Gauge problem in the gravitational self-force: First post-Newtonian force in the Regge-Wheeler gauge

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We discuss the gravitational self-force on a particle in a black hole space-time. For a point particle, the full (bare) self-force diverges. It is known that the metric perturbation induced by a particle can be divided into two parts, the direct part (or the S part) and the tail part (or the R part), in the harmonic gauge, and the regularized self-force is derived from the R part which is regular and satisfies the source-free perturbed Einstein equations. In this paper, we consider a gauge transformation from the harmonic gauge to the Regge-Wheeler gauge in which the full metric perturbation can be calculated, and present a method to derive the regularized self-force for a particle in circular orbit around a Schwarzschild black hole in the Regge-Wheeler gauge. As a first application of this method, we then calculate the self-force to first post-Newtonian order. We find the correction to the total mass of the system due to the presence of the particle is correctly reproduced in the force at the Newtonian order.

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I. INTRODUCTION

Thanks to recent advances in technology, an era of gravitational wave astronomy has arrived. There are already several large-scale laser interferometric gravitational wave detectors that are in operation in the world. Among them are the Laser Interferometric Gravitational Wave Observatory (LIGO) [1], GEO-600 [2] and TAMA300 [3]. VIRGO [4] expected to start its operation soon. The primary targets for these ground-based detectors are inspiraling compact binaries, and they are expected to be detected in the near future.

On the other hand, there is a future space-based interferometric detector project the Laser Interferometer Space Antenna (LISA) [5] that can detect gravitational waves from solar-mass compact objects orbiting supermassive black holes. There is also a future plan called DECIGO [6]. To extract out physical information of such binary systems from detected gravitational wave signals, it is essential to know the theoretical gravitational waveforms accurately. The black hole perturbation approach is most suited for this purpose. In this approach, one considers gravitational waves emitted by a point particle that represents a compact object orbiting a black hole, assuming the mass of the particle (μ) is much less than that of the black hole (M); $\mu \ll M$.

In the lowest order in the mass ratio $(\mu/M)^0$, the orbit of the particle can be represented as a geodesic on the background geometry of a black hole. Already in this lowest order, by combining with the assumption of adiabatic orbital evolution, this approach has been proved to be very powerful for evaluating general relativistic corrections to the gravitational waveforms, even for neutron-star–neutron-star (NS-NS) binaries [7].

In the next order, the orbit deviates from the geodesic on the black hole background because the space-time is perturbed by the particle. We can interpret this deviation as the effect of the self-force on the particle itself. Since it is essential to take account of this deviation to predict the orbital evolution accurately, we have to derive the equation of motion that includes the self-force on the particle. The self-force is formally given by the tail part (or the R part by Detweiler and Whiting [8]) of the metric perturbation which is regular at the location of the particle.

The gravitational self-force is, however, not easily obtainable. There are two main reasons. First, the full (bare) metric perturbation due to a point particle diverges at the location of the particle, hence so does the self-force. As mentioned above, one has to identify the R part of the metric perturbation to obtain a meaningful self-force. However, the R part cannot be determined locally but depends on the whole history of the particle. Therefore, one usually identifies the divergent part which can be evaluated locally (called the S part) to a necessary order and subtract it from the full metric perturbation. This identification of the S part is sometimes called the subtraction problem. Second, the regularized self-force is formally defined only in the harmonic gauge because the form of the S part is known only in the harmonic gauge, whereas the metric perturbation of a black hole geometry can be calculated only in the ingoing or outgoing radiation gauge in the Kerr background, or in the Regge-Wheeler gauge in the Schwarzschild background. Hence, one has to find a gauge transformation to express the full metric perturbation and the divergent part in the same gauge. This is called the gauge problem.

In this paper, as a first step toward a complete derivation of the gravitational self-force, we consider a particle orbiting a Schwarzschild black hole, and propose a method to calculate the regularized self-force by solving the subtraction and gauge problems simultaneously. Namely, we develop a method to regularize the self-force in the Regge-Wheeler gauge. The regularization is done by the "mode decomposition regularization" [9], which is effectively the same in the present case as the "mode-sum regularization" developed in [10–12].

Recently, Barack and Ori [13] proposed what they call the intermediate gauge approach to the gauge problem. Applying this method, the gravitational self-force for an orbit plunging straight into a Schwarzschild black hole was calculated by Barack and Lousto [14]. It is noted that, although their approach is philosophically quite different from our present approach, practically both approaches turn out to give the same result as far as the Regge-Wheeler gauge calculations are concerned.

As for the case of the Kerr background, the only known gauge in which the metric perturbation can be evaluated is the radiation gauge formulated by Chrzanowski [15]. However, the Chrzanowski construction of the metric perturbation becomes ill-defined in the neighborhood of the particle, i.e., the Einstein equations are not satisfied there [13]. Some progress was made by Ori [16] to obtain the correct, full metric perturbation in the Kerr background. The regularization parameters in the mode-sum regularization for the Kerr case are calculated by Barack and Ori [17].

The paper is organized as follows. In Sec. II we briefly review the situation of the self-force problem and explain our strategy. In Sec. III we give the regularization prescription under the Regge-Wheeler gauge condition. In Sec. IV we calculate the full metric perturbation and the full force in the Regge-Wheeler gauge with the Regge-Wheeler-Zerilli formalism. In Sec. V we evaluate the singular, divergent part in the harmonic gauge by local analysis at the particle location and expand it in the Fourier-harmonic form. In Sec. VI we calculate the S part under the Regge-Wheeler gauge condition by using the gauge transformation. By subtracting this S part from the full force evaluated in Sec. IV, we obtain the regularized gravitational self-force in Sec. VII. Finally, we summarize our calculation and discuss the future work in Sec. VIII. Some details of the calculations as well as discussions on the $\ell=0$ and 1 modes are given in Appendixes A–F.

II. GAUGE PROBLEM

We consider the linearized metric perturbation

$$h_{\mu\nu}=\tilde{g}_{\mu\nu}-g_{\mu\nu}, \tag{2.1}$$

where $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ is the background and the perturbed metric, respectively. Here we define the force due to the metric perturbation as the part that gives rise to a deviation from the background geodesic:

$$\frac{d^2 z^\alpha}{d\tau^2}+\Gamma^\alpha_{\mu\nu}\frac{dz^\mu}{d\tau}\frac{dz^\nu}{d\tau}=\frac{1}{\mu}F^\alpha[h], \tag{2.2}$$

where $z^\alpha(\tau)$ is an orbit of the particle parametrized by the background proper time [i.e., $g_{\mu\nu}(dz^\mu/d\tau)(dz^\nu/d\tau)=-1$]. From the geodesic equation on $\tilde{g}_{\mu\nu}$, we obtain

$$F^\alpha[h]=-\mu P^\alpha_\beta\biggl(\bar{h}_{\beta\gamma;\delta}-\frac{1}{2}g_{\beta\gamma}\bar{h}^\epsilon{}_{\epsilon;\delta}-\frac{1}{2}\bar{h}_{\gamma\delta;\beta}+\frac{1}{4}g_{\gamma\delta}\bar{h}^\epsilon{}_{\epsilon;\beta}\biggr)u^\gamma u^\delta, \tag{2.3}$$

where $P_\alpha{}^\beta=\delta_\alpha{}^\beta+u_\alpha u^\beta$, $\bar{h}_{\alpha\beta}=h_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}h_\mu{}^\mu$ and $u^\alpha=dz^\alpha/d\tau$.

In the case that the perturbation is produced by a point particle, however, we face the problem that $h_{\mu\nu}$ diverges at the location of the particle, and so does the force. Therefore, we cannot naively apply the above calculation to obtain the self-force of the particle. Mino, Sasaki and Tanaka [18] and Quinn and Wald [19] gave a formal answer to this problem by considering the metric perturbation in the harmonic gauge. According to them, the metric perturbation in the vicinity of the orbit can be divided into two parts: the direct part and the tail part. The direct part has support only on the past null cone of the field point x^μ and diverges in the limit $x^\mu\to z^\mu(\tau)$. The tail part has support inside the past null cone and gives the physical self-force which is regular at the location of the particle. But it is almost impossible to calculate the tail part of the metric perturbation directly, because it depends on the global structure of the space-time as well as on the history of the particle motion. In contrast, the direct part can be evaluated locally in terms of geometrical quantities. Hence, instead of directly calculating the tail part, we consider the subtraction of the direct part from the full metric perturbation, where the latter can be calculated in principle by the Regge-Wheeler-Zerilli or Teukolsky formalism for black hole perturbations [20–24].

From the fact that F^α is a linear differential operator on $h_{\mu\nu}$ (with a suitable extension of u^μ off the particle trajectory), we can calculate the self-force by subtracting the direct part from the full force under the harmonic gauge as

$$\lim_{x\to z(\tau)}F_\alpha[h^{\mathrm{tail,H}}(x)]=\lim_{x\to z(\tau)}\{F_\alpha[h^{\mathrm{full,H}}(x)]-F_\alpha[h^{\mathrm{dir,H}}(x)]\}, \tag{2.4}$$

where the superscript H stands for the harmonic gauge. When we perform this subtraction, the full metric perturbation and the direct part must be evaluated in the harmonic gauge because this division is meaningful only in this gauge.

However, it is difficult to obtain the full metric perturbation directly in the harmonic gauge. In order to overcome this difficulty, one possibility is to perform the gauge transformation to the harmonic gauge from the gauge in which the full metric perturbation is obtained. In our previous paper [23], we investigated this problem for the Schwarzschild case, namely, we formulated a method to perform the gauge transformation from the Regge-Wheeler (RW) gauge to the harmonic gauge. We expressed the gauge transformation equations in the Fourier-harmonic expanded form and derived a set of decoupled equations for the coefficients of each mode. Applications of this method are now under study.

Recently, Detweiler and Whiting found a slight but important modification of the above division of the metric perturbation [8]. The new direct part, called the S part, $h^{\mathrm{S,H}}_{\mu\nu}$, is constructed to be an inhomogeneous solution of the linearized Einstein equations (in the harmonic gauge) as

$$\overline{h}^{\mathrm{full/S,H};\alpha}_{\mu\nu;\alpha}+2R_{\mu}{}^{\alpha}{}_{\nu}{}^{\beta}\overline{h}^{\mathrm{full/S,H}}_{\alpha\beta}=-16\pi T_{\mu\nu}. \tag{2.5}$$

The new tail part, called the R part, $h^{\mathrm{R,H}}_{\mu\nu}$, is then a homogeneous solution. Since the S and R parts are both the solutions of the Einstein equations, we can define the S and R parts in another gauge, which are also the solutions of the Einstein equations, by performing the gauge transformation of each part. Therefore, we can consider the subtraction procedure under some other convenient gauge by transforming the S part from the harmonic gauge to the desired gauge. Thus, another, perhaps more promising, possibility is to formulate a method to derive the S part in the Regge-Wheeler or radiation gauge, where we have formalisms to evaluate the full metric perturbation, and to obtain the regularized self-force by subtracting the S part in this gauge. In this paper we focus on the Schwarzschild case and consider the subtraction in the Regge-Wheeler gauge.

To subtract the S part, we adopt the mode decomposition regularization [9]. In this method, the subtraction procedure (2.4) is done at each harmonic mode. The full force is obtained in the form of the Fourier-harmonic expansion. The Fourier (frequency) integral can be easily done in the case of circular orbits. On the other hand, the S part is known only in the vicinity of the particle. Hence, one has to extend it over the sphere to obtain its harmonic coefficients. This procedure introduces some ambiguity in the harmonic expansion of the S part. In particular, each harmonic mode obtained by this extension has no physical significance by itself. The physical significance is recovered only after we sum over all the modes. Because of this ambiguity, we have to treat the $\ell=0$ and 1 modes with special care, as will be shown later.

III. SELF-FORCE IN THE REGGE-WHEELER GAUGE

The Schwarzschild metric is given in the standard Schwarzschild coordinates as

$$g_{\mu\nu}dx^{\mu}dx^{\nu}=-f(r)dt^2+f(r)^{-1}dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2),\quad f(r)=1-\frac{2M}{r}. \tag{3.1}$$

We denote the location of the particle at its proper time $\tau=\tau_0$ as

$$\{z_0^{\alpha}\}=\{z^{\alpha}(\tau_0)\}=\{t_0,r_0,\theta_0,\phi_0\}. \tag{3.2}$$

Formally, the gravitational self-force acting on the particle is given by the tail part in the harmonic gauge, as expressed in the left-hand side of Eq. (2.4). Using the notions of the S and R parts introduced by Detweiler and Whiting [8], it may be rewritten as

$$\begin{aligned}F^{\mathrm{H}}_{\alpha}(\tau)&=\lim_{x\to z(\tau)}F_{\alpha}[h^{\mathrm{R,H}}_{\mu\nu}](x)\\&=\lim_{x\to z(\tau)}F_{\alpha}[h^{\mathrm{full,H}}_{\mu\nu}-h^{\mathrm{S,H}}_{\mu\nu}](x)\\&=\lim_{x\to z(\tau)}\{F_{\alpha}[h^{\mathrm{full,H}}_{\mu\nu}](x)-F_{\alpha}[h^{\mathrm{S,H}}_{\mu\nu}](x)\},\end{aligned} \tag{3.3}$$

where $h^{\mathrm{S,H}}_{\mu\nu}$ and $h^{\mathrm{R,H}}_{\mu\nu}$ denote the S and R parts, respectively, of the metric perturbation in the harmonic gauge. The S part can be calculated by the local coordinate expansion [9].

Now, we consider the gauge transformation from the harmonic gauge to the RW gauge defined by

$$x^{\mathrm{H}}_{\mu}\to x^{\mathrm{RW}}_{\mu}=x^{\mathrm{H}}_{\mu}+\xi^{\mathrm{H}\to\mathrm{RW}}_{\mu}, \tag{3.4}$$

$$h^{\mathrm{H}}_{\mu\nu}\to h^{\mathrm{RW}}_{\mu\nu}=h^{\mathrm{H}}_{\mu\nu}-2\nabla_{(\mu}\xi^{\mathrm{H}\to\mathrm{RW}}_{\nu)}, \tag{3.5}$$

where $\xi^{\mathrm{H}\to\mathrm{RW}}_{\mu}$ is the generator of the gauge transformation. Then the self-force in the RW gauge is given by

$$\begin{aligned}F^{\mathrm{RW}}_{\alpha}(\tau)&=\lim_{x\to z(\tau)}F_{\alpha}[h^{\mathrm{R,RW}}]\\&=\lim_{x\to z(\tau)}F_{\alpha}[h^{\mathrm{R,H}}-2\nabla\xi^{\mathrm{H}\to\mathrm{RW}}[h^{\mathrm{R,H}}]](x)\\&=\lim_{x\to z(\tau)}F_{\alpha}[h^{\mathrm{full,H}}-h^{\mathrm{S,H}}\\&\quad-2\nabla\xi^{\mathrm{H}\to\mathrm{RW}}[h^{\mathrm{full,H}}-h^{\mathrm{S,H}}]](x)\\&=\lim_{x\to z(\tau)}F_{\alpha}[h^{\mathrm{full,H}}-2\nabla\xi^{\mathrm{H}\to\mathrm{RW}}[h^{\mathrm{full,H}}]-h^{\mathrm{S,H}}\\&\quad+2\nabla\xi^{\mathrm{H}\to\mathrm{RW}}[h^{\mathrm{S,H}}]](x)\\&=\lim_{x\to z(\tau)}\{F_{\alpha}[h^{\mathrm{full,RW}}](x)\\&\quad-F_{\alpha}[h^{\mathrm{S,H}}-2\nabla\xi^{\mathrm{H}\to\mathrm{RW}}[h^{\mathrm{S,H}}]](x)\},\end{aligned} \tag{3.6}$$

where we have omitted the space-time indices of $h_{\mu\nu}$ and $\nabla_{(\mu}\xi_{\nu)}$ for notational simplicity. The full metric perturbation $h^{\mathrm{full,RW}}_{\mu\nu}$ can be calculated by using the Regge-Wheeler-Zerilli formalism, while the S part $h^{\mathrm{S,H}}_{\mu\nu}$ can be obtained with sufficient accuracy by the local analysis near the particle location. Thus the remaining issue is if we can unambiguously determine the gauge transformation

$$\xi^{\mathrm{S,H}\to\mathrm{RW}}_{\alpha}=\xi^{\mathrm{H}\to\mathrm{RW}}_{\alpha}[h^{\mathrm{S,H}}_{\mu\nu}]. \tag{3.7}$$

Note that the self-force (3.6) is almost identical to the expression obtained in the intermediate gauge approach [13], if we replace the S and R parts by the direct and tail parts, respectively. The only difference is that the S and R parts are now solutions of the inhomogeneous and homogeneous Einstein equations, respectively. Hence the S part in the RW gauge is (at least formally) well-defined provided that the gauge transformation of the S part, Eq. (3.7), is unique. As will be shown later in Eqs. (6.4), this turns out to be indeed the case. Therefore one may identify the self-force (3.6) to be actually the one evaluated in the RW gauge [25], not in some intermediate gauge.

IV. FULL METRIC PERTURBATION AND ITS FORCE

In this section we consider the full metric perturbation and its self-force in the case of a circular orbit. First, the metric perturbation is calculated by the Regge-Wheeler-Zerilli formalism in which a Fourier-harmonic expansion is used because of the symmetry of the background space-time. Next, we derive the self-force by acting force operators and represent it in terms of ℓ mode coefficients after summing over ω and m for the Fourier-harmonic series.

A. Regge-Wheeler-Zerilli formalism

On the Schwarzschild background, the metric perturbation $h_{\mu\nu}$ can be expanded in terms of tensor harmonics as

$$\begin{aligned}\boldsymbol{h}=\sum_{\ell m}\Bigg[& f(r)H_{0\ell m}(t,r)\boldsymbol{a}^{(0)}_{\ell m}-i\sqrt{2}H_{1\ell m}(t,r)\boldsymbol{a}^{(1)}_{\ell m}\\&+\frac{1}{f(r)}H_{2\ell m}(t,r)\boldsymbol{a}_{\ell m}-\frac{i}{r}\sqrt{2\ell(\ell+1)}h^{(e)}_{0\ell m}(t,r)\boldsymbol{b}^{(0)}_{\ell m}\\&+\frac{1}{r}\sqrt{2\ell(\ell+1)}h^{(e)}_{1\ell m}(t,r)\boldsymbol{b}_{\ell m}\\&+\sqrt{\frac{1}{2}\ell(\ell+1)(\ell-1)(\ell+2)}G_{\ell m}(t,r)\boldsymbol{f}_{\ell m}\\&+\left(\sqrt{2}K_{\ell m}(t,r)-\frac{\ell(\ell+1)}{\sqrt{2}}G_{\ell m}(t,r)\right)\boldsymbol{g}_{\ell m}\\&-\frac{\sqrt{2\ell(\ell+1)}}{r}h_{0\ell m}(t,r)\boldsymbol{c}^{(0)}_{\ell m}+\frac{i\sqrt{2\ell(\ell+1)}}{r}h_{1\ell m}(t,r)\boldsymbol{c}_{\ell m}\\&+\frac{\sqrt{2\ell(\ell+1)(\ell-1)(\ell+2)}}{2r^2}h_{2\ell m}(t,r)\boldsymbol{d}_{\ell m}\Bigg],\end{aligned}\tag{4.1}$$

where $\boldsymbol{a}^{(0)}_{\ell m}$, $\boldsymbol{a}_{\ell m}$ ... are the tensor harmonics introduced by Zerilli [21]. The energy-momentum tensor of a point particle takes the form

$$\begin{aligned}T^{\mu\nu}&=\mu\int_{-\infty}^{+\infty}\delta^{(4)}[x-z(\tau)]\frac{dz^\mu}{d\tau}\frac{dz^\nu}{d\tau}d\tau\\&=\mu\frac{1}{u^t}u^\mu u^\nu\frac{\delta[r-r_0(t)]}{r^2}\delta^{(2)}[\boldsymbol{\Omega}-\boldsymbol{\Omega}_0(t)],\end{aligned}\tag{4.2}$$

where the orbit has been expressed as

$$x^\mu=z^\mu(\tau)=\{t_0(\tau),r_0(\tau),\theta_0(\tau),\phi_0(\tau)\},\tag{4.3}$$

with τ being regarded as a function of time determined by $t=T(\tau)$. The RW gauge is defined by the conditions on the metric perturbation as

$$h_2^{\rm RW}=h_0^{(e)\rm RW}=h_1^{(e)\rm RW}=G^{\rm RW}=0.\tag{4.4}$$

The Regge-Wheeler and Zerilli equations are obtained by plugging the metric perturbation (4.1) in the linearized Einstein equations and Fourier decomposing them. (Recently, the Regge-Wheeler-Zerilli formalism is improved by Jhingan and Tanaka [24].)

