$, the density has the standard expression as function of the scale factor.

We actually do not have to carry out in full detail the program described in the Appendix. This is because our choices of the bulk and brane metrics already constrain the embedding, leading to the above extrinsic curvature. Then a shortcut would be to employ Eqs. (A4) and (A12) in order to express $\dot{a}^2$ and $\ddot{a}$ algebraically. (As the sign ambiguity was verified to cancel out from all equations, from now on we mean by Δ the differences between quantities taken from the R and L regions.) First we find by pure algebra that

$$\overline{B} = -\frac{\widetilde{\kappa}^2 a}{6}(\rho+\lambda),$$

$$\Delta B = \frac{3\Delta A + \widetilde{\kappa}^2 a C}{\widetilde{\kappa}^2(\rho+\lambda)}, \tag{77}$$

then

$$\frac{\dot{a}^2+\overline{f}}{a^2} = \frac{\kappa^2\lambda}{6} + \frac{\kappa^2\rho}{3}\left(1+\frac{\rho}{2\lambda}\right) + \frac{(\Delta B)^2}{4a^2}, \tag{78}$$

$$\frac{\overline{A}}{a} = \frac{\kappa^2\lambda}{6} - \frac{\kappa^2}{6}\left[\rho\left(1+\frac{2\rho}{\lambda}\right)+3p\left(1+\frac{\rho}{\lambda}\right)\right] + \frac{C\Delta B}{2a(\rho+\lambda)} + \frac{3(p-\lambda)(\Delta B)^2}{4a^2(\rho+\lambda)}. \tag{79}$$

By C we have denoted the source term in Eq. (A4), multiplied by $\mp$, in the present case

$$C = \frac{6\overline{q}\Delta q}{a^6} + \Delta\widetilde{\Lambda}. \tag{80}$$

By employing the definition of the metric function (68), and Eqs. (72), (80), we obtain

$$\Delta B = \frac{12a^2\Delta m - 12\widetilde{\kappa}^2\overline{q}\Delta q + \widetilde{\kappa}^2 a^6 \Delta\widetilde{\Lambda}}{2\widetilde{\kappa}^2 a^5(\rho+\lambda)}, \tag{81}$$

and the generalized Friedmann and Raychaudhuri equations

$$\frac{\dot{a}^2+k}{a^2} = \frac{\Lambda_0}{3} + \frac{\kappa^2\rho}{3}\left(1+\frac{\rho}{2\lambda}\right) + \frac{2\overline{m}}{a^4} - \frac{\widetilde{\kappa}^2\overline{q^2}}{a^6} - \frac{\widetilde{\kappa}^2(\Delta q)^2}{4a^6} + \frac{(\Delta B)^2}{4a^2}, \tag{82}$$

$$\frac{\ddot{a}}{a} = \frac{\Lambda_0}{3} - \frac{\kappa^2}{6}\left[\rho\left(1+\frac{2\rho}{\lambda}\right)+3p\left(1+\frac{\rho}{\lambda}\right)\right] - \frac{2\overline{m}}{a^4} + \frac{2\widetilde{\kappa}^2\overline{q^2}}{a^6} + \frac{\widetilde{\kappa}^2(\Delta q)^2}{2a^6} + \frac{C\Delta B}{2a(\rho+\lambda)} + \frac{3(p-\lambda)(\Delta B)^2}{4a^2(\rho+\lambda)}, \tag{83}$$

where Λ_0 is a true constant given as

$$2\Lambda_0 = \kappa^2\lambda + \widetilde{\kappa}^2\overline{\widetilde{\Lambda}}. \tag{84}$$

