Nonequilibrium evolution of correlation functions: A canonical approach

Supratim Sengupta* and F. C. Khanna†
Theoretical Physics Institute, Department of Physics, University of Alberta, Edmonton, Alberta, Canada T6G 2J1 and TRIUMF, 4004 Wesbrook Mall, Vancouver, British Columbia, Canada V6T 2A3

Sang Pyo Kim‡
Department of Physics, Kunsan National University, Kunsan 573-701, Korea and Asia Pacific Center for Theoretical Physics, Pohang 790-784, Korea

(Received 14 January 2003; revised manuscript received 20 August 2003; published 26 November 2003)

We study nonequilibrium evolution in a self-interacting quantum field theory invariant under space translation only by using a canonical approach based on the recently developed Liouville–von Neumann formalism. The method is first used to obtain the correlation functions both in and beyond the Hartree approximation, for the quantum mechanical analogue of the ϕ^4 model. The technique involves representing the Hamiltonian in a Fock basis of annihilation and creation operators. By separating it into a solvable Gaussian part involving quadratic terms and a perturbation of quartic terms, it is possible to find the improved vacuum state to any desired order. The correlation functions for the field theory are then investigated in the Hartree approximation, and those beyond the Hartree approximation are obtained by finding the improved vacuum state corrected up to $O(\lambda^2)$. These correlation functions take into account next-to-leading and next-to-next-to-leading order effects in the coupling constant. We also use the Heisenberg formalism to obtain the time evolution equations for the equal-time, connected correlation functions beyond the leading order. These equations are derived by including the connected four-point functions in the hierarchy. The resulting coupled set of equations form part of an infinite hierarchy of coupled equations relating the various connected *n*-point functions. The connection with other approaches based on the path integral formalism is established, and the physical implications of the set of equations are discussed with particular emphasis on thermalization.

DOI: 10.1103/PhysRevD.68.105014 PACS number(s): 11.10.Ef, 11.10.Wx, 11.15.Bt

I. INTRODUCTION

In the past few years, a lot of attention has been focused on the investigation of classical and quantum fields evolving out of equilibrium. Such interest is physically well motivated because the very early history of the Universe provides many scenarios where nonequilibrium effects may have played an important role. The reheating of the Universe after inflation [1], the formation and growth of domains in any generic spontaneous symmetry breaking phase transition [2], the formation of topological defects [3–8], and the possible formation of a quark-gluon plasma during the deconfinement transition or disoriented chiral condensates during the chiral phase transition [9] are just some instances where the proper understanding of the physical process may crucially depend on our understanding of nonequilibrium quantum fields. The experimental accessibility of some of these phenomena, such as the formation of a quark-gluon plasma, made possible through heavy-ion colliders at the BNL Relativistic Heavy Ion Collider (RHIC) and CERN Large Hadron Collider (LHC), has also been a strong motivating factor behind the revival of interest in nonequilibrium evolution of quantum fields. The development of new theoretical techniques and the availability of more efficient computational resources have also made it possible to investigate in some detail the nonlinear effects which play such a crucial role in the evolution dynamics.

Another important issue in this context is that of thermalization in closed quantum systems. Is it possible for macroscopic irreversible behavior to manifest itself starting from microscopic reversible (unitary) quantum dynamics [10–13]? In other words, would it be possible for a closed quantum system to thermalize when it is perturbed from its initial thermal state? Can the process of thermalization be adequately described within the mean field description? If not, what is the role of interactions (nonlinearities) in bringing about thermalization? Is it possible to develop a consistent theoretical framework to address these important issues? Some of these questions have only recently begun to be addressed using newly developed theoretical tools for dealing with nonequilibrium quantum fields [10–15].

Until recently, the issue of thermalization of a closed quantum system was typically addressed by a separation of the system into a subsystem made up of the nonthermal soft modes (with longer thermalization time scales) and an environment consisting of the thermal hard modes (which thermalize on much shorter time scales compared to the soft modes) [10–12]. The subsequent interaction between the soft modes and the environment (often treated stochastically) leads to the eventual thermalization of the subsystem made up of soft modes. Mean field theory (Hartree approximation) has also been extensively and successfully used to study the dynamics of nonequilibrium quantum fields [16–19] and has yielded valuable insights into the early-time dynamical behavior. However, since the mean field (Hartree) approxima-

*Electronic address: sengupta@phys.ualberta.ca
†Electronic address: khanna@phys.ualberta.ca
‡Electronic address: sangkim@kunsan.ac.kr

 ©2003 The American Physical Society

tion is essentially a linear approximation, it fails to correctly capture long-time dynamical behavior where nonlinear effects play a dominant role. Moreover, in the mean field approximation, the different field modes interact with a spatially homogeneous mean field with equal strength and therefore the effects of direct scattering (which is responsible for redistribution and eventual equipartition of energy among different modes) are neglected. Hence thermalization of the system cannot be achieved [13,15]. Recently, there have been some attempts [20,21] to explore the possibility of thermalization within the mean field scheme by considering spatially inhomogeneous mean fields. The motivation was to see if the scattering of the fluctuation modes with the modes of the inhomogeneous mean field aids in thermalization. Salle, Smit, and Vink [20] proposed a density matrix describing an ensemble of pure Gaussian initial states. By defining the mean field to be an ensemble average over a set of Gaussian density matrices, and considering the interaction between the quantum fluctuation modes and the inhomogeneous mean field, they showed that approximate thermalization is observed over intermediate time scales in the sense of particle distribution of low-momentum modes approaching a Bose-Einstein form for a spontaneously broken theory. For long times, the particle distributions were found to tend toward a classical Boltzmann form. However, in a contrasting study, Bettencourt *et al.* [21] showed that by considering the dynamics in the presence of a spatially inhomogeneous mean field, but *without* ensemble averaging over a certain class of initial conditions, the Hartree approximation fails to establish a thermal Bose-Einstein (BE) particle distribution even at late times. In a more recent paper [22], Salle and Smit pointed out that the energy densities used in the simulations of [21] were not large enough to ensure even approximate thermalization in the BE sense over the time scales observed. They found that by choosing a large enough energy density and a different set of initial conditions, BE behavior was exhibited by the low-momentum modes at intermediate time scales, even for a symmetric theory. However, even the approximate thermalization time scale was much larger in comparison to that of a spontaneously broken theory. In none of these works was quantum thermalization observed for all modes and all energy densities, over the time scales observed.

In order to understand the complicated process of thermalization of closed quantum field systems, one needs to go beyond the mean field (Hartree or leading order large-N) approximation schemes. This has been attempted recently by many different groups [13–15,23,24]. More recently, time-reversal invariant equations for correlation functions of nonequilibrium scalar fields have been derived [13] based on a three-loop expansion of the two-particle irreducible (2PI) effective action [25,26]. This method has also been extended to O(N) symmetric field theories by carrying out a systematic $1/N$ expansion of the 2PI effective action. The three-loop approximation (for the single real scalar field case) and the next-to-leading-order (NLO) expansion for the O(N) model incorporates the effect of direct scattering and shows promising evidence of late-time thermalization. A detailed analysis of the nonequilibrium dynamics shows the evidence of three distinct temporal regimes [13] characterized by an early-time exponential damping, an intermediate hydrodynamic regime, and a late-time regime manifest through exponential approach to thermal equilibrium. An alternative approximation scheme has been developed by Cooper and collaborators focusing on resummation methods based on the exact Schwinger-Dyson equations [15]. This scheme, called the bare vertex approximation scheme (first studied by Kraichnan [27]) involves obtaining the exact Schwinger-Dyson equations and then neglecting the vertex corrections. All the methods discussed above are based on path-integral techniques within the Keldysh-Schwinger closed time path formalism [28], and the basic dynamical equations can be quite complicated to derive.

Our main aim in this paper is to make use of canonical formalism to study the nonequilibrium evolution of quantum fields. The Liouville–von Neumann (LvN) formalism was first applied (within the Hartree approximation scheme) to study the early-time growth of domains during a quenched second order phase transition [29]. The LvN formalism which solves directly the quantum LvN equation, is another quantum picture [30,31] in addition to the Schrödinger and Heisenberg pictures. Further, it is shown that the LvN formalism provides a convenient and powerful method for analyzing nonequilibrium systems such as time-dependent oscillators [30–32] and quenched phase transitions [29,33–35]. We use this formalism to obtain the Gaussian vacuum and thermal evolution equations in the Hartree approximation. We develop a method for obtaining the improved vacuum state and explicitly show how to obtain equations for the first and second order coefficients which lead to the improved vacuum state correct to $O(\lambda^2)$. We then derive expressions for the two-point functions by taking the expectation value of products of field operators with respect to this improved vacuum state. This method clearly indicates that non-Gaussian effects first make their appearance at $O(\lambda^2)$.

An alternative canonical approach based on the Heisenberg formalism is then employed to obtain a set of nonequilibrium evolution equations for the correlation functions. After clarifying the relation between the LvN formalism and the Heisenberg picture, we take the vacuum expectation values of the Heisenberg equations for all possible combinations of products of field operators and obtain a hierarchy of coupled equations for the ordinary n-point correlators. To obtain the dynamical equations for the connected equal-time correlators beyond the leading order, we make use of the method of cluster expansion, which allows us to express the ordinary n-point correlators in terms of their connected counterparts. This method provides an alternative nonperturbative approach for going beyond the mean field approximation. We compare our results with those obtained in the literature using other methods [24,36]. In this context we consider two recent approaches used to obtain the evolution equations for the partition function [36] and correlators in a quantum mechanical model discussed by Ryzhov and Yaffe (RY) [24]. We show that the canonical approach used in this paper yields the appropriate evolution equations, and thereby establish a connection between various approaches for obtain-

ing the nonequilibrium evolution equations for *equal-time* correlation functions.

The paper is organized as follows. In the next section we briefly outline the Liouville–von Neumann formalism which will be subsequently used to study the nonequilibrium dynamics of the correlation functions in and beyond the Hartree approximation. In Sec. III, we describe in detail a quantum mechanical model of anharmonic oscillators and use the LvN formalism to obtain the evolution equations for the vacuum and thermal correlation functions in the Hartree approximation. In Sec. IV, we develop the LvN formalism to study the non-Gaussian dynamics of the quantum mechanical anharmonic oscillator model. This section sets the stage for the application of the LvN formalism to study non-Gaussian dynamics in the more complicated field theory model. The issue of stability of the LvN method, in the context of the anharmonic oscillator, is discussed in Sec. IV B. In Sec. V, the evolution equations for a *nonlinear* self-interacting scalar field theory are first derived in the Hartree approximation, both for the nonequilibrium as well as for the thermal equilibrium case. Nonequilibrium evolution beyond the leading order is discussed in detail Sec. VI which contains the most important results of this paper. The LvN formalism is used to investigate nonequilibrium dynamics beyond the leading order in Sec. VI A. In Sec. VI B, we make use of the Heisenberg formalism to obtain a hierarchy of nonequilibrium evolution equations for the connected *n*-point functions. A comparison between our approach and other methods used in the literature is carried out in Sec. VII. Section VIII contains a summary of our main results and a discussion of the physical implications of the results in the context of issues like thermalization in quantum field theory and phase transitions. The method of cluster expansion [37] is outlined in Appendix A, where we also write down the set of equations for the connected correlators in configuration space. The equivalence between the RY method and the LvN formalism is established in Appendix B.

II. LIOUVILLE–VON NEUMANN PICTURE FOR TIME-DEPENDENT SYSTEMS

The systems under study in this paper have Hamiltonians whose coupling constants (parameters) depend on time explicitly. These systems describe nonequilibrium processes in the sense that the Hamiltonians do not give the correct density operators $e^{-\beta\hat{H}(t)}/Z_{\mathrm{H}}$. To properly find the Hilbert space and density operators for such time-dependent systems, we have to clarify the picture of the quantum evolution. Let the Hamiltonian be defined in terms of the Schrödinger operators for a quantum mechanical system

$$\hat{H}(t)=\hat{H}(\hat{p}_{\mathrm{S}},\hat{q}_{\mathrm{S}},t), \tag{1}$$

and for a quantum field theory

$$\hat{H}(t)=\hat{H}(\hat{\pi}_{\mathrm{S}},\hat{\phi}_{\mathrm{S}},t). \tag{2}$$

Here the systems depend on time only through time-dependent coupling constants (parameters). In addition to the well-known Schrödinger, Heisenberg, and interaction pictures there is another picture for such nonequilibrium systems [30,31].

First, in the Schrödinger picture, the (functional) Schrödinger equation (in units of $\hbar=1$)

$$i\frac{\partial}{\partial t}|\Psi(t)\rangle=\hat{H}(t)|\Psi(t)\rangle \tag{3}$$

has the exact quantum state

$$|\Psi(t)\rangle=\hat{U}(t)|\Psi\rangle_{\mathrm{S}} \tag{4}$$

determined by the unitary evolution operator

$$i\frac{\partial}{\partial t}\hat{U}(t)=\hat{H}(t)\hat{U}(t). \tag{5}$$

For the time-dependent case, in contrast with the time-independent case, it is not easy to find the evolution operator, which is formally defined as

$$\hat{U}(t)=T\exp\left[-i\int^{t}\hat{H}(t')dt'\right], \tag{6}$$

where T denotes the standard time-ordered operator. Second, the Heisenberg operators

$$\hat{O}_{\mathrm{H}}(t)=\hat{U}^{\dagger}(t)\hat{O}_{\mathrm{S}}\hat{U}(t) \tag{7}$$

satisfy the Heisenberg equation of motion

$$i\frac{\partial}{\partial t}\hat{O}_{\mathrm{H}}(t)+[\hat{H}_{\mathrm{H}}(t),\hat{O}_{\mathrm{H}}(t)]=0, \tag{8}$$

where $\hat{H}_{\mathrm{H}}(t)$ is the time-dependent Hamiltonian Heisenberg operator. The Heisenberg operator $\hat{H}_{\mathrm{H}}(t)$ is simply given by replacing the Schrödinger operators $\hat{p}_{\mathrm{S}}$ and $\hat{q}_{\mathrm{S}}$ by $\hat{p}_{\mathrm{H}}(t)$ and $\hat{q}_{\mathrm{H}}(t)$ in Eq. (1) and $\hat{\pi}_{\mathrm{S}}$ and $\hat{\phi}_{\mathrm{S}}$ by $\hat{\pi}_{\mathrm{H}}(t)$ and $\hat{\phi}_{\mathrm{H}}(t)$ in Eq. (2). However, the explicit form of $\hat{p}_{\mathrm{H}}(t),\hat{q}_{\mathrm{H}}(t)$ and $\hat{\pi}_{\mathrm{H}}(t),\hat{\phi}_{\mathrm{H}}(t)$ requires either the exact knowledge of the evolution operator (6) in advance or the solution to Eq. (8). Third, we may introduce the Liouville operators

$$\hat{O}_{\mathrm{L}}(t)=\hat{U}(t)\hat{O}_{\mathrm{S}}\hat{U}^{\dagger}(t). \tag{9}$$

Thus the Liouville operators evolve the Schrödinger operators backward in time [38]. It follows then that the Liouville operators satisfy the quantum LvN equation

$$i\frac{\partial}{\partial t}\hat{O}_{\mathrm{L}}(t)+[\hat{O}_{\mathrm{L}}(t),\hat{H}(t)]=0. \tag{10}$$

We can show that any eigenstate of the Liouville operator

$$\hat{O}_{\mathrm{L}}(t)|\Psi_{\lambda},t\rangle=\lambda|\Psi_{\lambda},t\rangle, \tag{11}$$

where λ is the eigenvalue of the corresponding Schrödinger operator $\hat{O}_{\mathrm{S}}$, satisfies the Schrödinger equation (3). In fact, it follows that [30]

$$|\Psi(t)\rangle=\sum_{\lambda} C_{\lambda}\exp\left\{ i\int dt\langle\Psi_{\lambda},t|\left[i\frac{\partial}{\partial t}-\hat{H}(t)\right]|\Psi_{\lambda},t\rangle\right\} \times|\Psi_{\lambda},t\rangle \tag{12}$$

where the C_{λ}'s are constants.

The essential idea of the LvN method [29–33] is that the quantum LvN equation provides all the quantum and statistical information of nonequilibrium systems. Technically, the linearity of the LvN equation allows any functional of an operator satisfying the LvN equation to be another operator. Thus we may use some suitable operator $\hat{O}_{\mathrm{L}}(t)$ to define the density operator $\hat{\rho}(t)=e^{-\beta\hat{O}_{\mathrm{L}}(t)}/Z_O$ for the time-dependent system, provided $\hat{O}_{\mathrm{L}}(t)$ satisfies the LvN equation. In this sense the LvN method unifies quantum statistical mechanics with quantum mechanics. The LvN method treats the time-dependent, nonequilibrium system exactly in the same way as the time-independent, equilibrium one. Moreover, the LvN method can be applied to nonequilibrium fermion systems with a minimal modification [39].

