Vaidya's "Kerr-Einstein" metric cannot be matched to the Kerr metric

D. P. G. Cox*

School of Mathematical Sciences, Queen Mary University of London, Mile End Road, London E1 4NS, United Kingdom

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An attempt is made to match the Kerr metric with Vaidya's description of a black hole in a cosmological background. The attempt is unsuccessful and shows that the object at the center of Vaidya's metric is certainly non-Kerr like.

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I. INTRODUCTION

This work is motivated by the need for solutions of Einstein's equation that might characterize a black hole in a background which is not asymptotically flat. Specifically, a class of space-times found by Vaidya [1] is investigated. Vaidya's Einstein-Kerr (VEK) metric has the two interesting properties of either tending to a Kerr metric as the cosmological radius approaches infinity or tending to a Friedmann-Lemaître-Robertson-Walker universe as the central mass vanishes. Nayak *et al.* [2] have successfully matched the nonrotating case of Vaidya's solution with a Schwarzschild interior and an Einstein exterior. The physical properties of the composite space-time have subsequently been investigated [3].

Here is an attempt to generalize the previous work by first trying to match the VEK space-time to a Kerr interior. Sadly this matching is not possible. This shows that the Vaidya's space-time does not represent "Kerr-in-Einstein." Without some interior matching it does not represent a black hole in any useful sense because the equation of state is not physically acceptable in this region. This is a pity because possible models of black holes in nonflat spaces are scarce despite their importance in physical situations. The purposes of this article are to serve as a caution for those who might try the matching and to present a first test for any solution that might be a candidate for the interior region.

The overall strategy will be to assume that a matching is possible and seek a relationship between the parameter spaces of the two metrics. Subsequent sections are arranged as follows: first the metrics are presented in the forms which are convenient for the problem; it is shown that the metrics can be matched in only one way at the limit of stationarity; an examination of the first fundamental form shows that no acceptable relationship exists between the Kerr metric and Vaidya's metric.

II. THE METRICS

The first step is to transform the Vaidya metric into a convenient form [4],

$$\begin{aligned} ds^2 &= (1-2m\mu)dt^2 - 4m\mu a\sin^2\alpha\, dt\, d\beta \\ &\quad - \frac{D}{D(1-2m\mu)+a^2\sin^2\alpha}dr^2 - \frac{D}{1-\dfrac{a^2}{R^2}\sin^2\alpha}d\alpha^2 \\ &\quad - ((1+2m\mu)a^2\sin^4\alpha + D\sin^2\alpha)d\beta^2, \end{aligned}$$

where $\mu = (R/D)\sin(r/R)\cos(r/R)$ and $D = (R^2 - a^2)\sin^2(r/R) + a^2\cos^2\alpha$. Here a is interpreted as the angular momentum, m as the central mass and R as the cosmological radius parameter. It can be seen that this tends to the familiar Boyer-Lindquist version of Kerr (Eq. 2.13 in [5]),

$$\begin{aligned} ds^2 &= \left(1 - \frac{2M\bar{r}}{\bar{r}^2 + A^2\cos^2\theta}\right)d\bar{t}^2 - \frac{4M\bar{r}A\sin^2\theta}{\bar{r}^2 + A^2\cos^2\theta}d\bar{t}\,d\phi \\ &\quad - \frac{\bar{r}^2 + A^2\cos^2\theta}{\bar{r}^2 + A^2 - 2M\bar{r}}d\bar{r}^2 - (\bar{r}^2 + A^2\cos^2\theta)d\theta^2 \\ &\quad - \left((\bar{r}^2 + A^2)\sin^2\theta + \frac{2M\bar{r}A^2\sin^4\theta}{\bar{r}^2 + A^2\cos^2\theta}\right)d\phi^2, \end{aligned}$$

in the limit $R \to \infty$. In this limit the coordinates (t, r, α, β) and parameters (m, a) of the VEK form become simply the coordinates $(\bar{t}, \bar{r}, \theta, \phi)$ and parameters (M, A) of the Kerr metric.