A comparison with Eqs. (A9) and (A10) gives the cosmological "constant" and the potential U_0:

$$\frac{\Lambda}{3} = \frac{\Lambda_0}{3} + \frac{\widetilde{\kappa}^2\overline{q^2}}{2a^6} + \frac{\widetilde{\kappa}^2(\Delta q)^2}{8a^6} + \frac{C\Delta B}{4a(\rho+\lambda)} + \frac{(\rho+3p-2\lambda)(\Delta B)^2}{8a^2(\rho+\lambda)}, \tag{85}$$

$$\frac{\kappa^2}{3}U_0\left(\frac{a_0}{a}\right)^4 = \frac{2\overline{m}}{a^4} - \frac{3\widetilde{\kappa}^2\overline{q^2}}{2a^6} - \frac{3\widetilde{\kappa}^2(\Delta q)^2}{8a^6} - \frac{C\Delta B}{4a(\rho+\lambda)} + \frac{(\rho-3p+4\lambda)(\Delta B)^2}{8a^2(\rho+\lambda)}. \tag{86}$$

The Friedmann equation (82) and Raychaudhuri equation (83), after suitable conversion of notation, reduce to the corresponding results of [13,6] in the case $q=0$. When the cosmological constant is the same in both bulk regions ($\Delta\widetilde{\Lambda}=0$), earlier results [8] are recovered.

In the Z_2-symmetric limit we recover the Friedmann equation given in [26]. In the absence of charge, the Friedmann and Raychaudhuri equations given in [4] and [3] emerge.

VI. CHARGED VAIDYA–ANTI-de SITTER BULK

The generalization of the 4-dimensional charged Vaidya solution [30] in a cosmological context was discussed in [31]. We will do the same here in 5 dimensions. Let us start with the bulk metric written in Eddington-Finkelstein type coordinates,

$$d\widetilde{s}^2 = -f(v,r;k)dv^2 + 2\epsilon dv dr + r^2[d\chi^2 + \mathcal{H}^2(\chi;k)(d\theta^2+\sin^2\theta d\phi^2)], \tag{87}$$

where $\epsilon=1$ holds for an outgoing v coordinate (the $v=\text{const}$ lines are ingoing), while $\epsilon=-1$ for ingoing v ($v=\text{const}$ lines outgoing). It can also be written as

$$\widetilde{g}_{ab} = -u_a u_b + n_a n_b + r^2 h_{ab}, \tag{88}$$

where

$$u = \frac{\partial}{\partial\tau} = \dot{v}\frac{\partial}{\partial v} + \dot{r}\frac{\partial}{\partial r}, \tag{89}$$

has unit negative norm, implying

$$f\dot{v} = \epsilon\dot{r} + (\dot{r}^2+f)^{1/2}. \tag{90}$$

We have again chosen the $\dot{v}>0$ root and a dot again denotes derivatives with respect to τ. Then the unit normal 1-form to both h_{ab} and u_a becomes

$$n = \pm(-1)^\sigma(-\dot{r}dv + \dot{v}dr). \tag{91}$$

Finally the 1-form field u_a is

$$u = -(\dot{r}^2 + f)^{1/2}dv + \epsilon\dot{v}dr = -d\tau. \tag{92}$$

We suppose that the bulk contains radiation (geometrical optics limit: null dust) with energy-momentum tensor

$$\widetilde{T}^{ND}_{ab} = \frac{3\beta(v,r)}{\widetilde{\kappa}^2 r^3} l_a l_b . \tag{93}$$

Here $\beta(v,r)$ determines the energy density (it has the dimension of a linear density of mass) and l is a null 1-form:

$$l = dv = \dot{v}[\pm\epsilon(-1)^\sigma n - u]. \tag{94}$$

Such radiation is ingoing for $\epsilon = 1$ and outgoing for $\epsilon = -1$. In the bulk there is also an electromagnetic contribution,

$$\widetilde{T}^{EM}_{ab} = \frac{3q^2(v)}{r^6}(u_a u_b - n_a n_b + r^2 h_{ab}), \tag{95}$$

generated by a null 5-potential

$$A_a = \frac{q(v)}{r^2} l_a . \tag{96}$$

Then the bulk Einstein equation for the metric ansatz (87), with the source term

$$\widetilde{\Pi}_{cd} = -\widetilde{\Lambda}\widetilde{g}_{ab} + \widetilde{T}^{ND}_{ab} + \widetilde{T}^{EM}_{ab} , \tag{97}$$

is solved by

$$\epsilon\beta = \frac{dm}{dv} - \frac{\widetilde{\kappa}^2 q}{r^2}\frac{dq}{dv} \tag{98}$$

and

$$f(v,r;k) = k - \frac{1}{r^2}\left[2m(v) + \frac{\widetilde{\kappa}^2\widetilde{\Lambda}}{6}r^4 - \frac{q^2(v)}{r^2}\right]. \tag{99}$$

The functions $m(v)$ and $q(v)$ are freely specifiable.