III. ANHARMONIC OSCILLATOR IN THE HARTREE APPROXIMATION

As a precursor to the investigation of ϕ^4 field theory, we apply the LvN method to a simple quantum mechanical model of anharmonic oscillators

$$\hat{H}=\frac{1}{2}\hat{p}^2\pm\frac{\omega^2}{2}\hat{x}^2+\frac{\lambda}{4!}\hat{x}^4 \tag{13}$$

and derive the evolution equations for the coherent state expectation value of position and momentum variables as well as the subtracted two-point correlators. The anharmonic oscillator with the lower sign is a quantum mechanical analogue for the second order phase transition. All the time-dependent operators in Secs. III, IV, and V will denote Liouville operators, whose subscript L will be dropped.

The main idea behind the LvN method is to require the pair of invariant operators [32,40] defined as

$$\hat{a}(t)=i[u^*(t)\hat{p}-\dot{u}^*(t)\hat{x}],\quad \hat{a}^{\dagger}(t)=-i[u(t)\hat{p}-\dot{u}(t)\hat{x}] \tag{14}$$

to satisfy the LvN equation, i.e.,

$$i\frac{\partial\hat{a}}{\partial t}+[\hat{a},\hat{H}(t)]=0,\quad i\frac{\partial\hat{a}^{\dagger}}{\partial t}+[\hat{a}^{\dagger},\hat{H}(t)]=0. \tag{15}$$

Here $u(t)$ and $u^*(t)$ are auxiliary variables in terms of which the two-point correlators will be expressed. The invariant operators may be made the annihilation and creation operators satisfying the standard commutation relation at equal times

$$[\hat{a}(t),\hat{a}^{\dagger}(t)]=1, \tag{16}$$

which leads to the Wronskian condition

$$\dot{u}^*(t)u(t)-u^*(t)\dot{u}(t)=i. \tag{17}$$

All the other commutation relations vanish:

$$[\hat{a}(t),\hat{a}(t)]=[\hat{a}^{\dagger}(t),\hat{a}^{\dagger}(t)]=0. \tag{18}$$

Equation (14) can be inverted to express the position and momentum operators $\hat{x}$ and $\hat{p}$ in terms of the annihilation and creation operators as

$$\hat{x}(t)=u(t)\hat{a}(t)+u^*(t)\hat{a}^{\dagger}(t),$$

$$\hat{p}(t)=\dot{u}(t)\hat{a}(t)+\dot{u}^*(t)\hat{a}^{\dagger}(t). \tag{19}$$

The coherent state is defined either as the eigenstate of $\hat{a}(t)$

$$\hat{a}(t)|\alpha,t\rangle=\alpha|\alpha,t\rangle \tag{20}$$

with a complex eigenvalue α or as the displaced state of the vacuum state given by

$$|\alpha,t\rangle=\hat{D}^{\dagger}(\alpha)|0,t\rangle=e^{-\alpha^*\alpha/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n,t\rangle, \tag{21}$$

where $\hat{D}$ is the displacement operator

$$\hat{D}(\alpha)=e^{-\alpha\hat{a}^{\dagger}(t)+\alpha^*\hat{a}(t)}. \tag{22}$$

The coherent state can also be found using the variational principle [41].

The coherent state then leads to the expectation values of $\hat{x}$, $\hat{p}$, $\hat{x}^2$, and $\hat{p}^2$:

$$\overline{x}\equiv\langle\alpha,t|\hat{x}|\alpha,t\rangle=\alpha u(t)+\alpha^*u^*(t),$$

$$\overline{p}\equiv\langle\alpha,t|\hat{p}|\alpha,t\rangle=\alpha\dot{u}(t)+\alpha^*\dot{u}^*(t),$$

$$\langle\alpha,t|\hat{x}^2|\alpha,t\rangle=\overline{x^2}+u^*(t)u(t),$$

$$\langle\alpha,t|\hat{p}^2|\alpha,t\rangle=\overline{p^2}+\dot{u}^*(t)\dot{u}(t). \tag{23}$$

The subtracted two-point correlators [24] defined below are then given by

$$g_{xx}(t)=\langle\hat{x}^2\rangle-\overline{x^2}=u^*(t)u(t),$$

$$g_{pp}(t)=\langle\hat{p}^2\rangle-\overline{p^2}=\dot{u}^*(t)\dot{u}(t),$$

$$g_{xp}(t)=\langle\hat{x}\hat{p}\rangle-\overline{xp}=\dot{u}^*(t)u(t),$$

$$g_{px}(t)=g_{xp}^*(t)=u^*(t)\dot{u}(t), \tag{24}$$

from which we obtain the evolution equations for the two-point correlators,

$$\dot{g}_{xx}(t)=g_{xp}(t)+g_{px}(t),$$

$$\dot{g}_{pp}(t)=\ddot{u}^*(t)\dot{u}(t)+\dot{u}^*(t)\ddot{u}(t),$$

$$\dot{g}_{xp}(t)=\ddot{u}^*(t)u(t)+\dot{u}^*(t)\dot{u}(t). \tag{25}$$

A. Correlation functions in coherent state

The expectation value of the Hamiltonian with respect to the coherent state (20) leads to an effective Hamiltonian

$$H_{\rm eff}\equiv\langle\hat{H}\rangle_{\rm cs}=\frac{1}{2}\overline{p^2}\pm\frac{\omega^2}{2}\overline{x^2}+\frac{\lambda}{4!}\overline{x^4}+\frac{1}{2}g_{xx}^{(2)}\left(\pm\omega^2+\frac{\lambda}{2}\overline{x^2}\right)+\frac{3\lambda}{4!}g_{xx}^{(2)2}+O(\lambda^3). \tag{26}$$

The auxiliary field $u(t)$, after differentiating Eq. (15) with respect to $\hat{x}$ and taking the coherent state expectation value, satisfies the equation

$$\ddot{u}(t)\pm\omega^2u(t)+\langle\alpha,t|\frac{\delta^2\hat{H}}{\delta\hat{x}^2}|\alpha,t\rangle u(t)=0, \tag{27}$$

as does its complex conjugate $u^*(t)$. Inserting the expectation values obtained in Eq. (23) into Eq. (27), one obtains the equation for the complex $u(t)$,

$$\ddot{u}(t)+\left[\pm\omega^2+\frac{\lambda}{2}\overline{x^2}+\frac{\lambda}{2}u^*u\right]u(t)=0. \tag{28}$$

Using the above equation and the definition Eq. (24) of the subtracted two-point correlators, we get the following coupled set of evolution equations for the subtracted two-point correlators:

$$\begin{aligned}\dot{g}_{xx}(t)&=g_{xp}(t)+g_{px}(t),\\ \dot{g}_{pp}(t)&=-\left(\pm\omega^2+\frac{\lambda}{2}\overline{x^2}\right)(g_{xp}+g_{px})+O(\lambda^2),\\ \dot{g}_{xp}(t)&=g_{pp}-g_{xx}\left(\pm\omega^2+\frac{\lambda}{2}\overline{x^2}\right)+O(\lambda^2).\end{aligned} \tag{29}$$

It is important to note that these equations are correct only up to $O(\lambda^2)$ except for the first one, which is exact. This is due to the quartic term appearing in the potential, unlike the case of the simple harmonic oscillator where the quadratic (Gaussian) form of the Hamiltonian leads to the vanishing of all correlators greater than second order. In this case, even if one starts from a Gaussian state peaked at $\overline{x}=0$ for which all the correlators greater than second order vanished, the subsequent evolution of the coupled set of equations would induce the appearance of nonvanishing values for higher order correlators at $O(\lambda^3)$ and higher [24]. This will be discussed later when we obtain the evolution equations for a ϕ^4 field theory beyond the leading order.

B. Correlation functions in thermal state

For the case of the anharmonic oscillator in an initial thermal equilibrium with the positive sign for the unbroken symmetry, the evolution equations for the auxiliary variable and the two-point correlators up to $O(\lambda^2)$ can be obtained by taking the expectation value with respect to the coherent thermal state:

$$\hat{\rho}_{\rm CT}=\frac{1}{Z_{\rm CT}}\exp[-\beta\{\Omega\hat{a}^\dagger(t)\hat{a}(t)+\delta\hat{a}^\dagger(t)+\delta^*\hat{a}(t)+\epsilon_0\}], \tag{30}$$

where $\epsilon_0=\Omega/2+|\delta|^2/(\Omega)$. As $\hat{a}(t)$ and $\hat{a}^\dagger(t)$ approximately satisfy the quantum LvN equation, Eq. (15), so does the density operator (30). We choose Ω to satisfy the gap equation for the unbroken symmetry

$$\Omega^2=\omega^2+\frac{\lambda}{4\Omega}. \tag{31}$$

As discussed in [29], the displacement operator (22) with $\alpha=\delta/(\Omega)$ unitarily transforms the coherent thermal density matrix to a thermal one,

$$\hat{D}^\dagger(\alpha)\hat{\rho}_{\rm CT}\hat{D}(\alpha)=\frac{1}{Z_{\rm T}}\exp[-\beta\Omega\hat{a}^\dagger(t)\hat{a}(t)]=\hat{\rho}_{\rm T}. \tag{32}$$

By making use of the unitary transformation

$$\begin{aligned}\hat{D}^\dagger(\alpha)\hat{a}(t)\hat{D}(\alpha)&=\hat{a}(t)-\alpha,\\ \hat{D}^\dagger(\alpha)\hat{a}^\dagger(t)\hat{D}(\alpha)&=\hat{a}^\dagger(t)-\alpha^*,\end{aligned} \tag{33}$$

it is easy to show that

$$\begin{aligned}\langle\hat{x}^2\rangle_{\rm CT}&=\overline{x^2}+\coth\left(\frac{\beta\Omega}{2}\right)u^*(t)u(t),\\ \langle\hat{p}^2\rangle_{\rm CT}&=\overline{p^2}+\coth\left(\frac{\beta\Omega}{2}\right)\dot{u}^*(t)\dot{u}(t),\end{aligned} \tag{34}$$

where now

$$\begin{aligned}\overline{x}&=\langle\alpha,t|\hat{x}|\alpha,t\rangle=-(\alpha u+\alpha^*u^*),\\ \overline{p}&=\langle\alpha,t|\hat{p}|\alpha,t\rangle=-(\alpha\dot{u}+\alpha^*\dot{u}^*).\end{aligned} \tag{35}$$

The expectation value of Eq. (15) with respect to the coherent thermal state (30) leads to the equation for the complex $u(t)$:

$$\ddot{u}(t)+\left[\pm\omega^2+\frac{\lambda}{2}\overline{x^2}+\frac{\lambda}{2}\coth\left(\frac{\beta\Omega}{2}\right)u^*u\right]u(t)=0. \tag{36}$$

The corresponding thermal two-point correlators are then given by

$$\begin{aligned}g_{{\rm T}xx}(t)&=\langle\hat{x}^2\rangle_{\rm CT}-\overline{x^2}=\coth\left(\frac{\beta\Omega}{2}\right)u^*(t)u(t),\\ g_{{\rm T}pp}(t)&=\langle\hat{p}^2\rangle_{\rm CT}-\overline{p^2}=\coth\left(\frac{\beta\Omega}{2}\right)\dot{u}^*(t)\dot{u}(t),\\ g_{{\rm T}xp}(t)&=\langle\hat{x}\hat{p}\rangle_{\rm CT}-\overline{xp}=\coth\left(\frac{\beta\Omega}{2}\right)\dot{u}^*(t)u(t),\\ g_{{\rm T}px}(t)&=g^*_{{\rm T}xp}(t)=\coth\left(\frac{\beta\Omega}{2}\right)u^*(t)\dot{u}(t),\end{aligned} \tag{37}$$

from which we obtain the evolution equations for the thermal two-point correlators:

$$\dot{g}_{\mathrm{T}xx}(t)=g_{\mathrm{T}xp}(t)+g_{\mathrm{T}px}(t),$$

$$\dot{g}_{\mathrm{T}pp}(t)=-\left[\pm\omega^2+\frac{\lambda}{2}\overline{x^2}+\frac{\lambda}{2}\coth\left(\frac{\beta\Omega}{2}\right)u^*u\right]\times(g_{\mathrm{T}xp}+g_{\mathrm{T}px})+O(\lambda^2),$$

$$\dot{g}_{\mathrm{T}xp}(t)=g_{\mathrm{T}pp}-g_{\mathrm{T}xx}\left[\pm\omega^2+\frac{\lambda}{2}\overline{x^2}+\frac{\lambda}{2}\coth\left(\frac{\beta\Omega}{2}\right)u^*u\right]+O(\lambda^2). \tag{38}$$

In the $T=0$ limit, Eqs. (37) and (38) reduce to the evolution equations (28) and (29). Expressions for the subtracted two-point correlators can then be obtained by solving Eqs. (36) for $u(t)$ perturbatively in powers of λ.

IV. ANHARMONIC OSCILLATOR BEYOND THE HARTREE APPROXIMATION

In this section we apply the LvN method to the anharmonic oscillator in Sec. III, to go beyond the Hartree approximation. Our stratagem is to represent the full Hamiltonian (13) in the Fock space basis (14) and to follow the standard perturbation theory by taking the quadratic terms as an unperturbed part and the quartic terms as a perturbation. For the convenience of computation, we express the representation in normal ordering where all $\hat{a}^\dagger$ stand to the left of $\hat{a}$.

We divide $\hat{H}$ into a Gaussian part $\hat{H}_{\mathrm{G}}$, the quadratic part, and a perturbation Hamiltonian $\hat{H}_{\mathrm{P}}$, the remaining quartic part,

$$\hat{H}=\hat{H}_{\mathrm{G}}+\lambda\hat{H}_{\mathrm{P}}, \tag{39}$$

where

$$\hat{H}_{\mathrm{G}}=\frac{1}{2}:\hat{p}^2:\pm\frac{\omega^2}{2}:\hat{x}^2:+\frac{6\lambda}{4!}\langle\hat{x}^2\rangle_{\mathrm{G}}:\hat{x}^2:+E_{\mathrm{G}}, \tag{40}$$

$$\hat{H}_{\mathrm{P}}=\frac{1}{4!}:\hat{x}^4:. \tag{41}$$

Here E_{G} is the vacuum expectation value

$$E_{\mathrm{G}}=\frac{1}{2}\langle\hat{p}^2\rangle_{\mathrm{G}}\pm\frac{\omega^2}{2}\langle\hat{x}^2\rangle_{\mathrm{G}}+\frac{3\lambda}{4!}\langle\hat{x}^2\rangle_{\mathrm{G}}^2. \tag{42}$$

Using the normal ordered operators

$$:(u\hat{a}+u^*\hat{a}^\dagger)^n:\ =\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}u^{*(n-k)}u^k\hat{a}^{\dagger(n-k)}\hat{a}^k, \tag{43}$$

we obtain the expectation value with respect to the Gaussian vacuum state that is annihilated by $\hat{a}(t)$,

$$E_{\mathrm{G}}(t)=\frac{1}{2}\left[\dot{u}^*\dot{u}\pm\omega^2u^*u+\frac{\lambda}{4}(u^*u)^2\right]. \tag{44}$$

A few comments are in order. First, the separation of the Hamiltonian into quadratic and quartic parts in Eqs. (40) and (41) is reminiscent of the Caswell-Killingbeck method [42], which separates the Hamiltonian into a solvable part and a perturbation. In fact, as we shall show below, the quadratic part is solvable via the LvN method even for explicitly time-dependent systems. Second, the quadratic part (40) involves a term proportional to the coupling constant λ, which makes any perturbation theory based on it reliable even in the strong coupling limit of λ. This term is the same as the Hartree approximation $\hat{q}^4\rightarrow 6\langle\hat{q}^2\rangle\hat{q}^2$. As will be shown below, the wave function(al)s of the Hamiltonian (40) are the same as those from the Gaussian effective potential method by Chang and Stevenson [43]. The vacuum state is also the same as the Hartree approximation. The equivalence of the vacuum state between the LvN and Gaussian effective potential will be shown below (see also Ref. [29]).