In the VEK metric, the only place where the Ricci tensor vanishes is at the stationary limit (where $g_{tt} = 0$) and so this is the only surface suitable for matching with a vacuum solution such as Kerr. The stationary limit is at $r = r_{sl}$ where

$$\tan\left(\frac{r_{sl}}{R}\right) = \frac{mR + \sqrt{m^2R^2 - [R^2 - a^2\sin^2\alpha]a^2\cos^2\alpha}}{R^2 - a^2\sin^2\alpha} =: q(\alpha).$$

Here the function $q(\alpha)$ is introduced because it is much easier to work with than surds. The explicit appearance of α can be removed by noting

$$a^2\cos^2\alpha = \frac{2mRq - q^2(R^2 - a^2)}{1 + q^2},$$

$$a^2\sin^2\alpha = \frac{a^2 - 2mRq + q^2R^2}{1 + q^2}.$$

*Email address: d.cox@qmul.ac.uk

Writing $q' = dq/d\alpha$ note that

$$q'^2 = \frac{q(q(a^2-R^2)+2mR)(q^2+1)^2(a^2-2mRq+q^2R^2)}{[q(a^2-R^2)+mR(1-q^2)]^2}.$$

By substituting, $dr^2 \to R^2 q'^2 (1+q^2)^{-2} d\alpha^2$, the following metric on the stationary limit 3-surface is obtained:

$$ds^2 = -2\frac{a^2-2mRq+q^2R^2}{a(1+q^2)}dt\,d\beta - 2\frac{(a^2-2mRq+q^2R^2)(a^2-mRq+q^2R^2)}{a^2(1+q^2)^2}d\beta^2 - 2\frac{R^5m^3q(1+q^2)^2}{(R^2-a^2+2mRq)(-qa^2+mRq^2+qR^2-mR)^2}d\alpha^2. \tag{1}$$

Now go through the same steps with the Kerr metric. Here we use capital *M* and *A* rather than the familiar *m* and *a*, so that the parameters in the two metrics considered are clearly distinguishable. The stationary limit is where $g_{\overline{tt}} = 0$, or where

$$\overline{r_{sl}} = M + \sqrt{M^2 - A^2\cos^2\theta} =: Q(\theta)$$

say. Writing $Q' = dQ/d\theta$ note that

$$Q'^2 = \frac{(2MQ-Q^2)(A^2-2MQ+Q^2)}{M^2-2MQ+Q^2}.$$

Substituting $d\overline{r^2} \to Q'^2 d\theta^2$, we obtain the metric on the 3-surface,

$$ds^2 = -2\frac{A^2-2MQ+Q^2}{A}d\bar{t}\,d\phi - 2\frac{(A^2-2MQ+Q^2)(A^2-MQ+Q^2)}{A^2}d\phi^2 - 2\frac{M^3Q}{M^2-2MQ+Q^2}d\theta^2. \tag{2}$$

III. COVARIANT DESCRIPTION

The next procedure shows that if the metrics can be matched at the limit of stationarity, the coordinates α and θ are essential and can be matched in only one way. Ignoring for a moment the associations implied by the labelling, how should the basis forms of the three-surfaces relate?

Both Eqs. (1) and (2) are of the form

$$ds^2 = -2J\sin^2(x^1)dx^0dx^2 - 2K(x^1)\sin^2(x^1)(dx^2)^2 - L^2(x^1)(dx^1)^2, \tag{3}$$

where J is the momentum parameter a or A, the coordinates are $\{x^i\} = \{t,\alpha,\beta\}$ or $\{x^I\} = \{\bar{t},\theta,\phi\}$ and K and L are given functions for each metric. From Eq. (3) it is clear the three-surface can be described as a foliation of timelike two-surfaces of constant x^1. In order to obtain covariant descriptions which we can compare, it is necessary to examine the Killing vectors. At any point the two-surfaces have a tangent space with a basis $\{\partial_{x^0}, \partial_{x^2}\}$ composed of Killing vectors. There might be a third Killing vector in this tangent space. If there were such a vector, basis vectors in the two-surface found by the (three-dimensional version of the) classification procedure described, for example, in Chap. 9 of [6], could only be covariantly determined up to a boost at each point. The quantity ω in the following form of Eq. (3) parametrizes this possible boost:

$$ds^2 = -2(J\omega dx^0 + K\omega dx^2)[\omega^{-1}\sin^2(x^1)dx^2] - (L\,dx^1)^2.$$

In this expression, the one-forms containing ω are the null one-forms in the surface $x^1 = \text{const}$, and will therefore be covariantly determined, up to the boost parameter, if the surfaces $x^1 = \text{const}$ themselves are covariantly determined. The corresponding choice of basis vectors is then

$$\left\{ e_{\hat{0}} = \frac{1}{J\omega}\partial_{x^0}, e_{\hat{1}} = -\frac{K\omega}{J\sin^2(x^1)}\partial_{x^0} + \frac{\omega}{\sin^2(x^1)}\partial_{x^2}, e_{\hat{2}} = \frac{1}{L}\partial_{x^1} \right\}.$$

