Modified gravity with negative and positive powers of curvature: Unification of inflation and cosmic acceleration

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A modified gravity, which eliminates the need for dark energy and which seems to be stable, is considered. The terms with positive powers of curvature support the inflationary epoch while the terms with negative powers of curvature serve as effective dark energy, supporting current cosmic acceleration. The equivalent scalar-tensor gravity may be compatible with the simplest solar system experiments.

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I. INTRODUCTION

The evidence is mounting that the Universe is undergoing a phase of accelerated expansion at the present epoch. The indications of cosmic acceleration appeared not only from the high redshift surveys of type Ia supernovae [1] but also from the anisotropy power spectrum of the cosmic microwave background [2]. The favored explanation for this behavior is that the Universe is presently dominated by some form of dark energy. However, none of existing dark energy models is completely satisfactory. Moreover it is very hard to construct a theoretical basis for the origin of this exotic matter, which is seen precisely at the current epoch when one needs the source for cosmic acceleration.

In the recent papers [3,4] (see also [5,6]) a gravitational alternative for dark energy was suggested. It goes as follows. The standard Einstein action is modified at low curvature by the terms that dominate precisely at low curvature. The simplest possibility of this sort is a $1/R$ term; other negative powers of curvature may be introduced as well. Moreover, more complicated terms may be suggested such as $1/(\Box R+\text{const})$, etc. The only condition is to dominate over R and to produce cosmic acceleration consistent with current astrophysical data. The interesting feature of the theories with negative powers of curvature is that they may be expected from some time-dependent compactification of string or M theory, as was demonstrated in [7]. Moreover, quantum fluctuations in nearly flat spacetime may induce such terms in the same way as the expansion of the effective action at large curvature predicts the terms with positive power of (nonlocal combinations of) curvature invariants (for a recent review, see [8]).

Clearly, having the gravitational foundation for description of current cosmic acceleration seems to be much more natural than the introduction by hand of the mysterious dark energy, cosmic fluid with negative pressure. However, as any other theory pretending to describe current Friedmann-Robertson-Walker (FRW) universe such a modified gravitational theory should pass a number of consistency checks to be considered as the realistic theory. For instance, as any other higher derivative theory, the theory with $1/R$ may develop instabilities [9] (see, however, [10] and discussion in Sec. VI). As cosmic acceleration from such a theory is also instable [3] the question arises: which of the instabilities is more realistic and are some more modifications of the theory required? From another point of view, the gravitational action which is the function of only curvature may be presented in a number of ways in the equivalent form as scalar-tensor theory with one (or several) scalar(s). Then the fundamental question is: which action corresponds to our physical world? For instance, it has been mentioned [11] that for the model [3] the equivalent Brans-Dicke (BD) action is not acceptable as it is ruled out by the solar system experiments. This goes against the consideration of the initial modified gravity with the $1/R$ a term as physical theory. In such circumstances it may be a good idea to try to search for other variants of then modified gravitational theory which still excludes the need for dark energy. Moreover, one can try to construct the theory which predicts inflation at very early Universe and cosmic acceleration currently in the combined setup.

In the present paper we suggest a new model of modified gravity which contains positive and negatives powers of curvature. Symbolically, the Lagrangian looks like $\mathcal{L}=R+R^m+1/R^n$ where n, m are positive (not necessary integer) numbers. The theory may be presented in an equivalent form as some scalar-tensor gravity. At large curvature, the terms of the sort R^m dominate. If $1<m<2$ the power law inflation occurs at early times. If $m=2$ the anomaly driven (Starobinsky) inflation occurs at early times. At an intermediate region, the theory is Einstein gravity. Currently, at low curvature the terms of the sort $1/R^n$ dominate. These terms serve as a gravitational alternative to dark energy and produce cosmic acceleration. It is remarkable that such modified gravity

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does not suffer from the instabilities pointed out for the version with the Lagrangian $L=R+1/R$. Moreover, it may pass the solar system tests for scalar-tensor gravity.

The paper is organized as follows. In the next section we discuss various forms of the action for modified gravity, both in Jordan and in Einstein frames. It is also shown that such a theory may have the origin in the braneworld scenario by fine-tuning of surface counterterms. Section III is devoted to the study of simplest de Sitter solutions for modified gravity. The occurrence of two deSitter phases is mentioned. The properties of the scalar potential in the scalar-tensor formulation of theory where scalar should be identified with curvature, are investigated. In Sec. IV, FRW cosmology for such a model (with or without matter) is discussed. It is shown that the model naturally admits the unification of the inflation at early times and the cosmic acceleration at late times. In Sec. VI we demonstrate that higher derivative terms make the dangerous instabilities of the original $1/R$ theory to become much less essential at cosmological scales. Moreover, the model easily passes solar system constraints to scalar-tensor gravity. Some summary and outlook is presented in the final section.

II. ACTIONS FOR THE MODIFIED GRAVITATIONAL THEORY

Let us start from the rather general 4-dimensional action:

$$S=\frac{1}{\kappa^2}\int d^4x\sqrt{-g}f(R). \tag{1}$$

Here R is the scalar curvature and $f(R)$ is some arbitrary function. Introducing the auxiliary fields A and B, one may rewrite action (1) as follows:

$$S=\frac{1}{\kappa^2}\int d^4x\sqrt{-g}\{B(R-A)+f(A)\}. \tag{2}$$

By the variation over B, $A=R$ follows. Substituting it into Eq. (2), action (1) can be reproduced. Making the variation with respect to A first, we obtain

$$B=f'(A), \tag{3}$$

which may be solved with respect to A as

$$A=g(B). \tag{4}$$

Eliminating A in Eq. (2) with the help of Eq. (4), we obtain

$$S=\frac{1}{\kappa^2}\int d^4x\sqrt{-g}\{B[R-g(B)]+f(g(B))\}. \tag{5}$$

Instead of A, one may eliminate B by using Eq. (3) and obtain

$$S=\frac{1}{\kappa^2}\int d^4x\sqrt{-g}\{f'(A)(R-A)+f(A)\}. \tag{6}$$

At least classically, the two actions (5) and (6) are equivalent to each other.