For odd parity waves that are defined by the parity $(-1)^{\ell+1}$ under the transformation $(\theta,\phi)\to(\pi-\theta,\phi+\pi)$, we introduce a new radial function $R^{(\rm odd)}_{\ell m\omega}(r)$ in terms of which the two radial functions of the metric perturbation are expressed as

$$\begin{aligned}h^{\rm RW}_{1\ell m\omega}&=\frac{r^2}{(r-2M)}R^{(\rm odd)}_{\ell m\omega},\\h^{\rm RW}_{0\ell m\omega}&=\frac{i}{\omega}\frac{d}{dr^*}(rR^{(\rm odd)}_{\ell m\omega})\\&\quad-\frac{8\pi r(r-2M)}{\omega\left[\frac{1}{2}\ell(\ell+1)(\ell-1)(\ell+2)\right]^{1/2}}D_{\ell m\omega}.\end{aligned}\tag{4.5}$$

The new radial function $R^{(\rm odd)}_{\ell m\omega}(r)$ satisfies the Regge-Wheeler equation,

$$\begin{aligned}\frac{d^2}{dr^{*2}}R^{(\rm odd)}_{\ell m\omega}+[\omega^2-V_\ell(r)]R^{(\rm odd)}_{\ell m\omega}&=\frac{8\pi i}{\left[\frac{1}{2}\ell(\ell+1)(\ell-1)(\ell+2)\right]^{1/2}}\frac{r-2M}{r^2}\\&\quad\times\Bigg\{-r^2\frac{d}{dr}\left[\left(1-\frac{2M}{r}\right)D_{\ell m\omega}\right]\\&\quad+(r-2M)[(\ell-1)(\ell+2)]^{1/2}Q_{\ell m\omega}\Bigg\},\end{aligned}\tag{4.6}$$

where $r^*=r+2M\log(r/2M-1)$, and the potential V_ℓ is given by

$$V_\ell(r)=\left(1-\frac{2M}{r}\right)\left(\frac{\ell(\ell+1)}{r^2}-\frac{6M}{r^3}\right).\tag{4.7}$$

The source term $Q_{\ell m\omega}$ vanishes in the case of a circular orbit and

$$\begin{aligned}D_{\ell m\omega}(r)=&\left[\frac{1}{2}\ell(\ell+1)(\ell-1)(\ell+2)\right]^{-1/2}\\&\times\mu\frac{(u^\phi)^2}{u^t}\delta(r-r_0)m\partial_\theta Y^*_{\ell m}(\theta_0,\phi_0),\end{aligned}\tag{4.8}$$

where the orbit is given by

$$z^\alpha(\tau)=\left\{u^t\tau,r_0,\frac{\pi}{2},u^\phi\tau\right\},\quad u^t=\sqrt{\frac{r_0}{r_0-3M}},$$

$$u^\phi=\frac{1}{r_0}\sqrt{\frac{M}{r_0-3M}}=\Omega u^t, \tag{4.9}$$

where $\Omega=\sqrt{M/r_0^3}$ is the orbital frequency. The orbit is assumed to be on the equatorial plane without loss of generality.

For even parity waves with the parity $(-1)^\ell$, we introduce a new radial function $R^{(Z)}_{\ell m\omega}(r)$ in terms of which the four radial functions of the metric perturbation are expressed as

$$K^{\rm RW}_{\ell m\omega}=\frac{\lambda(\lambda+1)r^2+3\lambda Mr+6M^2}{r^2(\lambda r+3M)}R^{(Z)}_{\ell m\omega}+\frac{r-2M}{r}\frac{d}{dr}R^{(Z)}_{\ell m\omega}-\frac{r(r-2M)}{\lambda r+3M}\tilde{C}_{1\ell m\omega}+\frac{i(r-2M)^2}{r(\lambda r+3M)}\tilde{C}_{2\ell m\omega},$$

$$H^{\rm RW}_{1\ell m\omega}=-i\omega\frac{\lambda r^2-3\lambda Mr-3M^2}{(r-2M)(\lambda r+3M)}R^{(Z)}_{\ell m\omega}-i\omega r\frac{d}{dr}R^{(Z)}_{\ell m\omega}+\frac{i\omega r^3}{\lambda r+3M}\tilde{C}_{1\ell m\omega}+\frac{\omega r(r-2M)}{r(\lambda r+3M)}\tilde{C}_{2\ell m\omega},$$

$$H^{\rm RW}_{0\ell m\omega}=\frac{\lambda r(r-2M)-\omega^2r^4+M(r-3M)}{(r-2M)(\lambda r+3M)}K^{\rm RW}_{\ell m\omega}+\frac{M(\lambda+1)-\omega^2r^3}{i\omega r(\lambda r+3M)}H^{\rm RW}_{1\ell m\omega}+\tilde{B}_{\ell m\omega},$$

$$H^{\rm RW}_{2\ell m\omega}=H^{\rm RW}_{0\ell m\omega}-16\pi r^2\left[\frac{1}{2}\ell(\ell+1)(\ell-1)(\ell+2)\right]^{-1/2}F_{\ell m\omega}, \tag{4.10}$$

where

$$\lambda=\frac{1}{2}(\ell-1)(\ell+2), \tag{4.11}$$

and the source terms are given by

$$\tilde{B}_{\ell m\omega}=\frac{8\pi r^2(r-2M)}{\lambda r+3M}\left\{A_{\ell m\omega}+\left[\frac{1}{2}\ell(\ell+1)\right]^{-1/2}B_{\ell m\omega}\right\}-\frac{4\pi\sqrt{2}}{\lambda r+3M}\frac{Mr}{\omega}A^{(1)}_{\ell m\omega},$$

$$\tilde{C}_{1\ell m\omega}=\frac{8\pi}{\sqrt{2}\omega}A^{(1)}_{\ell m\omega}+\frac{1}{r}\tilde{B}_{\ell m\omega}-16\pi r\left[\frac{1}{2}\ell(\ell+1)(\ell-1)(\ell+2)\right]^{-1/2}F_{\ell m\omega},$$

$$\tilde{C}_{2\ell m\omega}=-\frac{8\pi r^2}{i\omega}\frac{\left[\frac{1}{2}l(l+1)\right]^{-1/2}}{r-2M}B^{(0)}_{\ell m\omega}-\frac{ir}{r-2M}\tilde{B}_{\ell m\omega}+\frac{16\pi ir^3}{r-2M}\left[\frac{1}{2}\ell(\ell+1)(\ell-1)(\ell+2)\right]^{-1/2}F_{\ell m\omega}. \tag{4.12}$$

Here the harmonic coefficients of the source terms $A_{\ell m\omega}$, $A^{(1)}_{\ell m\omega}$, and $B_{\ell m\omega}$ vanish in the circular case and

$$B^{(0)}_{\ell m\omega}=\left[\frac{\ell(\ell+1)}{2}\right]^{-1/2}\mu u^\phi\left(1-\frac{2M}{r}\right)\frac{1}{r}\times\delta(r-r_0)mY^*_{\ell m}(\theta_0,\phi_0),$$

$$F_{\ell m\omega}=\frac{1}{2}\left[\frac{\ell(\ell+1)(\ell-1)(\ell+2)}{2}\right]^{-1/2}\mu\frac{(u^\phi)^2}{u^t}\times\delta(r-r_0)[\ell(\ell+1)-2m^2]Y^*_{\ell m}(\theta_0,\phi_0). \tag{4.13}$$

The new radial function $R^{(Z)}_{\ell m\omega}(r)$ obeys the Zerilli equation,

$$\frac{d^2}{dr^{*2}}R^{(Z)}_{\ell m\omega}+[\omega^2-V^{(Z)}_\ell(r)]R^{(Z)}_{\ell m\omega}=S^{(Z)}_{\ell m\omega}, \tag{4.14}$$

where

$$V^{(Z)}_\ell(r)=\left(1-\frac{2M}{r}\right)\times\frac{2\lambda^2(\lambda+1)r^3+6\lambda^2Mr^2+18\lambda M^2r+18M^3}{r^3(\lambda r+3M)^2} \tag{4.15}$$

and

$$S^{(Z)}_{\ell m\omega}=-i\frac{r-2M}{r}\frac{d}{dr}\left[\frac{(r-2M)^2}{r(\lambda r+3M)}\times\left(\frac{ir^2}{r-2M}\tilde{C}_{1\ell m\omega}+\tilde{C}_{2\ell m\omega}\right)\right]+i\frac{(r-2M)^2}{r(\lambda r+3M)^2}\left[\frac{\lambda(\lambda+1)r^2+3\lambda Mr+6M^2}{r^2}\tilde{C}_{2\ell m\omega}+i\frac{\lambda r^2-3\lambda Mr-3M^2}{(r-2M)}\tilde{C}_{1\ell m\omega}\right]. \tag{4.16}$$

The Zerilli equation can be transformed to the Regge-Wheeler equation by the Chandrasekhar transformation if desired, as shown in Appendix A. However, here, we treat the original Zerilli equation.

B. Full metric perturbation

The homogeneous solutions of the Regge-Wheeler equation are discussed in detail by Mano et al. [26] and in Appendix A. By constructing the retarded Green function from the homogeneous solutions with appropriate boundary conditions, namely, the two independent solutions with the ingoing and up-going wave boundary conditions, we can solve the Regge-Wheeler and Zerilli equations to obtain the full metric perturbation in the RW gauge. Here, we consider the radial functions up to the first post-Newtonian (1PN) order.

The radial function for the odd part of the metric perturbation is obtained as

$$R^{(\mathrm{odd})}_{\ell m\omega}(r) = \begin{cases} \dfrac{16 i\pi\mu\Omega^2 m r}{(2\ell+1)\ell(\ell+1)(\ell+2)}\left(\dfrac{r}{r_0}\right)^{\ell}\partial_\theta Y^*_{\ell m}(\theta_0,\phi_0) & \text{for } r<r_0 \\ -\dfrac{16 i\pi\mu\Omega^2 m r_0}{(2\ell+1)(\ell-1)\ell(\ell+1)}\left(\dfrac{r_0}{r}\right)^{\ell}\partial_\theta Y^*_{\ell m}(\theta_0,\phi_0) & \text{for } r>r_0, \end{cases} \tag{4.17}$$

where $\Omega = u^\phi/u^t$. For the even part, the radial function is obtained as

$$R^{(\mathrm{Z})}_{\ell m\omega} = \frac{8\Omega m\pi u^t\mu}{(2\ell+1)(\ell+2)(\ell+1)\omega}\times\left[\left(-\frac{r^3}{(2\ell+3)r_0}+\frac{(\ell^2-\ell+4)r_0 r}{\ell(2\ell-1)(\ell-1)}\right)\omega^2+2\frac{r}{r_0}+2\frac{(\ell^2-2\ell-1)Mr}{(\ell-1)r_0^2}-\frac{2(\ell^4+\ell^3-6\ell^2-4\ell-4)M}{\ell(\ell-1)(\ell+2)r_0}\right]\times\left(\frac{r}{r_0}\right)^{\ell}Y^*_{\ell m}(\theta_0,\phi_0)\quad \text{for } r<r_0, \tag{4.18}$$

$$R^{(\mathrm{Z})}_{\ell m\omega} = \frac{8\Omega m\pi u^t\mu}{(2\ell+1)\ell(\ell-1)\omega}\times\left[\left(\frac{r^2}{2\ell-1}-\frac{(\ell^2+3\ell+6)r_0^2}{(\ell+1)(2\ell+3)(\ell+2)}\right)\omega^2+2-2\frac{(\ell^2+4\ell+2)M}{(\ell+2)r_0}+\frac{2(\ell^4+3\ell^3-3\ell^2-7\ell-6)M}{(\ell+1)(\ell-1)(\ell+2)r}\right]\times\left(\frac{r_0}{r}\right)^{\ell}Y^*_{\ell m}(\theta_0,\phi_0)\quad \text{for } r>r_0.$$

The metric perturbation in the RW gauge is obtained from Eqs. (4.5) and (4.10).

C. Full force

Formally, the force derived from the full metric perturbation is given by

$$F^{\mu}_{\mathrm{full,RW}}(z) = -\frac{\mu}{2}(g^{\mu\nu}+u^\mu u^\nu)(2h^{\mathrm{full,RW}}_{\nu\alpha;\beta}-h^{\mathrm{full,RW}}_{\alpha\beta;\nu})u^\alpha u^\beta. \tag{4.19}$$

If we decompose the above into harmonic modes, each mode becomes finite at the location of the particle though the sum over the modes diverges. We therefore apply the "mode decomposition regularization" method, in which the force is decomposed into harmonic modes and subtract the harmonic-decomposed S part mode by mode before the coincidence limit $x\to z(\tau)$ is taken.

Since the orbit under consideration is circular, the source term contains the factor $\delta(\omega-m\Omega)$, and the frequency integral can be trivially performed. Hence we can calculate the harmonic coefficients of the full metric perturbation in the time domain. This is a great advantage of the circular orbit case, since the S part can be given only in the time domain. We also note that the θ component of the force vanishes because of the symmetry, and $F^\phi(z) = [(r_0-2M)/(r_0^3\Omega)]F^t$ for a circular orbit.

The even and odd parity parts of the full self-force are expressed in terms of the metric perturbation as

$$F^{t\ \mathrm{RW}}_{(\mathrm{even})} = \sum_{\ell m}\frac{i\mu m\Omega r_0}{2(r_0-3M)(r_0-2M)}[(r_0-2M)H^{\mathrm{RW}}_{0\ell m,m\Omega}(r_0)+MK^{\mathrm{RW}}_{\ell m,m\Omega}(r_0)]Y_{\ell m}(\theta_0,\phi_0),$$

$$F^{r\ \mathrm{RW}}_{(\mathrm{even})} = \sum_{\ell m}\frac{\mu(r_0-2M)}{2r_0^2(r_0-3M)}\left(2MH^{\mathrm{RW}}_{0\ell m,m\Omega}(r_0)+2MK^{\mathrm{RW}}_{\ell m,m\Omega}(r_0)+r_0(r_0-2M)\frac{d}{dr}H^{\mathrm{RW}}_{0\ell m,m\Omega}(r_0)+r_0M\frac{d}{dr}K^{\mathrm{RW}}_{\ell m,m\Omega}(r_0)\right)Y_{\ell m}(\theta_0,\phi_0),$$

$$F^{t\ \mathrm{RW}}_{(\mathrm{odd})} = \sum_{\ell m}\frac{i\mu M m}{r_0(r_0-3M)(r_0-2M)}h^{\mathrm{RW}}_{0\ell m,m\Omega}(r_0)\times\partial_\theta Y_{\ell m}(\theta_0,\phi_0),$$

$$F^{r\ \mathrm{RW}}_{(\mathrm{odd})} = \sum_{\ell m}\frac{\mu(r_0-2M)\Omega}{r_0-3M}\left(\frac{d}{dr}h^{\mathrm{RW}}_{0\ell m,m\Omega}(r_0)\right)\times\partial_\theta Y_{\ell m}(\theta_0,\phi_0). \tag{4.20}$$

It is understood that the derivatives appearing in the above expressions are taken before the coincidence limit. It may be noted that there is no contribution from the components H_1^{RW}

and $H_2^{\rm RW}$ to the even force and no contribution from $h_1^{\rm RW}$ to the odd force for a circular orbit.

Inserting the metric perturbation under the RW gauge to the above, and performing the summation over m, we find

$$F^t_{\rm full,RW}|_\ell=0,$$

$$F^{r(+)}_{\rm full,RW}|_\ell=-\frac{(\ell+1)\mu^2}{r_0^2}+\frac{1}{2}\frac{\mu^2(12\ell^3+25\ell^2+4\ell-21)M}{r_0^3(2\ell+3)(2\ell-1)},$$

$$F^{r(-)}_{\rm full,RW}|_\ell=\frac{\ell\mu^2}{r_0^2}-\frac{1}{2}\frac{\mu^2(12\ell^3+11\ell^2-10\ell+12)M}{(2\ell-1)(2\ell+3)r_0^3},$$

$$F^\theta_{\rm full,RW}|_\ell=0,$$

$$F^\phi_{\rm full,RW}|_\ell=0. \tag{4.21}$$

We see that the only nonvanishing component is the radial component as expected because there is no radiation reaction effect at 1PN order. In the above, the indices $(+)$ and $(-)$ denote that the coincidence limit is taken from outside ($r>r_0$) of the orbit and inside ($r<r_0$) of the orbit, respectively, and the vertical bar suffixed with ℓ,

$$\cdots|_\ell,$$

denotes the coefficient of the ℓ mode in the coincidence limit. The formulas for the summation over m are shown in Appendix F.

We note that the above result is valid for $\ell\geqslant 2$. Although the $\ell=0$ and 1 modes do not contribute to the self-force formally, because of our inability to know the exact form of the S part, it turns out that we do need to calculate the contributions from the $\ell=0$ and 1 modes. These modes are treated in Appendix E.

V. *S* PART OF THE METRIC PERTURBATION AND FORCE

In this section we calculate the S part of the metric perturbation and its self-force (S force) by using the local coordinate expansion. The S part of the metric perturbation in the harmonic gauge is given covariantly as

$$\bar{h}^{\rm S,H}_{\mu\nu}=4\mu\left[\frac{\bar{g}_{\mu\alpha}(x,z_{\rm ret})\bar{g}_{\nu\beta}(x,z_{\rm ret})u^\alpha(\tau_{\rm ret})u^\beta(\tau_{\rm ret})}{\sigma_{;\gamma}(x,z_{\rm ret})u^\gamma(\tau_{\rm ret})}\right]$$
$$+2\mu(\tau_{\rm adv}-\tau_{\rm ret})\bar{g}_\mu{}^\alpha(x,z_{\rm ret})\bar{g}_\nu{}^\beta(x,z_{\rm ret})R_{\gamma\alpha\delta\beta}(z_{\rm ret})$$
$$\times u^\gamma(\tau_{\rm ret})u^\delta(\tau_{\rm ret})+O(y^2), \tag{5.1}$$

where $z_{\rm ret}=z(\tau_{\rm ret})$, $\tau_{\rm ret}$ is the retarded proper time defined by the past light cone condition of the field point x, $\tau_{\rm adv}$ is the advanced proper time defined by the future light cone condition of the field point x, $\bar{g}_{\mu\alpha}$ is the parallel displacement bi-vector, and y is the expansion parameter of the local expansion, which may be taken to be the difference of the coordinates between x and z_0, $y^\mu=x^\mu-z_0^\mu$. Details of the local expansion are given in [9]. The difference between the S part and the direct part appears in the terms of $O(y)$, i.e., the second term on the right-hand side of Eq. (5.1). In the local coordinate expansion of the S part, it is convenient to use the quantities

$$\epsilon:=(r_0^2+r^2-2r_0r\cos\Theta\cos\Phi)^{1/2},$$

$$T:=t-t_0,\quad R:=r-r_0,$$

$$\Theta:=\theta-\frac{\pi}{2},\quad \Phi:=\phi-\phi_0. \tag{5.2}$$

A. *S* part of the metric perturbation

Using the variables defined in Eqs. (5.2), it is straightforward to calculate the S part to 1PN order. Here we note that, in general, we have to evaluate the S part up through the accuracy of $O(y)$, because the force is given by first derivatives of the metric components. The result takes the form,

$$h^{\rm S,H}_{\mu\nu}=\mu\sum_{m,n,p,q,r}c_{m,n,p,q,r}\frac{T^mR^n\Theta^p\Phi^q}{\epsilon^r}, \tag{5.3}$$

where m, n, p, q and r are positive integers. The explicit expressions for the components are shown in Appendix D, Eqs. (D1).

B. Tensor harmonics expansion of the *S* part

In the preceding section, we calculated the S part of the metric perturbation in the local coordinates expansion. In order to use them in the mode decomposition regularization, it is necessary to expand them in terms of tensor spherical harmonics, which involves an extension of the locally expanded S part to a quantity defined over the sphere. Since the only requirement is to recover the local behavior near the orbit correctly, there exists much freedom in the way of extending the locally known S part to a globally defined (but only approximate) S part on the whole sphere. To guarantee the accuracy of $h^{\rm S,H}_{\mu\nu}$ up through $O(y)$ in the local expansion, because the leading term diverges as $1/y$, a spherical extension must be accurate enough to recover the behavior at $O(y^2)$ beyond the leading order. Below, using one of such extensions as given in Appendix B, we derive the harmonic coefficients of the S part.

Once we fix the method of spherical extension, it is possible in principle to calculate the harmonic coefficients of the extended S part exactly. However, it is neither necessary nor quite meaningful because the extension is only approximate. In fact, corresponding to the fact that all the terms in positive powers of y vanish in the coincidence limit, it is known that all the terms of $O(1/L^2)$ or higher, where $L=\ell+1/2$, vanish when summed over ℓ [9] in the harmonic gauge. It should be noted, however, this result is obtained by expanding the force in the scalar spherical harmonics. In our present analysis, we employ the tensor spherical harmonic expansion. So, the meaning of the index ℓ is slightly different. Nevertheless, the same is found to be true. Namely, by expanding the S part of the metric perturbation in the tensor spherical harmonics, the S force in the harmonic gauge is found to have the form

$$F^{\mu(\pm)}_{\rm S,H}|_\ell = \pm A^\mu L + B^\mu + D^\mu_\ell, \tag{5.4}$$

where A^μ and B^μ are independent of ℓ, and the $\pm$ denotes that the limit to r_0 is taken from the greater or smaller value of r, and

$$D^\mu_\ell = \frac{d^\mu}{L^2-1} + \frac{e^\mu}{(L^2-1)(L^2-4)} + \frac{f^\mu}{(L^2-1)(L^2-4)(L^2-9)} + \cdots. \tag{5.5}$$

Then the summation of D^μ_ℓ over ℓ (from $\ell=0$ to ∞) vanishes. For convenience, let us call this the standard form. As we shall see later, the standard form of the S force is found to persist also in the RW gauge.

For the moment, let us assume the standard form of the S force both in the harmonic gauge and the RW gauge. Then we may focus our discussion on the divergent terms. When we calculate the S force in the RW gauge, we first transform the metric perturbation from the harmonic gauge to the RW gauge, and then take appropriate linear combinations of their first derivatives. We then find that the harmonic coefficients $h^{\rm S,H}_{2\ell m}$, $h^{\rm (e)S,H}_{0\ell m}$ and $h^{\rm (e)S,H}_{1\ell m}$ are differentiated two times, and $G^{\rm S,H}_{\ell m}$ is differentiated three times, while the rest are differentiated once, to obtain the S force. So, it is necessary and sufficient to perform the Taylor expansion of the harmonic coefficients up to $O(X^2)$ for $h^{\rm S,H}_{2\ell m}$, $h^{\rm (e)S,H}_{0\ell m}$ and $h^{\rm (e)S,H}_{1\ell m}$, and up to $O(X^3)$ for $G^{\rm S,H}_{\ell m}$, and the rest up to $O(X)$, where $X = T$ or R.