The Hartree approximation in Sec. III is equivalent to simply using the truncated quadratic part $\hat{H}_{\mathrm{G}}$ as the unperturbed part. Separating the quadratic part $\hat{H}_{\mathrm{G}}$ is the essence of the Hartree approximation [44] or the variational Gaussian approximation [43]. In fact, the perturbation $\lambda\hat{H}_{\mathrm{P}}$, although $\hat{H}_{\mathrm{G}}$ contains a term of the same order as the perturbation itself. Then the truncated quantum LvN equation

$$i\frac{\partial\hat{a}}{\partial t}+[\hat{a},\hat{H}_{\mathrm{G}}]=0,\quad i\frac{\partial\hat{a}^\dagger}{\partial t}+[\hat{a}^\dagger,\hat{H}_{\mathrm{G}}]=0 \tag{45}$$

leads to the mean field equation

$$\ddot{u}(t)+\left[\pm\omega^2+\frac{\lambda}{2}u^*u\right]u(t)=0. \tag{46}$$

The mean field equation above can also be obtained by minimizing E_{G} in Eq. (44), which proceeds by varying with respect to u^*, using $\delta\dot{u}^*/\delta u^*=\partial/\partial t$, and treating u^* and u independently. The equal-time commutation relation now is guaranteed by the Wronskian condition of Eq. (17). The Gaussian vacuum state is annihilated by $\hat{a}$

$$\hat{a}(t)|0,t\rangle_{\mathrm{G}}=0, \tag{47}$$

and the excited number states are obtained by applying $\hat{a}^\dagger$:

$$|n,t\rangle_{\mathrm{G}}=\frac{\hat{a}^{\dagger n}(t)}{\sqrt{n!}}|0,t\rangle_{\mathrm{G}}. \tag{48}$$

These are the exact quantum states of the time-dependent Schrödinger equation only for $\hat{H}_{\mathrm{G}}$:

$$i\frac{\partial}{\partial t}|n,t\rangle_{\rm G}=\hat{H}_{\rm G}(t)|n,t\rangle_{\rm G}. \tag{49}$$

By fixing the time-dependent phase factor [32] and including the factor from the *c*-number term in Eq. (40), the harmonic wave functions are given by

$$\Psi_{{\rm G},n}(x)=\frac{1}{\sqrt{2^2 n!}}\left(\frac{1}{2\pi u^* u}\right)^{1/4}\left(\frac{u}{\sqrt{u^* u}}\right)^{(2n+1)/2} \times H_n\left(\frac{q}{\sqrt{2u^* u}}\right)\exp\left[\frac{i}{2}\frac{\dot{u}^*}{u^*}q^2+i\frac{\lambda}{8}\int^t (u^* u)^2\right], \tag{50}$$

where $H_n(x)$ are the Hermite polynomials and the last *c*-number term in the exponent comes from the corresponding *c*-number term in Eq. (40). From now on we denote the wave functions (50) by $|n,t\rangle_{\rm G}$ without any loss of generality. These states form an orthonormal basis of the Fock space:

$${}_{\rm G}\langle n,t|m,t\rangle_{\rm G}=\delta_{nm}. \tag{51}$$

A. Beyond the Hartree approximation

To go beyond the Hartree approximation, we need to include the perturbation, which is now given by

$$\hat{H}_{\rm P}=\frac{1}{4!}(u^{*4}\hat{a}^{\dagger 4}+4u^{*3}u\hat{a}^{\dagger 3}\hat{a}+6u^{*2}u^2\hat{a}^{\dagger 2}\hat{a}^2 +4u^* u^3\hat{a}^\dagger\hat{a}^3+u^4\hat{a}^4). \tag{52}$$

There was an attempt in Ref. [45] to solve Eq. (15) for the full Hamiltonian including the perturbation (52) by improving $\hat{a}$ and $\hat{a}^\dagger$. Here we find the improved quantum states by directly solving the Schrödinger equation for the full Hamiltonian. The perturbation excites and deexcites any Gaussian number state $|n,t\rangle_{\rm G}$. As $\{|n,t\rangle_{\rm G}\}$ constitutes a Fock basis, we therefore expand the exact quantum states as [46]

$$|n,t\rangle=\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\lambda^l C^{(l)}_{n;m}(t)|m,t\rangle_{\rm G}, \tag{53}$$

where the lowest order coefficient is

$$C^{(0)}_{n;m}=\delta_{n,m}. \tag{54}$$

Using the fact that any state $|m,t\rangle_{\rm G}$ individually satisfies the Schrödinger equation (49) for $\hat{H}_{\rm G}$, the Schrödinger equation (3) for the full Hamiltonian (13) leads to the set of equations

$$\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} i\lambda^l\dot{C}^{(l)}_{n;m}(t)|m,t\rangle_{\rm G}=\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\lambda^l C^{(l)}_{n;m}(t)\lambda\hat{H}_{\rm P}|m,t\rangle_{\rm G}. \tag{55}$$

Comparing the powers of λ, we finally obtain a hierarchy of dynamical equations for the coefficients

$$\dot{C}^{(l)}_{n;m}(t)=-i\sum_{j=0}^{\infty} C^{(l-1)}_{n;j}(t)\,{}_{\rm G}\langle m,t|\hat{H}_{\rm P}(t)|j,t\rangle_{\rm G}. \tag{56}$$

Another expression for Eq. (53) may be obtained in a compact form using operators $\hat{a}^\dagger$ and $\hat{a}$. For each fixed *m*, we first sum over *l*,

$$C_{n;m}(t)=\sum_{l=0}^{\infty}\lambda^l C^{(l)}_{n;m}(t), \tag{57}$$

and then write any state $|m,t\rangle_{\rm G}$ as either an excited or a deexcited state of the given lowest order state $|n,t\rangle_{\rm G}$, which is realized by applying the creation or annihilation operators a certain number of times. Hence, Eq. (53) can be written by introducing an operator $\hat{U}_1$ as

$$|n,t\rangle=\sum_{m=0}^{\infty} C_{n;m}(t)|m,t\rangle_{\rm G}\equiv\hat{U}_{\rm I}[\hat{a}^\dagger(t),\hat{a}(t);t,\lambda]|n,t\rangle_{\rm G}, \tag{58}$$

and the Schrödinger equation leads to

$$\left[i\frac{\partial}{\partial t}\hat{U}_{\rm I}(t,\lambda)+[\hat{U}_{\rm I}(t,\lambda),\hat{H}_{\rm G}]-\lambda\hat{H}_{\rm P}\right]|n,t\rangle_{\rm G}=0.$$

Using Eq. (45) and technically assuming that all time derivatives act only on *c* numbers but not on the operators $\hat{a}^\dagger$ and $\hat{a}$, we obtain an interaction-picture-like equation for the operator $\hat{U}$:

$$i\frac{\partial}{\partial t}\hat{U}_{\rm I}(t,\lambda)=\lambda\hat{H}_{\rm P}\hat{U}_{\rm I}(t,\lambda). \tag{59}$$

We then obtain the formal solution

$$|n,t\rangle=\hat{U}_{\rm I}(t,\lambda)|n,t\rangle_{\rm G}(n=0,1,2,\ldots), \tag{60}$$

where

$$\hat{U}_{\rm I}(t,\lambda)=T\exp\left[-i\lambda\int\hat{H}_{\rm P}dt\right]. \tag{61}$$

Here *T* denotes a time ordering for the integral and $\hat{a}^\dagger(t)$ and $\hat{a}(t)$ are treated as if they are constant operators.

We now find the improved vacuum state up to any order either by solving Eq. (56) or by acting with the operator in Eq. (61) on the Gaussian vacuum state. For instance, the improved vacuum state to order λ^2 is given by

$$|0,t\rangle_{[2]}=|0,t\rangle_{\rm G}+\lambda\sum_{m=0} C^{(1)}_{0;m}(t)|m,t\rangle_{\rm G} +\lambda^2\sum_{m=0} C^{(2)}_{0;m}(t)|m,t\rangle_{\rm G}, \tag{62}$$

where the only nonvanishing coefficients are

$$C^{(1)}_{0;4}(t) = -i\frac{1}{\sqrt{4!}}\int^t u^{*4}(t'), \tag{63}$$

and

$$C^{(2)}_{0;8}(t) = (-i)^2\frac{\sqrt{70}}{4!}\int^t u^{*4}(t')\int^{t'} u^{*4}(t''),$$

$$C^{(2)}_{0;6}(t) = (-i)^2\frac{\sqrt{5}}{3}\int^t u^{*3}(t')u(t')\int^{t'} u^{*4}(t''),$$

$$C^{(2)}_{0;4}(t) = (-i)^2\frac{3}{\sqrt{4!}}\int^t u^{*2}(t')u^2(t')\int^{t'} u^{*4}(t''),$$

$$C^{(2)}_{0;2}(t) = (-i)^2\frac{1}{3\sqrt{2}}\int^t u^{*}(t')u^3(t')\int^{t'} u^{*4}(t''),$$

$$C^{(2)}_{0;0}(t) = (-i)^2\frac{1}{4!}\int^t u^{4}(t')\int^{t'} u^{*4}(t''). \tag{64}$$

The non-Gaussian nature of the vacuum state (62) can be exploited by calculating the kurtosis (higher moments). The two-point and four-point correlators with respect to the Gaussian vacuum state (50) are

$$\begin{aligned} {}_G\langle 0,t|\hat{x}^2|n,t\rangle_G &= u^*u, \\ {}_G\langle 0,t|\hat{x}^4|0,t\rangle_G &= 3(u^*u)^2, \end{aligned} \tag{65}$$

whereas those with respect to the improved vacuum state (62) are given by

$$\begin{aligned} {}_{[2]}\langle 0,t|\hat{x}^2|0,t\rangle_{[2]} &= u^*u + \lambda^2[\sqrt{2}(C^{(2)}_{0;2}u^2 + C^{(2)*}_{0;2}u^{*2}) \\ &\quad + (C^{(2)*}_{0;0} + C^{(2)}_{0;0} + 9C^{(1)*}_{0;4}C^{(1)}_{0;4})u^*u] \\ &\quad + O(\lambda^3), \end{aligned} \tag{66}$$

$$\begin{aligned} &{}_{[2]}\langle 0,t|\hat{x}^4|0,t\rangle_{[2]} \\ &= 3(u^*u)^2 + \sqrt{4!}\lambda(C^{(1)}_{0;4}u^4 + C^{(1)*}_{0;4}u^{*4}) \\ &\quad + \lambda^2\{\sqrt{4!}(C^{(2)}_{0;4}u^4 + C^{(2)*}_{0;4}u^{*4}) \\ &\quad + 6\sqrt{2}[C^{(2)}_{0;2}(u^*u)u^2 + C^{(2)*}_{0;2}(u^*u)u^{*2}] \\ &\quad + (123C^{(1)*}_{0;4}C^{(1)}_{0;4} + 3C^{(2)*}_{0;0} + 3C^{(2)}_{0;0})(u^*u)^2\} \\ &\quad + O(\lambda^3). \end{aligned} \tag{67}$$

B. Stability of the LvN method

The stability of the LvN method should be checked since it is a time-dependent perturbation theory. That is, any secular coefficient in Eq. (53) should be removed systematically to ensure a physically meaningful solution. For that purpose, we compare the LvN method with the standard perturbation theory for the well-known anharmonic oscillator with unbroken symmetry [positive sign in Eq. (13)]. In that case we find the solution to the auxiliary mean field equation (46),

$$u(t) = \frac{1}{\sqrt{2\Omega}}e^{-i\Omega t}, \tag{68}$$

where Ω is given by the gap equation

$$\Omega^2 = \omega^2 + \frac{\lambda}{4\Omega}. \tag{69}$$

Then the time-dependent wave function (50) is given by

$$\begin{aligned} \Psi_{G,n}(x,t) &= \exp\left[-i\left(\Omega(n+1/2) - \frac{\lambda}{32\Omega^2}\right)t\right] \\ &\quad \times \frac{1}{\sqrt{2^2 n!}}\left(\frac{\Omega}{\pi}\right)^{1/4} H_n(\sqrt{\Omega}q)\exp\left[-\frac{\Omega}{2}q^2\right] \\ &= \exp\left[-i\left(\Omega(n+1/2) - \frac{\lambda}{32\Omega^2}\right)t\right]\Psi_{G,n}(x), \end{aligned} \tag{70}$$

where $\Psi_{G,n}(x)$ denotes the harmonic oscillator wave function.

Substituting the solution (68) into Eqs. (63) and (64), we find the improved vacuum state corrected to $O(\lambda^2)$:

$$\begin{aligned} |0,t\rangle_{[2]} &= \exp\left[-i\left(\frac{\Omega}{2} - \frac{\lambda}{32\Omega^2}\right)t\right]\left[\left(1 + i\frac{\lambda^2}{2^9\times 3\Omega^5}t\right)|0\rangle_G\right. \\ &\quad + \frac{\lambda^2}{2^7\times 3\sqrt{2}\Omega^6}|2\rangle_G - \left(\frac{\lambda}{2^5\sqrt{6}\Omega^3} - \frac{\sqrt{3}\lambda^2}{2^9\sqrt{2}\Omega^6}\right)|4\rangle_G \\ &\quad \left. + \frac{\lambda^2}{2^7\times 3^2\Omega^6}|6\rangle_G + \frac{\sqrt{70}\lambda^2}{2^{12}\times 3\Omega^6}|8\rangle_G\right] + O(\lambda^3). \end{aligned} \tag{71}$$

Note that $C^{(2)}_{0;0}$ originating from the four quanta creation and the subsequent annihilation leads to a secular term increasing as t. This is not a drawback of the LvN method but just a consequence of the time-dependent perturbation theory searching time-dependent states. As the coefficient of $|0,t\rangle_G$ is the first two terms of $\exp[i\lambda^2/(2^9\times 3\Omega^5)t]$, it can be approximately absorbed into the overall time-dependent factor to $O(\lambda^2)$:

$$\exp\left[-i\left(\frac{\Omega}{2} - \frac{\lambda}{32\Omega^2} - \frac{\lambda^2}{2^9\times 3\Omega^5}\right)t\right]. \tag{72}$$

This factor coincides with the time-dependent solution to the full Schrödinger equation obtained from time-independent perturbation method [47], where the corrected energy at $O(\lambda^2)$ is

$$E_{[2]} = \frac{\Omega}{2} - \frac{\lambda}{32\Omega^2} - \frac{\lambda^2}{2^9\times 3\Omega^5} + O(\lambda^3). \tag{73}$$

The higher order terms that come from the creation of even numbers of quanta and subsequent annihilation of equal quanta, or vice versa, also contain secular terms proportional to powers of t depending on the number of such processes. All these terms will provide the correct energy to the Schrödinger equation.

Another way to understand this phase factor and thereby secular terms is to use the formal solution (61). The operator has an exponential form [48]

$$\hat{U}_{\rm I}(t,\lambda)=\exp\Bigg(-i\lambda\int^{t}\hat{H}_{\rm P}(t')+(-i\lambda)^2 \times\Bigg[\int^{t}dt'\hat{H}(t'),\int^{t'}dt''\hat{H}_{\rm P}(t'')\Bigg]+O(\lambda^3)\Bigg). \tag{74}$$

The c-number term from the commutator

$$\exp\Bigg[-\frac{\lambda^2}{(4!)^2}\int^{t}dt'u^4(t')\int^{t'}dt''u^{*4}(t'')[\hat{a}^4,\hat{a}^{\dagger 4}]\Bigg] \rightarrow\exp\Bigg[i\frac{\lambda^2}{2^9\times 3\Omega^5}t\Bigg] \tag{75}$$

is nothing but the phase factor (72). Now the time-dependent vacuum state to $O(\lambda^2)$ *does not* involve any secular term as shown:

$$|0,t\rangle_{[2]}=e^{iE_{[2]}t}\Bigg\{|0\rangle_{\rm G}+e^{-i(\lambda^2/2^9\times 3\Omega^5)t}\Bigg[\frac{\lambda^2}{2^7\times 3\sqrt{2}\Omega^6}|2\rangle_{\rm G} -\Bigg(\frac{\lambda}{2^5\sqrt{6}\Omega^3}-\frac{\sqrt{3}\lambda^2}{2^9\sqrt{2}\Omega^6}\Bigg)|4\rangle_{\rm G}+\frac{\lambda^2}{2^7\times 3^2\Omega^6}|6\rangle_{\rm G} +\frac{\sqrt{70}\lambda^2}{2^{12}\times 3\Omega^6}|8\rangle_{\rm G}\Bigg]\Bigg\}+O(\lambda^3). \tag{76}$$

We thus have shown that the seemingly secular behavior can be removed systematically by taking the proper time-dependent phase factor for the wave function. This phase factor yields the correct energy for the anharmonic oscillator (13) to any desired order. The idea of removing the secular terms of higher order corrections by absorbing them into the corrected energy is equivalent to removing the secular terms by renormalizing the frequency in multiple-scale perturbation theory [49]. We note that the LvN method proves very accurate because the lowest order vacuum state is a Gaussian state that extremizes the Hamiltonian and the corrected vacuum state is expanded in the Fock basis [47]. Further, the LvN method is a powerful tool for finding the quantum states for nonequilibrium systems that are undergoing phase transitions [29]. For the field theory case, the elimination of secular terms can be achieved using multiple-scale perturbation theory and this aspect has been discussed in detail in [33].