Action (5) or (6) may be called the Jordan frame action with auxiliary fields. A more convenient Einstein frame theory may be worked out as well. Under the conformal transformation

$$g_{\mu\nu}\to e^{\sigma}g_{\mu\nu}, \tag{7}$$

d-dimensional scalar curvature is transformed as

$$R^{(d)}\to e^{-\sigma}\left(R^{(d)}-(d-1)\Box\sigma - \frac{(d-1)(d-2)}{4}g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma\right). \tag{8}$$

Then for $d=4$, by choosing

$$\sigma=-\ln f'(A), \tag{9}$$

action (6) is rewritten as

$$S_E=\frac{1}{\kappa^2}\int d^4x\sqrt{-g}\left\{R-\frac{3}{2}\left(\frac{f''(A)}{f'(A)}\right)^2 g^{\rho\sigma}\partial_\rho A\partial_\sigma A - \frac{A}{f'(A)}+\frac{f(A)}{f'(A)^2}\right\}. \tag{10}$$

Using $\sigma=-\ln f'(A)=-\ln B$, the Einstein frame action looks like

$$S_E=\frac{1}{\kappa^2}\int d^4x\sqrt{-g}\left(R-\frac{3}{2}g^{\rho\sigma}\partial_\rho\sigma\partial_\sigma\sigma-V(\sigma)\right). \tag{11}$$

Here

$$V(\sigma)=e^{\sigma}g(e^{-\sigma})-e^{2\sigma}f(g(e^{-\sigma}))=\frac{A}{f'(A)}-\frac{f(A)}{f'(A)^2}. \tag{12}$$

This is the standard form of the scalar-tensor theories where scalar field is fictitious one. Although $S_E=S$, we denote the action given in the Einstein frame by S_E. In Eq. (9), it is assumed $f'(A)>0$. Even if $f'(A)<0$, $\sigma=-\ln|f'(A)|$ may be defined. Then the sign in front of the scalar curvature becomes negative. In other words, antigravity could be generated.

As a specific choice, we consider

$$f(R)=R-\frac{a}{(R-\Lambda_1)^n}+b(R-\Lambda_2)^m. \tag{13}$$

Here we assume the coefficients $n,m,a,b>0$ but n,m may be fractional. Two last terms may be changed by the sum which includes the terms with various negative and positive powers of curvature:

$$f(R)=\sum_n a_nR^n. \tag{14}$$

Here n can run from negative to positive values. In general, n needs not to be integer. It may also be (positive or negative) an irrational number, that is, $F(R)$ can be rather arbitrary function.

When the first term is absent, $n=m>1$, $-a=b$ and $\Lambda_1=\Lambda_2$ the evident duality symmetry appears. At very high curvature and supposing both cosmological constants to be small, the higher derivative terms dominate. Subsequently, with the decrease of curvature the Einstein gravity dominates. For low curvature (depending on the choice of Λ_1) the negative power of curvature may become the leading contribution to the theory.

If A or R is large,

$$e^{-\sigma}=B\sim bm(A-\Lambda_2)^{m-1}. \tag{15}$$

Then if $m>1$ and $e^{-\sigma}\rightarrow+\infty$,

$$V(\sigma)\sim-(bm)^{-1/(m-1)}\left(1+\frac{1}{m}\right)e^{(m-2)/(m-1)\sigma}. \tag{16}$$

On the other hand, when $A-\Lambda_1$ or $R-\Lambda_1$ is small, we obtain

$$e^{-\sigma}\sim an(A-\Lambda_1)^{-n-1}. \tag{17}$$

When $e^{-\sigma}\rightarrow+\infty$,

$$V(\sigma)\sim-(an)^{1/(n+1)}\left(1-\frac{1}{n}\right)e^{(n+2)/(n+1)\sigma}. \tag{18}$$

The above arguments indicate that $V(\sigma)$ is not a single-valued function of σ but, at least, there are two branches for $m>2$. The above modified gravitational action will be our starting point in the attempt to construct the Universe where both phases—early time inflation and late time cosmic acceleration—occur.

III. PROPERTIES OF THE SCALAR POTENTIAL

Let us discuss the properties of the scalar potential for $F(R)$ given by Eq. (13). In this section, we concentrate on the case in which that the matter contribution can be neglected.

With no matter and for the Ricci tensor $R_{\mu\nu}$ being covariantly constant, the equation of motion corresponding to action (1) is

$$0=2f(R)-Rf'(R), \tag{19}$$

which is the algebraic equation with respect to R. For action (13)

$$0=-R+\frac{(n+2)a}{(R-\Lambda_1)^n}+(m-2)b(R-\Lambda_2)^m. \tag{20}$$

Especially when $n=1$ and $m=2$, one gets

$$R=R_{\pm}=\frac{\Lambda_1\pm\sqrt{\Lambda_1^2+12a}}{2}. \tag{21}$$

If $a>0$, one solution corresponds to de Sitter space and another to anti–de Sitter. If $-\Lambda_1^2/12<a<0$ and $\Lambda_1>0$, both solutions express the de Sitter space. We may consider other cases, such as $n=1$ and $m=3$. For simplicity, we chosen $\Lambda_1=\Lambda_2=0$. Then Eq. (20) becomes

$$0=-R+\frac{3a}{R}+bR^3. \tag{22}$$

For positive a and b, Eq. (22) has two solutions in general. If we further assume a and b is small, there is a solution with large R_l and the one with small R_s:

$$R_l\sim b^{-1/4},\quad R_s\sim\sqrt{3a}. \tag{23}$$

Since the square root of the scalar curvature corresponds to the expansion rate of the de Sitter universe, the inflation may be generated by a solution R_l and the present cosmic speed-up may be generated by R_s. One arrives at a very interesting picture of the Universe evolution where modification of gravity at high and low curvatures predicts both (early and late time) phases of the accelerated expansion.