In this triad the nonvanishing components of the Ricci tensor are

$$R_{\hat{2}\hat{2}} = \frac{2}{L^2} + \frac{2\cos(x^1)}{L^3\sin(x^1)}\frac{dL}{dx^1},$$

$$R_{\hat{0}\hat{1}} = -\frac{\cos^2(x^1)}{L^2\sin^2(x^1)} + \frac{R_{\hat{2}\hat{2}}}{2},$$

$$R_{\hat{1}\hat{1}} = -\frac{2\omega^2\cos(x^1)}{L^2\sin^3(x^1)}\frac{dK}{dx^1} - \frac{\omega^2}{L^2\sin^2(x^1)}\frac{d^2K}{d(x^1)^2} + \frac{\omega^2}{L^3\sin^2(x^1)}\frac{dK}{dx^1}\frac{dL}{dx^1}. \tag{4}$$

The invariant Ricci scalar depends only on x^1

$$R=\frac{2\cos^2(x^1)}{L^2\sin^2(x^1)}-\frac{4}{L^2}-\frac{4\cos(x^1)}{L^3\sin(x^1)}\frac{dL}{dx^1}.$$

Since the forms of L in the metrics under consideration imply that this is not constant for any interval in x^1, there is no Killing vector with a component in the x^1 direction and the surfaces $x^1=\mathrm{const}$ are uniquely defined by the geometry as the orbits of the isometry group [i.e., the coordinate x^1 used above is in fact geometrically defined up to $x^1\to f(x^1)$]. Therefore the vectors $\partial/\partial\alpha$ and $\partial/\partial\theta$ have the same significance as the vectors orthogonal to those surfaces, and must be parallel to one another: thus $\alpha=F(\theta)$ for some function F. Restricting now our attention to the two-surfaces of constant x^1, the noninvariance of $R_{\hat{1}\hat{1}}$ under change of ω shows that there is no boost symmetry so we have only two Killing vectors. Furthermore, the uniqueness of the surfaces $x^1=\mathrm{const}$, and of null directions in a timelike plane, show that the basis vectors chosen are uniquely geometrically determined up to the boost ω. Hence we can use the invariance of $R_{\hat{0}\hat{1}}$ and $R_{\hat{2}\hat{2}}$ under boosts to show that $\cos^2(x^1)/L^2\sin^2(x^1)$ is an invariant [from Eq. (4)]. Evaluating this for the two metrics implies that at the junction

$$g_{\alpha\alpha}\tan^2(\alpha)=g_{\theta\theta}\tan^2(\theta), \tag{5}$$

which will be useful in determining $F(\theta)$.

IV. FIRST FUNDAMENTAL FORMS

This section completes the discussion of how the coordinate systems of the two metrics must relate in the first fundamental forms of the metrics. The vectors ∂_{x^2} in each case must be identified because they are the unique Killing vectors with closed circular orbits of period 2π in the two solutions. Adopting the notation

$$\Lambda^{x^I}_{x^i}=\frac{\partial x^I}{\partial x^i}$$

this can be stated as $\Lambda^{\phi}_{\beta}=1$. The tangency of the two Killing vectors to surfaces of constant x^1 can also be put as $\Lambda^{\theta}_{t}=0$, $\Lambda^{\theta}_{\beta}=0$.

If the matching is to work, the 3-surfaces must describe the same embedded metric (or first fundamental form) and so should be subject to the transformation law

$$g_{x^ix^j}=g_{x^Ix^J}\Lambda^{x^I}_{x^i}\Lambda^{x^J}_{x^j}.$$

From this,

$$g_{\beta\beta}=2g_{\bar{t}\phi}\Lambda^{\bar{t}}_{\beta}+g_{\phi\phi}.$$

Since we require $g_{\beta\beta}=g_{\phi\phi}$ and we cannot have $g_{\bar{t}\phi}=0$ for all values of θ then clearly,

$$\Lambda^{\bar{t}}_{\beta}=0.$$

Also

$$g_{t\beta}=g_{\bar{t}\phi}\Lambda^{\bar{t}}_{t}+g_{\phi\phi}\Lambda^{\phi}_{t},$$

so that

$$\Lambda^{\bar{t}}_{t}=\frac{g_{t\beta}-g_{\phi\phi}\Lambda^{\phi}_{t}}{g_{\bar{t}\phi}}.$$