There is another kind of instability originating from the dynamics of the system itself. Phase transitions from an explicit symmetry breaking in time provide such a dynamical instability. The anharmonic oscillator (13) whose quadratic potential changes signs from positive to negative can be a quantum mechanical analogue for the phase transition. After the symmetry is broken, the mean field equation

$$\ddot{u}+\Bigg[-\omega^2+\frac{\lambda}{2}u^*u\Bigg]u=0 \tag{77}$$

may have a period when the quadratic term $(-\omega^2)$ dominates over the quartic one $(\lambda u^*u/2)$. Then the u grows exponentially as $u\approx e^{\omega t}/\sqrt{2\omega}$ until the quartic term grows and becomes comparable to the quadratic one. During this period all higher order corrections $C^{(l)}$ beyond the Hartree approximation grow exponentially as powers of u and u^*. This dynamical instability ceases when the state reaches the true vacuum state and oscillates over it. Thus the dynamical instability for a limited period does not cause any serious secular behavior as for the static system.

V. ϕ^4 FIELD THEORY IN THE HARTREE APPROXIMATION

Now we will apply the LvN method to the ϕ^4 field theory, but first we work out correlation functions within the Hartree approximation in this section. The ϕ^4 field theory to be considered in this paper has the Hamiltonian in D space dimensions

$$\hat{H}(t)=\int d^Dx\Bigg[\frac{1}{2}\hat{\pi}^2+\frac{1}{2}(\nabla\hat{\phi})^2+\frac{m^2}{2}\hat{\phi}^2+\frac{\lambda}{4!}\hat{\phi}^4\Bigg], \tag{78}$$

where $\hat{\pi}(x)=\dot{\hat{\phi}}(x)$ is the conjugate momentum operator. We divide the quantum field $\hat{\phi}$ into a classical background and quantum fluctuations over this background, $\hat{\phi}(x)=\phi_c(x)+\hat{\phi}_f(x)$, where the classical background (mean) field $\phi_c(x)$ is in general considered to be spatially homogeneous. Then the Hamiltonian can be decomposed as

$$\hat{H}(t)=H_{\rm c}(t)+\hat{H}_{\rm f}(t)+\hat{H}_{\rm int}(t)+\delta\hat{H}_{\rm int}(t), \tag{79}$$

where

$$H_{\rm c}(t)=\int d^Dx\Bigg[\frac{1}{2}\pi_c^2+\frac{1}{2}(\nabla\phi_c)^2+\frac{m^2}{2}\phi_c^2+\frac{\lambda}{4!}\phi_c^4\Bigg],$$

$$\hat{H}_{\rm f}(t)=\int d^Dx\Bigg[\frac{1}{2}\hat{\pi}_f^2+\frac{1}{2}(\nabla\hat{\phi}_f)^2+\frac{m^2}{2}\hat{\phi}_f^2+\frac{\lambda}{4!}\hat{\phi}_f^4\Bigg],$$

$$\hat{H}_{\rm int}(t)=\int d^Dx\Bigg[\frac{\lambda}{4}\phi_c^2\hat{\phi}_f^2\Bigg],$$

$$\delta\hat{H}_{\rm int}(t)=\int d^Dx\Bigg[\pi_c\hat{\pi}_f+m^2\phi_c\hat{\phi}_f+\frac{\lambda}{3!}\phi_c(\phi_c^2+\hat{\phi}_f^2)\hat{\phi}_f\Bigg]. \tag{80}$$

Here $H_{\rm c}$ and $\hat{H}_{\rm f}$ are the purely classical and quantum parts. We have divided the interaction into $\hat{H}_{\rm int}$ with even powers of $\hat{\phi}_f$ and $\delta\hat{H}_{\rm int}$ with odd powers of $\hat{\pi}_f$ and $\hat{\phi}_f$. The $\phi_c(x)$ is

nonvanishing only for the case when symmetry is spontaneously broken. We treat individually and collectively the modes in the momentum space of quantum fluctuations and the inhomogeneous background field. The momentum modes are given by the Fourier transform of a field (operator) and its inverse transform,

$$F(x)=\int [dk]F(k)e^{ik\cdot x},$$

$$F(k)=\int d^Dx F(x)e^{-ik\cdot x}, \tag{81}$$

where F denotes either $\hat{\phi}_f(x)$ and $\hat{\pi}_f(x)$ or $\phi_c(x)$ and $\pi_c(x)$, and

$$[dk]=\frac{d^Dk}{(2\pi)^D}. \tag{82}$$

Then the quadratic integral takes the form

$$\int d^Dx\hat{F}^2(x)=\int [dk]\hat{F}(k)\hat{F}(-k). \tag{83}$$

The Fourier modes of $\hat{\phi}_f$ and $\hat{\pi}_f$ will be denoted by

$$\hat{\Phi}_k=\int d^Dx\hat{\phi}_f(x)e^{-ik\cdot x},$$

$$\hat{\Pi}_k=\int d^Dx\hat{\pi}_f(x)e^{-ik\cdot x}. \tag{84}$$

The Hermiticity of $\hat{\phi}_f$ and $\hat{\pi}_f$ implies that $\hat{\Phi}_k^\dagger=\hat{\Phi}_{-k}$ and $\hat{\Pi}_k^\dagger=\hat{\Pi}_{-k}$.

The commutation relation of the fields

$$[\hat{\phi}_f(x),\hat{\pi}_f(y)]=i\delta(\mathbf{x}-\mathbf{y}) \tag{85}$$

leads to those of modes in the momentum space,

$$[\hat{\Phi}_{k'},\hat{\Pi}_k]=i(2\pi)^D\delta(k'+k). \tag{86}$$

In terms of the annihilation and creation operators satisfying the equal-time commutation relation

$$[\hat{a}_{k'}(t),\hat{a}_k^\dagger(t)]=(2\pi)^D\delta(k'-k), \tag{87}$$

the momentum space operators $\hat{\Phi}_k$ and $\hat{\Pi}_k$ may be expressed as

$$\hat{\Phi}_k=\varphi_k(t)\hat{a}_k(t)+\varphi_{-k}^*(t)\hat{a}_{-k}^\dagger(t),$$

$$\hat{\Pi}_k=\dot{\varphi}_k(t)\hat{a}_k(t)+\dot{\varphi}_{-k}^*(t)\hat{a}_{-k}^\dagger(t). \tag{88}$$

Here it is assumed that $\varphi_{-k}(t)=\varphi_k(t)$ and

$$\dot{\varphi}_k^*(t)\varphi_k(t)-\varphi_k^*(t)\dot{\varphi}_k(t)=i. \tag{89}$$

Then the annihilation and creation operators are also expressed as

$$\hat{a}_k(t)=i[\varphi_k^*(t)\hat{\Pi}_k-\dot{\varphi}_k^*(t)\hat{\Phi}_k],$$

$$\hat{a}_k^\dagger(t)=-i[\varphi_{-k}(t)\hat{\Pi}_{-k}-\dot{\varphi}_{-k}(t)\hat{\Phi}_{-k}]. \tag{90}$$

Note that the momentum space operators $\hat{\Phi}_k$ and $\hat{\Pi}_k$ are regarded as time-independent ones whereas $\hat{a}_k(t)$ and $\hat{a}_k^\dagger(t)$ are time-dependent Liouville ones in the LvN picture. The Gaussian vacuum is the state annihilated by all $\hat{a}_k(t)$:

$$\hat{a}_k(t)|0,t\rangle_{\mathrm{G}}=0, \tag{91}$$

or the product of the Gaussian vacuum state for each $\hat{a}_k(t)$:

$$|0,t\rangle_{\mathrm{G}}=\prod_k |0_k,t\rangle_{\mathrm{G}}. \tag{92}$$

In the Hartree approximation we consider only those quadratic terms from $\hat{H}_{\mathrm{f}}$ and $\hat{H}_{\mathrm{int}}$ whose Gaussian vacuum expectation values do not vanish. Then the interaction term has the Fourier modes

$$\begin{aligned}\hat{H}_{\mathrm{int}}&=\frac{\lambda}{4}\int [dk_1]\int [dk_2]\int [dk_3]\int [dk_4](2\pi)^D\\&\quad\times\delta(k_1+k_2+k_3+k_4)\phi_c(k_1)\phi_c(k_2)\hat{\Phi}_{k_3}\hat{\Phi}_{k_4}\\&\to\frac{\lambda}{4}\int [dk_1]\int [dk_2](2\pi)^D\\&\quad\times\delta(k_1+k_2)\phi_c(k_1)\phi_c(k_2)\int [dk_3]\hat{\Phi}_{k_3}\hat{\Phi}_{-k_3}\\&=\frac{\lambda}{4}\phi_c^2(x)\int [dk]\hat{\Phi}_k\hat{\Phi}_{-k}.\end{aligned} \tag{93}$$

The quartic term in the Hamiltonian for the fluctuation field ϕ_f can be approximated as

$$\hat{\phi}_f^4\to 6\langle\hat{\phi}_f^2\rangle_{\mathrm{G}}\hat{\phi}_f^2. \tag{94}$$

Then the resulting quadratic part takes the form

$$\begin{aligned}\hat{H}_{\mathrm{G}}=\int [dk]\Bigg[&\frac{1}{2}\hat{\Pi}_k\hat{\Pi}_{-k}+\frac{1}{2}\Bigg(\omega_k^2+\frac{\lambda}{2}\phi_c^2\\&+\frac{\lambda}{2}\int [dk']\langle\hat{\Phi}_{k'}\hat{\Phi}_{-k'}\rangle_{\mathrm{G}}\Bigg)\hat{\Phi}_k\hat{\Phi}_{-k}\Bigg]\quad(\omega_k^2=m^2+k^2).\end{aligned} \tag{95}$$

Under the field redefinition $\hat{\Pi}_{\pm k}=(\hat{\Pi}_k\pm\hat{\Pi}_{-k})/2$ and $\hat{\Phi}_{\pm k}=(\hat{\Phi}_k\pm\hat{\Phi}_{-k})/2$, this Hamiltonian is equivalent to that of harmonic oscillators,

$$\hat{H}_{\rm G}(t)=\sum_{\kappa}\left[\frac{1}{2}\hat{\Pi}^2_{\kappa}(t)+\frac{1}{2}\Omega^2_{\kappa}(t)\hat{\Phi}^2_{\kappa}(t)\right], \tag{96}$$

where

$$\Omega^2_{\kappa}(t)=\omega^2_k+\frac{\lambda}{2}\phi^2_c(x)+\frac{\lambda}{2}\int[dk']\langle\hat{\Phi}_{k'}\hat{\Phi}_{-k'}\rangle_{\rm G}. \tag{97}$$

We may identify the classical background ϕ_c and π_c either with the vacuum expectation values of $\hat{\phi}$ and $\hat{\pi}$ or with the coherent state expectation values of $\hat{\phi}_f$ and $\hat{\pi}_f$, respectively:

$$\phi_c(x)=\langle\hat{\phi}(x)\rangle_{\rm vac}=\langle\hat{\phi}_f(x)\rangle_{\rm cs}\equiv\int[dk]\langle\hat{\Phi}_k\rangle_{\rm cs}e^{ik\cdot x},$$

$$\pi_c(x)=\langle\hat{\pi}(x)\rangle_{\rm vac}=\langle\hat{\pi}_f(x)\rangle_{\rm cs}\equiv\int[dk]\langle\hat{\Pi}_k(t)\rangle_{\rm cs}e^{ik\cdot x}, \tag{98}$$

where $\langle\cdots\rangle_{\rm cs}$ denotes the expectation value taken with respect to the coherent state:

$$\langle\hat{\Phi}_k\rangle_{\rm cs}=\alpha_k\varphi_k(t)+\alpha^*_{-k}\varphi^*_{-k}(t),$$

$$\langle\hat{\Pi}_k\rangle_{\rm cs}=\alpha_k\dot{\varphi}_k(t)+\alpha^*_{-k}\dot{\varphi}^*_{-k}(t). \tag{99}$$

Now the equations for the auxiliary field variables φ_k and φ^*_k can easily be obtained in the Hartree approximation by making use of the LvN equations for $\hat{a}_k(t)$ and $\hat{a}^\dagger_k(t)$ for the Hamiltonian (96). Then the LvN equations

$$i\frac{\partial\hat{a}^\dagger_k(t)}{\partial t}+[\hat{a}^\dagger_k(t),\hat{H}_{\rm G}(t)]=0,$$

$$i\frac{\partial\hat{a}_k(t)}{\partial t}+[\hat{a}_k(t),\hat{H}_{\rm G}(t)]=0 \tag{100}$$

lead to the equations

$$\ddot{\varphi}_k(t)+\left[\omega^2_k+\frac{\lambda}{2}\phi^2_c+\frac{\lambda}{2}\left(\int[dk']\varphi^*_{k'}\varphi_{k'}\right)\right]\varphi_k(t)=0,$$

$$\ddot{\varphi}^*_k(t)+\left[\omega^2_k+\frac{\lambda}{2}\phi^2_c+\frac{\lambda}{2}\left(\int[dk']\varphi^*_{k'}\varphi_{k'}\right)\right]\varphi^*_k(t)=0. \tag{101}$$

The equation for the classical background $\phi_c(t)$ is obtained from the effective classical Hamiltonian $H_{\rm c}(t)+\langle\hat{H}_{\rm int}(t)\rangle_{\rm G}$ as

$$\ddot{\phi}_c(\mathbf{x},t)-\nabla^2\phi_c(\mathbf{x},t)+\left[m^2+\frac{\lambda}{3!}\phi^2_c(\mathbf{x},t)+\frac{\lambda}{2}\left(\int[d^Dk']\varphi^*_{k'}\varphi_{k'}\right)\right]\phi_c(\mathbf{x},t)=0. \tag{102}$$

It is more advantageous to work with the evolution equations for correlation functions, rather than the field equations. First, the two-point correlation functions in thermal equilibrium are related to the Bose-Einstein distribution function (for a quantum theory) and the temperature of the system (for a classical theory) in a fairly simple way; thereby allowing us to use them as benchmarks to track the evolution of the system toward thermal equilibrium. Second, it is more convenient to make systematic improvements to the mean field description, by working with equations for the correlation functions. As we shall see in Sec. VI, the mean field equations for the two-point correlation functions are part of an infinite hierarchy of evolution equations for the connected, equal-time, *n*-point correlators and are obtained by truncating this hierarchy at the level of the two-point functions.