We now investigate potential (12) for action (13):

$$V(A)=\frac{\frac{a\{(n+1)A-\Lambda_1\}}{(A-\Lambda_1)^{n+1}}+b\{(m-1)A+\Lambda_2\}(A-\Lambda_2)^{m-1}}{\left\{1+\frac{an}{(A-\Lambda_1)^{n+1}}+bm(A-\Lambda_2)^{m-1}\right\}^2}. \tag{24}$$

As is clear from Eq. (2) that $A=R$, if we regard the scalar curvature R in action (1) with the physical one, A is nothing but the physical curvature itself (after equation of motion for A is satisfied).

When $A\rightarrow\pm\infty$,

$$V(A)\rightarrow\frac{m-1}{bm^2}A^{-m+2}. \tag{25}$$

Therefore if $m<2$, $V(A)\rightarrow\infty$, if $m=2$, $V(A)\rightarrow 1/4b$, and if $m>0$, $V(A)\rightarrow 0$. On the other hand, when $A\rightarrow\Lambda_1$, one finds

$$V(A)\rightarrow\frac{\Lambda_1}{an}(A-\Lambda_1)^{n+1}\rightarrow 0. \tag{26}$$

Since potential (24) is rather complicated, in the following, we only consider the case which $n=1$, $m=2$, $\Lambda_1=\Lambda_2=0$:

$$V(A)=\frac{\frac{2a}{A}+bA^2}{\left(1+\frac{a}{A^2}+2bA\right)^2}. \tag{27}$$

When $A\rightarrow\pm\infty$, $V(A)\rightarrow 1/4b$. $V(A)$ vanishes at

$$A=0,\quad -\left(\frac{2a}{b}\right)^{1/3}. \tag{28}$$

The denominator of potential (27) vanishes at

$$A=A_0\equiv-\frac{1}{6b}+\alpha_+ +\alpha_- , \tag{29}$$

$$\alpha_\pm^3\equiv\frac{1}{2}\left\{-\frac{1}{54b^3}-\frac{a}{2b}\pm\frac{1}{2b}\sqrt{\left(a+\frac{1}{18b^2}\right)\left(a+\frac{1}{54b^2}\right)}\right\}.$$

When $A\to A_0$, the potential diverges. We should note that $A_0<0$ if $a,b<0$. Since

$$V'(A)=\frac{2\left(-\frac{a}{A^2}+bA\right)\left(1-\frac{3a}{A^2}\right)}{\left(1+\frac{a}{A^2}+2bA\right)^3}, \tag{30}$$

$V(A)$ has stationary points, where $V'(A)=0$, at

$$A=\left(\frac{a}{b}\right)^{1/3}, \quad \pm\sqrt{3a}. \tag{31}$$

Then by summarizing the above analysis, we find the rough shape of the potential: The asymptotic value of the potential is $1/4b$. The potential vanishes twice when A vanishes and A takes a negative value $A=-(2a/b)^{1/3}$ in Eq. (28). The potential is singular when A takes a negative value, $A=A_0$ in Eq. (29). The potential has three extrema. Two of them are given by positive A: $A=(a/b)^{1/3}$, $\sqrt{3a}$ and one of the three extrema is given by negative A: $A=-\sqrt{3a}$.

When A is small, Eq. (9) shows that

$$e^{-\sigma}=\frac{a}{A^2}+2bA\sim\frac{a}{A^2}, \tag{32}$$

that is

$$A=\sqrt{a}e^{(1/2)\sigma}. \tag{33}$$

Since $A\to 0$ corresponds to $\sigma\to-\infty$ one may only consider the region where A and also physical scalar curvature are positive. Then the singularity at $A=A_0$ Eq. (29) does not appear.

Since potential (27) has the form

$$V(A)\sim\frac{2}{a^2}A^3\sim\frac{2}{\sqrt{a}}e^{(3/2)\sigma}, \tag{34}$$

in the region where A is small, the Universe evolves as the power law inflation, as we will see later, Eq. (54). Equation (34) shows that the potential is an increasing function for small A. Since the potential is almost flat but monotonically increasing for large A, the following cosmological scenario can be considered. Let us assume that the Universe starts from large but finite positive A. Since A corresponds to the physical scalar curvature R, the Universe expands rapidly because $\sqrt{R}$ is the expansion rate of the de Sitter space. After that, A rolls down to the small value and the rate of the expansion becomes small. When A reaches the small A region, the Universe begins the power law expansion.

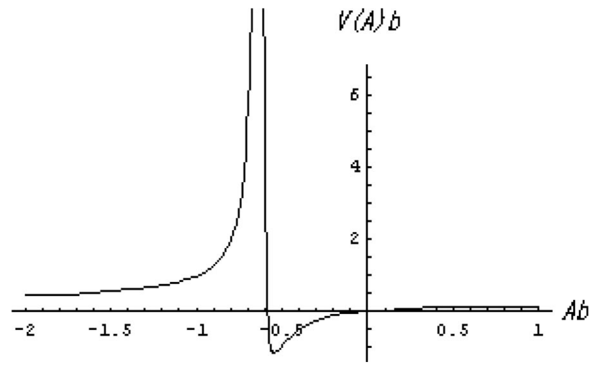

FIG. 1. $V(A)b$ vs Ab for $ab^2=\frac{1}{10}$.