To the accuracy mentioned above, the harmonic coefficients of the S part are found in the form

$$h^{\rm S,H}_{0\ell m}(t,r) = \frac{2}{L}\pi\mu\left[\frac{4iTmr_0(L^2-2)(u^\phi)^2}{\mathcal{L}^{(2)}(L^2-1)} + \cdots\right] \times \partial_\theta Y^*_{\ell m}(\theta_0,\phi_0),$$

$$h^{\rm S,H}_{1\ell m}(t,r) = \frac{2}{L}\pi\mu\left[\frac{-2ir_0m(2r_0+R)(u^\phi)^2}{\mathcal{L}^{(2)}(L^2-1)}\right]\partial_\theta Y^*_{\ell m}(\theta_0,\phi_0),$$

$$h^{\rm S,H}_{2\ell m}(t,r) = \frac{2}{L}\pi\mu\left[-\frac{1}{6}\frac{r_0m(72r_0RL^4+48r_0RL^5+\cdots)}{\mathcal{L}^{(4)}(L^2-1)(L^2-4)} \times (u^\phi)^2\right]\partial_\theta Y^*_{\ell m}(\theta_0,\phi_0), \tag{5.6}$$

etc., where we have defined

$$\mathcal{L}^{(2)} = \ell(\ell+1) = \left(L^2 - \frac{1}{4}\right),$$

$$\mathcal{L}^{(4)} = \ell(\ell+1)(\ell-1)(\ell+2) = \left(L^2-\frac{1}{4}\right)\left(L^2-\frac{9}{4}\right). \tag{5.7}$$

The explicit expressions for the coefficients are given in Appendix D, Eqs. (D2). Shown there are the coefficients in the case when we approach the orbit from inside ($r<r_0$). The results in the case of approaching from outside ($r>r_0$) are obtained in the same manner. For readers' convenience, these are placed at the web page: http://www2.yukawa.kyoto-u.ac.jp/~misao/BHPC/.

Now we consider the S force in the harmonic gauge. It is noted that the t, θ and ϕ components of the S force vanish after summing over m modes. The r component of the S force is derived as

$$F^{r(-)}_{\rm S,H}|_\ell = \sum_m \frac{2\pi\mu^2}{L}\Bigg[\Bigg(\frac{2L-1}{2r_0^2} + \frac{M(10L^3+11L^2-10L-17)}{4r_0^3(L^2-1)} - \frac{M(64L^5+28L^4-320L^3-695L^2+256L+442)m^2}{16r_0^3\mathcal{L}^{(2)}(L^2-1)(L^2-4)} - \frac{M(156L^2-179)m^4}{4r_0^3\mathcal{L}^{(2)}(L^2-1)(L^2-4)(L^2-9)}\Bigg)|Y_{\ell m}(\theta_0,\phi_0)|^2 + \left(\frac{13Mm^2}{r_0^3\mathcal{L}^{(2)}(L^2-1)(L^2-4)} - \frac{M(2L-1)(2L^2+2L-1)}{r_0^3\mathcal{L}^{(2)}(L^2-1)}\right)|\partial_\theta Y_{\ell m}(\theta_0,\phi_0)|^2\Bigg]. \tag{5.8}$$

The formulas for summation over m are summarized in Appendix F. For example, we have

$$\sum_m \frac{2\pi}{L} m^2 |Y_{\ell m}(\pi/2,0)|^2 = \frac{\mathcal{L}^{(2)}}{2},$$

$$\sum_m \frac{2\pi}{L}|\partial_\theta Y_{\ell m}(\pi/2,0)|^2 = \frac{\mathcal{L}^{(2)}}{2}. \tag{5.9}$$

Using these formulas, we obtain

$$F^t_{\rm S,H}|_\ell = 0,$$

$$F^{r(\pm)}_{\rm S,H}|_\ell = \mp\frac{1}{2}\frac{\mu^2(2r_0-3M)}{r_0^3}L - \frac{1}{8}\frac{\mu^2(4r_0-7M)}{r_0^3} + \frac{\mu^2 M(172L^4-14784L^2+299)}{128r_0^3(L^2-1)(L^2-4)(L^2-9)}$$
$$= \mp\frac{1}{2}\frac{\mu^2(2r_0-3M)}{r_0^3}L - \frac{1}{8}\frac{\mu^2(4r_0-7M)}{r_0^3} + O\left(\frac{1}{L^2}\right),$$
$$F^{\theta}_{\rm S,H}|_\ell = 0,$$
$$F^{\phi}_{\rm S,H}|_\ell = 0. \tag{5.10}$$

This is indeed of the standard form. In particular, the factor $\mathcal{L}^{(2)}$ which is present in the denominators before summing over m turns out to be cancelled by the same factor that arises from summation over m. If it were present in the final result, we would not be able to conclude that the summation of D^μ_ℓ over ℓ vanishes. We note that, apart from the fact that the denominator of the D^μ_ℓ term takes the standard form, the numerical coefficients appearing in the numerator should not be taken rigorously. This is because our calculation is accurate only to $O(y^0)$ of the S force, while the numerical coefficients depend on the $O(y)$ behavior of it (an example is shown in Appendix C). It is also noted that the $O(1/L)$ terms are absent in the S force, implying the absence of logarithmic divergence.

It is important to note that ℓ in the above runs from 0 to ∞. Although there are some tensor harmonics that do not exist for $\ell=0$ and/or $\ell=1$, we note that the corresponding harmonic coefficients contribute to the B^μ and D^μ_ℓ terms of the S force individually, with $B^\mu + D^\mu_\ell = 0$. That is, we set the contributions to A^μ to zero and adjust the D^μ_ℓ term in such a way that $D^\mu_\ell = -B^\mu$ for these special coefficients while keeping the standard form for D^μ_ℓ.

VI. *S* PART IN THE REGGE-WHEELER GAUGE

Now, we transform the S part of the metric perturbation from the harmonic gauge to the RW gauge. The gauge transformation functions are given in the tensor-harmonic expansion form as

$$\xi^{(\rm odd)}_\mu = \sum_{\ell m} \Lambda^{\rm S,H\to RW}_{\ell m}(t,r) \times\left\{0,0,\frac{-1}{\sin\theta}\partial_\phi Y_{\ell m}(\theta,\phi), \sin\theta\partial_\theta Y_{\ell m}(\theta,\phi)\right\},$$
$$\xi^{(\rm even)}_\mu = \sum_{\ell m} \{M^{\rm S,H\to RW}_{0\ell m}(t,r)Y_{\ell m}(\theta,\phi),\ M^{\rm S,H\to RW}_{1\ell m}(t,r)Y_{\ell m}(\theta,\phi),\ M^{\rm S,H\to RW}_{2\ell m}(t,r)\partial_\theta Y_{\ell m}(\theta,\phi),\ M^{\rm S,H\to RW}_{2\ell m}(t,r)\partial_\phi Y_{\ell m}(\theta,\phi)\}. \tag{6.1}$$

There is one degree of gauge freedom for the odd part and three degrees for the even part. To satisfy the RW gauge condition (4.4), we obtain the equations for the gauge functions that are found to be rather simple:

$$h^{\rm S,H}_{2\ell m}(t,r) = -2i\Lambda^{\rm S,H\to RW}_{\ell m}(t,r),$$
$$h^{(\rm e)S,H}_{0\ell m}(t,r) = -M^{\rm S,H\to RW}_{0\ell m}(t,r) - \partial_t M^{\rm S,H\to RW}_{2\ell m}(t,r),$$
$$h^{(\rm e)S,H}_{1\ell m}(t,r) = -M^{\rm S,H\to RW}_{1\ell m}(t,r) - r^2\partial_r\left(\frac{M^{\rm S,H\to RW}_{2\ell m}(t,r)}{r^2}\right),$$
$$G^{\rm S,H}_{\ell m}(t,r) = -\frac{2}{r^2}M^{\rm S,H\to RW}_{2\ell m}(t,r). \tag{6.2}$$

We therefore find

$$\Lambda^{\rm S,H\to RW}_{\ell m}(t,r) = \frac{i}{2}h^{\rm S,H}_{2\ell m}(t,r), \tag{6.3}$$

$$M^{\rm S,H\to RW}_{2\ell m}(t,r) = -\frac{r^2}{2}G^{\rm S,H}_{\ell m}(t,r),$$
$$M^{\rm S,H\to RW}_{0\ell m}(t,r) = -h^{(\rm e)S,H}_{0\ell m}(t,r) - \partial_t M^{\rm S,H\to RW}_{2\ell m}(t,r),$$
$$M^{\rm S,H\to RW}_{1\ell m}(t,r) = -h^{(\rm e)S,H}_{1\ell m}(t,r) - r^2\partial_r\left(\frac{M^{\rm S,H\to RW}_{2\ell m}(t,r)}{r^2}\right). \tag{6.4}$$

We note that it is not necessary to calculate any integration with respect to t or r. It is also noted that the gauge functions are determined uniquely. This is because the RW gauge is a gauge in which there is no residual gauge freedom (for $\ell \geqslant 2$).

Then the S part of the metric perturbation in the RW gauge is expressed in terms of those in the harmonic gauge as follows. The odd parity components are found as

$$h^{\rm S,RW}_{0\ell m}(t,r) = h^{\rm S,H}_{0\ell m}(t,r) + \partial_t\Lambda^{\rm S,H\to RW}_{\ell m}(t,r),$$
$$h^{\rm S,RW}_{1\ell m}(t,r) = h^{\rm S,H}_{1\ell m}(t,r) + r^2\partial_r\left(\frac{\Lambda^{\rm S,H\to RW}_{\ell m}(t,r)}{r^2}\right), \tag{6.5}$$

and the even parity components are found as

$$H^{\rm S,RW}_{0\ell m}(t,r)=H^{\rm S,H}_{0\ell m}(t,r)+\frac{2r}{r-2M}\bigg[\partial_t M^{\rm S,H\to RW}_{0\ell m}(t,r)-\frac{M(r-2M)}{r^3}M^{\rm S,H\to RW}_{1\ell m}(t,r)\bigg],$$

$$H^{\rm S,RW}_{1\ell m}(t,r)=H^{\rm S,H}_{1\ell m}(t,r)+\bigg[\partial_t M^{\rm S,H\to RW}_{1\ell m}(t,r)+\partial_r M^{\rm S,H\to RW}_{0\ell m}(t,r)-\frac{2M}{r(r-2M)}M^{\rm S,H\to RW}_{0\ell m}(t,r)\bigg],$$

$$H^{\rm S,RW}_{2\ell m}(t,r)=H^{\rm S,H}_{2\ell m}(t,r)+\frac{2(r-2M)}{r}\bigg[\partial_r M^{\rm S,H\to RW}_{1\ell m}(t,r)+\frac{M}{r(r-2M)}M^{\rm S,H\to RW}_{1\ell m}(t,r)\bigg],$$

$$K^{\rm S,RW}_{\ell m}(t,r)=K^{\rm S,H}_{\ell m}(t,r)+\frac{2(r-2M)}{r^2}M^{\rm S,H\to RW}_{1\ell m}(t,r), \tag{6.6}$$

where the gauge functions are given by Eqs. (6.3) and (6.4).

A. Gauge transformation and the *S* part in the RW gauge

Inserting the results obtained in the previous section to Eqs. (6.3) and (6.4), we obtain the gauge functions that transform the S part from the harmonic gauge to the RW gauge. They are shown in Appendix D, Eqs. (D3). It may be noted that the gauge functions do not contribute to the metric at the Newtonian order. In other words, both the RW gauge and the harmonic gauge reduce to the same (Newtonian) gauge in the Newtonian limit.

The S part of the metric perturbation in the RW gauge is now found in the form,

$$h^{\rm S,RW}_{0\ell m}(t,r)=\frac{2}{L}\pi\mu\bigg[\frac{4iTmr_0(L^2-2)(u^\phi)^2}{\mathcal{L}^{(2)}(L^2-1)}+\cdots\bigg]\times\partial_\theta Y^*_{\ell m}(\theta_0,\phi_0),$$

$$h^{\rm S,RW}_{1\ell m}(t,r)=\frac{2}{L}\pi\mu\bigg[-\frac{imr_0}{3}\frac{(-60r_0L^3+174r_0L^2+\cdots)(u^\phi)^2}{\mathcal{L}^{(4)}(L^2-1)(L^2-4)}\bigg]\times\partial_\theta Y^*_{\ell m}(\theta_0,\phi_0), \tag{6.7}$$

etc.

The explicit expressions are given in Appendix D, Eqs. (D4).

B. *S* force

Next we calculate the S part of the self-force. Of course, it diverges in the coincidence limit. However, as we noted several times, in the mode decomposition regularization in which the regularization is done for each harmonic mode, the harmonic coefficients of the S part are finite.

The calculation is straightforward. Expanding the formula for the self-force (2.3) in terms of the tensor harmonics, we obtain

$$\begin{aligned}F^{t\ \rm RW}_{\rm (even)}=\sum_{\ell m}\frac{\mu r_0}{2(r_0-3M)^2(r_0-2M)}\{&-r_0M[\partial_t H^{\rm RW}_{0\ell m}(t_0,r_0)]\\&+2M^2[\partial_t H^{\rm RW}_{0\ell m}(t_0,r_0)]+imr_0^2\Omega H^{\rm RW}_{0\ell m}(t_0,r_0)\\&-6imMr_0\Omega H^{\rm RW}_{0\ell m}(t_0,r_0)+8imM^2\Omega H^{\rm RW}_{0\ell m}(t_0,r_0)\\&-imr_0M\Omega K^{\rm RW}_{\ell m}(t_0,r_0)+2imM^2\Omega K^{\rm RW}_{\ell m}(t_0,r_0)\\&+5M^2[\partial_t K^{\rm RW}_{\ell m}(t_0,r_0)]\\&-2r_0M[\partial_t K^{\rm RW}_{\ell m}(t_0,r_0)]\}Y_{\ell m}(\theta_0,\phi_0),\end{aligned}$$

$$\begin{aligned}F^{r\ \rm RW}_{\rm (even)}=\sum_{\ell m}-\frac{\mu(r_0-2M)}{2r_0^2(r_0-3M)}\{&2r_0^2[\partial_t H^{\rm RW}_{1\ell m}(t_0,r_0)]\\&-2MH^{\rm RW}_{0\ell m}(t_0,r_0)+2imr_0^2\Omega H^{\rm RW}_{1\ell m}(t_0,r_0)\\&-2MK^{\rm RW}_{\ell m}(t_0,r_0)-r_0^2[\partial_r H^{\rm RW}_{0\ell m}(t_0,r_0)]\\&+2r_0M[\partial_r H^{\rm RW}_{0\ell m}(t_0,r_0)]\\&-r_0M[\partial_r K^{\rm RW}_{\ell m}(t_0,r_0)]\}Y_{\ell m}(\theta_0,\phi_0),\end{aligned}$$

$$\begin{aligned}F^{t\ \rm RW}_{\rm (odd)}=\sum_{\ell m}\frac{-i\mu\Omega r_0^2}{(r_0-3M)^2(r_0-2M)}\{&\Omega mMh^{\rm RW}_{0\ell m}(t_0,r_0)\\&-i(r_0-2M)[\partial_t h^{\rm RW}_{0\ell m}(t_0,r_0)]\}\partial_\theta Y_{\ell m}(\theta_0,\phi_0),\end{aligned}$$

$$\begin{aligned}F^{r\ \rm RW}_{\rm (odd)}=\sum_{\ell m}\frac{\mu\Omega(r_0-2M)}{r_0-3M}\{&[\partial_r h^{\rm RW}_{0\ell m}(t_0,r_0)]\\&-[\partial_t h^{\rm RW}_{1\ell m}(t_0,r_0)]\\&-i\Omega mh^{\rm RW}_{1\ell m}(t_0,r_0)\}\partial_\theta Y_{\ell m}(\theta_0,\phi_0).\end{aligned} \tag{6.8}$$

Substituting the S part of the metric components in the RW gauge as shown in Eqs. (6.7), given explicitly in Eqs. (D4), into the above, we find that the t, θ and ϕ components of the S force vanish after summing over m. The r component of the S force inside the particle trajectory is derived as

$$F^{r(-)}_{\mathrm{S,RW}}|_\ell=\sum_m \frac{2\pi\mu^2}{L}\Bigg[\Bigg(\frac{2L-1}{2r_0^2}+\frac{M(10L^3+11L^2-10L-17)}{4r_0^3(L^2-1)}-\frac{M(64L^5+28L^4-320L^3-695L^2+256L+442)m^2}{16r_0^3\mathcal{L}^{(2)}(L^2-1)(L^2-4)}$$
$$-\frac{M(156L^2-179)m^4}{4r_0^3\mathcal{L}^{(2)}(L^2-1)(L^2-4)(L^2-9)}\Bigg)|Y_{\ell m}(\theta_0,\phi_0)|^2$$
$$+\Bigg(\frac{13Mm^2}{r_0^3\mathcal{L}^{(2)}(L^2-1)(L^2-4)}-\frac{M(2L-1)(2L^2+2L-1)}{r_0^3\mathcal{L}^{(2)}(L^2-1)}\Bigg)|\partial_\theta Y_{\ell m}(\theta_0,\phi_0)|^2\Bigg]. \tag{6.9}$$

Summing the above over m, we obtain

$$F^t_{\mathrm{S,RW}}|_\ell=0,$$

$$F^{r(\pm)}_{\mathrm{S,RW}}|_\ell=\mp\frac{1}{2}\frac{\mu^2(2r_0-3M)}{r_0^3}L-\frac{1}{8}\frac{\mu^2(4r_0-7M)}{r_0^3}+\frac{\mu^2M(172L^4-14784L^2+299)}{128r_0^3(L^2-1)(L^2-4)(L^2-9)}$$
$$=\mp\frac{1}{2}\frac{\mu^2(2r_0-3M)}{r_0^3}L-\frac{1}{8}\frac{\mu^2(4r_0-7M)}{r_0^3}+O\left(\frac{1}{L^2}\right),$$

$$F^\theta_{\mathrm{S,RW}}|_\ell=0,$$

$$F^\phi_{\mathrm{S,RW}}|_\ell=0. \tag{6.10}$$

We now see that the S force in the RW gauge also has the standard form as in the case of the harmonic gauge and there is no $O(1/L)$ term. Note that, again with the same reason as we explained at the end of the preceding section, the final formula above should be regarded as valid for all ℓ from 0 to ∞.

VII. REGULARIZED GRAVITATIONAL SELF-FORCE

In the previous two sections we have calculated the full and S parts of the self-force in the RW gauge. Now we are ready to evaluate the regularized self-force. But there is one more issue to be discussed, namely, the treatment of the $\ell=0$ and 1 modes.

The full metric perturbation and its self-force are derived by the Regge-Wheeler-Zerilli formalism. This means they contain only the harmonic modes with $\ell\geqslant 2$. If we could know the exact S part, then the knowledge of the modes $\ell\geqslant 2$ would be sufficient to derive the regular R part of the self-force, because the R part of the metric perturbation is known to satisfy the homogeneous Einstein equations [8], and because there are no non-trivial homogeneous solutions in the $\ell=0$ and 1 modes. To be more precise, apart from the gauge modes that are always present, the $\ell=0$ homogeneous solution corresponds to a shift of the black hole mass and the $\ell=1$ odd parity by adding a small angular momentum to the black hole, both of which should be put to zero in the absence of an orbiting particle. As for the $\ell=1$ even mode, it is a pure gauge that corresponds to a dipolar shift of the coordinates. In other words, apart from possible gauge mode contributions, the $\ell=0$ and 1 modes of the full force should be exactly cancelled by those of the S part. In reality, however, what we have in hand is only an approximate S part. In particular, its individual harmonic coefficients do not have physical meaning. Let us denote the harmonic coefficients of the approximate S force by $F^{\mathrm{S,Ap}}_\ell$, while the exact S force and the full force by F^{S}_ℓ and F^{full}_ℓ, respectively. Then the R force F^{R} may be expressed as

$$F^{\mathrm{R}}=\sum_{\ell\geqslant 2}(F^{\mathrm{full}}_\ell-F^{\mathrm{S}}_\ell)$$
$$=\sum_{\ell\geqslant 0}(F^{\mathrm{full}}_\ell-F^{\mathrm{S}}_\ell)$$
$$=\sum_{\ell\geqslant 0}(F^{\mathrm{full}}_\ell-F^{\mathrm{S,Ap}}_\ell)-\sum_{\ell\geqslant 0}D_\ell$$
$$=\sum_{\ell\geqslant 2}(F^{\mathrm{full}}_\ell-F^{\mathrm{S,Ap}}_\ell)+\sum_{\ell=0,1}(F^{\mathrm{full}}_\ell-F^{\mathrm{S,Ap}}_\ell), \tag{7.1}$$

where $D_\ell=F^{\mathrm{S}}_\ell-F^{\mathrm{S,Ap}}_\ell$, and the last line follows from the fact that $F^{\mathrm{S,Ap}}_\ell$ are assumed to be obtained from a sufficiently accurate spherical extension of the local behavior of the S part to guarantee $\Sigma_{\ell\geqslant 0}D_\ell=0$. Thus, it is necessary to evaluate the $\ell=0$ and 1 modes of the full force to evaluate the self-force correctly.

First, we consider the contributions of $\ell\geqslant 2$ to the self-force. As noted before, for the 1PN calculation, the only r component of the full and S part of the self-force is non-zero. The ℓ mode coefficients corresponding to the first term in the last line of Eq. (7.1) are derived as

$$F^r_{\mathrm{RW}}|_\ell=F^r_{\mathrm{full,RW}}|_\ell-F^r_{\mathrm{S,RW}}|_\ell$$
$$=-\frac{45\mu^2M}{8(2\ell-1)(2\ell+3)r_0^3}. \tag{7.2}$$

Summing over ℓ modes, we obtain

$$F^r_{\mathrm{RW}}(\ell \geqslant 2) = -\frac{3\mu^2 M}{4 r_0^3}. \tag{7.3}$$

Next, we consider the $\ell=0$ and 1 modes. Detailed analyses are given in Appendix E. It is noted that the $\ell=0$ and $\ell=1$ odd modes, which describe the perturbation in the total mass and angular momentum, respectively, of the system due to the presence of the particle, are determinable in the harmonic gauge, with the retarded boundary condition. On the other hand, we were unable to solve for the $\ell=1$ even mode in the harmonic gauge. Since it is locally a gauge mode describing a shift of the center of mass coordinates, this gives rise to an ambiguity in the final result of the self-force. Nevertheless, we were able to resolve this ambiguity at Newtonian order, and hence to obtain an unambiguous interpretation of the resulting self-force.