We define the two-point subtracted correlation functions for the fields as

$$\begin{aligned}
g_{11}(x,x';t)&\equiv\langle[\hat{\phi}_f(x)-\phi_c(x)][\hat{\phi}_f(x')-\phi_c(x')]\rangle_{\rm cs}\\
&\equiv\langle\hat{\phi}_f(x)\hat{\phi}_f(x')\rangle_{\rm v},\\
g_{22}(x,x';t)&\equiv\langle[\hat{\pi}_f(x)-\pi_c(x)][\hat{\pi}_f(x')-\pi_c(x')]\rangle_{\rm cs}\\
&\equiv\langle\hat{\pi}_f(x)\hat{\pi}_f(x')\rangle_{\rm v},\\
g_{12}(x,x';t)&\equiv\langle[\hat{\phi}_f(x)-\phi_c(x)][\hat{\pi}_f(x')-\pi_c(x')]\rangle_{\rm cs}\\
&\equiv\langle\hat{\phi}_f(x)\hat{\pi}_f(x)\rangle_{\rm v},\\
g_{21}(x,x';t)&\equiv\langle[\hat{\pi}_f(x)-\pi_c(x)][\hat{\phi}_f(x')-\phi_c(x')]\rangle_{\rm cs}\\
&\equiv\langle\hat{\pi}_f(x)\hat{\phi}_f(x')\rangle_{\rm v}.
\end{aligned} \tag{103}$$

The subscript ''v'' implies the expectation value with respect to the vacuum state. Using the above expressions, we get

$$\begin{aligned}
g_{11}(x,x';t)&=\int[dk]\varphi^*_k(t)\varphi_k(t)e^{ik\cdot(x-x')},\\
g_{22}(x,x';t)&=\int[dk]\dot{\varphi}^*_k(t)\dot{\varphi}_k(t)e^{ik\cdot(x-x')},\\
g_{12}(x,x';t)&=\int[dk]\dot{\varphi}^*_k(t)\varphi_k(t)e^{ik\cdot(x-x')},\\
g_{21}(x,x';t)&=\int[dk]\varphi^*_k(t)\dot{\varphi}_k(t)e^{ik\cdot(x-x')}.
\end{aligned} \tag{104}$$

From Eq. (104), the equal-time two-point correlation functions $G_{ij}(k,t)$ in momentum space can be defined through the Fourier transform

$$g_{ij}(x,x';t)=\int[dk]G_{ij}(k,t)e^{ik\cdot(x-x')}, \tag{105}$$

where $i,j=1,2$. Taking the time derivative of the two-point correlation functions yields the following evolution equations:

$$\dot{g}_{11}(x,x';t)=g_{12}(x,x';t)+g_{21}(x,x';t),$$

$$\dot{g}_{22}(x,x';t)=\int [dk][\ddot{\varphi}_k^*(t)\dot{\varphi}_k(t)+\dot{\varphi}_k^*(t)\ddot{\varphi}_k(t)]e^{ik\cdot(x-x')},$$

$$\dot{g}_{12}(x,x';t)=\int [dk][\ddot{\varphi}_k^*(t)\varphi_k(t)+\dot{\varphi}_k^*(t)\dot{\varphi}_k(t)]e^{ik\cdot(x-x')}. \quad (106)$$

Substituting the expressions for $\ddot{\varphi}_k$ and $\ddot{\varphi}_k^*$ in Eq. (106) and making use of Eq. (105) results in the following evolution equations for the correlation functions in the momentum space in the Hartree approximation up to $O(\lambda^2)$:

$$\dot{G}_{11}(k,t)=G_{12}(k,t)+G_{21}(k,t),$$

$$\dot{G}_{22}(k,t)=-\left[\omega_k^2+\frac{\lambda}{2}\phi_c^2(t)+\frac{\lambda}{2}g_{11}(0,t)\right][G_{12}(k,t)+G_{21}(k,t)]+O(\lambda^2),$$

$$\dot{G}_{12}(k,t)=G_{22}(k,t)-\left[\omega_k^2+\frac{\lambda}{2}\phi_c^2(t)+\frac{\lambda}{2}g_{11}(0,t)\right]G_{11}(k,t)+O(\lambda^2). \quad (107)$$

As shown in [14,23], one can define a quantity $\gamma^{-2}(k)$ in terms of the momentum space two-point correlation functions, which is found to be conserved in the Hartree approximation,

$$\gamma^{-2}(k)=G_{11}(k,t)G_{22}(k,t)-\frac{1}{4}[G_{12}(k,t)+G_{21}(k,t)]^2$$
$$=-\frac{1}{4}(\dot{\varphi}_k^*\varphi_k-\varphi_k^*\dot{\varphi}_k)^2$$
$$=\frac{1}{4}, \quad (108)$$

as a direct consequence of the Wronskian condition, Eq. (17). Issues of renormalization of this model have been investigated in detail in Ref. [50].

It is also possible to define *thermal* two-point functions by taking the expectation value of the field operators with respect to an initial Gaussian thermal state,

$$\langle\hat{\phi}(x,t)\hat{\phi}(x',t)\rangle_T=\frac{1}{\mathcal{Z}}\mathrm{Tr}[e^{-\beta_0\hat{H}_G}\hat{\phi}_f(x)\hat{\phi}_f(x')], \quad (109)$$

where β_0 is the initial inverse temperature of the system and

$$\mathcal{Z}\equiv\mathrm{Tr}[e^{-\beta_0\hat{H}_G}]=\sum_{n_k=0}^{\infty}\langle n_k,t|e^{-\beta_0\Omega_k(\hat{a}_k^\dagger\hat{a}_k+1/2)}|n_k,t\rangle. \quad (110)$$

The corresponding equations for the auxiliary field variables, are again obtained from the LvN equations using the Hamiltonian (96) but with the expectation values taken with respect to the initial *thermal* Gaussian ensemble. This leads to the set of equations

$$\ddot{\varphi}_k(t)+\left[\omega_k^2+\frac{\lambda}{2}\phi_c^2+\frac{\lambda}{2}\left(\int [dk](2n_{k'}+1)\varphi_{k'}^*,\varphi_{k'}\right)\right]\varphi_k(t)=0,$$

$$\ddot{\varphi}_k^*(t)+\left[\omega_k^2+\frac{\lambda}{2}\phi_c^2+\frac{\lambda}{2}\left(\int [dk](2n_{k'}+1)\varphi_{k'}^*,\varphi_{k'}\right)\right]\varphi_k^*(t)=0,$$

$$\ddot{\phi}_c(x,t)-\nabla^2\phi_c(x,t)+\left[m^2+\frac{\lambda}{3!}\phi_c^2(x,t)+\frac{\lambda}{2}\left(\int [dk](2n_{k'}+1)\varphi_{k'}^*,\varphi_{k'}\right)\right]\phi_c(x,t)=0, \quad (111)$$

where $n_k(T)$ is the Bose-Einstein distribution for a field theory system at a temperature $T=1/\beta_0$:

$$n_k(T)=\frac{1}{e^{\beta_0\Omega_k}-1}. \quad (112)$$

Note that at $T=0$ the above set of equations reduces to Eqs. (101) and (102). The expressions for the thermal two-point equal-time correlation functions are also easily obtained:

$$g_{11}^T(x,x';t)\equiv\langle\hat{\phi}(x)\hat{\phi}(x')\rangle_{\rm T}=\int[dk]\coth\left(\frac{\beta_0\Omega_k}{2}\right)\varphi_k^*\varphi_k e^{ik\cdot(x-x')},$$

$$g_{22}^T(x,x';t)\equiv\langle\hat{\pi}(x)\hat{\pi}(x')\rangle_{\rm T}=\int[dk]\coth\left(\frac{\beta_0\Omega_k}{2}\right)\dot{\varphi}_k^*\dot{\varphi}_k e^{ik\cdot(x-x')},$$

$$g_{12}^T(x,x';t)\equiv\langle\hat{\phi}(x)\hat{\pi}(x')\rangle_{\rm T}=\int[dk]\coth\left(\frac{\beta_0\Omega_k}{2}\right)\dot{\varphi}_k^*\varphi_k e^{ik\cdot(x-x')},$$

$$g_{21}^T(x,x';t)\equiv\langle\hat{\pi}(x)\hat{\phi}(x')\rangle_{\rm T}=\int[dk]\coth\left(\frac{\beta_0\Omega_k}{2}\right)\varphi_k^*\dot{\varphi}_k e^{ik\cdot(x-x')},\tag{113}$$

from which it follows that

$$G_{11}^T(k,t)=\coth\left(\frac{\beta_0\Omega_k}{2}\right)\varphi_k^*\varphi_k,$$

$$G_{22}^T(k,t)=\coth\left(\frac{\beta_0\Omega_k}{2}\right)\dot{\varphi}_k^*\dot{\varphi}_k,$$

$$G_{12}^T(k,t)=\coth\left(\frac{\beta_0\Omega_k}{2}\right)\dot{\varphi}_k^*\varphi_k,$$

$$G_{21}^T(k,t)=\coth\left(\frac{\beta_0\Omega_k}{2}\right)\varphi_k^*\dot{\varphi}_k.\tag{114}$$

We can define another conserved quantity for the thermal two-point correlation functions

$$(\gamma^T)^{-2}(k)=G_{11}^T(k,t)G_{22}^T(k,t)-\frac{1}{4}[G_{12}^T(k,t)+G_{21}^T(k,t)]^2=\frac{1}{4}.\tag{115}$$

As in the case of nonequilibrium evolution, it is straightforward to obtain the evolution equation for the *thermal* two-point correlation functions in the Hartree approximation. Using Eqs. (114) and (111), we get

$$\dot{G}_{11}^T(k,t)=G_{12}^T(k,t)+G_{21}^T(k,t),$$

$$\dot{G}_{22}^T(k,t)=-\left[\omega_k^2+\frac{\lambda}{2}\phi_c^2(t)+\frac{\lambda}{2}g_{11}^T(0,t)\right][G_{12}^T(k,t)+G_{21}^T(k,t)]+O(\lambda^2),$$

$$\dot{G}_{12}(k,t)=G_{22}^T(k,t)-\left[\omega_k^2+\frac{\lambda}{2}\phi_c^2(t)+\frac{\lambda}{2}g_{11}^T(0,t)\right]G_{11}^T(k,t)+O(\lambda^2).\tag{116}$$

The above set of equations together with Eq. (111) describe self-consistently the evolution of the *thermal* two-point correlation functions in the Hartree approximation. It describes the evolution of the system initially in thermal equilibrium at a temperature T_0, after interactions are turned on at time $t=0$. They reduce to Eqs. (101) and (107) in the $T=0$ limit. It is also possible to obtain the set of equations (111)–(116) by taking the expectation value of the LvN equations, Eq. (100), with respect to the coherent thermal state with $H_{\rm G}$ being the Gaussian (quadratic) part of the Hamiltonian obtained after Hartree factorization of Eq. (78) (but without decomposing the field into a classical background and fluctuations) using the same techniques employed in the example of the anharmonic oscillator in Sec. III. The initial thermal ensemble can be described adequately by $\hat{H}_{\rm G}$ and the solution to Eq. (111) can be obtained as

$$\varphi_k(t)=\frac{1}{\sqrt{2\Omega_k}}e^{-i\Omega_k t},\tag{117}$$

where Ω_k is given by the gap equation

$$\Omega_k^2=\omega_k^2+\frac{\lambda}{2}\phi_c^2+\frac{\lambda}{4}\int[dk']\frac{1}{\Omega_{k'}}\coth\frac{\beta_0\Omega_{k'}}{2}.\tag{118}$$

For $t=0$, we then have $\varphi_k(0)=1/\sqrt{2\Omega_k}$. The initial thermal two-point correlation functions are then given by

$$G_{11}^T(k,0)=\frac{1}{2\Omega_k}\left[1+\frac{2}{e^{\beta_0\Omega_k}-1}\right],$$

$$G_{22}^T(k,0)=\frac{\Omega_k}{2}\left[1+\frac{2}{e^{\beta_0\Omega_k}-1}\right],$$

$$G_{12}^T(k,0)=\frac{i}{2}\left[1+\frac{2}{e^{\beta_0\Omega_k}-1}\right].\tag{119}$$

It is useful to compare our results with those obtained in [23,51], in the context of thermalization in classical field theories. After neglecting λ-dependent terms, the solution of

Eq. (111) is $\varphi_k(t)=e^{-i\omega_k t}/\sqrt{2\omega_k}$. With this form of φ_k, one can easily show that in the classical limit ($\hbar\to 0$) the two-point correlation functions reduce to

$$G^T_{11}(k,0)=\frac{T_0}{\omega_k^2},$$

$$G^T_{22}(k,0)=T_0,$$

$$G^T_{12}(k,0)=G^{T*}_{21}(k,0)=\frac{iT_0}{\omega_k}. \tag{120}$$

These useful relations, which are valid for a classical thermal system with unbroken symmetry, have been used as a benchmark to study thermalization in classical field theory [23,51]. It is possible to carry out numerically a spectral analysis of two-point, equal-time, field and momenta correlation functions and track their dynamical evolution. The flattening out (momentum independence) of the Fourier transform of the $\pi\pi$ equal-time correlation function would be an indicator of thermalization in classical field theory.

VI. NONEQUILIBRIUM EVOLUTION BEYOND THE LEADING ORDER

In this section we use two different approaches to discuss the nonequilibrium evolution beyond the leading order Hartree approximation. First, we use the LvN formalism to obtain expressions for the two-point and four-point functions correct to $O(\lambda^2)$ by including all quartic terms of the Hamiltonian and solving the time-dependent Schrödinger equation as in Sec. IV. This method unifies both the LvN formalism and the Schrödinger picture because the LvN formalism is used only in the Hartree approximation and all the non-Gaussian contributions from quartic terms are found in the Schrödinger picture. The second stage is similar to the interaction picture, although all states are expressed in terms of the time-dependent Hartree basis. It is found that the two-point functions have a non-Gaussian contribution at order $O(\lambda^2)$, confirming the result that the non-Gaussian effects appear only at $O(\lambda^2)$ [46]. Second, we make use of the Heisenberg formalism to obtain a set of evolution equations for the connected correlation functions beyond the leading order. We find that the set of equations form an infinite hierarchy, akin to the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy in statistical mechanics [52], and incorporating next-to-leading order effects amounts to truncating the hierarchy at the level of the four-point connected correlation functions.

The key differences between the two formalisms lie in the fact that while the LvN formalism (like the Schrödinger formalism) deals with evolution of states (with the field and conjugate momentum operators being time independent), the Heisenberg formalism is associated with evolution of field and conjugate momentum operators (with the quantum states being time independent). However, even in the LvN formalism, certain operators like the annihilation and creation operators and any function of them satisfy the LvN time-evolution equation, distinguished from the Heisenberg equation through a difference in sign. In fact, the LvN equations can be thought of as the backward time evolution of the Heisenberg equations [38]. Furthermore, the LvN approach provides a perturbative method for going beyond the Gaussian approximation and is useful for calculating non-Gaussian effects on domain growth in theories with perturbatively small self-coupling constant [46]. On the other hand, the Heisenberg formalism provides a systematic, nonperturbative approach for going beyond the leading order Hartree approximation in studying the nonequilibrium evolution of quantum fields. Inclusion of connected n-point functions in the hierarchy of evolution equations is effectively equivalent to a loop expansion in powers of $\hbar^{(n-1)}$ [24].

A. LvN formalism: Evolution beyond the leading order

In the LvN formalism discussed in the previous sections, the time-dependent vacuum state is found approximately at the lowest order as the state that is annihilated by the annihilation operators of all momenta in the Hartree approximation which satisfy the LvN equation. Going beyond the leading order amounts to determining the proper vacuum state of the full nonlinear, interacting field theory, which can be expressed in terms of a complete set of number states of the Gaussian Hamiltonian at the Hartree approximation. The Gaussian vacuum state can be improved to any desired order by including the perturbation part. The n-point functions are then obtained by taking the expectation value of the appropriate product of field and momentum operators with respect to the improved vacuum state.

The normal ordered Hamiltonian is decomposed into the quadratic Gaussian part and the quartic perturbation

$$\hat{H}=\hat{H}_{\rm G}+\lambda^2\hat{H}_{\rm P}, \tag{121}$$

where the Gaussian part is

$$\hat{H}_{\rm G}=\int [dk]\left[\frac{1}{2}\left\{\dot{\varphi}_{-k}\dot{\varphi}_k+\left(\omega_k^2+\frac{\lambda}{2}\int [dk_1]\varphi^*_{k_1}\varphi_{k_1}\right)\varphi_{-k}\varphi_k\right\}\hat{a}_{-k}\hat{a}_k+\frac{1}{2}\left\{\dot{\varphi}^*_{-k}\dot{\varphi}^*_k+\left(\omega_k^2+\frac{\lambda}{2}\int [dk_1]\varphi^*_{k_1}\varphi_{k_1}\right)\varphi^*_{-k}\varphi^*_k\right\}\hat{a}^\dagger_{-k}\hat{a}^\dagger_k\right.$$
$$\left.+\left\{\dot{\varphi}^*_k\dot{\varphi}_k+\left(\omega_k^2+\frac{\lambda}{2}\int [dk_1]\varphi^*_{k_1}\varphi_{k_1}\right)\varphi^*_k\varphi_k\right\}\hat{a}^\dagger_k\hat{a}_k\right] \tag{122}$$

and the perturbation

$$\hat{H}_P=\frac{1}{4!}\int [dk_1][dk_2][dk_3][dk_4]\delta(k_1+k_2+k_3+k_4)[\varphi_{k_1}\varphi_{k_2}\varphi_{k_3}\varphi_{k_4}\hat{a}_{k_1}\hat{a}_{k_2}\hat{a}_{k_3}\hat{a}_{k_4}+4\varphi^*_{-k_1}\varphi_{k_2}\varphi_{k_3}\varphi_{k_4}\hat{a}^\dagger_{-k_1}\hat{a}_{k_2}\hat{a}_{k_3}\hat{a}_{k_4}$$
$$+6\varphi^*_{-k_1}\varphi^*_{-k_2}\varphi_{k_3}\varphi_{k_3}\hat{a}^\dagger_{-k_1}\hat{a}^\dagger_{-k_2}\hat{a}_{k_3}\hat{a}_{k_4}+4\varphi^*_{-k_1}\varphi^*_{-k_2}\varphi^*_{-k_3}\varphi_{k_4}\hat{a}^\dagger_{-k_1}\hat{a}^\dagger_{-k_2}\hat{a}^\dagger_{-k_3}\hat{a}_{k_4}+\varphi^*_{-k_1}\varphi^*_{-k_2}\varphi^*_{-k_3}\varphi^*_{-k_4}\hat{a}^\dagger_{-k_1}\hat{a}^\dagger_{-k_2}\hat{a}^\dagger_{-k_3}\hat{a}^\dagger_{-k_4}]. \tag{123}$$