We now give some typical shapes of $V(A)$. In Fig. 1 the shape of the potential $V(A)b$ vs Ab for $ab^2=\frac{1}{10}$ is given. When $A\to\pm\infty$, $V(A)$ approaches the constant value $1/4b$. As mentioned in Eq. (28), $V(A)$ vanishes at $A=0$ and $A=-(2a/b)^{1/3}<0$. When $A\to A_0<0$ in Eq. (4), the potential diverges. The negative A minimum corresponds to $A=-\sqrt{3a}$. In Fig. 2, the region $A\sim 0.5$ is considered. Then two extrema occur, which correspond to $A=(a/b)^{1/3}$ and $A=\sqrt{3a}[(a/b)^{1/3}<\sqrt{3a}]$. If the Universe starts at $A=\sqrt{3a}$, which is locally stable, the Universe is in the de Sitter phase. If by some mechanism, such as the thermal fluctuations, etc., A becomes smaller, the Universe evolves to the power law inflation. The Universe may start with $A=(a/b)^{1/3}$, where it is unstable. We should note that the potential with $ab^2=\frac{1}{10}$ is bounded from below. In Fig. 3, the shape of the potential $V(A)b$ vs Ab for $ab^2=\frac{1}{3}$ is given. The behavior when $A\geqslant 0$ is qualitatively not so changed from the $ab^2=\frac{1}{10}$ case. In the region $A<0$, however, the potential becomes unbounded from below. When $ab^2=\frac{1}{10}$, the potential is bounded from below since $A_0<-\sqrt{3a}$. Note that $A=-\sqrt{3a}$ corresponds to the minimum. When $ab^2=\frac{1}{3}$, we have $A_0<-\sqrt{3a}$ and $A=-\sqrt{3a}$ corresponds to the maximum, then the potential becomes unbounded from below. Note once more that the above results have physical meaning in terms of our modified gravity with negative powers of curvature only when the

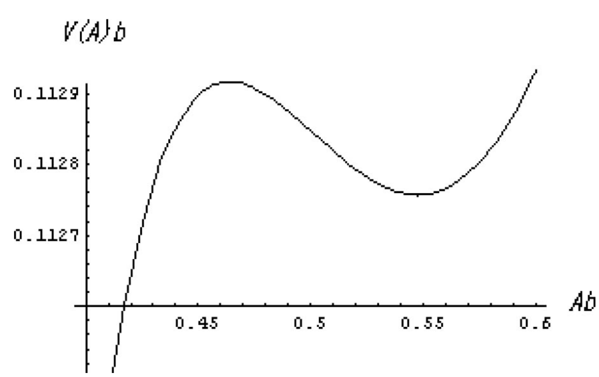

FIG. 2. $V(A)b$ vs Ab for $ab^2=\frac{1}{10}$. The behavior in the region $Ab\sim 0.5$.

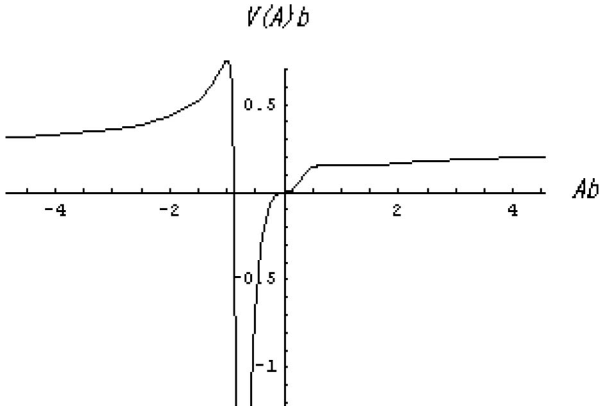

FIG. 3. $V(A)b$ vs Ab for $ab^2=\frac{1}{3}$.

equation of motion for the field A itself is satisfied. (Only in this case A may be identified with the curvature of the physical metric.)

Thus, we discussed the properties of the scalar potential corresponding to a quite simple version of the modified gravity. Similarly, the properties of other models of such sort may be analyzed.

IV. FRW COSMOLOGY IN MODIFIED GRAVITY

Some simple properties of FRW cosmology in modified gravity may be easily addressed. One may add the matter to action (1) with the matter action denoted by $S_{(m)}$. Then the energy-momentum tensor $T_{(m)\mu\nu}$ can be defined by

$$T_{(m)\mu\nu}=-\frac{2}{\sqrt{-g}}\frac{\delta S_{(m)}}{\delta g^{\mu\nu}}. \tag{35}$$

After rescaling metric (7), the energy-momentum tensor $T_{(mE)\mu\nu}$ in the Einstein frame (11) is related to $T_{(m)\mu\nu}$ by

$$T_{(mE)\mu\nu}=\mathrm{e}^{\sigma}T_{(m)\mu\nu}. \tag{36}$$

Defining the matter energy density $\rho_{(m)}$ and the pressure $p_{(m)}$ by

$$T_{(m)00}=-\rho_{(m)}g_{00}, \quad T_{(m)ij}=p_{(m)}g_{ij}, \tag{37}$$

the corresponding quantities $\rho_{(mE)}$ and $p_{(mE)}$ in the Einstein frame (11) are given by

$$\rho_{(mE)}=\mathrm{e}^{2\sigma}\rho_{(m)}, \quad p_{(mE)}=\mathrm{e}^{2\sigma}p_{(m)}. \tag{38}$$