The correction to the regularized self-force that arises from the $\ell=0$ and 1 modes, corresponding to the second term in the last line of Eq. (7.1), is found as

$$\delta F^r_{\mathrm{RW}}(\ell=0,1) = \frac{2\mu^2}{r_0^2} - \frac{41\mu^2 M}{4 r_0^3}. \tag{7.4}$$

Finally, adding Eqs. (7.3) and (7.4), we obtain the regularized gravitational self-force to the 1PN order as

$$F^r_{\mathrm{RW}} = \frac{2\mu^2}{r_0^2} - \frac{11\mu^2 M}{r_0^3}. \tag{7.5}$$

Since there will be no effect of the gravitational radiation at the 1PN order, i.e., the t and ϕ components are zero, the above force describes the correction to the radius of the orbit that deviates from the geodesic on the unperturbed background. It is noted that the first term proportional to μ^2 is just the correction to the total mass of the system at the Newtonian order, where r_0 is interpreted as the distance from the center of mass of the system to the particle.

VIII. CONCLUSION

In this paper we proposed a new method to derive the regularized gravitational self-force on a point particle in circular orbit around the Schwarzschild black hole, and, as a demonstration, we derived the regularized self-force analytically to the first post-Newtonian (1PN) order. The regularization of the gravitational self-force may be divided into the two problems, the subtraction problem and the gauge problem. To regularize and subtract the divergent part, we employed the "mode decomposition regularization," in which everything is expanded in the spherical harmonics and the regularization is performed at each ℓ mode. As for the gauge problem, utilizing the recent discovery by Detweiler and Whiting that the regularized force may be derived from the R part of the metric perturbation that satisfies the source-free Einstein equations, we considered the regularized force in the Regge-Wheeler gauge.

In the present paper actual calculations were done only for circular orbit and to the 1PN order. However, there remains a problem for the even parity $\ell=1$ mode. In this metric perturbation approach, there inevitably remains ambiguity of the gauge in the resulting self-force. To circumvent this difficulty, the only way seems to be to regularize at the level of the Weyl scalar ψ_4 or the Hertz potential Ψ, which are free from the $\ell=0$ and 1 modes. As another problem, to make our method applicable to general cases, it is therefore necessary to extend to general orbits and to higher PN orders. Some progress in this direction, based on analytical methods, is under way [27]. It will also be necessary to incorporate numerical techniques if we are to treat completely general orbits. Some development is done by Fujita et al. [28].

Our final goal is to derive the self-force on the Kerr background. Recently, Mino [29] has proposed a new approach to the radiation reaction problem by using the radiative Green function. In his method, assuming the validity of the adiabatic approximation, the radiation reaction to the conserved quantities including the Carter constant can be calculated from the radiative Green function, which is free from any singular, divergent behavior. This is a great computational advantage. However, this method cannot treat the self-force for a completely general orbit because of the assumption of adiabaticity. It is therefore still necessary to derive the self-force in the general case. One possibility is to consider the regularization of the Weyl scalar Ψ_4 and construct the R part of the metric perturbation in the radiation gauge by using the Chrzanowski method. Investigations in this direction are also in progress [30].

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APPENDIX A: ANALYSIS OF MANO ET AL.

In this appendix we summarize the analysis of Mano et al. [26] which we use in order to derive the full metric perturbation for $\ell \geqslant 2$ modes.

1. Homogeneous solutions

We investigate the analytic expression of the Regge-Wheeler functions, and generate these functions in an explicit manner up to $O(v^2)$ corrections relative to the leading order in the slow-motion expansion, i.e., first post-Newtonian order. (More detail analysis is given in [31].) Here v is a characteristic velocity of the particle. The Regge-Wheeler equation is

$$\left[\frac{d}{dr}\left(1-\frac{2M}{r}\right)\frac{d}{dr}+\left(1-\frac{2M}{r}\right)^{-1}[\omega^2-V_\ell(r)]\right]R_{\ell m\omega}^{(\text{even/odd})}(r)$$
$$=\left(1-\frac{2M}{r}\right)^{-1}S_{\ell m\omega}^{(\text{even/odd})}(r). \tag{A1}$$

The source term $S_{\ell m\omega}^{(\text{even})}$ is expressed in terms of the source terms of the Zerilli equations [21] as

$$S_{\ell m\omega}^{(\text{even})}=\left(\lambda(\lambda+1)+\frac{9M^2(r-2M)}{r^2(\lambda r+3M)}\right)S_{\ell m\omega}^{(\text{Z})}$$
$$-3M\left(1-\frac{2M}{r}\right)\frac{d}{dr}S_{\ell m\omega}^{(\text{Z})}, \tag{A2}$$

and the Zerilli function $R_{\ell m\omega}^{(\text{Z})}$ is derived from $R_{\ell m\omega}^{(\text{even})}$ [32] as

$$R_{\ell m\omega}^{(\text{Z})}=\frac{1}{\lambda^2(\lambda+1)^2+9\omega^2M^2}$$
$$\times\left[\left(\lambda(\lambda+1)+\frac{9M^2(r-2M)}{r^2(\lambda r+3M)}\right)R_{\ell m\omega}^{(\text{even})}\right.$$
$$\left.+3M\left(1-\frac{2M}{r}\right)\frac{d}{dr}R_{\ell m\omega}^{(\text{even})}\right]. \tag{A3}$$

So, we may focus on the Regge-Wheeler function. The Regge-Wheeler equation is rewritten as

$$\frac{d^2}{dz^2}X(z)+\left[\frac{1}{z-\epsilon}-\frac{1}{z}\right]\frac{d}{dz}X(z)+\left[1+\frac{2\epsilon}{z-\epsilon}+\frac{\epsilon^2}{(z-\epsilon)^2}\right.$$
$$\left.-\frac{\ell(\ell+1)}{z(z-\epsilon)}+\frac{3\epsilon}{z^2(z-\epsilon)}\right]X(z)=\left(1-\frac{\epsilon}{z}\right)^{-2}S(z). \tag{A4}$$

Here $z=\omega r$ and $\epsilon=2M\omega$, and we use the symbol $X(z)$ for $R_{\ell m\omega}^{(\text{even/odd})}(r)$, $S(z)$ for $S_{\ell m\omega}^{(\text{even/odd})}(r)$. In the post-Newtonian expansion, both z and ϵ are assumed to be small, while only ϵ is considered to be small in the post-Minkowskian expansion. We note that $z\sim O(v)$ and $\epsilon\sim O(v^3)$ in the post-Newtonian expansion.

First, we consider a homogeneous Regge-Wheeler function in the form of a series of the Coulomb wave functions, $X_C{}^\nu$ [see Eqs. (3.4) and (3.6) in Ref. [26]].

$$X_C{}^\nu(z)=\left(1-\frac{\epsilon}{z}\right)^{-i\epsilon}\sum_{n=-\infty}^{\infty}i^n$$
$$\times\frac{\Gamma(n+\nu-1-i\epsilon)\Gamma(n+\nu+1-i\epsilon)}{\Gamma(n+\nu+1+i\epsilon)\Gamma(n+\nu+3+i\epsilon)}a_n{}^\nu F_{\nu+n}(z),$$

$$F_{n+\nu}(z)=e^{-iz}(2z)^{n+\nu}z\frac{\Gamma(n+\nu+1+i\epsilon)}{\Gamma(2n+2\nu+2)}{}_1$$
$$\times F_1(n+\nu+1+i\epsilon;2n+2\nu+2;2iz), \tag{A5}$$

where ${}_1F_1$ is the confluent hypergeometric function, and the expansion coefficients $a_n{}^\nu$ are determined by the three-term recurrence relation [see (2.5) and below in Ref. [26]]

$$\alpha_n{}^\nu a_{n+1}{}^\nu+\beta_n{}^\nu a_n{}^\nu+\gamma_n{}^\nu a_{n-1}{}^\nu=0,$$

$$\alpha_n{}^\nu=-i\epsilon\frac{(n+\nu-1+i\epsilon)(n+\nu-1-i\epsilon)(n+\nu+1-i\epsilon)}{(n+\nu+1)(2n+2\nu+3)},$$

$$\beta_n{}^\nu=(n+\nu)(n+\nu+1)-\ell(\ell+1)+2\epsilon^2$$
$$+\frac{\epsilon^2(4+\epsilon^2)}{(n+\nu)(n+\nu+1)},$$

$$\gamma_n{}^\nu=i\epsilon\frac{(n+\nu+2+i\epsilon)(n+\nu+2-i\epsilon)(n+\nu+i\epsilon)}{(n+\nu)(2n+2\nu-1)}, \tag{A6}$$

and ν, which is called the renormalized angular momentum, is determined by requiring the convergence of the series expansion in $X_C{}^\nu$. Replacing ν by $-\nu-1$, one obtains the other independent solution $X_C{}^\nu$. It is important to note that the renormalized angular momentum in the post-Minkowskian expansion becomes

$$\nu=\ell+O(\epsilon^2)=\ell+O(v^6). \tag{A7}$$

Hence $\nu=\ell$ to 1PN order.

The post-Minkowskian expansion of the coefficients $a_n{}^\nu$ is also discussed in Ref. [26]. With the normalization $a_0{}^\nu=1$, they are found for $\ell\geqslant 2$,

$$a_n{}^\nu\sim O(\epsilon^{|n|})\quad(n\geqslant-\ell+2),$$
$$a_{-\ell+1}{}^\nu\sim O(\epsilon^{\ell+1}),$$
$$a_{-\ell}{}^\nu\sim O(\epsilon^{\ell+2}),$$
$$a_{-\ell-1}{}^\nu\sim O(\epsilon^{\ell+2}),$$
$$a_n{}^\nu\sim O(\epsilon^{|n|+1})\quad(-\ell-2\geqslant n\geqslant-2\ell),$$
$$a_n{}^\nu\sim O(\epsilon^{|n|-1})\quad(-2\ell-1\geqslant n). \tag{A8}$$

The post-Minkowskian expansion of the coefficients $a_n{}^{-\nu-1}$ can be obtained by using the symmetry,

$$a_n{}^{\nu}=a_{-n}{}^{-\nu-1}. \tag{A9}$$

[See (2.13) in Ref. [26].]

The leading terms in the Regge-Wheeler functions in the slow-motion expansion become

$$X_C{}^{\nu}\sim O(z^{\ell+1}\epsilon^0),$$
$$X_C{}^{-\nu-1}\sim O(z^{-\ell}\epsilon^0). \tag{A10}$$

Then, for instance, if we consider 1PN order, it is sufficient to take account of the a_0^{ν} and a_{-1}^{ν} terms in $X_C{}^{\nu}$ and $X_C{}^{-\nu-1}$ $(\ell\geqslant 2)$.

In Ref. [26] the homogeneous Regge-Wheeler functions with the in-going and up-going boundary conditions are derived in the form of linear combinations of $X_C{}^{\nu}$ and $X_C{}^{-\nu-1}$. The in-going boundary condition is that waves are purely in-going at the black hole horizon, and the up-going boundary condition is that waves are purely out-going at infinity

$$X_{\rm in}{}^{\nu}=K_{\nu}X_C{}^{\nu}+K_{-\nu-1}X_C{}^{-\nu-1},$$

$$K_{\nu}=-\frac{\pi 2^{-\nu}\epsilon^{-\nu-1}}{\Gamma(\nu+1+i\epsilon)\Gamma(\nu-1+i\epsilon)\Gamma(\nu+3+i\epsilon)\sin\pi(\nu+i\epsilon)}\sum_{n=0}^{\infty}\frac{\Gamma(n+\nu-1+i\epsilon)\Gamma(n+2\nu+1)}{n!\,\Gamma(n+\nu+3-i\epsilon)}a_n{}^{\nu}$$
$$\times\left[\sum_{n=-\infty}^{0}\frac{\Gamma(n+\nu-1-i\epsilon)\Gamma(n+\nu+1-i\epsilon)}{(-n)!\,\Gamma(n+\nu+1+i\epsilon)\Gamma(n+\nu+3+i\epsilon)\Gamma(n+2\nu+2)}a_n{}^{\nu}\right]^{-1}, \tag{A11}$$

$$X_{\rm up}{}^{\nu}=\frac{1}{e^{2i\pi\nu}+\dfrac{\sin\pi(\nu+i\epsilon)}{\sin\pi(\nu-i\epsilon)}}\left[\frac{\sin\pi(\nu+i\epsilon)}{\sin\pi(\nu-i\epsilon)}X_C{}^{\nu}-ie^{i\pi\nu}X_C^{-\nu-1}\right]. \tag{A12}$$

The leading order of K_{ν} and $K_{-\nu-1}$ for $\ell\geqslant 2$ becomes

$$K_{\nu}\sim O(\epsilon^{-\ell-2}),$$
$$K_{-\nu-1}\sim O(\epsilon^{\ell-2}). \tag{A13}$$

Then we find

$$\frac{K_{-\nu-1}X_C^{-\nu-1}}{K_{\nu}X_C{}^{\nu}}\sim O(\epsilon^{2\ell}z^{-2\ell-1})=O(v^{4\ell-1}). \tag{A14}$$

Therefore, we may replace $X_{\rm in}{}^{\nu}$ by $X_C{}^{\nu}$ to 3PN order. As for $X_{\rm up}{}^{\nu}$, we find

$$\frac{\dfrac{\sin\pi(\nu+i\epsilon)}{\sin\pi(\nu-i\epsilon)}X_C{}^{\nu}}{-ie^{i\pi\nu}X_C^{-\nu-1}}\sim O(z^{2\ell+1})=O(v^{2\ell+1}). \tag{A15}$$

Thus, we may replace $X_{\rm up}{}^{\nu}$ by $X_C{}^{-\nu-1}$ to 2PN order.

For convenience, we define the homogeneous solutions $\tilde{X}_C{}^{\nu}$ and $\tilde{X}_C^{-\nu-1}$ normalized as

$$\tilde{X}_C{}^{\nu}(z)=\frac{\Gamma(\nu+3+i\epsilon)\Gamma(2\nu+2)}{\Gamma(\nu-1-i\epsilon)\Gamma(\nu+1-i\epsilon)}X_C{}^{\nu}$$
$$=z^{\ell+1}[1+O(v)], \tag{A16}$$

$$\tilde{X}_C{}^{-\nu-1}(z)=\frac{\Gamma(-\nu+2+i\epsilon)\Gamma(-2\nu)}{\Gamma(-\nu-2-i\epsilon)\Gamma(-\nu-i\epsilon)}X_C^{-\nu-1}$$
$$=z^{-\ell}[1+O(v)]. \tag{A17}$$

These are expanded to $O(v^2)$ as

$$\tilde{X}_C{}^{\nu}(z)=z(2z)^{\nu}\left(1-\frac{1}{2}\frac{z^2}{3+2\ell}-\frac{1}{2}\frac{(\ell-2)(\ell+2)\epsilon}{\ell z}+O(v^3)\right), \tag{A18}$$

$$\tilde{X}_C{}^{-\nu-1}(z)=z(2z)^{-\nu-1}\left(1+\frac{1}{2}\frac{z^2}{2\ell-1}+\frac{1}{2}\frac{(\ell-1)(\ell+3)\epsilon}{(\ell+1)z}+O(v^3)\right), \tag{A19}$$

where $\nu=\ell+O(v^6)$.

To summarize, for the in-going homogeneous solution normalized as

$$\tilde{X}_{\rm in}{}^{\nu}(z)=\tilde{X}_C{}^{\nu}+\beta_{\nu}\tilde{X}_C{}^{-\nu-1}, \tag{A20}$$

all the coefficients β_{ν} can be set to zero up through 3PN order, while, for the up-going solution normalized as

$$\tilde{X}_{\rm up}{}^{\nu}(z)=\tilde{X}_C{}^{-\nu-1}+\gamma_{\nu}\tilde{X}_C{}^{\nu}, \tag{A21}$$

we may put $\gamma_{\nu}=0$ up through 2PN order.

2. Retarded Green function

Using the Regge-Wheeler functions ${X_{\rm in}}^{\nu}$ and ${X_{\rm up}}^{\nu}$, the retarded Green function is constructed as

$$ {G_{\rm ret}}^{\nu}(z,z') = \frac{1}{W({X_{\rm in}}^{\nu}(z'),{X_{\rm up}}^{\nu}(z'))}[{X_{\rm in}}^{\nu}(z){X_{\rm up}}^{\nu}(z') \times \theta(z'-z) + {X_{\rm up}}^{\nu}(z){X_{\rm in}}^{\nu}(z')\theta(z-z')], \tag{A22} $$

where $W(X_1,X_2)$ is the Wronskian,

$$ W(X_1(z'),X_2(z')) \equiv -\left(1-\frac{\epsilon}{z'}\right)\left(X_1(z')\frac{d}{dz'}X_2(z') - X_2(z')\frac{d}{dz'}X_1(z')\right) = \text{const.} \tag{A23} $$

This Green function satisfies

$$ \left\{\partial_z^2 + \left[\frac{1}{z-\epsilon}-\frac{1}{z}\right]\partial_z + \left[1+\frac{2\epsilon}{z-\epsilon}+\frac{\epsilon^2}{(z-\epsilon)^2}-\frac{\ell(\ell+1)}{z(z-\epsilon)} + \frac{3\epsilon}{z^2(z-\epsilon)}\right]\right\}{G_{\rm ret}}^{\nu}(z,z') = -\left(1-\frac{\epsilon}{z}\right)^{-1}\delta(z-z'). \tag{A24} $$

Then the Regge-Wheeler function with the source term $S^{(\text{even/odd})}_{\ell m\omega}(r)$ is given by

$$ R^{(\text{even/odd})}_{\ell m\omega}(r) = -\int_{2M}^{\infty} dr'\, {G_{\rm ret}}^{\nu}(r,r')\frac{1}{\omega}\left(1-\frac{2M}{r'}\right)^{-1} \times S^{(\text{even/odd})}_{\ell m\omega}(r'). \tag{A25} $$

Here we are only interested in the Green function accurate to 1PN order. A numerical method to construct an accurate Green function based on this Mano-Suzuki-Takasugi method is discussed in Ref. [28].

APPENDIX B: SPHERICAL EXTENSION OF THE *S* PART

In this appendix we consider the tensor harmonic decomposition of the *S* part. First, we give the decomposition of ϵ^n where

$$ \epsilon = (r^2 + r_0^2 - 2r_0 r\cos\mathbf{\Omega}\cdot\mathbf{\Omega}_0)^{1/2}, \tag{B1} $$

and $\mathbf{\Omega}_0$ is taken to be on the equatorial plane, $(\pi/2,\phi_0)$. Extending $\mathbf{\Omega}$ over the whole sphere, we have

$$ \epsilon^p = \sum_{\ell m}\frac{4\pi}{2\ell+1}E_\ell^{(p)}(r,r_0)Y_{\ell m}(\mathbf{\Omega})Y^*_{\ell m}(\mathbf{\Omega}_0), \tag{B2} $$

where the detail derivation as well as the coefficients $E_\ell^{(p)}$ are given in Appendix D of [9].