Now

$$g_{tt}=0=2g_{\bar{t}\phi}\Lambda^{\phi}_{t}\Lambda^{\bar{t}}_{t}+g_{\phi\phi}(\Lambda^{\phi}_{t})^2$$

becomes

$$0=\Lambda^{\phi}_{t}(2g_{t\beta}-g_{\phi\phi}\Lambda^{\phi}_{t}).$$

Hence we must consider two cases. Either $\Lambda^{\phi}_{t}=0$ and $\Lambda^{\bar{t}}_{t}=g_{t\beta}/g_{\bar{t}\phi}$ or else $\Lambda^{\phi}_{t}=2g_{t\beta}/g_{\phi\phi}$ and $\Lambda^{\bar{t}}_{t}=-g_{t\beta}/g_{\bar{t}\phi}$. However, in both cases, we find the equations

$$g_{\alpha\beta}=0=g_{\bar{t}\phi}(\Lambda^{\bar{t}}_{\alpha}+\Lambda^{\phi}_{\alpha}\Lambda^{\bar{t}}_{\beta})+g_{\phi\phi}\Lambda^{\phi}_{\alpha},$$

$$g_{t\alpha}=0=g_{\bar{t}\phi}(\Lambda^{\bar{t}}_{t}\Lambda^{\phi}_{\alpha}+\Lambda^{\phi}_{t}\Lambda^{\bar{t}}_{\alpha})+g_{\phi\phi}\Lambda^{\phi}_{t}\Lambda^{\phi}_{\alpha}$$

imply that $\Lambda^{\phi}_{\alpha}=\Lambda^{\bar{t}}_{\alpha}=0$. The remaining equation is then

$$g_{\alpha\alpha}=g_{\theta\theta}(\Lambda^{\theta}_{\alpha})^2.$$

Together with Eq. (5) we obtain $\alpha=\pm\theta+\mathrm{const}$, and α and θ both have range $[0,\pi]$. Without loss of generality we can take $\alpha=\theta$. To show that this cannot work, substitute $\alpha=\theta$ into any of the above equations relating the nonzero metric coefficients. Using $g_{\beta\beta}=g_{\phi\phi}$ I obtained the following quartic in q:

$$(-M^2a^2A^2R^2-M^2a^4A^2+2A^2R^4a^2-a^4R^4+2M^2a^4R^2-A^4R^4)q^4-2mR(a-A)(a+A)(-a^2R^2+M^2a^2+2A^2R^2)q^3$$
$$+[-4A^4m^2R^2+2M^2a^4R^2+2M^2a^6-M^2a^2A^2R^2-2a^6R^2+4A^2R^2a^4+4A^2m^2R^2a^2-a^4m^2R^2-3M^2a^4A^2$$
$$-2A^4R^2a^2]q^2-2a^2mR(a-A)(a+A)(-a^2+2A^2+M^2)q+a^4(a-A)(a+A)(-a^2+A^2+2M^2)=0$$

having divided by an overall factor of $a^2(1+q^2)$. By comparing the coefficient of q and the constant term it is clear that $a^2=A^2$. Using this information, the coefficient of q^4 can only vanish for the very special case of $R^2=a^2$. Since we are only interested in relating the two parameter spaces, such a restriction means that the two metrics cannot be matched.

V. CONCLUSIONS

The matching of Kerr with Vaidya's VEK solution is certainly not possible, but this does not preclude other interior vacuum solutions. In the spherically symmetric subcase, the nonrotating VEK case studied in [2], the only possible vacuum interior is the Schwarzschild metric, since it is the unique spherically symmetric solution of the vacuum Einstein's equations. Kerr is the unique solution which is stationary, axisymmetric, nonsingular at the Killing horizon and is asymptotically flat. However, a vacuum interior for VEK need not be asymptotically flat, and hence solutions other than Kerr could arise, i.e., there could be another stationary axisymmetric vacuum which is a black hole, in the sense of possessing a Killing horizon and/or an ergosphere, and which does match VEK. The existence of some such interior solution would certainly be of interest as the VEK spacetime itself becomes unphysical in between the limit of stationarity and the event horizon (for example the pressure of the fluid becomes negative in this region).

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