The brane is given by the embedding relations $v = v(\tau)$ [through Eq. (90)] and $r = a(\tau)$, thus we replace $(r,\dot{r})$ with $(a,\dot{a})$ in the above formulas. By construction, n is the normal to the brane. The induced metric is Eq. (41) and the extrinsic curvature becomes

$$K_{ab} = \mp(-1)^\sigma\left[\frac{2\ddot{a} + \dfrac{\partial f}{\partial a} - \epsilon\dot{v}^2\dfrac{\partial f}{\partial v}}{2(\dot{a}^2 + f)^{1/2}} u_a u_b - (\dot{a}^2 + f)^{1/2} a h_{ab}\right]. \tag{100}$$

We note that the differences with respect to the Reissner–Nordström–anti-de Sitter case arise in the definition of the function A:

$$2A_I = 2\ddot{a} + \frac{\partial f_I}{\partial a} - \epsilon_I\dot{v}_I^2\frac{\partial f_I}{\partial v}, \tag{101}$$

and in an additional term contained by C:

$$C = \frac{6\overline{q}\Delta q}{a^6} + \Delta\widetilde{\Lambda} - \frac{3\Delta(\beta\dot{v}^2)}{2\widetilde{\kappa}^2 a^3}. \tag{102}$$

Since, by virtue of Eq. (98) the condition $2\epsilon_I\beta_I + a^2\partial f_I/\partial v = 0$ holds, the expression (81) for ΔB, as well as the generalized Friedmann equation (82), are unchanged relative to the Reissner–Nordström–anti-de Sitter case. The Raychaudhuri equation (besides containing a different C) acquires two new terms on the right-hand side. These are

$$2\mathcal{K} = -\frac{\overline{\beta\dot{v}^2}}{a^3} - \frac{3\Delta B}{2\widetilde{\kappa}^2 a^4(\rho + \lambda)}\Delta(\beta\dot{v}^2). \tag{103}$$

Then the cosmological "constant" and the potential in the charged Vaidya–anti-de Sitter case are found immediately from the ones characterizing the Reissner–Nordström–anti-de Sitter case:

$$\left(\frac{\Lambda}{3}\right)_{chV\,AdS5} = \left(\frac{\Lambda}{3}\right)_{RN\,AdS5} + \mathcal{K}, \tag{104}$$

$$(U_0)_{chV\,AdS5} = (U_0)_{RN\,AdS5} - \frac{3}{\kappa^2}\left(\frac{a}{a_0}\right)^4\mathcal{K}. \tag{105}$$

[Remember that Λ and U_0 on the RHS should be computed with C given by Eq. (102).] In the Z_2-symmetric limit the Friedmann equation reduces to the one given in [14].

Equations (A1) and (A3) are again satisfied, while Eq. (A2) this time differs from the ordinary continuity equation:

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = \frac{3}{\widetilde{\kappa}^2 a^3}\Delta[\epsilon(-1)^\sigma\beta\dot{v}^2]$$

$$= \frac{3}{\widetilde{\kappa}^2 a^3}\sum_{I=L,R}\epsilon_I(-1)^{\eta_I}\beta_I\dot{v}_I^2 . \tag{106}$$

The global sign $\epsilon_I(-1)^{\eta_I}$ of the two terms in the sum above is negative for radiation leaving the brane and positive for radiation arriving to the brane. For a given equation of state $p(\rho)$, this time the expression $\rho = \rho(a)$ is different from the standard one. This is due to the fact that the brane radiates or is irradiated; thus there is no brane-energy conservation. (The Z_2-symmetric, uncharged limit of the case $\eta_L = \eta_R = 1 = \epsilon_L = \epsilon_R$ was discussed in [15].)