We now assume the metric in the physical (Jordan) frame is given in the FRW form:

$$ds^2=-dt^2+\hat{a}(t)^2\sum_{i,j=1}^{3}\hat{g}_{ij}dx^i dx^j. \tag{39}$$

Here $\hat{g}_{ij}$ is the metric of the Einstein manifold, which is defined by the Ricci tensor $\hat{R}_{ij}$ constructed from $\hat{g}_{ij}$ by $\hat{R}_{ij}=k\hat{g}_{ij}$ with a constant k. From the conservation of the energy-momentum tensor $\nabla^{\mu}T_{\mu\nu}=0$ one gets

$$0=\dot{\rho}_{(m)}+3H(\rho_{(m)}+p_{(m)}). \tag{40}$$

Here

$$H=\frac{\dot{\hat{a}}}{\hat{a}}, \tag{41}$$

and it is assumed $\rho_{(m)}$, $p_{(m)}$ and also σ do not depend on the spatial coordinates but on the time coordinate. The matter is chosen to be the perfect fluid, which satisfies

$$p_{(m)}=w\rho_{(m)}. \tag{42}$$

Equation (40) can be solved in the usual way

$$\rho_{(m)}=C\hat{a}^{-3(1+w)}, \tag{43}$$

with a constant of integration C. In the Einstein frame, from Eq. (38), we find

$$\rho_{(mE)}=C\hat{a}_E^{-3(1+w)}\mathrm{e}^{-[(3w-1)/2]\sigma}. \tag{44}$$

Here $\hat{a}_E$ is the scale factor in the Einstein frame,

$$\hat{a}_E=\mathrm{e}^{-\sigma/2}\hat{a}. \tag{45}$$

The FRW equation in the Einstein frame has the following form:

$$3H_E^2+\frac{3k}{2\hat{a}_E^2}=\frac{\kappa^2}{2}(\rho_{(\sigma E)}+\rho_{(mE)}). \tag{46}$$

Here

$$H_E\equiv\frac{\dot{\hat{a}}_E}{\hat{a}_E} \tag{47}$$

and $\rho_{(\sigma E)}$ expresses the contribution from the σ field,

$$\rho_{(\sigma E)}\equiv\frac{1}{\kappa^2}\left(\frac{3}{2}\dot{\sigma}^2+V(\sigma)\right). \tag{48}$$

With the Einstein frame metric denoted as $g_{E\mu\nu}$, one obtains

$$\frac{1}{\sqrt{-g_E}}\frac{\delta S_{(m)}}{\delta\sigma}=-\frac{1}{\sqrt{-g_E}}g^{\mu\nu}\frac{\delta S_{(m)}}{\delta g^{\mu\nu}}=\frac{1}{2}g^{\mu\nu}T_{\mu\nu}\sqrt{\frac{g}{g_E}}. \tag{49}$$

Further using Eqs. (37), (42), (43), and (45), we find

$$\frac{1}{\sqrt{-g_E}}\frac{\delta S_{(m)}}{\delta\sigma}=\frac{3w-1}{2}\rho_{(m)}\mathrm{e}^{2\sigma}$$
$$=\frac{3w-1}{2}C\hat{a}_E^{-3(1+w)}\mathrm{e}^{-[(3w-1)/2]\sigma}. \tag{50}$$

Then the equation of motion for σ in the Einstein frame, corresponding to the action $S_E+S_{(m)}$ [S_E is given in Eq. (11)] has the following form:

$$0=3(\ddot{\sigma}+3H_E\dot{\sigma})+V'(\sigma)-\frac{3w-1}{2}\kappa^2C\hat{a}_E^{-3(1+w)}\mathrm{e}^{-[(3w-1)/2]\sigma}. \tag{51}$$

The question now is to solve the system of equations (46) and (51). Before going to consider the coupling with matter, we first consider the case of the vacuum where $C=0$. Especially when the potential is given by Eq. (34) as $A=R$ is small, a solution is given by

$$\hat{a}_E=\hat{a}_{E0}\left(\frac{t_E}{t_0}\right)^{4/3}, \quad \sigma=-\frac{4}{3}\ln\frac{t_E}{t_0}. \tag{52}$$

Here t_E is the time coordinate in the Euclidean frame, which is related to the time coordinate t in the (physical) Jordan frame by $\mathrm{e}^{\sigma/2}dt_E=dt$. As a result

$$3t_E^{1/3}=t, \tag{53}$$

and even in the physical (Jordan) frame the power law inflation occurs

$$\hat{a}=\mathrm{e}^{\sigma/2}\hat{a}_E\propto t_E^{2/3}\propto t^2, \tag{54}$$

which is consistent with the result in [3].[1] Hence, at small curvature the (instable) cosmic acceleration is predicted by the terms containing inverse curvature. If the present Universe corresponds to the above power law inflation, the curvature of the present Universe should be small compared with that of the de Sitter Universe solution in Eq. (21) with $\Lambda_1=0$. As the Hubble constant in the present universe is $(10^{-33}\ \mathrm{eV})^{-1}$, the parameter a, which corresponds to μ^4 in [3], should be much larger than $(10^{-33}\ \mathrm{eV})^4$.

One may also consider the case in which curvature is large. When A or curvature is large, potential (27) becomes a constant: $V(A)\rightarrow 1/4b$. Then from Eq. (51) we may assume, if we neglect the contribution from matter, that $\dot{\phi}$ is small. From Eq. (46) it follows

$$3H_E^2+\frac{3k}{2\hat{a}_E^2}\sim\frac{\kappa^2}{8b}, \tag{55}$$

which shows that the spacetime is de Sitter. For the case of $k=0$, H_E becomes a constant:

$$H_E=\sqrt{\frac{\kappa^2}{24b}}. \tag{56}$$

Then Universe expands exponentially.