In terms of the coefficients $E_\ell^{(p)}$, the formulas needed to decompose the *S* part are derived as

$$ \frac{1}{\epsilon} = \sum_{\ell m}\frac{4\pi}{2\ell+1}E_\ell^{(-1)}Y_{\ell m}(\mathbf{\Omega})Y^*_{\ell m}(\mathbf{\Omega}_0), $$

$$ \frac{\Phi}{\epsilon} = \sum_{\ell m}\frac{4\pi}{2\ell+1}\frac{imE_\ell^{(1)}}{r_0 r}Y_{\ell m}(\mathbf{\Omega})Y^*_{\ell m}(\mathbf{\Omega}_0), $$

$$ \frac{\Phi^2}{\epsilon} = \sum_{\ell m}\frac{4\pi}{2\ell+1}\left[-\frac{E_\ell^{(1)}}{r_0^2}-\frac{1}{3}\frac{m^2E_\ell^{(3)}}{r_0^4}\right]Y_{\ell m}(\mathbf{\Omega})Y^*_{\ell m}(\mathbf{\Omega}_0), $$

$$ \frac{1}{\epsilon^3} = \sum_{\ell m}\frac{4\pi}{2\ell+1}E_\ell^{(-3)}Y_{\ell m}(\mathbf{\Omega})Y^*_{\ell m}(\mathbf{\Omega}_0), $$

$$ \frac{\Phi}{\epsilon^3} = \sum_{\ell m}\frac{4\pi}{2\ell+1}\left[-\frac{1}{2}\frac{imE_\ell^{(1)}}{r_0^3 r}-\frac{1}{2}\frac{im(-R^2+2r_0^2)E_\ell^{(-1)}}{r_0^3 r}-\frac{1}{9}\frac{im^3E_\ell^{(3)}}{r_0^5 r}\right]Y_{\ell m}(\mathbf{\Omega})Y^*_{\ell m}(\mathbf{\Omega}_0), $$

$$ \frac{\Phi^2}{\epsilon^3} = \sum_{\ell m}\frac{4\pi}{2\ell+1}\left[\frac{2}{45}\frac{m^4E_\ell^{(5)}}{r_0^7 r}+\frac{1}{2}\frac{(2r_0^2r+2m^2r_0^3-2R^2m^2r-r_0^3)E_\ell^{(1)}}{r_0^5r^2}+\frac{1}{2}\frac{(-2R^2r+2r_0^2r+R^2r_0)E_\ell^{(-1)}}{r_0^3r^2}+\frac{m^2E_\ell^{(3)}}{r_0^5 r}\right] \times Y_{\ell m}(\mathbf{\Omega})Y^*_{\ell m}(\mathbf{\Omega}_0), $$

$$ \frac{\Phi^3}{\epsilon^3} = \sum_{\ell m}\frac{4\pi}{2\ell+1}\left[3\frac{imE_\ell^{(1)}}{r_0^4}+\frac{1}{3}\frac{im^3E_\ell^{(3)}}{r_0^6}\right]Y_{\ell m}(\mathbf{\Omega})Y^*_{\ell m}(\mathbf{\Omega}_0), $$

$$\frac{\Phi^4}{\epsilon^3}=\sum_{\ell m}\frac{4\pi}{2\ell+1}\left[-\frac{1}{15}\frac{m^4E_\ell^{(5)}}{r_0^8}-3\frac{E_\ell^{(1)}}{r_0^4}-2\frac{m^2E_\ell^{(3)}}{r_0^6}\right]Y_{\ell m}(\mathbf{\Omega})Y^*_{\ell m}(\mathbf{\Omega}_0),$$

$$\frac{1}{\epsilon^5}=\sum_{\ell m}\frac{4\pi}{2\ell+1}E_\ell^{(-5)}Y_{\ell m}(\mathbf{\Omega})Y^*_{\ell m}(\mathbf{\Omega}_0),$$

$$\frac{\Phi^2}{\epsilon^5}=\sum_{\ell m}\frac{4\pi}{2\ell+1}\left[\frac{1}{6}\frac{(-3R^2r+2Rr_0r-4Rr_0^2+2r_0^2r)E_\ell^{(-3)}}{r_0^4r}-\frac{1}{3}\frac{m^2E_\ell^{(1)}}{r_0^6}-\right.$$
$$\left.\times\frac{1}{6}\frac{(-r_0^2r^2-4Rm^2r_0^3-4R^2m^2r^2+2m^2r_0^2r^2)E_\ell^{(-1)}}{r_0^6r^2}-\frac{2}{27}\frac{m^4E_\ell^{(3)}}{r_0^8}\right]Y_{\ell m}(\mathbf{\Omega})Y^*_{\ell m}(\mathbf{\Omega}_0),$$

$$\frac{\Phi^4}{\epsilon^5}=\sum_{\ell m}\frac{4\pi}{2\ell+1}\left[2\frac{m^2E_\ell^{(1)}}{r_0^{10}}+\frac{E_\ell^{(-1)}}{r_0^4}+\frac{1}{9}\frac{m^4E_\ell^{(3)}}{r_0^8}\right]Y_{\ell m}(\mathbf{\Omega})Y^*_{\ell m}(\mathbf{\Omega}_0),$$

$$\frac{\Theta}{\epsilon}=\sum_{\ell m}\frac{4\pi}{2\ell+1}\left[-\frac{E_\ell^{(1)}}{r_0^2}\right]Y_{\ell m}(\mathbf{\Omega})\partial_\theta Y^*_{\ell m}(\mathbf{\Omega}_0),$$

$$\frac{\Theta}{\epsilon^3}=\sum_{\ell m}\frac{4\pi}{2\ell+1}\left[-\frac{1}{2}\frac{E_\ell^{(1)}}{r_0^3r}+\frac{1}{6}\frac{(-R^2+6r_0^2)E_\ell^{(-1)}}{r_0^3r}-\frac{1}{9}\frac{m^2E_\ell^{(3)}}{r_0^5r}\right]Y_{\ell m}(\mathbf{\Omega})\partial_\theta Y^*_{\ell m}(\mathbf{\Omega}_0),$$

$$\frac{\Theta\Phi}{\epsilon^3}=\sum_{\ell m}\frac{4\pi}{2\ell+1}\frac{imE_\ell^{(1)}}{r_0^2r^2}Y_{\ell m}(\mathbf{\Omega})\partial_\theta Y^*_{\ell m}(\mathbf{\Omega}_0),$$

$$\frac{\Theta\Phi^2}{\epsilon^3}=\sum_{\ell m}\frac{4\pi}{2\ell+1}\left[-\frac{E_\ell^{(1)}}{r_0^4}-\frac{1}{3}\frac{m^2E_\ell^{(3)}}{r_0^6}\right]Y_{\ell m}(\mathbf{\Omega})\partial_\theta Y^*_{\ell m}(\mathbf{\Omega}_0),$$

$$\frac{\Theta\Phi}{\epsilon^5}=\sum_{\ell m}\frac{4\pi}{2\ell+1}\left[-\frac{1}{6}\frac{im(2r_0^2+3r^2-4r_0r)E_\ell^{(1)}}{r_0^4r^4}+\frac{1}{9}\frac{im(-3r_0^3+2R^2r)E_\ell^{(-1)}}{r_0^5r^2}-\frac{1}{18}\frac{im^3E_\ell^{(3)}}{r_0^6r^2}\right]Y_{\ell m}(\mathbf{\Omega})\partial_\theta Y^*_{\ell m}(\mathbf{\Omega}_0),$$

$$\frac{\Theta\Phi^3}{\epsilon^5}=\sum_{\ell m}\frac{4\pi}{2\ell+1}\left[\frac{imE_\ell^{(1)}}{r_0^3r^3}+\frac{1}{9}\frac{im^3E_\ell^{(3)}}{r_0^4r^4}\right]Y_{\ell m}(\mathbf{\Omega})\partial_\theta Y^*_{\ell m}(\mathbf{\Omega}_0). \tag{B3}$$

Note that these formulas are valid only in the sense of the spherical extension given by Eq. (B2).

APPENDIX C: $O(y^2)$ CORRECTION

In this appendix as an example to clarify how the standard form is recovered and why it is necessary to include the $\ell=0,1$ modes even if some of the tensor harmonics are identically zero for these modes, we consider the S part of the metric components $h_{t\theta}$ and $h_{t\phi}$ and analyze the contribution of their $O(y^2)$ terms to the self-force in the harmonic gauge.

These metric components give rise to the coefficient $h_{0\ell m}^{(\mathrm{e})}$ of the vector harmonic proportional to $(\partial_\theta Y_{\ell m},\partial_\phi Y_{\ell m})$. Note that this vanishes identically for $\ell=0$. Since the contribution of the $O(y^2)$ terms to the self-force is zero, the sum of $h_{0\ell m}^{(\mathrm{e})}$ over ℓ should vanish. We show that it indeed has the standard form for general ℓ. However, to guarantee that the sum over ℓ is zero, it is necessary to include the contribution from $\ell=0$ as well. This implies $B^\mu+D^\mu_\ell=0$ for $\ell=0$ as discussed at the end of Sec. V.

The local expansion of the S part of the metric components $h_{t\theta}$ and $h_{t\phi}$ takes the form

$$^{(2)}h_{t\theta}^{\mathrm{S,H}}=\left[\frac{\Phi^{2n+2}\Theta}{\epsilon^{2n+1}},\frac{\Phi^{2n+1}R\Theta}{\epsilon^{2n+1}}\right],$$
$$^{(2)}h_{t\phi}^{\mathrm{S,H}}=\left[\frac{\Phi^{2n+3}}{\epsilon^{2n+1}},\frac{\Phi^{2n+2}R}{\epsilon^{2n+1}}\right], \tag{C1}$$

where we have retained only terms that may contribute to the self-force, and the superscript (2) means $O(y^2)$. The tensor harmonic coefficients $h_{0\ell m}^{(e)}$ are given by

$$h^{(e)S,H}_{0\ell m}(t,r)=\frac{-1}{\ell(\ell+1)}\int [h_{t\theta}\partial_\theta Y^*_{\ell m}(\theta,\phi)+h_{t\phi}\partial_\phi Y^*_{\ell m}(\theta,\phi)]d\mathbf{\Omega}, \tag{C2}$$

where $\ell\neq 0$. For the $O(y^2)$ terms of the form (C1), we have

$$^{(2)}h^{(e)S,H}_{0\ell m}(t,r)=\frac{-1}{\ell(\ell+1)}\int\left[\frac{\Phi^{2n+2}}{\epsilon^{2n+1}},\frac{\Phi^{2n+1}R}{\epsilon^{2n+1}}\right]\times Y^*_{\ell m}(\theta,\phi)d\mathbf{\Omega}. \tag{C3}$$

The force is given by

$$F^{r(\pm)}_{S,H}[h^{(e)S,H}_0]=\sum_{\ell m}\frac{\mu(r_0-2M)}{2r_0^3(r_0-3M)}\times[-2im\Omega\partial_r h^{(e)S,H}_{0\ell m}(t_0,r_0)]Y_{\ell m}(\pi/2,0),$$

$$\Omega=\frac{u^\phi}{u^t}. \tag{C4}$$

Here since the terms of interest are already of $O(y^2)$, we may use the leading order formulas for the spherical extension of the local coordinate expansion [9]. We have

$$\epsilon^{2n-1}\sim\sum_{\ell m}\frac{2\pi}{L}\frac{\kappa_n}{(L^2-1)(L^2-2^2)\cdots(L^2-n^2)}\times\left(\frac{r_<}{r_>}\right)^\ell Y_{\ell m}(\theta,\phi)Y^*_{\ell m}(\pi/2,0), \tag{C5}$$

$$\frac{\Phi}{\epsilon}\sim\partial_\phi\epsilon, \tag{C6}$$

where $n\geqslant 1$, $r_>=\max\{r,r_0\}$, $r_<=\min\{r,r_0\}$, $L=\ell+1/2$ and κ_n is a constant independent of L. Therefore, Eq. (C3) is evaluated as

$$\frac{-1}{\ell(\ell+1)}\int\frac{\Phi^{2n+2}}{\epsilon^{2n+1}}Y^*_{\ell m}(\theta,\phi)d\mathbf{\Omega}$$
$$\sim\frac{1}{\mathcal{L}^{(2)}}\int\partial_\phi^{2n+2}\epsilon^{2n+3}Y^*_{\ell m}(\theta,\phi)d\mathbf{\Omega}$$
$$\sim\frac{2\pi}{L}\frac{1}{\mathcal{L}^{(2)}}\frac{m^{2n+2}}{(L^2-1)(L^2-2^2)\cdots[L^2-(n+2)^2]}\left(\frac{r_<}{r_>}\right)^\ell\times Y^*_{\ell m}(\pi/2,0),$$

$$\frac{-1}{\ell(\ell+1)}\int\frac{\Phi^{2n+1}R}{\epsilon^{2n+1}}Y^*_{\ell m}(\theta,\phi)d\mathbf{\Omega}$$
$$\sim\frac{1}{\mathcal{L}^{(2)}}\int R\partial_\phi^{2n+1}\epsilon^{2n+1}Y^*_{\ell m}(\theta,\phi)d\mathbf{\Omega}$$
$$\sim\frac{2\pi}{L}\frac{1}{\mathcal{L}^{(2)}}\frac{m^{2n+1}(r-r_0)}{(L^2-1)(L^2-2^2)\cdots[L^2-(n+1)^2]}\times\left(\frac{r_<}{r_>}\right)^\ell Y^*_{\ell m}(\pi/2,0), \tag{C7}$$

where $n\geqslant 0$ and

$$\mathcal{L}^{(2)}=\ell(\ell+1)=\left(L^2-\frac{1}{4}\right). \tag{C8}$$

Using Eq. (C4), and retaining only the terms that will remain after summing over m, we have

$$F^{r(\pm)}_{S,H}[h^{(e)S,H}_0]\sim\sum_{\ell m}\frac{2\pi}{L}\frac{1}{\mathcal{L}^{(2)}}\times\frac{m^{2n+2}}{(L^2-1)(L^2-2^2)\cdots[L^2-(n+1)^2]}\times|Y_{\ell m}(\pi/2,0)|^2. \tag{C9}$$

The m summation gives

$$\sum_m\frac{2\pi}{L}\frac{m^{2n+2}}{\mathcal{L}^{(2)}}|Y_{\ell m}(\pi/2,0)|^2=\sum_{k=0}^n\alpha_k L^{2k}. \tag{C10}$$

Thus, the $O(y^2)$ terms contribute to the D^μ_ℓ term in the form of the standard form, and the sum over ℓ vanishes provided we include the $\ell=0$ term in the summation. Since the $O(y^2)$ terms do not contribute to the force anyway, it then follows that we may adjust the numerators of the D^μ_ℓ term so as to give $D^\mu_0=-B^\mu$.

APPENDIX D: CALCULATION OF THE S PART

In this appendix we show the S part of the metric perturbation and its gauge transformation. The S part of the metric perturbation under the harmonic gauge are obtained in the local coordinate expansion as

$$h_{tt}^{\mathrm{S,H}}=\mu\Bigg[2\frac{1}{\epsilon}+\left(+2\frac{\Phi T r_0^2}{\epsilon^3}+\frac{\Phi T R^2}{\epsilon^3}-\frac{\Phi T}{\epsilon}+\frac{2}{3}\frac{\Phi^3 T r_0^2}{\epsilon^3}+2\frac{\Phi T r_0 R}{\epsilon^3}\right)u^{\phi}$$

$$+\left(-\frac{1}{2}\frac{\epsilon}{r_0^3}-\frac{RT^2}{\epsilon^3 r_0^2}-4\frac{1}{\epsilon r_0}-2\frac{R^2}{\epsilon^3 r_0}+\frac{R^3}{\epsilon^3 r_0^2}-\frac{R^4}{\epsilon^3 r_0^3}-\frac{1}{2}\frac{R^2T^2}{\epsilon^3 r_0^3}+\frac{1}{2}\frac{T^2}{\epsilon r_0^3}-\frac{5}{2}\frac{R^2}{\epsilon r_0^3}+4\frac{R}{\epsilon r_0^2}\right)M$$

$$-\left(-2\frac{r_0^4\Phi^4T^2}{\epsilon^5}+\frac{r_0^2T^2}{\epsilon^3}-\frac{r_0^2\Phi^2}{\epsilon}-3\frac{r_0^4\Phi^2T^2}{\epsilon^5}-6\frac{r_0^2\Phi^2R^2T^2}{\epsilon^5}+\frac{2}{3}\frac{r_0^4\Phi^4}{\epsilon^3}-4\frac{r_0^2}{\epsilon}-6\frac{r_0^3\Phi^2T^2R}{\epsilon^5}\right.$$

$$\left.+3\frac{r_0^2\Phi^2T^2}{\epsilon^3}+2\frac{r_0^3\Phi^2R}{\epsilon^3}+2\frac{r_0^2\Phi^2R^2}{\epsilon^3}+\frac{r_0^4\Phi^2}{\epsilon^3}\right)(u^{\phi})^2\Bigg]+O(y^2),$$

$$h_{tr}^{\mathrm{S,H}}=h_{rt}^{\mathrm{S,H}}=\mu\left[-4\frac{\Phi r_0}{\epsilon}u^{\phi}+\left(4\frac{T}{\epsilon r_0^2}-4\frac{TR}{\epsilon r_0^3}\right)M+\left(-4\frac{r_0^3\Phi^2T}{\epsilon^3}-4\frac{r_0^2\Phi^2TR}{\epsilon^3}\right)(u^{\phi})^2\right]+O(y^2),$$

$$h_{t\theta}^{\mathrm{S,H}}=h_{\theta t}^{\mathrm{S,H}}=\mu\left[-2\frac{TM\Theta}{\epsilon r_0}+4\frac{r_0^2\Phi\Theta}{\epsilon}u^{\phi}+4\frac{r_0^4\Phi^2T\Theta}{\epsilon^3}(u^{\phi})^2\right]+O(y^2),$$

$$h_{t\phi}^{\mathrm{S,H}}=h_{\phi t}^{\mathrm{S,H}}=\mu\Bigg[\left(2\epsilon-2\frac{R^2}{\epsilon}-4\frac{r_0R}{\epsilon}-4\frac{r_0^2}{\epsilon}\right)u^{\phi}-2\frac{\Phi T}{\epsilon r_0}M$$

$$+\left(4\frac{r_0^2\Phi T}{\epsilon}-8\frac{r_0^2\Phi TR^2}{\epsilon^3}-8\frac{r_0^3\Phi TR}{\epsilon^3}-4\frac{r_0^4\Phi T}{\epsilon^3}-\frac{4}{3}\frac{r_0^4\Phi^3T}{\epsilon^3}\right)(u^{\phi})^2\Bigg]+O(y^2),$$

$$h_{rr}^{\mathrm{S,H}}=\mu\Bigg[2\frac{1}{\epsilon}+\left(2\frac{\Phi T r_0^2}{\epsilon^3}+\frac{\Phi TR^2}{\epsilon^3}-\frac{\Phi T}{\epsilon}+\frac{2}{3}\frac{\Phi^3Tr_0^2}{\epsilon^3}+2\frac{\Phi Tr_0R}{\epsilon^3}\right)u^{\phi}$$

$$+\left(-\frac{17}{2}\frac{\epsilon}{r_0^3}-4\frac{R}{\epsilon r_0^2}-\frac{RT^2}{\epsilon^3r_0^2}+4\frac{1}{\epsilon r_0}-2\frac{R^2}{\epsilon^3r_0}+\frac{R^3}{\epsilon^3r_0^2}-\frac{R^4}{\epsilon^3r_0^3}-\frac{1}{2}\frac{R^2T^2}{\epsilon^3r_0^3}+\frac{1}{2}\frac{T^2}{\epsilon r_0^3}+\frac{11}{2}\frac{R^2}{\epsilon r_0^3}\right)M$$

$$-\left(-2\frac{r_0^4\Phi^4T^2}{\epsilon^5}-5\frac{r_0^2\Phi^2}{\epsilon}-3\frac{r_0^4\Phi^2T^2}{\epsilon^5}-6\frac{r_0^2\Phi^2R^2T^2}{\epsilon^5}+\frac{r_0^2T^2}{\epsilon^3}+\frac{2}{3}\frac{r_0^4\Phi^4}{\epsilon^3}-6\frac{r_0^3\Phi^2T^2R}{\epsilon^5}+3\frac{r_0^2\Phi^2T^2}{\epsilon^3}\right.$$

$$\left.+2\frac{r_0^3\Phi^2R}{\epsilon^3}+2\frac{r_0^2\Phi^2R^2}{\epsilon^3}+\frac{r_0^4\Phi^2}{\epsilon^3}\right)(u^{\phi})^2\Bigg]+O(y^2),$$

$$h_{r\theta}^{\mathrm{S,H}}=h_{\theta r}^{\mathrm{S,H}}=O(y^2),$$

$$h_{r\phi}^{\mathrm{S,H}}=h_{\phi r}^{\mathrm{S,H}}=\mu\left(4\frac{r_0^2\Phi R}{\epsilon}+4\frac{r_0^3\Phi}{\epsilon}\right)(u^{\phi})^2+O(y^2),$$

$$h_{\theta\theta}^{\mathrm{S,H}}=\mu\Bigg[2\frac{r_0^2}{\epsilon}+4\frac{r_0R}{\epsilon}+2\frac{R^2}{\epsilon}+\left(-\frac{r_0^2\Phi T}{\epsilon}+7\frac{r_0^2\Phi TR^2}{\epsilon^3}+6\frac{r_0^3\Phi TR}{\epsilon^3}+\frac{2}{3}\frac{r_0^4\Phi^3T}{\epsilon^3}+2\frac{r_0^4\Phi T}{\epsilon^3}\right)u^{\phi}$$

$$+\left(-\frac{5}{2}\frac{R^2T^2}{\epsilon^3r_0}+\frac{1}{2}\frac{T^2}{\epsilon r_0}-\frac{R^4}{\epsilon^3r_0}-2\frac{r_0R^2}{\epsilon^3}-\frac{RT^2}{\epsilon^3}-3\frac{R^3}{\epsilon^3}+\frac{7}{2}\frac{\epsilon}{r_0}+\frac{3}{2}\frac{R^2}{\epsilon r_0}\right)M$$

$$+\left(12\frac{r_0^5\Phi^2T^2R}{\epsilon^5}+2\frac{r_0^6\Phi^4T^2}{\epsilon^5}+3\frac{r_0^6\Phi^2T^2}{\epsilon^5}-\frac{r_0^6\Phi^2}{\epsilon^3}-\frac{2}{3}\frac{r_0^6\Phi^4}{\epsilon^3}+\frac{r_0^4\Phi^2}{\epsilon}-\frac{r_0^2R^2T^2}{\epsilon^3}-2\frac{r_0^3RT^2}{\epsilon^3}-\frac{r_0^4T^2}{\epsilon^3}\right.$$

$$-4\frac{r_0^5\Phi^2R}{\epsilon^3}+21\frac{r_0^4\Phi^2R^2T^2}{\epsilon^5}-3\frac{r_0^4\Phi^2T^2}{\epsilon^3}-7\frac{r_0^4\Phi^2R^2}{\epsilon^3}(u^\phi)^2+O(y^2),$$

$$h^{\rm S,H}_{\theta\phi}=h^{\rm S,H}_{\phi\theta}=-4\mu\frac{r_0^4\Phi\Theta}{\epsilon}(u^\phi)^2+O(y^2),$$

$$h^{\rm S,H}_{\phi\phi}=\mu\Bigg[-2\epsilon+2\frac{r_0^2}{\epsilon}+2\frac{r_0^2\Phi^2}{\epsilon}+4\frac{r_0R}{\epsilon}+4\frac{R^2}{\epsilon}+\left(\frac{8}{3}\frac{r_0^4\Phi^3T}{\epsilon^3}-3\frac{r_0^2\Phi T}{\epsilon}+9\frac{r_0^2\Phi TR^2}{\epsilon^3}+6\frac{r_0^3\Phi TR}{\epsilon^3}+2\frac{r_0^4\Phi T}{\epsilon^3}\right)u^\phi$$
$$+\left(-2\frac{r_0\Phi^2R^2}{\epsilon^3}-\frac{5}{2}\frac{R^2T^2}{\epsilon^3r_0}+\frac{1}{2}\frac{T^2}{\epsilon r_0}-3\frac{R^4}{\epsilon^3r_0}-2\frac{r_0R^2}{\epsilon^3}-\frac{RT^2}{\epsilon^3}-3\frac{R^3}{\epsilon^3}+\frac{7}{2}\frac{\epsilon}{r_0}+\frac{7}{2}\frac{R^2}{\epsilon r_0}\right)M$$
$$+\left(4\frac{r_0^4}{\epsilon}+\frac{r_0^2T^2}{\epsilon}-4r_0^2\epsilon+8\frac{r_0^3R}{\epsilon}+8\frac{r_0^2R^2}{\epsilon}-\frac{r_0^4T^2}{\epsilon^3}-4\frac{r_0^5\Phi^2R}{\epsilon^3}+24\frac{r_0^4\Phi^2R^2T^2}{\epsilon^5}-7\frac{r_0^4\Phi^2T^2}{\epsilon^3}-8\frac{r_0^4\Phi^2R^2}{\epsilon^3}\right.$$
$$\left.+12\frac{r_0^5\Phi^2T^2R}{\epsilon^5}+5\frac{r_0^6\Phi^4T^2}{\epsilon^5}+3\frac{r_0^6\Phi^2T^2}{\epsilon^5}-\frac{r_0^6\Phi^2}{\epsilon^3}-\frac{5}{3}\frac{r_0^6\Phi^4}{\epsilon^3}+2\frac{r_0^4\Phi^2}{\epsilon}-2\frac{r_0^2R^2T^2}{\epsilon^3}-2\frac{r_0^3RT^2}{\epsilon^3}\right)(u^\phi)^2\Bigg]$$
$$+O(y^2). \tag{D1}$$