Obviously, the expression $\beta\dot{v}^2$ is needed for the last four equations. Of course, β depends on the freely specifiable functions $m(v)$ and $q(v)$, as given by Eq. (98), while $\dot{v}$ is determined by Eqs. (73) and (90). At this point it is useful to note, that there is no natural normalization condition for a null vector, therefore l can be freely rescaled $l \rightarrow \sigma l$ at the price that $\beta \rightarrow \beta/\sigma^2$ is also rescaled accordingly. Employing this freedom one can even shift all information about the null dust into the null vector, by choosing $\sigma = \sqrt{\beta}$. Alternatively, in the present case a simple way to proceed would be to choose $\sigma = \dot{v}^{-1}$ which has the consequence that the new linear density of mass is

$$\alpha = \beta \dot{v}^2. \tag{107}$$

Then one can interpret α in Eqs. (103)–(106) as a freely specifiable parameter. This choice was followed in [15].

However we will follow a third route, guided by the desire to have freely imposable functions with obvious meaning on the brane. The arbitrary metric functions $m(v)$ and $q(v)$, due to the embedding relation $v = v(\tau)$, can already be interpreted from a brane point of view as arbitrary functions of time $m(\tau)$ and $q(\tau)$. Their time derivatives, defined as $\dot{m} = \dot{v}\, dm/dv$ and $\dot{q} = \dot{v}\, dq/dv$, have again a natural interpretation for an observer living on the brane. One can therefore define a third linear density of mass by choosing $\sigma = (\dot{v})^{-1/2}$ as

$$\gamma = \beta \dot{v} = \epsilon \left(\dot{m} - \frac{\tilde{\kappa}^2 q}{a^2} \dot{q} \right), \tag{108}$$

and rewrite Eqs. (103)–(106) in terms of γ, which has immediate interpretation for a brane observer. Then $\dot{v}$ arises linearly in the term $\beta \dot{v}^2 = \gamma \dot{v}$ and we obtain the following expression:

$$\beta_I \dot{v}_I^2 = \dot{a} \frac{\gamma_I}{f_I} + \epsilon_I (-1)^{\eta_I} \frac{\gamma_I B_I}{f_I}. \tag{109}$$

This can be expressed in terms of $\overline{f}$, $\overline{B}$, $\overline{\gamma}$, Δf, ΔB and $\Delta \gamma$ or equivalently in terms of average values and jumps of m, $\dot{m}$, q, $\dot{q}$ and $\tilde{\Lambda}$. If this latter interpretation is chosen, an interesting feature which emerges, is the occurrence of an $\dot{a}$ term in both the Raychaudhuri and continuity equations. This can be avoided in the first choice (107), but then the relation between α and the set $dm/dv, dq/dv$ will contain it.

VII. CONCLUDING REMARKS

We have given a generic decomposition of the Einstein equations, in which the tensorial, vectorial and scalar projections are equivalent to the effective Einstein, the Codazzi and the twice contracted Gauss equation. The junction conditions applied across a brane separating two nonidentical space-times give rise to the final form of these equations. The effective Einstein equation contains a varying cosmological constant, and extra terms beyond the standard Z_2-symmetric case, which characterize the nonsymmetric embedding and the bulk matter.

The formalism can be applied for any situation. Of particular interest would be the cases of branes containing black holes or obeying cosmological symmetries. We have discussed the latter case here.

When the brane has cosmological symmetries and a perfect fluid, obeying the same symmetries, the effective Einstein equations decouple into generalized Friedmann and generalized Raychaudhuri equations. These were given in a form insensitive to the particular embedding. Only the cosmological "constant" and the potential U_0 depend on the details of the embedding and bulk matter. While Λ can be found algebraically, the potential U_0 is determined by a first order ordinary differential equation. An algorithm was given in the Appendix to study cosmology on such generalized Friedmann branes.