The matter contribution to the energy-momentum tensor may be accounted for in the same way. When it is dominant compared with that from σ, Eqs. (46) and (51) can be reduced as

$$0=3H_E^2+\frac{3k}{2\hat{a}_E^2}-\frac{\kappa^2}{2}C\hat{a}_E^{-3(1+w)}\mathrm{e}^{-[(3w-1)/2]\sigma}, \tag{57}$$

$$0=3(\ddot{\sigma}+3H_E\dot{\sigma})-\frac{3w-1}{2}\kappa^2C\hat{a}_E^{-3(1+w)}\mathrm{e}^{-[(3w-1)/2]\sigma}. \tag{58}$$

When $w=\frac{1}{3}$, which corresponds to the radiation, from Eq. (58) one finds

$$\dot{\sigma}=\dot{\sigma}_0\left(\frac{\hat{a}_{E0}}{\hat{a}_E}\right)^3. \tag{59}$$

Here $\dot{\sigma}_0$ and $\hat{a}_{E0}$ are constants. Equation (59) expresses the redshift of $\dot{\sigma}$. When $w=0$, which corresponds to the dust, if we can regard σ is almost constant, we find

$$\dot{\sigma}=\dot{\sigma}_0\left(\frac{\hat{a}_{E0}}{\hat{A}_E}\right)^3.-\frac{C\kappa^2(t-t_0)}{6}\hat{a}_E^{-3(1+w)}\mathrm{e}^{-[(3w-1)/2]\sigma}. \tag{60}$$

The obtained results are not changed from those in [3] where the possibility of cosmic acceleration in the $1/R$ model was established.

When $n=1$ or in general n is an odd integer, as is clear from Eq. (18), $A=0$ is not a minimum nor local minimum although $V'(A)=0$ at $A=0$. Of course, since $A\rightarrow 0$ corresponds to $\sigma\rightarrow +\infty$, it might be unnecessary to consider the region $A<0$. In the region $A\geqslant 0$, $A=0$ can give a minimum of the potential. On the other hand, when n is an even integer, $A=0$ is at least local minimum. Then if inflation occurs in the region $A\sim 0$, inflation should be stable.

We now consider the general case wherein $f(A)$ is given by Eq. (13). When $R=A\sim\Lambda_1$ is small, the potential is given by Eq. (18). Neglecting the contribution from the matter fields, solving Eqs. (46) and (51), we obtain, instead of Eq. (52),

$$\hat{a}_E=\hat{a}_{E0}\left(\frac{t_E}{t_0}\right)^{(n+1)(2n+1)/(n+2)^2},$$

$$\sigma=-\frac{2(n+1)}{(n+2)}\ln\frac{t_E}{t_0}. \tag{61}$$

Instead of Eq. (53), the physical time is

$$(n+2)t_E^{1/(n+2)}=t. \tag{62}$$

Then the power law cosmic acceleration occurs in the physical (Jordan) frame:

$$\hat{a}\propto t^{(n+1)(2n+1)/n+2}. \tag{63}$$

It is quite remarkable that actually any negative power of curvature supports cosmic acceleration. This gives the freedom in modification of the model to achieve the consistency with experimental tests of Newtonian gravity.

[1] ϕ in [3] can be identified as $\sigma=-\sqrt{2/3}\,\phi/M_p$.

On the other hand, if the scalar curvature $R=A$ is large, the potential is given by Eq. (16). Solving Eqs. (46) and (51) again, one obtains

$$\hat{a}_E=\hat{a}_{E0}\left(\frac{t_E}{t_0}\right)^{(m-1)(2m-1)/(m-2)^2},$$

$$\sigma=-\frac{2(m-1)}{(m-2)}\ln\frac{t_E}{t_0},\quad (m-2)t_E^{-1/(m-2)}=t. \tag{64}$$

Thus in the physical (Jordan) frame, the Universe shrinks with the power law if $m>2$:[2]

$$\hat{a}\propto t^{-(m-1)(2m-1)/m-2}. \tag{65}$$

Of course, if we change the arrow of time by $t\to t_0-t$, inflation occurs with the inverse power law and at $t=t_0$, the size of the Universe diverges. It is remarkable that when m is fractional (or irrational) and $1<m<2$, Eqs. (64) and (65) are still valid. Then the power in Eq. (65) becomes positive, the Universe evolves with the (fractional) power law expansion.

Anyway, it is interesting that our model may unify both phases: early time inflation (for $m<2$ or $m=2$) and current cosmic acceleration.

In [3], from the dimensional analysis it is predicted for $\mu^4=a$ (a for $n=1$ case) to be $\mu\sim H_0\sim 10^{-33}$ eV. Here H_0 is the Hubble constant in the present Universe. From the dimensional analysis, one may choose

$$a=(10^{-33}\ \text{eV})^{2(n+1)},\quad b=(10^{-33}\ \text{eV})^{2(-m+1)}. \tag{66}$$

In principle, these parameters should not be small ones, unless predicted by experimental constraints.

Adding the (dominant) matter contribution to the energy-momentum tensor, the situation is not changed from the $n=1$ and $m=1$ case and the case of [3]. We obtain Eqs. (59) and (60) again. Thus, the possibility of cosmic acceleration in the model with negative powers of curvature is demonstrated. Moreover, terms with positive powers of curvature in such a model may realize inflation at early times.

V. SIMPLEST TESTS FOR MODIFIED GRAVITY

In this section, we discuss the (in)stability of our higher derivative model under perturbations. The simplest constraint to the theory parameters from the equivalent BD-type gravity is also analyzed.

In [9], a small gravitational object similar the Earth or the Sun in the model [3] is considered. It has been shown that the system quickly becomes instable.