The harmonic coefficients of the above S part are calculated as

$$h^{\rm S,H}_{0\ell m}(t,r)=\frac{2}{L}\pi\mu\Bigg[\frac{4iTmr_0(L^2-2)(u^\phi)^2}{\mathcal{L}^{(2)}(L^2-1)}-(8r_0-6r_0m^2-18r_0L^2+4r_0L^4-4R+16RL-13Rm^2$$
$$-7RL^2-20RL^3+2RL^4+4RL^5)u^\phi/[\mathcal{L}^{(2)}(L^2-1)(L^2-4)]\Bigg]\partial_\theta Y^*_{\ell m}(\theta_0,\phi_0),$$

$$h^{\rm S,H}_{1\ell m}(t,r)=\frac{2}{L}\pi\mu\left[\frac{-2ir_0m(2r_0+R)(u^\phi)^2}{\mathcal{L}^{(2)}(L^2-1)}\right]\partial_\theta Y^*_{\ell m}(\theta_0,\phi_0),$$

$$h^{\rm S,H}_{2\ell m}(t,r)=\frac{2}{L}\pi\mu\Bigg[-\frac{1}{6}r_0m(72r_0RL^4+48r_0RL^5-240r_0RL^3-288r_0Rm^2+4108R^2+1056r_0^2-240R^2L^3-1488r_0R$$
$$+1392R^2m^2+24R^2L^6-1147R^2L^2+192R^2L-66R^2L^4+48R^2L^5+84r_0RL^2+192r_0RL-456r_0^2L^2-48R^2m^2L^2$$
$$+48r_0^2L^4+288r_0^2m^2)(u^\phi)^2/[\mathcal{L}^{(4)}(L^2-1)(L^2-4)]\Bigg]\partial_\theta Y^*_{\ell m}(\theta_0,\phi_0),$$

$$H^{\rm S,H}_{0\ell m}(t,r)=\frac{2}{L}\pi\mu\Bigg[\frac{1}{4}(-504r_0+144r_0m^2+12r_0L^6+614r_0L^2-62Rm^2L^2+2Rm^2L^4-170r_0L^4-529RL^2-10RL^6$$
$$-168RL^5+588RL^3+143RL^4+396Rm^2+40Rm^4-432RL+12RL^7+468R-52r_0m^2L^2$$
$$+4r_0m^2L^4)(u^\phi)^2/[(L^2-1)(L^2-4)(L^2-9)]-\frac{2imTu^\phi}{r_0}+\frac{1}{4}\frac{(2r_0+R-8RL+8RL^3)M}{r_0^3(L^2-1)}$$
$$+\frac{(2r_0-R+2RL)}{r_0^2}\Bigg]Y^*_{\ell m}(\theta_0,\phi_0),$$

$$H^{\mathrm{S,H}}_{1\ell m}(t,r)=\frac{2}{L}\pi\mu\left[-2\frac{T(-108+111L^2+2L^6-29L^4+44m^2L^2-2m^2L^4-234m^2-20m^4)(u^\phi)^2}{(L^2-1)(L^2-4)(L^2-9)}+\frac{2im(-R+2r_0)u^\phi}{(L^2-1)r_0}+\frac{4TM}{r_0^3}\right]Y^*_{\ell m}(\theta_0,\phi_0),$$

$$H^{\mathrm{S,H}}_{2\ell m}(t,r)=\frac{2}{L}\pi\mu\left[-\frac{1}{4}(-648r_0-576r_0m^2+4r_0L^6+378r_0L^2+134Rm^2L^2-2Rm^2L^4-70r_0L^4+241RL^2+2RL^6-56RL^5+196RL^3-39RL^4-1044Rm^2-40Rm^4-144RL+4RL^7-468R+100r_0m^2L^2-4r_0m^2L^4)(u^\phi)^2/[(L^2-1)(L^2-4)(L^2-9)\right]-\frac{2imTu^\phi}{r_0}+\frac{1}{4}\frac{(2r_0+R-8RL+8RL^3)M}{r_0^3(L^2-1)}+\frac{(2r_0-R+2RL)}{r_0^2}Y^*_{\ell m}(\theta_0,\phi_0),$$

$$h^{(\mathrm{e})\mathrm{S,H}}_{0\ell m}(t,r)=\frac{2}{L}\pi\mu\Big[T(72r_0+392Rm^2L^3-288Rm^2L-112Rm^2L^5-120r_0m^4+48Rm^4L^2+8Rm^2L^7+4Rm^2L^6+8r_0m^2L^6-828r_0m^2-8r_0L^6-314r_0L^2+172Rm^2L^2-50Rm^2L^4+106r_0L^4-941RL^2-20RL^6+277RL^4-414Rm^2-612Rm^4+612R+344r_0m^2L^2-100r_0m^2L^4)(u^\phi)^2/[2\mathcal{L}^{(2)}(L^2-1)(L^2-4)(L^2-9)]+i(8R^2L^6+16r_0RL^5+16r_0^2L^4+8r_0RL^4-46R^2L^4-80r_0RL^3+55R^2L^2-56r_0^2L^2-52r_0RL^2+64r_0RL-32r_0^2+80r_0R+16r_0Rm^2+4R^2+40R^2m^2)mu^\phi/[4r_0\mathcal{L}^{(2)}(L^2-1)(L^2-4)]+\frac{T(-R+2r_0)M}{r_0^3\mathcal{L}^{(2)}}\Big]Y^*_{\ell m}(\theta_0,\phi_0),$$

$$h^{(\mathrm{e})\mathrm{S,H}}_{1\ell m}(t,r)=\frac{2}{L}\pi\mu\{-(20r_0RL^4+3528r_0Rm^2-756R^2+288r_0^2-464r_0Rm^2L^2+8r_0Rm^2L^4+720r_0R+1062R^2m^2+20R^2L^6+993R^2L^2-281R^2L^4+160m^4r_0^2-260r_0RL^2-104r_0^2L^2-16R^2m^4L^2+684R^2m^4+102R^2m^2L^4-8R^2m^2L^6-388R^2m^2L^2+560r_0Rm^4+8r_0^2L^4+1872r_0^2m^2+16r_0^2m^2L^4-352r_0^2m^2L^2)(u^\phi)^2/[4\mathcal{L}^{(2)}(L^2-1)(L^2-4)(L^2-9)]\}Y^*_{\ell m}(\theta_0,\phi_0),$$

$$K^{\mathrm{S,H}}_{\ell m}(t,r)=\frac{2}{L}\pi\mu\Big[\frac{1}{192}(12744r_0-9408Rm^2L^3+6912Rm^2L+2688Rm^2L^5-120RL^8+144r_0L^8+144RL^9+960r_0m^4-288Rm^4L^2-192Rm^2L^7+120Rm^2L^6-144r_0m^2L^6-648r_0m^2-2172r_0L^6-15402r_0L^2+6462Rm^2L^2-1758Rm^2L^4+9438r_0L^4+16455RL^2+1938RL^6+9800RL^5-14788RL^3-9453RL^4-3240Rm^2+7272Rm^4+7056RL-2212RL^7-7884R-4140r_0m^2L^2+1764r_0m^2L^4)(2L-1)(2L+1)(u^\phi)^2/[\mathcal{L}^{(4)}(L^2-1)(L^2-4)(L^2-9)]-\frac{2imTu^\phi}{r_0}+\frac{1}{4}\frac{(2r_0+R-8RL+8RL^3)M}{r_0^3(L^2-1)}+\frac{(2r_0-R+2RL)}{r_0^2}\Big]Y^*_{\ell m}(\theta_0,\phi_0),$$

$$G^{\mathrm{S,H}}_{\ell m}(t,r)=\frac{2}{L}\pi\mu\Big[\frac{1}{48}(-28296r_0R^2-5184r_0^3m^2+30780R^3+9792r_0^2Rm^2L^2+192r_0^2Rm^2L^6-384r_0^2Rm^2L^7-6048r_0^2R-1104r_0^3L^2-2544r_0^3L^6+7488r_0^3L^4+192r_0^3L^8+1920r_0^3m^4-384r_0^3m^2L^6-8064r_0^3m^2L^2+4416r_0^3m^2L^4+13416r_0^2RL^2-15536r_0^2RL^3+1560r_0^2RL^6-2864r_0^2RL^7-7680r_0^2RL^4+11872r_0^2RL^5+32R^3L^{11}-71343R^3L^2$$

$$+10047R^3L^6+38356R^3L^5+15650R^3L^3+39596R^3L^4-13182R^3L^7-5976R^3L^8-104004R^3m^2+688R^3L^9$$
$$+35688R^3m^4+496R^3L^{10}-41544R^3L-13824r_0R^2m^2L-192r_0R^2m^2L^8+18816r_0R^2m^2L^3+384r_0R^2m^2L^7$$
$$+43266r_0R^2L^2-9480r_0R^2m^2L^4-4416r_0R^2m^4L^2-1536r_0^2Rm^4L^2-5376r_0R^2m^2L^5+2832r_0R^2m^2L^6$$
$$-11088r_0R^2m^2L^2-18816r_0^2Rm^2L^3+5376r_0^2Rm^2L^5+13824r_0^2Rm^2L-96r_0^2RL^8+192r_0^2RL^9-2592r_0^2Rm^2$$
$$+16704r_0^2Rm^4+6336r_0^2RL+8742r_0R^2L^6-22624r_0R^2L^5+53168r_0R^2L^3-24816r_0R^2L^4+3632r_0R^2L^7$$
$$-1584r_0R^2L^8+62856r_0R^2m^2-192r_0R^2L^9+40464r_0R^2m^4+96r_0R^2L^{10}-33984r_0R^2L-4256R^3m^2L^5$$
$$-1560R^3m^2L^6+108648R^3m^2L^2+5248R^3m^4L-20076R^3m^2L^4-8416R^3m^4L^2+128R^3m^4L^3+12672R^3m^2L$$
$$+4496R^3m^2L^3+256R^3m^4L^4+976R^3m^2L^7-64R^3m^2L^9+288R^3m^2L^8-2784r_0^2Rm^2L^4$$
$$-1728r_0^3)(u^\phi)^2/[r_0^2\mathcal{L}^{(4)}(L^2-1)(L^2-4)(L^2-9)]\Bigg]Y^*_{\ell m}(\theta_0,\phi_0). \tag{D2}$$

The gauge transformation from the harmonic gauge to the RW gauge is given by

$$M^{\mathrm{S,H}\to\mathrm{RW}}_{0\ell m}(t,r)=\frac{2}{L}\pi\mu\Bigg[-T(72r_0-112Rm^2L^5+392Rm^2L^3-288Rm^2L-120r_0m^4+8Rm^2L^7+4Rm^2L^6+48Rm^4L^2$$
$$+8r_0m^2L^6-828r_0m^2-8r_0L^6-314r_0L^2+172Rm^2L^2-50Rm^2L^4+106r_0L^4-941RL^2-20RL^6+277RL^4$$
$$-414Rm^2-612Rm^4+612R+344r_0m^2L^2-100r_0m^2L^4)(u^\phi)^2/[2\mathcal{L}^{(2)}(L^2-1)(L^2-4)(L^2-9)]$$
$$-i(8R^2L^6+16r_0RL^5-46R^2L^4+8r_0RL^4+16r_0^2L^4-80r_0RL^3-56r_0^2L^2+55R^2L^2-52r_0RL^2+64r_0RL$$
$$+4R^2+40R^2m^2+80r_0R+16r_0Rm^2-32r_0^2)mu^\phi/[4r_0\mathcal{L}^{(2)}(L^2-1)(L^2-4)]$$
$$-\frac{T(-R+2r_0)M}{r_0^3\mathcal{L}^{(2)}}\Bigg]Y^*_{\ell m}(\theta_0,\phi_0),$$

$$M^{\mathrm{S,H}\to\mathrm{RW}}_{1\ell m}(t,r)=\frac{2}{L}\pi\mu\Bigg[\frac{1}{96}(-21504r_0RL^5+75264r_0RL^3-72312r_0RL^4-69984r_0Rm^2+13932R^2-21600r_0^2$$
$$-36096r_0Rm^2L^4+107136r_0Rm^2L^2-96L^8r_0^2-107568r_0R+50850R^2L^4+58845R^2L^6-120528R^2m^2$$
$$-99315R^2L^2-3456m^2LR^2+6240m^2L^6r_0R+384m^4L^3R^2-254232R^2L+21084r_0L^6R+15744m^4LR^2$$
$$+1536m^4L^2r_0R-28896m^2L^5R^2+69936m^2L^3R^2+1536r_0L^7R-27882L^7R^2+8064m^4r_0^2+192r_0^2L^9$$
$$+1488L^9R^2-3360L^8r_0R+192L^{10}r_0R+144684r_0RL^2+6336r_0^2L-55296r_0RL+25944r_0^2L^2$$
$$+4080m^2L^7R^2+384m^4L^4R^2-192R^2m^2L^9+96R^2L^{11}-96R^2m^2L^8-384r_0Rm^2L^8+244086R^2L^3$$
$$+36444R^2L^5+1872L^{10}R^2-23880L^8R^2+1752r_0^2L^6-10608r_0^2L^4+11872r_0^2L^5-15536r_0^2L^3-103680r_0^2m^2$$
$$-12096r_0^2m^2L^4-18816r_0^2m^2L^3+13824r_0^2m^2L+73728r_0^2m^2L^2+84096r_0Rm^4+9720R^2m^2L^6$$
$$+5376r_0^2m^2L^5+337824R^2m^2L^2-115752R^2m^2L^4-27168R^2m^4L^2+248688R^2m^4-2864r_0^2L^7$$
$$+576r_0^2m^2L^6-384r_0^2m^2L^7+2304m^4r_0^2L^2)(u^\phi)^2/[\mathcal{L}^{(4)}(L^2-1)(L^2-4)(L^2-9)]\Bigg]Y^*_{\ell m}(\theta_0,\phi_0),$$

$$M^{\mathrm{S,H}\to\mathrm{RW}}_{2\ell m}(t,r)=\frac{2}{L}\pi\mu\Bigg[-\frac{1}{96}(11208r_0^2RL^2-4768R^3m^2L^5+4416r_0^3m^2L^4-8064r_0^3m^2L^2-384r_0^3m^2L^6+7296r_0^2RL^4$$
$$-42120r_0R^2-2864r_0^2RL^7-3528r_0^2RL^6-15536r_0^2RL^3+20544r_0^2Rm^4-12960r_0^2Rm^2+192r_0^2RL^9$$
$$+288r_0^2RL^8+11872r_0^2RL^5+6336r_0^2RL+9318r_0R^2L^6+68994r_0R^2L^2+96r_0R^2L^{10}-6336r_0^2Rm^2L^2$$

$$+75792r_0R^2m^4+192r_0R^2L^9+52488r_0R^2m^2-1584r_0R^2L^8-2096r_0R^2L^7-384r_0^2Rm^2L^7-576r_0^2Rm^2L^6$$
$$-32688r_0R^2L^4+22096r_0R^2L^3+1120r_0R^2L^5-656R^3L^9+6048r_0^2Rm^2L^4-18816r_0^2Rm^2L^3-21312r_0R^2L$$
$$+13824r_0^2Rm^2L+5376r_0^2Rm^2L^5+432r_0R^2m^2L^2-192r_0R^2m^2L^8+2832r_0R^2m^2L^6+5376r_0R^2m^2L^5$$
$$-1536r_0^2Rm^4L^2+13824r_0R^2m^2L-7488r_0R^2m^4L^2-10632r_0R^2m^2L^4-64R^3m^2L^9+1040R^3m^2L^7$$
$$+768R^3m^4L^4+48R^3m^2L^3+47520R^3m^2L-7168R^3m^4L^3-6240R^3m^4L^2-11868R^3m^2L^4+75712R^3m^4L$$
$$-384r_0R^2m^2L^7+18504R^3m^2L^2+1992R^3m^2L^6-18816r_0R^2m^2L^3-96R^3m^2L^8-5184r_0^3m^2+32R^3L^{11}$$
$$-24492R^3L^5+52594R^3L^3+33004R^3L^4+5426R^3L^7-152R^3L^8-27540R^3m^2-35448R^3m^4+48R^3L^{10}$$
$$-30600R^3L+19980R^3-50019R^3L^2-5597R^3L^6-1104r_0^3L^2-2544r_0^3L^6+7488r_0^3L^4+192r_0^3L^8$$
$$+1920r_0^3m^4-9504r_0^2R-1728r_0^3)(u^\phi)^2/[\mathcal{L}^{(4)}(L^2-1)(L^2-4)(L^2-9)]\Bigg]Y^*_{\ell m}(\theta_0,\phi_0),$$

$$\Lambda^{\mathrm{S,H}\to\mathrm{RW}}_{\ell m}(t,r)=\frac{2}{L}\pi\mu\Bigg[-\frac{1}{12}i(48r_0RL^5-240r_0RL^3+72r_0RL^4-288r_0Rm^2+4108R^2+1056r_0^2-1488r_0R-66R^2L^4$$
$$+24R^2L^6+1392R^2m^2-1147R^2L^2+192R^2L+84r_0RL^2+192r_0RL-456r_0^2L^2-240R^2L^3+48R^2L^5$$
$$+48r_0^2L^4+288r_0^2m^2-48R^2m^2L^2)r_0m(u^\phi)^2/[\mathcal{L}^{(4)}(L^2-1)(L^2-4)]\Bigg]\partial_\theta Y^*_{\ell m}(\theta_0,\phi_0). \tag{D3}$$