With a definite choice of the bulk and embedding, the situation becomes even simpler, the integration of the first order differential equation being replaced by pure algebra. We have employed this advantage first in the case of a Reissner–Nordström–anti-de Sitter bulk, then for a charged Vaidya–anti-de Sitter bulk. In both cases we have matched across the brane inner-outer regions of the bulk space-times, finding the appropriate generalized Friedmann and Raychaudhuri equations. The Raychaudhuri equation acquires peculiar terms in the radiating case, which implies a nonstandard dependence of the density on the scale factor. Our equations allow for different mass and charge functions as well as bulk cosmological constants on the two sides of the brane. However the junction does not allow for different values of k on the two sides, as k is the curvature index in the induced metric, required to be continuous. If Maxwell equations have to be satisfied in 5D, then the two charges have to be equal in magnitude but of opposite sign [32].

The equations characterizing cosmological evolution both in the charged and in the radiating case being given, the arena opens for imposing constraints from experimental data on the nonsymmetric character of the embedding as well. In the low energy regime ($\rho \ll \lambda$), for example, ΔB contributes with the radiation-like term $(3/2\tilde{\kappa}^2\lambda^2)(\Delta m \Delta \tilde{\Lambda}/a^4)$ to the Friedmann equation and it is expected that CMB-anisotropy data will constrain its magnitude, thus implicitly the non-Z_2-symmetric features of the bulk too.

While in the charged case only 3 cases should be considered (junctions of inner-inner, inner-outer and outer-outer regions), in the radiating case the direction of the radiation flow further diversifies the situation, leading to a total of 10 cases to be discussed. Some of these will be ruled out by energy conditions to be satisfied on the brane. Further subtleties of the radiating case include whether in the inner regions there are radiating stellar objects, black holes or naked singularities (encountered in the 4D Vaidya solution as well). Investigations into these issues are under way.

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APPENDIX: DYNAMICS OF PERFECT FLUID FRIEDMANN BRANES IN ARBITRARY BULK. AN ALGORITHM

Let us consider the physically interesting case of $d=4$. For a Friedmann brane with perfect fluid, embedded in a non-Z_2 way into a bulk containing some energy-momentum tensor $\widetilde{\Pi}_{cd}$, we present here an algorithmic way of solving the relevant equations.

Constraints on the bulk matter:

$$h_{ab}\Delta(g^{ac}n^d\widetilde{\Pi}_{cd})=0, \tag{A1}$$

$$\Delta(u^c n^d\widetilde{\Pi}_{cd})=\dot{\rho}+3\frac{\dot{a}}{a}(\rho+p). \tag{A2}$$

Once solved by some choice for the matter fields in the bulk, we can pass to the *constraints on the embedding:*

$$\widetilde{\kappa}^2\overline{(g_a^c n^d\widetilde{\Pi}_{cd})}=\nabla_c\overline{K}_a^c-\nabla_a\overline{K}, \tag{A3}$$

$$\Delta(n^a n^b\widetilde{\Pi}_{ab})=(p-\lambda)\overline{K}+(\rho+p)u_a u_b\overline{K}^{ab}. \tag{A4}$$

The mean value of the extrinsic curvature, once found, gives rise to the trace and tracefree parts of the extrinsic curvature term $\overline{L}_{ab}$:

$$\overline{L}=\overline{K}_{ab}\overline{K}^{ab}-\overline{K}^2, \tag{A5}$$

$$\overline{L}_{ab}^{TF}=\overline{K}_{ab}\overline{K}-\overline{K}_{ac}\overline{K}_b^c+\frac{\overline{L}}{4}g_{ab}, \tag{A6}$$

and therefore to the *cosmological constant*

$$\Lambda=\frac{\kappa^2\lambda}{2}-\frac{\overline{L}}{4}-\frac{\widetilde{\kappa}^2}{2}\overline{(n^c n^d\widetilde{\Pi}_{cd})}. \tag{A7}$$

The first order differential equation determines *the unknown potential* U_0:

$$\kappa^2\left(\frac{a_0}{a}\right)^4\dot{U}_0+\dot{\Lambda}+\kappa^2\left(1+\frac{\rho}{\lambda}\right)\Delta(u^c n^d\widetilde{\Pi}_{cd})=0. \tag{A8}$$