The general[3] equation of motion corresponding action (1) with matter is given by

$$\frac{1}{2}g_{\mu\nu}f(R)-R_{\mu\nu}f'(R)-g_{\mu\nu}\Box f'(R)+\nabla_\mu\nabla_\nu f'(R)=-\frac{\kappa^2}{2}T_{(m)\mu\nu}. \tag{67}$$

Here $T_{(m)\mu\nu}$ is the energy-momentum tensor of the matter. By multiplying $g^{\mu\nu}$ to (67), one arrives at[4]

$$\Box R+\frac{f^{(3)}(R)}{f^{(2)}(R)}\nabla_\rho R\nabla^\rho R+\frac{f'(R)R}{3f^{(2)}(R)}-\frac{2f(R)}{3f^{(2)}(R)}=\frac{\kappa^2}{6f^{(2)}(R)}T. \tag{68}$$

Here $T=T^\rho_{(m)\rho}$. Let $f(R)$ be given by (13). Then in the case of the Einstein gravity, where $a=b=0$, the solution of Eq. (68) is given by

$$R=R_0\equiv-\frac{\kappa^2}{2}T. \tag{69}$$

The perturbation around solution (69) may be addressed

$$R=R_0+R_1,\quad (|R_1|\ll|R_0|). \tag{70}$$

Then by linearizing Eq. (68), we obtain

$$0=\Box R_0+\frac{f^{(3)}(R_0)}{f^{(2)}(R_0)}\nabla_\rho R_0\nabla^\rho R_0+\frac{f'(R_0)R_0}{3f^{(2)}(R_0)}-\frac{2f(R_0)}{3f^{(2)}(R_0)}-\frac{R_0}{3f^{(2)}(R_0)}+\Box R_1+2\frac{f^{(3)}(R_0)}{f^{(2)}(R_0)}\nabla_\rho R_0\nabla^\rho R_1+U(R_0)R_1. \tag{71}$$

Here

$$U(R_0)\equiv\left(\frac{f^{(4)}(R_0)}{f^{(2)}(R_0)}-\frac{f^{(3)}(R_0)^2}{f^{(2)}(R_0)^2}\right)\nabla_\rho R_0\nabla^\rho R_0+\frac{1}{3}R_0-\frac{f^{(1)}(R_0)f^{(3)}(R_0)R_0}{3f^{(2)}(R_0)^2}-\frac{f^{(1)}(R_0)}{f^{(2)}(R_0)}+\frac{2f(R_0)f^{(3)}(R_0)}{3f^{(2)}(R_0)^2}-\frac{R_0f^{(3)}}{f^{(2)}(R_0)^2}. \tag{72}$$

If $U(R_0)$ is negative, perturbation R_1 grows exponentially with time. The system becomes instable. In the following, we neglect the terms of the sort $\nabla_\rho R_0$. In the case of the model [3] where $b=\Lambda_1=\Lambda_2=0$,

$$U(R_0)=-R_0+\frac{R_0^3}{6a}. \tag{73}$$

[2]For $m=2$ the well-known anomaly driven (Starobinsky) inflation occurs.

[3]We consider the case wherein the Ricci tensor is not covariantly constant.

[4]The convention of the signature of the spacetime here is different from that in [9].

In order to describe the Universe acceleration in the present epoch, we may have [3]

$$\mu^{-1} \equiv a^{-1/4} \sim 10^{18}\ \text{sec} \sim (10^{-33}\ \text{eV})^{-1}. \tag{74}$$

[As discussed after Eq. (54), if the present Universe corresponds to the power law inflation, a can be much larger than $(10^{-33}\ \text{eV})^4$.] By using the estimations in [9],

$$\frac{R_0^3}{a} \sim (10^{-26}\ \text{sec})^{-2} \left(\frac{\rho_m}{\text{g cm}^{-3}} \right)^3,$$

$$R_0 \sim (10^3\ \text{sec})^{-2} \left(\frac{\rho_m}{\text{g cm}^{-3}} \right), \tag{75}$$

we find that the second term in Eq. (73) dominates. Here ρ_m is the mass density of the gravitating body. Since R_0 is negative from Eq. (69), $U(R_0)$ becomes negative and a microscopic gravitational instability occurs.

First we consider the case in which $n=1$, $m=2$, and $\Lambda_1 = \Lambda_2 = 0$. For

$$b \gg \frac{a}{|R_0^3|}, \tag{76}$$

one gets

$$U(R_0) \sim \frac{R_0}{3} < 0. \tag{77}$$

Then the instability shows up again. We should note, however, that since $|R_0/3| \ll R_0^3/6a$ from Eq. (75), the (macroscopic) instability development takes quite a long time. In fact, the time for the instability to occur is significantly improved (by the order of 10^{29}). Clearly, other higher derivative terms may significantly improve the estimation as is shown below.

Equations (74)–(76) indicate that $b^{-1} \ll (10^{11}\ \text{eV})^2$ if we assume $\rho_m \sim 1\ \text{g cm}^{-3}$. Then $\kappa^2/b \ll (10^{-17})^2$, which does not seem to be consistent with the bounds obtained from the observation [12] (see also [13] where inflation in the higher derivative (HD) gravity was discussed). The action of model in [13,12] contains the Einstein-Hilbert term and the R^2 term. There are two ways to overcome this bound. First of all, one can include the contributions from other terms containing some powers of curvatures and/or the contribution from the matter fields and the quantum effects like conformal anomaly during the inflation. For instance, an account of the conformal anomaly gives the way for the trace anomaly driven inflation. The bound might be changed. From another point of view, as it follows from the discussion after Eq. (54), if the present accelerating Universe corresponds to the power law inflation, a should be much larger than $(10^{-33}\ \text{eV})^4$. If we choose a to be large enough in Eq. (73) so that $a \gg R_0^2$, the first term in Eq. (73), which is positive, dominates. Then the instability [9] does not exist from the very beginning.

For $n=1$, $m=3$, and $\Lambda_1 = \Lambda_2 = 0$ one can assume

$$b \gg \frac{a}{R_0^4} \tag{78}$$

and

$$U(R_0) \sim -\frac{5R_0}{18} + \frac{1}{18bR_0}. \tag{79}$$

Then if

$$b < \frac{1}{5R_0^2}, \tag{80}$$

$U(R_0)$ is positive and the system becomes stable. From Eq. (75), we find $1/R_0^2 \gg a/R_0^4$. Then condition (80) is compatible with assumption (78). Thus, the addition of the terms with the positive powers of curvature may salvage the modified gravity as less instabilities occur or their development is macroscopic and other effects may prevent them.