And then, the coefficients of the S part under the RW gauge are calculated as

$$h^{\mathrm{S,RW}}_{0\ell m}(t,r)=\frac{2}{L}\pi\mu\Bigg[\frac{4iTmr_0(L^2-2)(u^\phi)^2}{\mathcal{L}^{(2)}(L^2-1)}-(8r_0-6r_0m^2-18r_0L^2+4r_0L^4-4R+16RL-13Rm^2-7RL^2-20RL^3+2RL^4$$
$$+4RL^5)u^\phi/[\mathcal{L}^{(2)}(L^2-1)(L^2-4)]\Bigg]\partial_\theta Y^*_{\ell m}(\theta_0,\phi_0),$$

$$h^{\mathrm{S,RW}}_{1\ell m}(t,r)=\frac{2}{L}\pi\mu\Bigg[-\frac{1}{3}i(-60r_0L^3+174r_0L^2+48Lr_0-792r_0+6r_0L^4+12r_0L^5-216r_0m^2+3380R-881RL^2-39RL^4$$
$$-24Rm^2L^2+984Rm^2+12RL^6)r_0m(u^\phi)^2/[\mathcal{L}^{(4)}(L^2-1)(L^2-4)]\Bigg]\partial_\theta Y^*_{\ell m}(\theta_0,\phi_0),$$

$$H^{\mathrm{S,RW}}_{0\ell m}(t,r)=\frac{2}{L}\pi\mu\Bigg[\frac{1}{16}(-648r_0+1792Rm^2L^5-6272Rm^2L^3+4608Rm^2L-40RL^8+48r_0L^8+48RL^9+1920r_0m^4$$
$$-128Rm^2L^7-56Rm^2L^6-608Rm^4L^2-112r_0m^2L^6+13104r_0m^2-564r_0L^6+2394r_0L^2-1106Rm^2L^2$$
$$+550Rm^2L^4+930r_0L^4+17457RL^2+902RL^6+2520RL^5-2316RL^3-6691RL^4+6228Rm^2+9752Rm^4$$
$$+432RL-684RL^7-10260R-4876r_0m^2L^2+1388r_0m^2L^4)(u^\phi)^2/[\mathcal{L}^{(2)}(L^2-1)(L^2-4)(L^2-9)]-\frac{2imTu^\phi}{r_0}$$
$$-\frac{1}{16}\frac{(-62r_0+56r_0L^2+33R-8RL-36RL^2+40RL^3-32RL^5)M}{r_0^3\mathcal{L}^{(2)}(L^2-1)}+\frac{(2r_0-R+2RL)}{r_0^2}\Bigg]Y^*_{\ell m}(\theta_0,\phi_0),$$

$$H^{\rm S,RW}_{1\ell m}(t,r)=\frac{2}{L}\pi\mu\Bigg[-T(360-742L^2-69L^6+375L^4-56m^2L^5-2m^2L^6-404m^2L^2+4L^8+64m^2L^4-90m^2$$
$$-16m^4L^2-296m^4-144m^2L+196m^2L^3+4m^2L^7)(u^\phi)^2/[\mathcal{L}^{(2)}(L^2-1)(L^2-4)(L^2-9)]$$
$$-im(4RL^6-21RL^4+19RL^2+4R+20Rm^2+4r_0L^5-2r_0L^4-20r_0L^3+4r_0L^2+16Lr_0$$
$$+16r_0+4r_0m^2)u^\phi/[r_0\mathcal{L}^{(2)}(L^2-1)(L^2-4)]+\frac{4TL^2M}{r_0^3\mathcal{L}^{(2)}}\Bigg]Y^*_{\ell m}(\theta_0,\phi_0),$$

$$H^{\rm S,RW}_{2\ell m}(t,r)=\frac{2}{L}\pi\mu\Bigg[\frac{1}{64}(-137592r_0-224Rm^2L^8-77056Rm^2L^5+186496Rm^2L^3-9216Rm^2L+4960RL^{10}-448r_0m^2L^8$$
$$-512Rm^2L^9-73728Lr_0+192RL^{11}+192r_0L^{10}+1024Rm^4L^3+41984Rm^4L+1664Rm^4L^4+2048m^4L^2r_0$$
$$-62976RL^8-3200r_0L^8+5024RL^9+100352r_0L^3-28672r_0L^5+112128r_0m^4+10880Rm^2L^7+23696Rm^2L^6$$
$$-74048Rm^4L^2+6560r_0m^2L^6+2048r_0L^7-88128r_0m^2+19228r_0L^6+163590r_0L^2+857898Rm^2L^2$$
$$-286590Rm^2L^4-70298r_0L^4-285729RL^2+151486RL^6+107832RL^5+643372RL^3+153079RL^4$$
$$-312012Rm^2+663528Rm^4-676656RL-79764RL^7+41364R+118908r_0m^2L^2-34876r_0m^2L^4)(u^\phi)^2/$$
$$[\mathcal{L}^{(4)}(L^2-1)(L^2-4)(L^2-9)]-\frac{2imTu^\phi}{r_0}+\frac{1}{4}\frac{(2r_0+R-8RL+8RL^3)M}{r_0^3(L^2-1)}+\frac{(2r_0-R+2RL)}{r_0^2}\Bigg]Y^*_{\ell m}(\theta_0,\phi_0),$$

$$K^{\rm S,RW}_{\ell m}(t,r)=\frac{2}{L}\pi\mu\Bigg[\frac{1}{192}(-99144r_0-1056Rm^2L^8-61824Rm^2L^5+112320Rm^2L^3-62208Rm^2L+288RL^{10}-1536r_0m^2L^7$$
$$-576r_0m^2L^8-768Rm^2L^9+25344Lr_0+768r_0L^9+576RL^{11}+576r_0L^{10}-1152Rm^4L^4+13056m^4L^2r_0$$
$$-5184RL^8-75264r_0m^2L^3+21504r_0m^2L^5+55296m^2Lr_0-9216r_0L^8-9760RL^9-62144r_0L^3+47488r_0L^5$$
$$+31296r_0m^4+12480Rm^2L^7+15504Rm^2L^6+26304Rm^4L^2+9504r_0m^2L^6-11456r_0L^7-414072r_0m^2$$
$$+46932r_0L^6+170154r_0L^2+114210Rm^2L^2-68394Rm^2L^4-113478r_0L^4+426969RL^2+37578RL^6$$
$$-202456RL^5+406212RL^3-171543RL^4+138024Rm^2+296856Rm^4-253584RL+59012RL^7-335988R$$
$$+296460r_0m^2L^2-66708r_0m^2L^4)(u^\phi)^2/[\mathcal{L}^{(4)}(L^2-1)(L^2-4)(L^2-9)]-\frac{2imTu^\phi}{r_0}$$
$$+\frac{1}{4}\frac{(2r_0+R-8RL+8RL^3)M}{r_0^3(L^2-1)}+\frac{(2r_0-R+2RL)}{r_0^2}\Bigg]Y^*_{\ell m}(\theta_0,\phi_0). \quad \text{(D4)}$$

APPENDIX E: $\ell=0$ AND 1 MODES

In this appendix we derive the contributions to the self-force in the RW gauge from the $\ell=0$ and 1 modes. As discussed at the beginning of Sec. VII, and described in Eq. (7.1), although there is no physical contribution from the $\ell=0$ and $\ell=1$ modes to the self-force in the rigorous sense, since we can calculate the *S* part only locally in the vicinity of the particle, its spherical extension inevitably contaminates each ℓ mode with other ℓ modes. Therefore, in particular, we have to take account of the corrections from the $\ell=0$ and $\ell=1$ modes to the self-force.

For the $\ell=0$ and $\ell=1$ odd modes, the RW gauge condition is automatically satisfied, since $h_0^{(e)}=h_1^{(e)}=G=0$ for $\ell=0$ and $h_2=0$ for $\ell=1$ odd modes. An appropriate choice of gauge is then to consider the perturbation under the retarded causal boundary condition in the harmonic gauge. In fact, if we recall the gauge transformation equations from the harmonic gauge to the RW gauge given by Eq. (6.2), we see that all the gauge transformation generators for $\ell=0$ and $\ell=1$ odd modes vanish. Thus, no gauge transformation is needed for the S part of these modes, and our task is to find the exact solutions in the harmonic gauge with the retarded boundary condition and perform the subtraction of the S part under the harmonic gauge.

For the $\ell=1$ even mode, the RW gauge condition is nontrivial and there is a gauge degree of freedom in the RW gauge, reflecting the fact that it is a pure gauge mode that describes a shift of the center of mass coordinates in the source-free case. On the other hand, the gauge transformation of this mode from the harmonic gauge to the RW gauge is uniquely determined. Thus, to determine the self-force unambiguously in the RW gauge, one first has to solve the perturbation equations in the harmonic gauge exactly (under the retarded boundary condition), transform the result to the RW gauge, and perform the subtraction of the S part. However, unfortunately, we were unable to solve for the $\ell=1$ even mode in the harmonic gauge due to a complicated structure of the perturbation equations (i.e., in the form of coupled hyperbolic equations). Thus, there remains a gauge ambiguity in the final result. Nevertheless, in the Newton limit when the coordinates can be defined globally, we can resolve the gauge ambiguity and give a definite meaning to the resulting self-force.

To summarize, the regularized self-force in the RW gauge is expressed as

$$F^{\mathrm{R,RW}}=\sum_{\ell\geq 2}(F_\ell^{\mathrm{full,RW}}-F_\ell^{\mathrm{S,RW,Ap}})+\delta F^{\mathrm{RW}}_{\ell=0,1}, \tag{E1}$$

where

$$\delta F^{\mathrm{RW}}_{\ell=0,1}=\delta F^{\mathrm{RW}}_{\ell=0}+\delta F^{\mathrm{RW}}_{\ell=1(\mathrm{odd})}+\delta F^{\mathrm{RW}}_{\ell=1(\mathrm{even})} \tag{E2}$$

with

$$\delta F^{\mathrm{RW}}_{\ell=0}=F^{\mathrm{full,H}}_{\ell=0}-F^{\mathrm{S,H,Ap}}_{\ell=0},$$

$$\delta F^{\mathrm{RW}}_{\ell=1(\mathrm{odd})}=F^{\mathrm{full,H}}_{\ell=1(\mathrm{odd})}-F^{\mathrm{S,H,Ap}}_{\ell=1(\mathrm{odd})},$$

$$\delta F^{\mathrm{RW}}_{\ell=1(\mathrm{even})}=F^{\mathrm{full,RW}}_{\ell=1(\mathrm{even})}-F^{\mathrm{S,RW,Ap}}_{\ell=1(\mathrm{even})}, \tag{E3}$$

where there remains a gauge ambiguity in $\delta F^{\mathrm{RW}}_{\ell=1(\mathrm{even})}$.

1. $\ell=0$ mode

First, we consider the $\ell=0$ mode of the full metric perturbation. It is noted that the $\ell=0$ mode consists of only the even parity part and all the derivatives of Y_{00} vanish. As noted above, this mode satisfies the RW gauge condition $h_0^{(e)}=h_1^{(e)}=G=0$ automatically. So, the appropriate choice of gauge is the harmonic gauge under the retarded boundary condition. To find the exact solution in this gauge, we consider a gauge transformation of the exact solution found by Zerilli.

This mode represents the perturbation in the total mass of the system and was analyzed by Zerilli. For the $\ell=0$ mode, there are two gauge degrees of freedom. The choice made by Zerilli is

$$H_1^{\mathrm{full,Z}}(t,r)=K^{\mathrm{full,Z}}(t,r)=0, \tag{E4}$$

which we call the Zerilli (Z) gauge and denote the quantities in it by the superscript Z. In the case of a circular orbit, the $\ell=0$ mode metric perturbation is solved to be

$$H_2^{\mathrm{full,Z}}(t,r)=\frac{a}{r-2M}\Theta(r-r_0),$$

$$H_0^{\mathrm{full,Z}}(t,r)=a\left[\frac{1}{r_0-2M}\Theta(r_0-r)+\frac{1}{r-2M}\Theta(r-r_0)\right]. \tag{E5}$$

Here we imposed the boundary condition that the black hole mass is unperturbed and the perturbation satisfies the asymptotic flatness. Note that the Zerilli gauge is singular in the sense that the metric has a discontinuity at $r=r_0$. The constant a is given by

$$a=2(4\pi)^{1/2}\mu u^t\left(1-\frac{2M}{r_0}\right). \tag{E6}$$

Note that the $\ell=0$ mode is independent of time. So we may write $H_2^{\mathrm{full,Z}}(t,r)=H_2^{\mathrm{full,Z}}(r)$.

Now we consider the gauge transformation from the above Z gauge to the harmonic gauge. The equations for the gauge transformation are formally written as

$$\xi_{\mu;\nu}{}^{\nu}=\overline{h^{\mathrm{Z}}_{\mu\nu}}{}^{;\nu},$$

$$\bar{h}_{\mu\nu}=h_{\mu\nu}-\frac{1}{2}g_{\mu\nu}h_\alpha{}^\alpha. \tag{E7}$$

Detailed discussions on the gauge transformation to the harmonic gauge are given in [23].

We set the gauge transformation generator ξ_μ as

$$\{\xi_\mu^{\mathrm{Z}\to\mathrm{H}}\}=\{M_0(r)Y_{00}(\theta,\phi),M_1(r)Y_{00}(\theta,\phi),0,0\}. \tag{E8}$$

In the circular case, the $\ell=0$ mode of Eq. (E7) is explicitly written down as

$$\left[\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}\right]M_0(r)=0, \tag{E9}$$

$$\left[\frac{r-2M}{r}\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{2(r-2M)}{r^3}\right]M_1(r)=S(r), \tag{E10}$$

where

$$S(r)=4\pi\frac{r^3}{(r-2M)^2}A_{00}^{(0)}(r)+\frac{M}{r(r-2M)}H_0^{\mathrm{Z}}(r)+\frac{2r-3M}{r(r-2M)}H_2^{\mathrm{Z}}(r)$$

$$=\frac{a}{2(r_0-2M)}\delta(r_0-r)+\frac{aM}{r(r-2M)(r_0-2M)}\Theta(r_0-r)+\frac{2a(r-M)}{r(r-2M)^2}\Theta(r-r_0). \tag{E11}$$

Since M_0 is independent of the source, we set it to zero in accordance with the retarded boundary condition. Thus we focus on the equation for M_1.

We employ the Green function method to solve Eq. (E10). Two independent homogeneous solutions are easily obtained as

$$M_1^{(\mathrm{homo},1)}=\frac{1}{r(r-2M)},\qquad M_1^{(\mathrm{homo},2)}=\frac{r^2}{r-2M}. \tag{E12}$$

Using the above homogeneous solutions, we construct the other two independent solutions M_1^{in} and M_1^{out} which are regular at the event horizon and infinity, respectively. We find

$$M_1^{\mathrm{in}}=-8M^3M_1^{(\mathrm{homo},1)}+M_1^{(\mathrm{homo},2)}=\frac{r^2+2Mr+4M^2}{r},\qquad M_1^{\mathrm{out}}=M_1^{(\mathrm{homo},1)}=\frac{1}{r(r-2M)}. \tag{E13}$$

Then the Green function is derived as

$$G(r,r')=\frac{1}{W}[M_1^{\mathrm{in}}(r)M_1^{\mathrm{out}}(r')\Theta(r'-r)+M_1^{\mathrm{out}}(r)M_1^{\mathrm{in}}(r')\Theta(r-r')],$$
$$W=(r-2M)^2[M_1^{\mathrm{in}}(r)\partial_rM_1^{\mathrm{out}}(r)-M_1^{\mathrm{out}}(r)\partial_rM_1^{\mathrm{in}}(r)]=-3, \tag{E14}$$

and M_1 is given by

$$M_1(r)=\int_{2M}^{\infty}G(r,r')r'(r'-2M)S(r')dr'. \tag{E15}$$

Although the integral can be performed without any approximation, we only show the result to 1PN order,

$$M_1(r)=\left[-\frac{5a}{6}\frac{r}{r_0}-\frac{a}{6}\frac{(13r_0+6r)M}{r_0^2}\right]\Theta(r_0-r)+\left[-\frac{a}{6}\frac{6r^2-r_0^2}{r^2}-\frac{a}{6}\frac{(30r^2-9rr_0-2r_0^2)M}{r^3}\right]\times\Theta(r-r_0). \tag{E16}$$

The metric perturbation transforms under the above gauge transformation as

$$H_0^{\mathrm{H}}(r)=H_0^{\mathrm{Z}}(r)+\frac{2M}{r^2}M_1(r),$$
$$H_1^{\mathrm{H}}(r)=-\frac{d}{dr}M_0(r)+\frac{2M}{r(r-2M)}M_0(r),$$
$$H_2^{\mathrm{H}}(r)=H_2^{\mathrm{Z}}(r)-2\left(1-\frac{2M}{r}\right)\times\left(\frac{d}{dr}M_1(r)+\frac{M}{r(r-2M)}M_1(r)\right),$$
$$K^{\mathrm{H}}(r)=-\frac{2(r-2M)}{r^2}M_1(r). \tag{E17}$$

Note that we have $H_1=0$ because $M_0=0$. (It may be noted that H_1 does not contribute to the force for a circular orbit even if it is non-zero.) Then the metric perturbation in the harmonic gauge is found as

$$H_0^{\mathrm{H}}=\left[a\frac{1}{r_0}+\frac{a}{3}\frac{(6r-5r_0)M}{rr_0^2}\right]\Theta(r_0-r)+\left[a\frac{1}{r}+\frac{a}{3}\frac{r_0^2M}{r^4}\right]\Theta(r-r_0),$$
$$H_2^{\mathrm{H}}=\left[\frac{5a}{3}\frac{1}{r_0}+\frac{a}{3}\frac{(6r-5r_0)M}{rr_0^2}\right]\Theta(r_0-r)+\left[\frac{a}{3}\frac{3r^2+2r_0^2}{r^3}-\frac{a}{3}\frac{(18r^2-18rr_0-r_0^2)M}{r^4}\right]\Theta(r-r_0),$$
$$K^{\mathrm{H}}=\left[\frac{5a}{3}\frac{1}{r_0}+a\frac{(r_0+2r)M}{rr_0^2}\right]\Theta(r_0-r)+\left[\frac{a}{3}\frac{6r^2-r_0^2}{r^3}+3a\frac{(2r-r_0)M}{r^3}\right]\Theta(r-r_0), \tag{E18}$$

and the full force is calculated as

$$F^r_{\mathrm{full,H}}(\ell=0)=\left[\frac{7\mu^2M}{r_0^3}\right]\Theta(r_0-r)+\left[-\frac{\mu^2}{r_0^2}+\frac{9\mu^2M}{2r_0^3}\right]\Theta(r-r_0). \tag{E19}$$

Next, we consider the S part of the metric perturbation. Its harmonic coefficients are given in Eqs. (5.6). Only the harmonic coefficients H_0, H_2 and K remain for the $\ell=0$ mode. To 1PN order, we have

$$H_0^{\mathrm{S,H}}(r)=\sqrt{4\pi}\mu\left\{\left[3\frac{M}{r_0^2}+\frac{2}{r_0}\right]\Theta(r_0-r)+\left[\frac{(-5R+3r_0)M}{r_0^3}+\frac{2(r_0-R)}{r_0^2}\right]\Theta(r-r_0)\right\},$$

$$H_2^{\mathrm{S,H}}(r)=\sqrt{4\pi}\mu\left\{\left[-\frac{M}{r_0^2}+\frac{2}{r_0}\right]\Theta(r_0-r)+\left[-\frac{(R+r_0)M}{r_0^3}+\frac{2(r_0-R)}{r_0^2}\right]\Theta(r-r_0)\right\},$$

$$K^{\mathrm{S,H}}(r)=\sqrt{4\pi}\mu\left\{\left[\frac{M}{r_0^2}+\frac{2}{r_0}\right]\Theta(r_0-r)+\left[\frac{(-3R+r_0)M}{r_0^3}+\frac{2(r_0-R)}{r_0^2}\right]\Theta(r-r_0)\right\}. \tag{E20}$$

The S force in the harmonic gauge is calculated as

$$F^r_{\mathrm{S,H}}(\ell=0)=\left[\frac{33\mu^2M}{8r_0^3}\right]\Theta(r_0-r)+\left[-\frac{\mu^2}{r_0^2}+\frac{13\mu^2M}{8r_0^3}\right]\Theta(r-r_0). \tag{E21}$$

From the above results, we obtain the contribution of the $\ell=0$ force as

$$\delta F^r_{\mathrm{RW}}(\ell=0)=\delta F^r_{\mathrm{H}}(\ell=0)=F^r_{\mathrm{full,H}}(\ell=0)-F^r_{\mathrm{S,H}}(\ell=0)=\frac{23\mu^2M}{8r_0^3}. \tag{E22}$$

2. $\ell=1$ odd parity mode

The $\ell=1$ odd mode represents the angular momentum perturbation added to the system. It also satisfies the odd parity RW gauge condition $h_2=0$ automatically. Therefore, as in the $\ell=0$ case, we look for the exact solution in the harmonic gauge with the retarded boundary condition.

The full metric perturbation consists of the two components, h_0^{full} and h_1^{full}. These were also solved by Zerilli. There is one gauge degree of freedom, and we may put $h_1=0$. The appropriate boundary condition is that the black hole angular momentum is unperturbed and the perturbation is well behaved at infinity. Then we find

$$h_0^{\mathrm{Z}}(t,r)=\left(b\frac{r^2}{r_0^3}\Theta(r_0-r)+b\frac{1}{r}\Theta(r-r_0)\right)\delta_{0,m}, \tag{E23}$$

where b is given by

$$b=2\sqrt{\frac{4\pi}{3}}\mu u^{\phi}r_0^2. \tag{E24}$$

Note that only the $m=0$ mode is non-zero, and it is time independent.

Next, we consider the gauge transformation to the harmonic gauge. We set

$$\xi_\mu=\Lambda_m^{\mathrm{Z}\to\mathrm{H}}(r)\left(0,0,-\frac{1}{\sin\theta}\partial_\phi Y_{1m}(\theta,\phi),\sin\theta\partial_\theta Y_{1m}(\theta,\phi)\right). \tag{E25}$$

The equation for $\Lambda_m^{\mathrm{Z}\to\mathrm{H}}$ becomes

$$\left[-\left(1-\frac{2M}{r}\right)^{-1}\partial_t^2+\partial_r\left(1-\frac{2M}{r}\right)\partial_r-\frac{2}{r^2}\right]\Lambda_m^{\mathrm{Z}\to\mathrm{H}}(r)=0. \tag{E26}$$

This is a source-free hyperbolic equation. So, with the retarded boundary condition, we find $\Lambda_m^{\mathrm{Z}\to\mathrm{H}}=0$, that is, the Zerilli gauge is equivalent to the harmonic gauge with the retarded boundary condition. The full force is then calculated as

$$F^{r(\mathrm{odd})}_{\mathrm{full,H}}(\ell=1)=\left[-\frac{4\mu^2M}{r_0^3}\right]\Theta(r_0-r)+\left[\frac{2\mu^2M}{r_0^3}\right]\Theta(r-r_0). \tag{E27}$$

The harmonic coefficients of the S part are given as

$$h_{01m}^{\mathrm{S,H}}(r)=-\sqrt{\frac{4\pi}{3}}\mu\left\{\left[-\frac{8}{9}(4R+2r_0)u^{\phi}\right]\Theta(r_0-r)+\left[-\frac{8}{9}(2r_0-2R)u^{\phi}\right]\Theta(r-r_0)\right\}\delta_{0,m},$$

$$h_{11m}^{\mathrm{S,H}}(r)=0, \tag{E28}$$

and the S force is obtained as

$$F^{r(\mathrm{odd})}_{\mathrm{S,H}}(\ell=1)=\left[-\frac{4\mu^2M}{r_0^3}\right]\Theta(r_0-r)+\left[\frac{2\mu^2M}{r_0^3}\right]\Theta(r-r_0). \tag{E29}$$

Subtracting the S part from the full force, we find

$$\delta F^{r(\mathrm{odd})}_{\mathrm{RW}}(\ell=1)=\delta F^{r(\mathrm{odd})}_{\mathrm{H}}(\ell=1)=F^{r(\mathrm{odd})}_{\mathrm{full,H}}(\ell=1)-F^{r(\mathrm{odd})}_{\mathrm{S,H}}(\ell=1)=0. \tag{E30}$$

Thus, our spherical extension turns out to be accurate enough to reproduce the correct $\ell=1$ odd mode up to 1PN order.

3. $\ell=1$ even parity mode

The $\ell=1$ even mode represents essentially a gauge mode that describes a shift of the center of momentum of the system. The coefficient G is absent from the beginning, while there is no loss in the gauge freedom. Hence there remains one degree of gauge freedom in the RW gauge. As mentioned at the beginning of this appendix, to fix the gauge completely it is necessary to solve the perturbation equations in the harmonic gauge with the retarded boundary condition, and to perform the gauge transformation to the RW gauge. However, because the perturbation equations become complicated, coupled hyperbolic equations in the harmonic gauge, we were unable to solve for this mode. Here, we therefore give up fixing the gauge unambiguously, but solve the perturbation equations in the RW gauge, imposing a gauge condition by hand.