Then Λ and U_0 can be inserted into the generalized *Friedmann and Raychaudhuri equations*:

$$\frac{\dot{a}^2+k}{a^2}=\frac{\Lambda}{3}+\kappa^2\frac{\rho}{3}\left(1+\frac{\rho}{2\lambda}\right)+\frac{\kappa^2}{3}\left(\frac{a_0}{a}\right)^4U_0, \tag{A9}$$

$$\frac{\ddot{a}}{a}=\frac{\Lambda}{3}-\frac{\kappa^2}{6}\left[\left(1+2\frac{\rho}{\lambda}\right)\rho+3\left(1+\frac{\rho}{\lambda}\right)p\right]-\frac{\kappa^2}{3}\left(\frac{a_0}{a}\right)^4U_0. \tag{A10}$$

Up to this point, only the mean value of the extrinsic curvature has to be solved for. (Still, its jump contributed implicitly to the functional form of the generalized Friedmann and Raychaudhuri equations.)

Finally, in order to study the *off-brane evolution*, the extrinsic curvature is needed on both sides on the brane

$$2K_{ab}^{\pm}=2\overline{K}_{ab}\pm\Delta K_{ab}. \tag{A11}$$

Thus we have to determine its jump from the *Lanczos equation*

$$3\Delta K_{ab}=-\widetilde{\kappa}^2[(2\rho+3p-\lambda)u_a u_b+(\rho+\lambda)a^2h_{ab}]. \tag{A12}$$

The evolution in the off-brane direction of the brane gravitational variables is determined by

$$\mathcal{L}_{\mathbf{n}}g_{ab}=2K_{ab}^{\pm}, \tag{A13}$$

$$\mathcal{L}_{\mathbf{n}}K_{ab}^{\pm}=K_{ac}^{\pm}K_b^{\pm c}-\frac{\widetilde{\kappa}^2}{3}(g_a^c g_b^d\widetilde{\Pi}_{cd}^{\pm})^{TF}-\overline{\mathcal{E}}_{ab}\mp\frac{1}{2}\Delta\mathcal{E}_{ab}-\frac{g_{ab}}{4}E^{\pm}+(\nabla_b\alpha_a-\alpha_b\alpha_a)^{\pm}, \tag{A14}$$

with

$$\overline{\mathcal{E}}_{ab}=\overline{L}_{ab}^{TF}+\frac{2\widetilde{\kappa}^2}{3}\overline{(g_a^c g_b^d\widetilde{\Pi}_{cd})^{TF}}-\kappa^2\left(\frac{a_0}{a}\right)^4U_0\left(u_a u_b+\frac{a^2}{3}h_{ab}\right), \tag{A15}$$

$$\Delta\mathcal{E}_{ab}=\frac{2\widetilde{\kappa}^2}{3}\Delta(g_a^c g_b^d\widetilde{\Pi}_{cd})^{TF}-\widetilde{\kappa}^2\left[(\rho+p)\overline{K}u_a u_b-2(\rho+p)u_{(a}\overline{K}_{b)}^c u_c-\frac{\rho+3p-2\lambda}{3}\overline{K}_{ab}\right]^{TF}, \tag{A16}$$

$$E^{\pm}=\widetilde{\kappa}^2\left(n^a n^b\widetilde{\Pi}_{ab}^{\pm}-\frac{1}{3}\widetilde{\Pi}^{\pm}\right), \tag{A17}$$

and $\alpha^b=n^c\widetilde{\nabla}_c n^b$ an acceleration-like quantity, the curvature of n^a, which can be freely specified.

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$$[H(\chi;k),\tau_{ab},g_{ab,}\kappa^2,\kappa^2/6\lambda,\tilde{\kappa}^2\tilde{\Lambda},\Lambda_0,\lambda,2m]$$

$$\rightarrow[R_k(\chi),t_{AB},h_{AB},8\pi G_4,\pi^4G_5^2,\Lambda=-6l^2,\Lambda_{eff},\rho_\lambda,\mu].$$

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