We define the improved vacuum state as

$$|0,t\rangle=\sum_{l=0}^{\infty}\sum_{n_1,n_2,\ldots,}\sum_{k_1,k_2,\ldots}\lambda^l C^{(l)}_{0,n_1,\ldots}|n_1,n_2,\ldots;t\rangle_G \tag{124}$$

where we have used a concise notation to represent n_1 particles with momentum k_1, n_2 particles with momentum k_2, and so on, for the state and the coefficient. The summation over momentum is for modes that have nonvanishing particle numbers. The subscript ''G'' refers to the Gaussian vacuum state. For the rest of this subsection, we will consider the momentum space to be a discrete set and consequently the Dirac delta functions will be replaced by the Kronecker delta functions. As we shall see, the coefficients $C^l_{0,n_1,\ldots}$, which determine the vacuum state of the full nonlinear theory, are not all nonvanishing. These coefficients are determined by requiring that the exact vacuum state $|0,t\rangle$ satisfies the Schrödinger equation for the full Hamiltonian, i.e.,

$$i\frac{\partial}{\partial t}|0;t\rangle=(\hat{H}_G+\lambda\hat{H}_P)|0;t\rangle. \tag{125}$$

Using Eq. (124) we get

$$\sum_{l=0}^{\infty}\sum_{n_1,n_2,\ldots,}\sum_{k_1,k_2,\ldots}i\lambda^l\dot{C}^{(l)}_{0,n_1,\ldots}|n_1,n_2,\ldots;t\rangle_G$$
$$=\sum_{l=0}^{\infty}\sum_{n_1,n_2,\ldots,k_1,k_2,\ldots}\lambda^{l+1}C^{(l)}_{0,n_1,\ldots}\hat{H}_P|n_1,n_2,\ldots;t\rangle_G. \tag{126}$$

Comparing the coefficients of λ^l on both sides, we get $C^{(0)}_{0;n_1,n_2,\ldots,n_j,\ldots}=\delta_{0,n_1}\delta_{0,n_2}\cdots\delta_{0,n_j}\cdots$. Then Eq. (126) reduces to

$$\sum_{l=1}^{\infty}\sum_{n_1,n_2,\ldots,}\sum_{k_1,k_2,\ldots}i\lambda^l\dot{C}^{(l)}_{0,n_1,\ldots}|n_1,n_2,\ldots;t\rangle_G$$
$$=\sum_{l=1}^{\infty}\sum_{n_1,n_2,\ldots,}\sum_{k_1,k_2,\ldots}\lambda^l C^{(l-1)}_{0,n_1,\ldots}\hat{H}_P|n_1,n_2,\ldots;t\rangle_G. \tag{127}$$

Equating the coefficients of the same power of λ on both sides, we get the following equation for the expansion coefficients:

$$\dot{C}^{(l)}_{0;n_1,n_2,\ldots}(t)=-i\sum_{m_{1'},\ldots,m_{j'},\ldots,}\sum_{k'_1,\ldots,k'_j,\ldots}C^{(l-1)}_{0;m_{1'},\ldots m_{j'},\ldots}(t)\,{}_G\langle n_1,\ldots,n_j,\ldots;t|\hat{H}_P|m_{1'},\ldots,m_{j'},\ldots;t\rangle_G. \tag{128}$$

The vacuum state correct to $O(\lambda^2)$ then becomes

$$|0;t\rangle_{[2]}=|0;t\rangle_G+\lambda\sum_{n_1,n_2,\ldots,n_j,\ldots,}\sum_{k_1,k_2,\ldots,k_j,\ldots}C^{(1)}_{0;n_1,\ldots,n_j,\ldots}(t)|n_1,\ldots,n_j,\ldots;t\rangle_G$$
$$+\lambda^2\sum_{n_1,n_2,\ldots,n_j,\ldots,}\sum_{k_1,k_2,\ldots,k_j,\ldots}C^{(2)}_{0;n_1,\ldots,n_j,\ldots}(t)|n_1,\ldots,n_j,\ldots;t\rangle_G. \tag{129}$$

From Eq. (123) and Eq. (128), the only nonvanishing contribution to $C^{(1)}_{0;n_1,\ldots,n_j,\ldots}(t)$ comes from the term $(1/4!)\int[dk_1]\times[dk_2][dk_3][dk_4]\delta(k_1+k_2+k_3+k_4)\varphi^*_{-k_1}\varphi^*_{-k_2}\varphi^*_{-k_3}\varphi^*_{-k_4}\hat{a}^\dagger_{-k_1}\hat{a}^\dagger_{-k_2}\hat{a}^\dagger_{-k_3}\hat{a}^\dagger_{-k_4}$, where the indices correspond to excitations in modes k_1,k_2,k_3. The most general form of the equation for the first order coefficient follows from Eq. (128). It can be written as

$$\dot{C}^{(1)}_{0;\{n_k\}} = -\frac{i}{4!}(N_f)(N_c)\sum_{k_1',k_2'\ldots}\langle n_1,\ldots,n_j,\ldots;t|\times\sum_{k_1'',k_2'',\ldots}\delta_{k_1''+k_2''+k_3''+k_4'',0}\varphi^*_{k_1''}\varphi^*_{k_2''}\varphi^*_{k_3''}\varphi^*_{k_4''}\times\hat{a}^\dagger_{k_1''}\hat{a}^\dagger_{k_2''}\hat{a}^\dagger_{k_3''}\hat{a}^\dagger_{k_4''}|0;t\rangle, \tag{130}$$

where $\{n_k\}\equiv n_{k_1},n_{k_2},n_{k_3},n_{k_4}$ and N_c and N_f are numerical coefficients that depend on the relationship between the modes k_1,k_2,k_3,k_4. N_c corresponds to the factor determined by the maximum power of a creation operator, whereas N_f is a combinatorial factor. It is important to note that *distinct* first order coefficients (with distinct values of n_{k_1},n_{k_2}, etc.) arise when constraints are imposed on the momentum modes k_1,k_2,k_3,k_4 which account for different types of scattering processes. In the absence of any constraints, $N_c=N_f=1$ and Eq. (130) gives

$$\dot{C}^{(1)}_{0;1_{k_1},1_{k_2},1_{k_3},1_{k_4}} = -\frac{i}{4!}\varphi^*_{k_1}\varphi^*_{k_2}\varphi^*_{k_3}\varphi^*_{k_4}\delta_{k_1+k_2+k_3+k_4,0}. \tag{131}$$

When the four modes appearing as indices in the creation operators are paired, resulting in two distinct creation operators (i.e., when $k_2''=k_1''$ and $k_4''=k_3''$), $N_c=(\sqrt{2})^2$ and $N_f=(4_{C_2}/2)\times 2_{C_2}$ (since there are three ways of forming two distinct pairs of creation operators); Eq. (130) becomes

$$\dot{C}^{(1)}_{0;2_{k_1},2_{k_2}} = -\frac{3i}{4!}(\sqrt{2})^2(\varphi^*_{k_1}\varphi^*_{k_2})^2\delta_{k_1+k_2,0}. \tag{132}$$

For a single pairing (i.e., when, for example, $k_2''=k_1''$ but $k_4''\neq k_3''$), $N_c=\sqrt{2}$ and $N_f=4_{C_2}$ (since there are 4_{C_2} ways of forming a single pair out of four creation operators) and Eq. (130) becomes

$$\dot{C}^{(1)}_{0;2_{k_1},1_{k_2},1_{k_3}} = -\frac{6i}{4!}\sqrt{2}(\varphi^{*2}_{k_1})\varphi^*_{k_2}\varphi^*_{k_3}\delta_{2k_1+k_2+k_3,0}. \tag{133}$$

From the form of the string of creation operators appearing in Eq. (130), it is clear that the number of different coefficients correspond to the different ways of partitioning of four-particle excitations in terms of excitations of lower order.

In order to derive non-Gaussian corrections to the two-point functions, it is necessary to obtain the corrected vacuum state, at least up to $O(\lambda^2)$. For this purpose, we need to determine the second order coefficients. Although there are many distinct second order coefficients, we will see that only a few of them will contribute to the two-point functions, thereby considerably simplifying the calculations. Nevertheless, for the sake of completeness, we explicitly outline below the general method for obtaining all possible second order coefficients. The equation for the second order coefficient can be written in the most general form as

$$\dot{C}^{(2)}_{0;\{n_k\}} = -i\sum_{k_1',k_2',\ldots}C^{(1)}_{0;\{m_{k'}\}}\delta_{k_1'+k_2'+k_3'+k_4',0}\times\langle n_1,\ldots,n_j,\ldots;t|\hat{H}_P\hat{a}^\dagger_{k_1'}\hat{a}^\dagger_{k_2'}\hat{a}^\dagger_{k_3'}\hat{a}^\dagger_{k_4'}|0;t\rangle. \tag{134}$$

It is clear from the string of annihilation and creation operators appearing in each term in Eq. (123) that the second order coefficients can be divided into five classes corresponding to zero-, two-, four-, six-, and eight-particle excitations, respectively. The zero-particle excitation case leads to just one second order coefficient given by the equation

$$\dot{C}^{(2)}_{0;0} = -\frac{i}{4!}\sum_{k_1',k_2',\ldots}C^{(1)}_{0;1_{k_1'},1_{k_2'},1_{k_3'},1_{k_4'}}\delta_{k_1'+k_2'+k_3'+k_4',0}\times\langle n_1,\ldots,n_j,\ldots;t|\sum_{k_1'',k_2'',\ldots}\delta_{k_1''+k_2''+k_3''+k_4'',0}\varphi_{k_1''}\times\varphi_{k_2''}\varphi_{k_3''}\varphi_{k_4''}\hat{a}_{k_1''}\hat{a}_{k_2''}\hat{a}_{k_3''}\hat{a}_{k_4''}\hat{a}^\dagger_{k_1'}\hat{a}^\dagger_{k_2'}\hat{a}^\dagger_{k_3'}\hat{a}^\dagger_{k_4'}|0;t\rangle. \tag{135}$$

By making use of the commutation relations between the annihilation and creation operators to normal order the above string of operators, it can be easily shown that the above equation reduces to

$$\dot{C}^{(2)}_{0;0} = -iC^{(1)}_{0;1_{k_1},1_{k_2},1_{k_3},1_{k_4}}\delta_{k_1+k_2+k_3+k_4,0}\varphi_{k_1}\varphi_{k_2}\varphi_{k_3}\varphi_{k_4}. \tag{136}$$

The second order coefficients corresponding to the two-particle excitation case, which is contributed by the second term in Eq. (123), can be written in general as

$$\dot{C}^{(2)}_{\{2_k\};0} = -4i\sum_{k_1',k_2',\ldots}C^{(1)}_{0;1_{k_1'},1_{k_2'},1_{k_3'},1_{k_4'}}\delta_{k_1'+k_2'+k_3'+k_4',0}\times\langle n_1,\ldots,n_j,\ldots;t|\sum_{k_1'',k_2'',\ldots}\delta_{-k_1''+k_2''+k_3''+k_4'',0}\times\delta_{k_2',k_2''}\delta_{k_3',k_3''}\delta_{k_4',k_4''}\varphi^*_{k_1''}\varphi_{k_2''}\varphi_{k_3''}\varphi_{k_4''}\hat{a}^\dagger_{k_1''}\hat{a}^\dagger_{k_1'}|0;t\rangle, \tag{137}$$

which leads to the equations for the following second-order coefficient

$$\dot{C}^{(2)}_{0;1_{k_1},1_{k_5}} = -4iC^{(1)}_{0;1_{k_1},1_{k_2},1_{k_3},1_{k_4}}\delta_{k_1+k_2+k_3+k_4,0}\times\delta_{k_2+k_3+k_4-k_5,0}\varphi^*_{k_5}\varphi_{k_2}\varphi_{k_3}\varphi_{k_4}. \tag{138}$$

The third term in Eq. (123) leads to the following general equation for the second-order coefficients associated with the four-particle excitation case:

$$\dot{C}^{(2)}_{\{4_k\};0} = -3i \sum_{k_1',k_2',\ldots} C^{(1)}_{0;1_{k_1'},1_{k_2'},1_{k_3'},1_{k_4'}} \delta_{k_1'+k_2'+k_3'+k_4',0} \times \langle n_1,\ldots,n_j,\ldots;t| \sum_{k_1'',k_2'',\ldots} \delta_{-k_1''-k_2''+k_3''+k_4'',0} \times \delta_{k_3',k_3''}\delta_{k_4',k_4''}\varphi^*_{k_1''}\varphi^*_{k_2''}\varphi_{k_3''}\varphi_{k_4''}\hat{a}^\dagger_{k_1''}\hat{a}^\dagger_{k_2''}\hat{a}^\dagger_{k_1'}\hat{a}^\dagger_{k_2'}|0;t\rangle, \tag{139}$$

which leads to the dynamical equations for the following four second-order coefficients:

$$\begin{aligned}
\dot{C}^{(2)}_{0;2_{k_1},2_{k_2}} &= -9i(\sqrt{2})^2 C^{(1)}_{0;1_{k_1},1_{k_2},1_{k_3},1_{k_4}} \times \delta_{k_1+k_2+k_3+k_4,0}\delta_{-k_1+k_2+k_3+k_4,0} \times \varphi^*_{k_1}\varphi^*_{k_2}\varphi_{k_3}\varphi_{k_4},\\
\dot{C}^{(2)}_{0;2_{k_1},1_{k_2},1_{k_5}} &= -18i\sqrt{2} C^{(1)}_{0;1_{k_1},1_{k_2},1_{k_3},1_{k_4}} \times \delta_{k_1+k_2+k_3+k_4,0}\delta_{-k_1+k_3+k_4-k_5,0} \times \varphi^*_{k_1}\varphi^*_{k_5}\varphi_{k_3}\varphi_{k_4},\\
\dot{C}^{(2)}_{0;1_{k_1},1_{k_2},1_{k_5},1_{k_6}} &= -3i C^{(1)}_{0;1_{k_1},1_{k_2},1_{k_3},1_{k_4}} \delta_{k_1+k_2+k_3+k_4,0} \times \delta_{k_3+k_4-k_5-k_6,0}\varphi^*_{k_5}\varphi^*_{k_6}\varphi_{k_3}\varphi_{k_4},\\
\dot{C}^{(2)}_{0;3_{k_1},1_{k_2}} &= -12i(\sqrt{6}) C^{(1)}_{0;1_{k_1},1_{k_2},1_{k_3},1_{k_4}} \times \delta_{-2k_1+k_3+k_4,0}\delta_{k_1+k_2+k_3+k_4,0} \times \varphi^{*2}_{k_1}\varphi_{k_3}\varphi_{k_4}.
\end{aligned} \tag{140}$$

Similarly, the second order coefficients corresponding to the six- and eight-particle excitations contributed by the last two terms in Eq. (123), respectively, can be easily obtained. The general structure of the second order coefficients corresponding to the six-particle excitation case is given by

$$\dot{C}^{(2)}_{\{6_k\};0} = -\frac{2i}{3} \sum_{k_1',k_2',\ldots} C^{(1)}_{0;1_{k_1'},1_{k_2'},1_{k_3'},1_{k_4'}} \delta_{k_1'+k_2'+k_3'+k_4',0} \times \langle n_1,\ldots,n_j,\ldots;t| \sum_{k_1'',k_2'',\ldots} \delta_{-k_1''-k_2''-k_3''+k_4'',0} \times \delta_{k_4',k_4''}\varphi^*_{k_1''}\varphi^*_{k_2''}\varphi^*_{k_3''}\varphi_{k_4''}\hat{a}^\dagger_{k_1''}\hat{a}^\dagger_{k_2''}\hat{a}^\dagger_{k_3''}\hat{a}^\dagger_{k_1'}\hat{a}^\dagger_{k_2'}\hat{a}^\dagger_{k_3'}|0;t\rangle, \tag{141}$$

and that associated with the eight-particle excitation case is given by

$$\dot{C}^{(2)}_{\{8_k\};0} = -\frac{i}{4!} \sum_{k_1',k_2',\ldots} C^{(1)}_{0;1_{k_1'},1_{k_2'},1_{k_3'},1_{k_4'}} \delta_{k_1'+k_2'+k_3'+k_4',0} \times \langle n_1,\ldots,n_j,\ldots;t| \sum_{k_1'',k_2'',\ldots} \delta_{k_1''+k_2''+k_3''+k_4'',0}\varphi^*_{k_1''} \times \varphi^*_{k_2''}\varphi^*_{k_3''}\varphi^*_{k_4''}\hat{a}^\dagger_{k_1''}\hat{a}^\dagger_{k_2''}\hat{a}^\dagger_{k_3''}\hat{a}^\dagger_{k_4''}\hat{a}^\dagger_{k_1'}\hat{a}^\dagger_{k_2'}\hat{a}^\dagger_{k_3'}\hat{a}^\dagger_{k_4'}|0;t\rangle. \tag{142}$$

However, none of those coefficients contribute to the expressions for the two-point functions correct to $O(\lambda^2)$. Hence, we do not list them explicitly.