It has been mentioned in Ref. [11] that the $1/R$ model, which is equivalent to some scalar-tensor gravity, is ruled out as a realistic theory due to the constraints to such theories. Let us study whether or not our theory can pass these constraints. The coupling of the σ-field with matter is always of the same order with gravity [14]. Then if σ field has a small mass, the present model cannot be realistic. One may consider the case that the present universe corresponds to that of solutions (21) with $\Lambda_1 = 0$, which corresponds to a minimum in Eq. (31) of the potential $V(A)$: $R = A = \sqrt{3a}$. We now calculate the square of scalar mass, which is proportional to $V''(\sigma)$. When $A = \sqrt{3a}$

$$\left. \frac{d^2V(\sigma)}{d\sigma^2} \right|_{A=\sqrt{3a}} = \left. \left\{ \left(\frac{d\sigma}{dA} \right)^{-2} \frac{d^2V(A)}{dA^2} \right\} \right|_{A=\sqrt{3a}}$$
$$= \frac{\sqrt{3a}\left(\frac{1}{3} + 2b\sqrt{3a}\right)^2}{\left(-\frac{1}{3} + b\sqrt{3a}\right)\left(\frac{4}{3} + 2b\sqrt{3a}\right)^3}. \tag{81}$$

Choosing

$$\beta \sim \frac{1}{3\sqrt{3a}}, \tag{82}$$

the mass of σ becomes large. Thus, the HD term may help to pass the solar system tests [17].

VI. DISCUSSION

In summary, we considered the modified gravity which naturally unifies two expansion phases of the Universe: inflation at early times and cosmic acceleration at the current epoch. The higher derivative terms with positive power of curvature are dominant at the early Universe providing the inflationary stage. The terms with negative power of curvature serve as a gravitational alternative for dark energy making possible the cosmic speed-up. It is shown that such a theory is stable unlike its more simple counterpart with only the Einstein and $1/R$ terms. Moreover, it may pass the simplest solar system constraint for scalar-tensor gravity equivalent to original modified gravity. Clearly, more checks of this theory should be done in order to conclude whether the model is realistic one or not. In any case, there is some space for modifications by adding the terms with other powers of curvature. Moreover, the use of other curvature invariants (Ricci and Riemann tensors) may be considered as well.

It seems very attractive that modified gravity which may originate from the string or M theory eliminates the need for dark energy. Clearly, deeper investigation in this extremely interesting direction connecting the cosmological constant problem, early time inflationary, and the current accelerating Universe is necessary. It is a challenge to find the solution for above fundamental cosmological problems from the first principles. The search for true gravitational action may be a step in this direction.

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APPENDIX: MODIFICATION OF THE GRAVITY FROM THE INFLATIONARY POTENTIALS

It is very interesting that modified gravity may be often predicted by the standard inflationary cosmology with inflaton field if the corresponding potential is not trivial. Let us consider the model coupled with the scalar field with exponential potential:

$$S=\int d^4x\sqrt{-g}\left(\frac{M_P^2}{16\pi}R-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-V_0 e^{\sqrt{8\pi}\alpha\phi/M_P}\right). \tag{A1}$$

Here M_P is the 4-dimensional Planck mass. It has been shown [15] that if $\alpha\leqslant\sqrt{2}$, the potential becomes shallow and supports the inflation. Recently there appeared some arguments towards to possibility of inflation and cosmic acceleration in the theory based on the braneworld scenario with similar exponential potentials [16]. If the metric is rescaled

$$g_{\mu\nu}\rightarrow e^{(4/M_P)\sqrt{(\pi/3)}\phi}g_{\mu\nu}, \tag{A2}$$

action (A1) can be rewritten as

$$S=\int d^4x\sqrt{-g}\left(\frac{M_P^2}{16\pi}e^{(4/M_P)\sqrt{(\pi/3)}\phi}R - V_0\exp\left[\sqrt{8\pi}\left(\alpha+\sqrt{\frac{8}{3}}\right)\frac{\phi}{M_P}\right]\right). \tag{A3}$$

Since the kinetic term for ϕ vanishes, we may regard ϕ to be an auxiliary field. By the variation over ϕ, one gets

$$0=\frac{M_P}{4\sqrt{3\pi}}e^{(4/M_P)\sqrt{(\pi/3)}\phi}R-\frac{V_0}{M_P}\sqrt{8\pi}\left(\alpha+\sqrt{\frac{8}{3}}\right)\times\exp\left[\sqrt{8\pi}\left(\alpha+\sqrt{\frac{8}{3}}\right)\frac{\phi}{M_P}\right], \tag{A4}$$

which may be solved as

$$\exp\left[\sqrt{8\pi}\left(\alpha+\sqrt{\frac{2}{3}}\right)\frac{\phi}{M_P}\right]=\frac{M_P^2}{8\pi\sqrt{6}(\alpha+\sqrt{\frac{8}{3}})V_0}R.$$

By substituting ϕ, (A5), into action (A3), we obtain

$$S=\frac{\sqrt{6}(\alpha+\sqrt{\frac{2}{3}})V_0}{2}\times\int d^4x\sqrt{-g}\left\{\frac{M_P^2R}{8\pi\sqrt{6}(\alpha+\sqrt{\frac{8}{6}})V_0}\right\}^{(\alpha+\sqrt{8/3})/(\alpha+\sqrt{2/3})}. \tag{A6}$$

Then action (A1) is equivalent to the modified gravity with the fractional power $(\alpha+\sqrt{8/3})/(\alpha+\sqrt{2/3})$ of the scalar curvature. If $-\sqrt{8/3}<\alpha<-\sqrt{2/3}$, the power of the curvature becomes negative again.

Similarly, one can consider other inflationary potentials. It is clear that many of them lead to modification of the gravitational theory, supporting the point of view that inflaton may not be physical but rather some fictitious scalar field.

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