To look for an exact solution in the RW gauge, following Zerilli, we choose $K=0$ in addition to $h_0^{(e)}=h_1^{(e)}=0$. Let us also call it the Zerilli gauge. The harmonic coefficients for the full metric perturbation in the Zerilli gauge are given by

$$H_{21m}^{\rm full,Z}(t,r)=\frac{1}{(r-2M)^2}f_m(t)\Theta(r-r_0),$$

$$H_{11m}^{\rm full,Z}(t,r)=-\frac{r}{(r-2M)^2}\partial_t f_m(t)\Theta(r-r_0),$$

$$H_{01m}^{\rm full,Z}(t,r)=\frac{1}{3(r-2M)^2}\left(f_m(t)+\frac{r^3}{M}\partial_t^2 f_m(t)\right)\Theta(r-r_0), \quad \text{(E31)}$$

where

$$f_m(t)=8\pi\mu u^t\frac{(r_0-2M)^2}{r_0}Y_{1m}^*(\theta_0(t),\phi_0(t)), \quad \text{(E32)}$$

and we have imposed the boundary condition that the perturbation is regular at horizon. It may be noted that although the $\ell=1$ even mode is locally a pure gauge, it is not so in the global sense because of the regularity at the horizon. Note that the $m=0$ components vanish because the orbit is on the equatorial plane. It is also noted that $H_1^{\rm full,Z}$ and $H_2^{\rm full,Z}$ are discontinuous at $r=r_0$, while $H_0^{\rm full,Z}$ is continuous because $\partial_t^2 f_m=-\Omega^2 f_m=-(M/r_0^3)f_m$ for $m=\pm1$, and the force depends only on $H_0^{\rm full,Z}$. The full force in this gauge is derived as

$$F_{\rm full,Z}^{r({\rm even})}(\ell=1)=\left[-\frac{3\mu^2}{r_0^2}-\frac{3\mu^2 M}{2r_0^3}\right]\Theta(r-r_0). \quad \text{(E33)}$$

The coefficient $H_{01m}^{\rm full,Z}$ in the above behaves as $\sim r$ at infinity. Without violating the RW gauge condition, it is possible to remove this singular behavior. Namely, we consider a gauge transformation,

$$H_{01m}^{\rm full,RW}(t,r)=H_{01m}^{\rm full,Z}(t,r)+\frac{2r}{r-2M}\left[\partial_t M_{01m}^{\rm full,Z\to RW}(t,r)-\frac{M(r-2M)}{r^3}M_{11m}^{\rm full,Z\to RW}(t,r)\right],$$

$$H_{11m}^{\rm full,RW}(t,r)=H_{11m}^{\rm full,Z}(t,r)+\left[\partial_t M_{11m}^{\rm full,Z\to RW}(t,r)+\partial_r M_{01m}^{\rm full,Z\to RW}(t,r)-\frac{2M}{r(r-2M)}M_{01m}^{\rm full,Z\to RW}(t,r)\right],$$

$$H_{21m}^{\rm full,RW}(t,r)=H_{21m}^{\rm full,Z}(t,r)+\frac{2(r-2M)}{r}\left[\partial_r M_{11m}^{\rm full,Z\to RW}(t,r)+\frac{M}{r(r-2M)}M_{11m}^{\rm full,Z\to RW}(t,r)\right],$$

$$K_{1m}^{\rm full,RW}(t,r)=\frac{2}{r^2}[2(r-2M)M_{11m}^{\rm full,Z\to RW}(t,r)-M_{21m}^{\rm full,Z\to RW}(t,r)],$$

$$h_{0\ 1m}^{(e)\rm full,RW}(t,r)=0=-M_{01m}^{\rm full,Z\to RW}(t,r)-\partial_t M_{21m}^{\rm full,Z\to RW}(t,r),$$

$$h_{1\ 1m}^{(e)\rm full,RW}(t,r)=0=-M_{11m}^{\rm full,Z\to RW}(t,r)-r^2\partial_r\left(\frac{M_{21m}^{\rm full,Z\to RW}(t,r)}{r^2}\right). \quad \text{(E34)}$$

As a solution of the above gauge equations that makes the metric perturbation regular at infinity, we choose

$$M_{01m}^{\rm full,Z\to RW}(t,r)=\frac{ir}{6m\Omega r_0^3}f_m(t),$$

$$M_{11m}^{\rm full,Z\to RW}(t,r)=\frac{1}{6m^2\Omega^2 r_0^3}f_m(t),$$

$$M_{21m}^{\rm full,Z\to RW}(t,r)=\frac{r}{6m^2\Omega^2 r_0^3}f_m(t). \quad \text{(E35)}$$

By the above gauge transformation, the r component of the force changes by

$$\delta F^{r(\mathrm{even})}_{\mathrm{full,Z\to RW}}(\ell=1)=\sum_{m=-1}^{1}\mu\Bigg(\frac{2M(r_0-2M)}{r_0^4}M^{\mathrm{full,Z\to RW}}_{11m}(t_0,r_0)+\frac{2imM\Omega}{(r_0-3M)r_0}M^{\mathrm{full,Z\to RW}}_{01m}(t_0,r_0)-\frac{(r_0-2M)}{(r_0-3M)}$$
$$\times\partial_t^2 M^{\mathrm{full,Z\to RW}}_{11m}(t_0,r_0)+\frac{2M}{r_0(r_0-3M)}\partial_t M^{\mathrm{full,Z\to RW}}_{01m}(t_0,r_0)-\frac{im\Omega(r_0-2M)}{(r_0-3M)}\partial_r M^{\mathrm{full,Z\to RW}}_{01m}(t_0,r_0)$$
$$-\frac{M(r_0-2M)^2}{(r_0-3M)r_0^3}\partial_r M^{\mathrm{full,Z\to RW}}_{11m}(t_0,r_0)-\frac{i\Omega m(r_0-2M)}{(r_0-3M)}\partial_t M^{\mathrm{full,Z\to RW}}_{11m}(t_0,r_0)\Bigg)Y_{1m}(\theta_0,\phi_0). \tag{E36}$$

So, to 1PN order, we find

$$\delta F^{r(\mathrm{even})}_{\mathrm{full,Z\to RW}}(\ell=1)$$
$$=\sum_{m=-1}^{1}\mu\left[\frac{(r_0-2M)^2}{2r_0^4(r_0-3M)}f_m(t)\right]Y_{1m}(\theta_0,\phi_0)$$
$$=\frac{3\mu^2}{r_0^2}-\frac{21\mu^2 M}{2r_0^3}. \tag{E37}$$

Thus, the full force in this RW gauge is given by

$$F^{r(\mathrm{even})}_{\mathrm{full,RW}}(\ell=1)=\left[\frac{3\mu^2}{r_0^2}-\frac{21\mu^2 M}{2r_0^3}\right]\Theta(r_0-r)$$
$$+\left[-\frac{12\mu^2 M}{r_0^3}\right]\Theta(r-r_0). \tag{E38}$$

It may be noted that, at Newtonian order, the r coordinate of the Zerilli gauge, in which the metric inside the orbit is unperturbed, corresponds to placing the black hole at $r=0$, while the gauge transformation that regularizes the asymptotic behavior at infinity makes r the radial coordinate measured in the center of mass coordinate system. In other words, r_0 in the Zerilli gauge gives the relative distance between the black hole and the particle, while r_0 after the transformation gives the distance from the center of mass to the particle. This explains the Newtonian part of the change in the force, $3\mu^2/r_0^2$. In this sense, the gauge freedom is under control at Newtonian order.

Now we turn to the S part. The harmonic coefficients in the harmonic gauge are given by

$$H^{\mathrm{S,H}}_{01m}(t,r)=\frac{4}{3}\pi\mu\Bigg\{\left[\frac{2r_0+2R}{r_0^2}+\frac{3MR}{r_0^3}-\frac{2iTmu^\phi}{r_0}+\frac{2}{9}\left(2r_0m^2+9R+Rm^2+\frac{27}{2}r_0\right)(u^\phi)^2\right]\Theta(r_0-r)$$
$$+\left[\frac{2r_0-4R}{r_0^2}-\frac{3MR}{r_0^3}-\frac{2iTmu^\phi}{r_0}+\frac{2}{9}\left(-\frac{63}{2}R+2r_0m^2+Rm^2+\frac{27}{2}r_0\right)(u^\phi)^2\right]\Theta(r-r_0)\Bigg\}Y^*_{1m}(\theta_0,\phi_0),$$

$$H^{\mathrm{S,H}}_{11m}(t,r)=\frac{4}{3}\pi\mu\Bigg\{\left[4\frac{TM}{r_0^3}+\frac{16}{9}imu^\phi-\frac{16}{9}\left(\frac{3}{2}-m\right)\left(\frac{3}{2}+m\right)(u^\phi)^2T\right]\Theta(r_0-r)$$
$$+\left[4\frac{TM}{r_0^3}+\frac{16}{9}imu^\phi-\frac{16}{9}\left(\frac{3}{2}-m\right)\left(\frac{3}{2}+m\right)(u^\phi)^2T\right]\Theta(r-r_0)\Bigg\}Y^*_{1m}(\theta_0,\phi_0),$$

$$H^{\mathrm{S,H}}_{21m}(t,r)=\frac{4}{3}\pi\mu\Bigg\{\left[\frac{2r_0+2R}{r_0^2}+\frac{3MR}{r_0^3}-\frac{2iTmu^\phi}{r_0}-\frac{2}{9}\left(9R-2r_0m^2-Rm^2+\frac{9}{2}r_0\right)(u^\phi)^2\right]\Theta(r_0-r)$$
$$+\left[\frac{2r_0-4R}{r_0^2}-\frac{3MR}{r_0^3}-\frac{2iTmu^\phi}{r_0}-\frac{2}{9}\left(\frac{9}{2}r_0-2r_0m^2-\frac{9}{2}R-Rm^2\right)(u^\phi)^2\right]\Theta(r-r_0)\Bigg\}Y^*_{1m}(\theta_0,\phi_0),$$

$$h^{(e)\mathrm{S,H}}_{01m}(t,r)=\frac{4}{3}\pi\mu\left\{\left[\frac{\frac{8}{9}im\left(\frac{9}{4}R^2+4r_0R+2r_0^2\right)u^\phi}{r_0}+\frac{8}{9}m^2T(4R+2r_0)(u^\phi)^2\right]\Theta(r_0-r)\right.$$
$$\left.+\left[\frac{\frac{8}{9}im\left(-2r_0R+\frac{9}{4}R^2+2r_0^2\right)u^\phi}{r_0}+\frac{8}{9}m^2T(2r_0-2R)(u^\phi)^2\right]\Theta(r-r_0)\right\}Y^*_{1m}(\theta_0,\phi_0),$$

$$h^{(e)\mathrm{S,H}}_{11m}(t,r)=\frac{4}{3}\pi\mu\left\{\left[-\frac{32}{81}m^2\left(-\frac{9}{4}R^2+2r_0^2+r_0R\right)(u^\phi)^2\right]\Theta(r_0-r)\right.$$
$$\left.+\left[-\frac{32}{81}m^2\left(-\frac{9}{4}R^2+2r_0^2+r_0R\right)(u^\phi)^2\right]\Theta(r-r_0)\right\}Y^*_{1m}(\theta_0,\phi_0),$$

$$K^{\mathrm{S,H}}_{1m}(t,r)=\frac{4}{3}\pi\mu\left\{\left[\frac{2r_0+2R}{r_0^2}+\frac{3MR}{r_0^3}-\frac{2iTmu^\phi}{r_0}+\frac{2}{9}\left(2r_0m^2+\frac{9}{2}r_0+Rm^2\right)(u^\phi)^2\right]\Theta(r_0-r)\right.$$
$$\left.+\left[\frac{2r_0-4R}{r_0^2}-\frac{3MR}{r_0^3}-\frac{2iTmu^\phi}{r_0}+\frac{2}{9}\left(-\frac{27}{2}R+2r_0m^2+Rm^2+\frac{9}{2}r_0\right)(u^\phi)^2\right]\Theta(r-r_0)\right\}Y^*_{1m}(\theta_0,\phi_0). \tag{E39}$$

We transform the above to the RW gauge, as discussed in Sec. V. Since G is absent from the beginning, Eqs. (6.4), which give the gauge transformation from the harmonic gauge to the RW gauge, are simplified as

$$M^{\mathrm{S,H}\to\mathrm{RW}}_{21m}(t,r)=0,\quad M^{\mathrm{S,H}\to\mathrm{RW}}_{01m}(t,r)=-h^{(e)\mathrm{S,H}}_{01m}(t,r),\quad M^{\mathrm{S,H}\to\mathrm{RW}}_{11m}(t,r)=-h^{(e)\mathrm{S,H}}_{11m}(t,r). \tag{E40}$$

The resulting harmonic coefficients in the RW gauge are expressed as those given in Eqs. (6.6), except for the gauge functions M_0 and M_1 that are now given by the above equations. From these, we find

$$H^{\mathrm{S,RW}}_{01m}(t,r)=\frac{4}{3}\pi\mu\left\{\left[\frac{2r_0+2R}{r_0^2}+\frac{3MR}{r_0^3}-\frac{2iTmu^\phi}{r_0}+\frac{2}{9}\left(-14r_0m^2+9R-31Rm^2+\frac{27}{2}r_0\right)(u^\phi)^2\right]\Theta(r_0-r)\right.$$
$$\left.+\left[\frac{2r_0-4R}{r_0^2}-\frac{3MR}{r_0^3}-\frac{2iTmu^\phi}{r_0}+\frac{2}{9}\left(-\frac{63}{2}R-14r_0m^2+17Rm^2+\frac{27}{2}r_0\right)(u^\phi)^2\right]\Theta(r-r_0)\right\}Y^*_{1m}(\theta_0,\phi_0),$$

$$H^{\mathrm{S,RW}}_{11m}(t,r)=\frac{4}{3}\pi\mu\left\{\left[4\frac{TM}{r_0^3}-\frac{8}{9}\left(2r_0+\frac{9}{2}R\right)imu^\phi-\frac{8}{9}\left(\frac{9}{2}+2m^2\right)(u^\phi)^2T\right]\Theta(r_0-r)\right.$$
$$\left.+\left[4\frac{TM}{r_0^3}+\frac{8}{9}\left(4r_0-\frac{9}{2}R\right)imu^\phi-\frac{8}{9}\left(\frac{9}{2}-4m^2\right)(u^\phi)^2T\right]\Theta(r-r_0)\right\}Y^*_{1m}(\theta_0,\phi_0),$$

$$H^{\mathrm{S,RW}}_{21m}(t,r)=\frac{4}{3}\pi\mu\left\{\left[\frac{2r_0+2R}{r_0^2}+\frac{3MR}{r_0^3}-\frac{2iTmu^\phi}{r_0}-\frac{2}{9}\left(9R-2r_0m^2+15Rm^2+\frac{9}{2}r_0\right)(u^\phi)^2\right]\Theta(r_0-r)\right.$$
$$\left.+\left[\frac{2r_0-4R}{r_0^2}-\frac{3MR}{r_0^3}-\frac{2iTmu^\phi}{r_0}-\frac{2}{9}\left(\frac{9}{2}r_0-2r_0m^2-\frac{9}{2}R+15Rm^2\right)(u^\phi)^2\right]\Theta(r-r_0)\right\}Y^*_{1m}(\theta_0,\phi_0),$$

$$K^{\mathrm{S,RW}}_{1m}(t,r)=\frac{4}{3}\pi\mu\left\{\left[\frac{2r_0+2R}{r_0^2}+\frac{3MR}{r_0^3}-\frac{2iTmu^\phi}{r_0}+\frac{2}{9}\left(2r_0m^2+\frac{9}{2}r_0+Rm^2\right)(u^\phi)^2\right]\Theta(r_0-r)\right.$$
$$\left.+\left[\frac{2r_0-4R}{r_0^2}-\frac{3MR}{r_0^3}-\frac{2iTmu^\phi}{r_0}+\frac{2}{9}\left(-\frac{27}{2}R+2r_0m^2+Rm^2+\frac{9}{2}r_0\right)(u^\phi)^2\right]\Theta(r-r_0)\right\}Y^*_{1m}(\theta_0,\phi_0). \tag{E41}$$

We note that only $H_1^{\rm S,RW}$ is discontinuous at $r=r_0$. However, as mentioned before, the force depends only on $H_0^{\rm S,RW}$ and $K^{\rm S,RW}$ which are continuous. We obtain the S force as

$$F_{\rm S,RW}^{r({\rm even})}(\ell=1)=\left[\frac{\mu^2}{r_0^2}+\frac{21\mu^2 M}{8r_0^3}\right]\Theta(r_0-r)+\left[-\frac{2\mu^2}{r_0^2}+\frac{9\mu^2 M}{8r_0^3}\right]\Theta(r-r_0). \tag{E42}$$

Subtracting the above from the full force (E38), we find

$$\delta F_{\rm RW}^{r({\rm even})}(\ell=1)=F_{\rm full,RW}^{r({\rm even})}(\ell=1)-F_{\rm S,RW}^{r({\rm even})}(\ell=1)=\frac{2\mu^2}{r_0^2}-\frac{105\mu^2 M}{8r_0^3}. \tag{E43}$$

We note that the Newtonian term, $2\mu^2/r_0^2$, is precisely the correction to the force at $O(\mu^2)$ when r_0 is the distance from the center of mass to the location of the particle.

If we recall the fact that both $H_1^{\rm full,Z}$ and $H_2^{\rm full,Z}$ are discontinuous at $r=r_0$ and the gauge transformation from the Zerilli gauge to a RW gauge given by Eq. (E35) does not change the discontinuity, while only $H_1^{\rm S,RW}$ is discontinuous for the S part, we see that the RW gauge we adopted to obtain the full force is different from the RW gauge for the S part obtained by the transformation from the harmonic gauge. Fortunately, however, because the force depends only on H_0 (and K) for circular orbits, and its discontinuity structure at $r=r_0$ happens to be the same in both gauges, the resulting force (E43) turns out to contain no discontinuity. Furthermore, as discussed above, the correct Newtonian force is recovered at $O(\mu^2)$. It is not clear if this desirable property holds because the orbit is circular or because only the 1PN order correction is considered. If this happens to be no longer the case for general orbits, it will be necessary to find a gauge transformation that remedies the discrepancy. In any case, except for the correction at Newtonian order, the gauge ambiguity remains in the final result, and its resolution is left for future work.

APPENDIX F: *m*-SUMMATION OF TENSOR HARMONICS

In this appendix we summarize the formulas for summing over m modes of the tensor harmonics for arbitrary ℓ. Specifically, the m sum we need to evaluate takes the form

$$\sum_{m=-\ell}^{\ell} m^N |Y_{\ell m}(\pi/2,0)|^2, \tag{F1}$$

where N is a non-negative integer. To perform the summation, we introduce the generating function

$$\Gamma_\ell(z)=\sum_{m=-\ell}^{\ell} e^{mz}|Y_{\ell m}(\pi/2,0)|^2. \tag{F2}$$

Then the sum (F1) may be evaluated as $\lim_{z\to 0}\partial_z^N\Gamma_\ell(z)$. The above function is calculated as

$$\Gamma_\ell(z)=\frac{2\ell+1}{4\pi}e^{\ell z}{}_2F_1\left(\frac{1}{2},-\ell;1;1-e^{-2z}\right), \tag{F3}$$

where ${}_2F_1$ is the hypergeometric function. This can be easily expanded to an arbitrary order of z. For example, to $O(z^6)$, we have

$$\Gamma_\ell(z)=\frac{2\ell+1}{4\pi}\left\{1+\left(\frac{\ell(\ell+1)}{2}\right)\frac{1}{2}z^2+\left(\frac{\ell(\ell+1)(3\ell^2+3\ell-2)}{8}\right)\frac{1}{4!}z^4+\left(\frac{\ell(\ell+1)(5\ell^4+10\ell^3-5\ell^2-10\ell+8)}{16}\right)\frac{1}{6!}z^6+O(z^8)\right\}. \tag{F4}$$

In the cases of the vector and tensor harmonics, it is necessary to evaluate the m sum of the form

$$\sum_{m=-\ell}^{\ell} m^N|\partial_\theta Y_{\ell m}(\pi/2,0)|^2. \tag{F5}$$

We introduce the generating function

$$\Delta_\ell(z)=\sum_{m=-\ell}^{\ell} e^{mz}|\partial_\theta Y_{\ell m}(\pi/2,0)|^2. \tag{F6}$$

This is expressed in terms of a hypergeometric function as

$$\Delta_\ell(z)=\frac{2\ell+1}{4\pi^2}e^{(\ell-1)z}\frac{\Gamma(\ell+1/2)\Gamma(3/2)}{\Gamma(\ell)}\times{}_2F_1\left(\frac{3}{2},-\ell+1;-\ell+\frac{1}{2};e^{-2z}\right). \tag{F7}$$

The sum (F5) is evaluated by taking the derivatives of the above generating function. Expanding in powers of z, the m sum (F5) is calculated as

$$\Delta_\ell(z)=\frac{2\ell+1}{4\pi}\left\{\left(\frac{\ell(\ell+1)}{2}\right)+\left(\frac{\ell(\ell+1)(\ell-1)(\ell+2)}{8}\right)\frac{1}{2}z^2+\left(\frac{\ell(\ell+1)(\ell-1)(\ell+2)(\ell^2+\ell-4)}{16}\right)\frac{1}{4!}z^4+\left(\frac{\ell(\ell+1)(\ell-1)(\ell+2)(5\ell^4+10\ell^3-45\ell^2-50\ell+136)}{128}\right)\frac{1}{6!}z^6+O(z^8)\right\}. \tag{F8}$$

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[2] GEO600 web page: http://www.geo600.uni-hannover.de/
[3] TAMA300 web page: http://tamago.mtk.nao.ac.jp/
[4] VIRGO web page: http://www.virgo.infn.it/
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