The normalization of the vacuum state correct to second order in λ is easily found in terms of the first and second order coefficients:

$$\begin{aligned}
{}_{[2]}\langle 0;t|0;t\rangle_{[2]} = 1+\lambda^2\Bigg[\Bigg\{ &\sum_{k_1,k_2} C^{(1)*}_{0;2_{k_1},2_{k_2}} C^{(1)}_{0;2_{k_1},2_{k_2}} + \sum_{k_1,k_2,k_3} C^{(1)*}_{0;2_{k_1},1_{k_2},1_{k_3}} C^{(1)}_{0;2_{k_1},1_{k_2},1_{k_3}} \\ &+ \sum_{k_1,k_2,k_3,k_4} C^{(1)*}_{0;1_{k_1},1_{k_2},1_{k_3},1_{k_4}} C^{(1)}_{0;1_{k_1},1_{k_2},1_{k_3}1_{k_4}} \Bigg\} + \{C^{(2)*}_{0;0} + C^{(2)}_{0;0}\}\Bigg] + O(\lambda^3).
\end{aligned} \tag{143}$$

We can now obtain expressions for the equal-time, two-point functions by taking the expectation values with respect to this improved, albeit perturbative, vacuum state, with the understanding that

$$G^{(2)}_{ab}(k) = {}_{[2]}\langle 0;t|\hat{\Phi}_a(k)\hat{\Phi}_b(-k)|0;t\rangle_{[2]}, \tag{144}$$

where the subscript indices $a,b,\ldots=1,2$ and $\hat{\Phi}_a$ is to be understood as the field operator $\hat{\Phi}(k)(a=1)$ or the conjugate momentum operator $\hat{\Pi}(k)(a=2)$. Moreover, we will be working with the symmetric theory which implies that $\phi_c=0$. It is important to note that in the LvN formalism the field and momentum operators are *time independent*. The time dependence of the two-point functions arise through the time dependence of the auxiliary field variable φ. It is then easy to show that the equal-time, two-point functions are given up to and including $O(\lambda^2)$ by the equations

$$G_{11}^{(2)}(k;t)=\varphi_k^*\varphi_k\Bigg[1+\lambda^2\Bigg\{\sum_{k_1,k_2}C^{(1)*}_{0;2_{k_1},2_{k_2}}C^{(1)}_{0;2_{k_1},2_{k_2}}+\sum_{k_1,k_2,k_3}C^{(1)*}_{0;2_{k_1},1_{k_2},1_{k_3}}C^{(1)}_{0;2_{k_1},1_{k_2},1_{k_3}}$$
$$+\sum_{k_1,k_2,k_3,k_4}C^{(1)*}_{0;1_{k_1},1_{k_2},1_{k_3},1_{k_4}}C^{(1)}_{0;1_{k_1},1_{k_2},1_{k_3},1_{k_4}}+(C^{(2)*}_{0;0}+C^{(2)}_{0;0})\Bigg\}\Bigg]$$
$$+\lambda^2\Bigg[2C^{(2)}_{0;1_{-k},1_k}\varphi_{-k}\varphi_k+4\sum_{k_1}(C^{(1)*}_{0;2_{-k},2_{k_1}}C^{(1)}_{0;2_{-k},2_{k_1}}\varphi^*_{-k}\varphi_{-k}+C^{(1)*}_{0;2_k,2_{k_1}}C^{(1)}_{0;2_k,2_{k_1}}\varphi^*_k\varphi_k)\Bigg]+O(\lambda^3), \tag{145}$$

$$G_{21}^{(2)}(k;t)=\dot{\varphi}_k^*\varphi_k\Bigg[1+\lambda^2\Bigg\{\sum_{k_1,k_2}C^{(1)*}_{0;2_{k_1},2_{k_2}}C^{(1)}_{0;2_{k_1},2_{k_2}}+\sum_{k_1,k_2,k_3}C^{(1)*}_{0;2_{k_1},1_{k_2},1_{k_3}}C^{(1)}_{0;2_{k_1},1_{k_2},1_{k_3}}$$
$$+\sum_{k_1,k_2,k_3,k_4}C^{(1)*}_{0;1_{k_1},1_{k_2},1_{k_3},1_{k_4}}C^{(1)}_{0;1_{k_1},1_{k_2},1_{k_3},1_{k_4}}+(C^{(2)*}_{0;0}+C^{(2)}_{0;0})\Bigg\}\Bigg]$$
$$+\lambda^2\Bigg[2C^{(2)}_{0;1_{-k},1_k}\dot{\varphi}_{-k}\varphi_k+4\sum_{k_1}(C^{(1)*}_{0;2_{-k},2_{k_1}}C^{(1)}_{0;2_{-k},2_{k_1}}\dot{\varphi}^*_{-k}\varphi_{-k}+C^{(1)*}_{0;2_k,2_{k_1}}C^{(1)}_{0;2_k,2_{k_1}}\varphi^*_k\dot{\varphi}_k)\Bigg]+O(\lambda^3), \tag{146}$$

$$G_{22}^{(2)}(k;t)=\dot{\varphi}_k^*\dot{\varphi}_k\Bigg[1+\lambda^2\Bigg\{\sum_{k_1,k_2}C^{(1)*}_{0;2_{k_1},2_{k_2}}C^{(1)}_{0;2_{k_1},2_{k_2}}+\sum_{k_1,k_2,k_3}C^{(1)*}_{0;2_{k_1},1_{k_2},1_{k_3}}C^{(1)}_{0;2_{k_1},1_{k_2},1_{k_3}}$$
$$+\sum_{k_1,k_2,k_3,k_4}C^{(1)*}_{0;1_{k_1},1_{k_2},1_{k_3},1_{k_4}}C^{(1)}_{0;1_{k_1},1_{k_2},1_{k_3},1_{k_4}}+(C^{(2)*}_{0;0}+C^{(2)}_{0;0})\Bigg\}\Bigg]$$
$$+\lambda^2\Bigg[2C^{(2)}_{0;1_{-k},1_k}\dot{\varphi}_{-k}\dot{\varphi}_k+4\sum_{k_1}(C^{(1)*}_{0;2_{-k},2_{k_1}}C^{(1)}_{0;2_{-k},2_{k_1}}\dot{\varphi}^*_{-k}\dot{\varphi}_{-k}+C^{(1)*}_{0;2_k,2_{k_1}}C^{(1)}_{0;2_k,2_{k_1}}\dot{\varphi}^*_k\dot{\varphi}_k)\Bigg]+O(\lambda^3). \tag{147}$$

It is clearly evident from this set of equations that the first term corresponds to the Hartree approximation result, and corrections to the Hartree approximation appear only at $O(\lambda^2)$. The auxiliary field mode variable φ_k is given by the solution of Eq. (101) with $\lambda=0$. The effect of scattering enters through the presence of the first and second order coefficients, which first appear at $O(\lambda^2)$. For example, four-particle scattering processes with overall momentum conservation are encoded in terms involving the first order coefficients $C^{(1)}_{0;1_{k_1},1_{k_2},1_{k_3},1_{k_4}}$. The LvN formalism thus provides a systematic perturbative method for computing non-Gaussian (beyond Hartree) corrections to the two-point function and higher n-point functions. Systematic corrections in powers of $\lambda\geqslant 3$ can also be derived in a similar manner after including terms of $O(\lambda^n)(n\geqslant 3)$ in the expression for the improved vacuum state. These expressions for the two-point functions are particularly suited to obtain non-Gaussian corrections to the domain size in systems undergoing a quenched second order phase transition. The four-point function and higher n-point functions in momentum space can also be computed in a similar manner, in terms of the auxiliary field modes and the expansion coefficients up to $O(\lambda^2)$. The λ-independent terms in the four-point functions correspond to the factorization of the four-point function in terms of products of two-point functions and thereby represent the Hartree approximation result.

B. Heisenberg formalism: Evolution beyond the leading order

In this subsection, we obtain the nonequilibrium evolution equations for the connected correlators, by taking the vacuum expectation value of the Heisenberg equations of motion for various combinations of products of the field and its conjugate momentum operators, after making use of the cluster expansion to express the ordinary n-point functions as sums of products of connected n-point functions of lower order. This technique was used to obtain the effective potential and investigate phase transitions in spontaneously broken ϕ^4 theory in $1+1$ and $2+1$ dimensions [37]. We note from Sec. II that the expectation value of a functional of Heisenberg operators with respect to the vacuum state becomes

$$\langle 0|\hat{F}_{\rm H}(t)|0\rangle=\langle 0,t|\hat{F}_{\rm S}|0,t\rangle, \tag{148}$$

where $\hat{F}_{\rm H}(t)$ is any general function(al) of operators in the Heisenberg picture and $\hat{F}_{\rm S}$ is the corresponding time-independent operator in the Schrödinger picture.

From now on, to take into account NLO effects and compare with other methods, we adopt an approach different from the LvN formalism. We take the expectation value of the Heisenberg equation for any functional of $\hat{\phi}_{\rm H}(t)$ and $\hat{\pi}_{\rm H}(t)$ as follows:

$$\langle 0|i\frac{\partial}{\partial t}\hat{F}_{\rm H}(t)|0\rangle=\langle 0|[\hat{F}_{\rm H}(t),\hat{H}_{\rm H}(t)]|0\rangle, \tag{149}$$

where $\hat{H}_{\rm H}(t)$ and $\hat{F}_{\rm H}(t)$ are time-dependent operators and $|0\rangle$ is the time-independent vacuum state. Even though the explicit form of $\hat{H}_{\rm H}(t)$ and $\hat{F}_{\rm H}(t)$ in terms of the Schrödinger operators requires knowledge of the unitary evolution operator, we will not be required to know their explicit forms for the purpose of the calculation below.

In general, the evolution equations for the ordinary n-point correlators which are defined as

$$g^{(n)}_{a,\ldots,b}(\mathbf{x}_1,\ldots,\mathbf{x}_n;t)=\langle\hat{\phi}_a(\mathbf{x}_1,t)\cdots\hat{\phi}_b(\mathbf{x}_n,t)\rangle \equiv\langle\hat{F}_{{\rm H};a,\ldots,b}(\mathbf{x}_1,\ldots,\mathbf{x}_n;t)\rangle \qquad (150)$$

are obtained by taking the vacuum expectation value of the Heisenberg equation (after dropping the subscript ''H'')

$$\frac{d\langle\hat{F}_{a,\ldots,b}\rangle}{dt}=\frac{1}{i}\langle[\hat{F}_{a,\ldots,b}(\mathbf{x}_1,\ldots,\mathbf{x}_n;t),\hat{H}]\rangle, \qquad (151)$$

where the subscript indices $a,b,\ldots=1,2$ for our model Hamiltonian and $\hat{\phi}_a$ is to be understood as the fluctuation field operator $\hat{\phi}_f(\mathbf{x},t)(a=1)$ or the conjugate momentum operator $\hat{\pi}_f(\mathbf{x},t)(a=2)$ in the Heisenberg picture. The subscript f has been removed for notational convenience. The fluctuation field and its conjugate momenta satisfy the usual commutation relations

$$[\hat{\phi}_a(\mathbf{x},t),\hat{\phi}_b(\mathbf{y},t)]=i\,\delta_{ab}\delta^D(\mathbf{x}-\mathbf{y}). \qquad (152)$$

For the unbroken symmetry case, $\phi_c=0$, only correlation functions of even order are nonvanishing. (For a theory exhibiting spontaneous symmetry breaking, a nonvanishing vacuum expectation value will induce a cubic interaction term because of which even *odd* n-point functions become nontrivial.) Using the cluster expansion [37] to express ordinary n-point functions in terms of the equal-time, connected n-point correlators, it is possible to obtain a set of evolution equations for the connected correlators. Because of the presence of the quartic coupling, the equations for the two-point functions would depend upon the connected four-point functions; the equations for the four-point functions would depend upon the six-point functions, and so on, thereby yielding an infinite hierarchy of evolution equations for the n-point correlators. To go beyond the Hartree approximation requires including the cluster expansion of the six-point function in terms of products of connected n-point functions of lower order and ignoring the effect of the connected six-point function in the expansion. This amounts to taking into account direct scattering effects and provides a systematic way of going beyond the leading-order mean field expansion. This then yields a set of closed equations which have been truncated at the four-point level. The leading order (Hartree approximation) result is easily recovered by expressing the four-point function as products of connected two-point functions and ignoring the term involving the connected four-point function in the cluster expansion. This yields a closed set of equations for the equal-time, connected two-point functions. In this scheme, incorporating NNLO effects would then amount to truncating the hierarchy of evolution equations at the six-point level. Appendix A lists the cluster expansion of n-point functions ($n\leqslant 4$) and the set of evolution equations for the equal-time, connected n-point functions ($n\leqslant 4$) in configuration space. It is often more convenient to work with the Fourier transforms of the evolution equations. For that purpose, we define the Fourier transforms of the two- and four-point functions as

$$g^{(2C)}_{ab}(x_1,x_2)=\int[dk_1][dk_2]\tilde{G}^{(2C)}_{ab}(k_1,k_2)\times e^{i(k_1\cdot x_1+k_2\cdot x_2)},$$

$$g^{(4C)}_{abcd}(x_1,x_2,x_3,x_4)=\int[dk_1][dk_2][dk_3][dk_4]\times\tilde{G}^{(4C)}_{abcd}(k_1,k_2,k_3,k_4)\times\exp\left[i\left(\sum_{i=1}^{i=4}k_i\cdot x_i\right)\right]. \qquad (153)$$

For translationally invariant theories,

$$\tilde{G}^{(2C)}_{ab}(k_1,k_2)=G^{(2C)}_{ab}(k_1)\,\delta(k_1+k_2),$$

$$\tilde{G}^{(4C)}_{abcd}(k_1,k_2,k_3,k_4)=G^{(4C)}_{abcd}(k_1,k_2,k_3)\,\delta(k_1+k_2+k_3+k_4), \qquad (154)$$

which is just an indication of the conservation of momentum at the vertices. The Fourier transform of Eq. (A1) and Eqs. (A3)–(A7) yields the following evolution equations for $G^{(2C)}_{ab}(k)$:

$$\dot{G}^{(2C)}_{11}(k_1)=G^{(2C)}_{12}(k_1)+G^{(2C)}_{21}(k_1),$$

$$\dot{G}^{(2C)}_{21}(k_1)=G^{(2C)}_{22}(k_1)-\left[\omega^2(k_1)+\frac{\lambda}{2}\int[dk']G^{(2C)}_{11}(k')\right]\times G^{(2C)}_{11}(k_1)-\frac{\lambda}{6}\int[dk'][dk'']G^{(4C)}_{1111}(k',k'',-k_1),$$

$$\dot{G}^{(2C)}_{22}(k_1)=-\left[\omega^2(k_1)+\frac{\lambda}{2}\int[dk']G^{(2C)}_{11}(k')\right]\times(G^{(2C)}_{12}(k_1)+G^{(2C)}_{21}(k_1))-\frac{\lambda}{6}\int[dk'][dk'']\times(G^{(4C)}_{1112}(k',k'',k_1)+G^{(4C)}_{2111}(k',k'',k_1)). \qquad (155)$$

By neglecting terms involving the connected four-point functions in the above equations and, setting $\phi_c=0$ (since we are discussing the symmetric theory), one recovers the leading order (Hartree approximation) evolution equations, Eq. (107), obtained in the last section. This also establishes the equivalence of the method used in this section to obtain NLO equations and the LvN method used in the previous sections. The evolution equations for the four-point functions $G^{(4C)}_{abcd}(k_1,k_2,k_3)$ obtained by taking the Fourier transform of Eqs. (A3)–(A7) are

$$\dot{G}^{(4C)}_{1111}(k_1,k_2,k_3)=G^{(4C)}_{2111}(k_1,k_2,k_3)+G^{(4C)}_{1211}(k_1,k_2,k_3)+G^{(4C)}_{1121}(k_1,k_2,k_3)+G^{(4C)}_{1112}(k_1,k_2,k_3), \tag{156}$$

$$\begin{aligned}\dot{G}^{(4C)}_{2111}(k_1,k_2,k_3)=&G^{(4C)}_{2211}(k_1,k_2,k_3)+G^{(4C)}_{2121}(k_1,k_2,k_3)+G^{(4C)}_{2112}(k_1,k_2,k_3)\\&-\omega^2(k_1+k_2+k_3)G^{(4C)}_{1111}(k_1,k_2,k_3)-\lambda G^{(2C)}_{11}(k_1)G^{(2C)}_{11}(k_2)G^{(2C)}_{11}(k_3)\\&-\frac{\lambda}{2}\Big[\int[dk'](G^{(2C)}_{11}(k')G^{(4C)}_{1111}(k_1,k_2,k_3)+G^{(2C)}_{11}(k_1)G^{(4C)}_{1111}(k',k_2,k_3)\\&+G^{(2C)}_{11}(k_2)G^{(4C)}_{1111}(k',k_1,k_3)+G^{(2C)}_{11}(k_3)G^{(4C)}_{1111}(k',k_1,k_2))\Big],\end{aligned} \tag{157}$$

$$\begin{aligned}\dot{G}^{(4C)}_{2211}(k_1,k_2,k_3)=&G^{(4C)}_{2221}(k_1,k_2,k_3)+G^{(4C)}_{2212}(k_1,k_2,k_3)-\omega^2(k_1+k_2+k_3)G^{(4C)}_{1211}(k_1,k_2,k_3)-\omega^2(k_1)G^{(4C)}_{2111}(k_1,k_2,k_3)\\&-\lambda(G^{(2C)}_{12}(k_1)G^{(2C)}_{11}(k_2)G^{(2C)}_{11}(k_3)+G^{(2C)}_{21}(k_1)G^{(2C)}_{11}(k_2)G^{(2C)}_{11}(k_3))\\&-\frac{\lambda}{2}\Big[\int[dk'](G^{(2C)}_{11}(k')G^{(4C)}_{1211}(k_1,k_2,k_3)+G^{(2C)}_{12}(k_1)G^{(4C)}_{1111}(k',k_2,k_3)\\&+G^{(2C)}_{11}(k_2)G^{(4C)}_{1121}(k',k_1,k_3)+G^{(2C)}_{11}(k_3)G^{(4C)}_{1121}(k',k_1,k_2))\\&+\int[dk'](G^{(2C)}_{11}(k')G^{(4C)}_{2111}(k_1,k_2,k_3)+G^{(2C)}_{21}(k_1)G^{(4C)}_{1111}(k',k_2,k_3)\\&+G^{(2C)}_{11}(k_2)G^{(4C)}_{2111}(k',k_1,k_3)+G^{(2C)}_{11}(k_3)G^{(4C)}_{2111}(k',k_1,k_2))\Big],\end{aligned} \tag{158}$$

$$\begin{aligned}\dot{G}^{(4C)}_{2221}(k_1,k_2,k_3)=&G^{(4C)}_{2222}(k_1,k_2,k_3)-\omega^2(k_1+k_2+k_3)G^{(4C)}_{1221}(k_1,k_2,k_3)-\omega^2(k_1)G^{(4C)}_{2121}(k_1,k_2,k_3)\\&-\omega^2(k_2)G^{(4C)}_{2211}(k_1,k_2,k_3)\\&-\lambda(G^{(2C)}_{12}(k_1)G^{(2C)}_{12}(k_2)G^{(2C)}_{11}(k_3)+G^{(2C)}_{21}(k_1)G^{(2C)}_{12}(k_2)G^{(2C)}_{11}(k_3)\\&+G^{(2C)}_{21}(k_1)G^{(2C)}_{21}(k_2)G^{(2C)}_{11}(k_3))\\&-\frac{\lambda}{2}\Big[\int[dk'](G^{(2C)}_{11}(k')G^{(4C)}_{1221}(k_1,k_2,k_3)+G^{(2C)}_{12}(k_1)G^{(4C)}_{1121}(k',k_2,k_3)\\&+G^{(2C)}_{12}(k_2)G^{(4C)}_{1121}(k',k_1,k_3)+G^{(2C)}_{11}(k_3)G^{(4C)}_{1122}(k',k_1,k_2))\\&+\int[dk'](G^{(2C)}_{11}(k')G^{(4C)}_{2121}(k_1,k_2,k_3)+G^{(2C)}_{21}(k_1)G^{(4C)}_{1121}(k',k_2,k_3)\\&+G^{(2C)}_{12}(k_2)G^{(4C)}_{2111}(k',k_1,k_3)+G^{(2C)}_{11}(k_3)G^{(4C)}_{2112}(k',k_1,k_2))\\&+\int[dk'](G^{(2C)}_{11}(k')G^{(4C)}_{2211}(k_1,k_2,k_3)+G^{(2C)}_{21}(k_1)G^{(4C)}_{2111}(k',k_2,k_3)\\&+G^{(2C)}_{21}(k_2)G^{(4C)}_{2111}(k',k_1,k_3)+G^{(2C)}_{11}(k_3)G^{(4C)}_{2211}(k',k_1,k_2))\Big],\end{aligned} \tag{159}$$

$$\dot{G}^{(4C)}_{2222}(k_1,k_2,k_3) = -(\omega^2(k_1+k_2+k_3)G^{(4C)}_{1222}(k_1,k_2,k_3)+\omega^2(k_1)G^{(4C)}_{2122}(k_1,k_2,k_3)+\omega^2(k_2)G^{(4C)}_{2212}(k_1,k_2,k_3)$$
$$+\omega^2(k_3)G^{(4C)}_{2221}(k_1,k_2,k_3))-\lambda(G^{(2C)}_{12}(k_1)G^{(2C)}_{12}(k_2)G^{(2C)}_{12}(k_3)+G^{(2C)}_{21}(k_1)G^{(2C)}_{12}(k_2)G^{(2C)}_{12}(k_3)$$
$$+G^{(2C)}_{21}(k_1)G^{(2C)}_{21}(k_2)G^{(2C)}_{12}(k_3)+G^{(2C)}_{21}(k_1)G^{(2C)}_{21}(k_2)G^{(2C)}_{21}(k_3))$$
$$-\frac{\lambda}{2}\Bigg[\int[dk'](G^{(2C)}_{11}(k')G^{(4C)}_{1222}(k_1,k_2,k_3)+G^{(2C)}_{12}(k_1)G^{(4C)}_{1122}(k',k_2,k_3)$$
$$+G^{(2C)}_{12}(k_2)G^{(4C)}_{1122}(k',k_1,k_3)+G^{(2C)}_{12}(k_3)G^{(4C)}_{1122}(k',k_1,k_2))$$
$$+\int[dk'](G^{(2C)}_{11}(k')G^{(4C)}_{2122}(k_1,k_2,k_3)+G^{(2C)}_{21}(k_1)G^{(4C)}_{1122}(k',k_2,k_3)$$
$$+G^{(2C)}_{12}(k_2)G^{(4C)}_{2112}(k',k_1,k_3)+G^{(2C)}_{12}(k_3)G^{(4C)}_{2112}(k',k_1,k_2))$$
$$+\int[dk'](G^{(2C)}_{11}(k')G^{(4C)}_{2212}(k_1,k_2,k_3)+G^{(2C)}_{21}(k_1)G^{(4C)}_{2112}(k',k_2,k_3)$$
$$+G^{(2C)}_{21}(k_2)G^{(4C)}_{2112}(k',k_1,k_3)+G^{(2C)}_{12}(k_3)G^{(4C)}_{2211}(k',k_1,k_2))$$
$$+\int[dk'](G^{(2C)}_{11}(k')G^{(4C)}_{2221}(k_1,k_2,k_3)+G^{(2C)}_{21}(k_1)G^{(4C)}_{2211}(k',k_2,k_3)$$
$$+G^{(2C)}_{21}(k_2)G^{(4C)}_{2211}(k',k_1,k_3)+G^{(2C)}_{21}(k_3)G^{(4C)}_{2211}(k',k_1,k_2))\Bigg]. \quad (160)$$

This set of equations completely determines the evolution of the correlation functions beyond the leading order. Numerical integration of this set of equations with nonequilibrium initial conditions and tracking of the subsequent dynamical evolution of the connected, *n*-point functions would make it possible to ascertain whether the proposed truncation scheme is good enough to ensure late-time thermalization of the system. The inclusion of the four-point correlators in the hierarchy amounts to considering terms of $O(\hbar^3)$ in the effective action. Since each connected *n*-point function $G^{(nC)} \sim O(\hbar^{n-1})$, inclusion of higher order connected *n*-point correlators in the hierarchy of evolution equations allows for a systematic way of incorporating quantum effects. The issue of truncation error arising from truncating the hierarchy at the four-point level needs to be addressed. This issue has been briefly discussed in Ref. [24] where it was argued that truncation error would build up with time and eventually invalidate the working assumption of the formal hierarchy of the connected correlators given above. For the quantum mechanical anharmonic oscillator and the O(N) vector model, the decoherence time scale for breakdown of the hierarchy was qualitatively argued to scale with $\hbar^{-1/2}$ and $\sqrt{N}$, respectively. However, that analysis was based on a simple $(0+1)$-dimensional quantum mechanical model, and quantitative treatment of this aspect for a system with infinite degrees of freedom (field theory) will certainly be more complicated. Although inclusion of higher order *n*-point functions in the hierarchy does not eliminate the instabilities, it does postpone the onset of these instabilities. This delay in the appearance of instabilities would make it possible to gain valuable insights into the nonequilibrium dynamics and approach to thermalization, in a temporal domain which lies beyond the domain of validity of the Hartree approximation. A possible way to avoid the instabilities arising due to the truncation of the hierarchy might involve adapting the direct interaction approximation developed by Kraichnan [27] in the context of turbulent fluid dynamics. We are currently exploring this possibility.

VII. COMPARISON WITH ALTERNATIVE METHODS

Several approaches have been developed to understand the dynamics of fields in nonequilibrium field theory. It would be useful to make a comparative study of the different approaches, specifying the relationship between the different methods and the relative merits and demerits of each approach. In this section, we will address this issue by comparing our approach with other methods that have been used to obtain evolution equations for the *equal-time* correlation functions. We will compare our canonical approach with two specific approaches [24,36] that have been recently discussed in the literature.

Wetterich's method [36] is based on deriving an evolution equation for the partition function (or the generating functional for 1PI graphs). The time evolution of the *n*-point correlation functions (or *n*-point vertex functions) is completely determined by the time evolution of the generating functional and the microscopic dynamical equations of motion. We will show below that the evolution equations for the equal-time correlation functions obtained using Wetterich's method match exactly with those derived in this paper using a canonical approach.

For a ϕ^4 theory without spontaneous symmetry breaking, the evolution equation for the generating functional can be cast in the form [36]

$$\partial_t Z[j(x),h(x);t]=(\mathcal{L}_{cl}+\mathcal{L}_{qm})Z[j(x),h(x);t], \quad (161)$$

where $\mathcal{L}_{cl}$ and $\mathcal{L}_{qm}$ are the classical and quantum parts of the Liouvillian operator, respectively, and are given by

$$\mathcal{L}_{cl}=\int dx\Bigg[j(x)\frac{\delta}{\delta h(x)} + h(x)\Bigg\{\nabla^2\frac{\delta}{\delta j(x)}-\Bigg(m^2\frac{\delta}{\delta j(x)}+\frac{\lambda}{6}\frac{\delta^3}{\delta j(x)^3}\Bigg)\Bigg\}\Bigg],$$

$$\mathcal{L}_{qm}=\frac{\lambda}{4!}\int dx\Bigg[h^3(x)\frac{\delta}{\delta j(x)}\Bigg]. \quad (162)$$

Here $j(x)$ and $h(x)$ are source terms for the field and conjugate momenta, respectively, and $Z[j(x),h(x);t]$ is the generating functional for n-point functions, $Z=\mathrm{Tr}(\exp\{\int dx[j(x)\phi(x)+h(x)\pi(x)]\}\rho)$, where ρ is the density matrix.

It is fairly straightforward to obtain the corresponding evolution equations for the generating functional for connected graphs $W[j(x),h(x);t]$ and the generating functional for 1PI graphs $\Gamma[\varphi_c(x),\pi_c(x);t]$. [$\varphi_c(x)$ and $\pi_c(x)$ are the vacuum expectation values of the field and conjugate momentum in the presence of sources $j(x)$ and $h(x)$, respectively.] $Z[j(x),h(x);t]$, $W[j(x),h(x);t]$, and $\Gamma[\varphi_c(x),\pi_c(x);t]$ are related by the equation $W[j(x),h(x);t]=\ln Z[j(x),h(x);t]$ and the Legendre transform $\Gamma[\varphi_c(x),\pi_c(x);t]=-\ln Z[j(x),h(x);t]+\int dx[j(x)\varphi_c(x)+h(x)\pi_c(x)]$. The evolution equation for $W[j(x),h(x);t]$, which completely determines the evolution of the connected n-point functions, then becomes

$$\partial_t W[j(x),h(x);t] = \int dx\Bigg[j(x)\frac{\delta W}{\delta h(x)}+h(x)(\nabla^2-m^2)\frac{\delta W}{\delta j(x)} -\frac{\lambda}{6}h(x)\Bigg\{\frac{\delta^3 W}{\delta j(x)^3}+3\frac{\delta W}{\delta j(x)}\frac{\delta^2 W}{\delta j(x)^2}+\Bigg(\frac{\delta W}{\delta j(x)}\Bigg)^3\Bigg\}\Bigg] + \frac{\lambda}{4!}\int dx\Bigg[h^3(x)\frac{\delta W}{\delta j(x)}\Bigg]. \quad (163)$$

For a symmetric theory $\varphi_c(x)|_{j=0}\equiv[\delta W/\delta j(x)]|_{j=0}=0$ and $\pi_c(x)|_{h=0}\equiv[\delta W/\delta h(x)]|_{h=0}=0$. Using Eq. (163) we find that

$$\dot{g}_{11}^{(2C)}(x_1,x_2)\equiv\frac{\delta^2}{\delta j(x_1)\delta j(x_2)}\partial_t W[j,h;t]\big|_{j=h=0} = g_{12}^{(2C)}(x_1,x_2)+g_{21}^{(2C)}(x_1,x_2),$$

$$\dot{g}_{21}^{(2C)}(x_1,x_2)\equiv\frac{\delta^2}{\delta h(x_1)\delta j(x_2)}\partial_t W[j,h;t]\big|_{j=h=0} = g_{22}^{(2C)}(x_1,x_2)+\nabla_1^2 g_{12}^{(2C)}(x_1,x_2) - m^2 g_{11}^{(2C)}(x_1,x_2) - \frac{\lambda}{2}g_{11}^{(2C)}(x_1,x_2)g_{11}^{(2C)}(x_1,x_1) - \frac{\lambda}{6}g_{1111}^{(4C)}(x_1,x_1,x_1,x_2),$$

$$\dot{g}_{22}^{(2C)}(x_1,x_2) \equiv\frac{\delta^2}{\delta h(x_1)\delta h(x_2)}\partial_t W[j,h;t]\big|_{j=h=0} = (\nabla_1^2-m^2)g_{12}^{(2C)}(x_1,x_2)+(\nabla_2^2-m^2)g_{21}^{(2C)}(x_1,x_2) - \frac{\lambda}{2}(g_{11}^{(2C)}(x_1,x_1)g_{12}^{(2C)}(x_1,x_2) + g_{11}^{(2C)}(x_2,x_2)g_{21}^{(2C)}(x_1,x_2)) - \frac{\lambda}{6}(g_{1112}^{(4C)}(x_1,x_1,x_1,x_2)+g_{2111}^{(4C)}(x_1,x_2,x_2,x_2)). \quad (164)$$

The above set of equations for the connected two-point function, derived using Eq. (163), is identical to Eq. (A1), which was derived using canonical methods and the cluster expansion for ordinary n-point functions. Similarly, one can show that the evolution for the connected four-point function obtained using the above method is the same as the ones obtained in Appendix A.

More recently, Ryzhov and Yaffe developed another method for obtaining the nonequilibrium, coupled set of evolution equations for the n-point correlators [24]. Their method is based on an expansion of the coherent state expectation value of the products of operators in terms of subtracted n-point functions and the assumption that the initial state is some coherent state. The central idea behind the RY method is the expansion of the Hamiltonian operator in terms of powers of the generators of the underlying coherence group, which is the Heisenberg group for the ϕ^4 theory. It consists of the operators $e^{(1)},e^{(2)},e^{(3)}\equiv\hat{\Phi}_1(k),\hat{\Phi}_2(k),\hat{1}$ satisfying the commutation relations $[e^{(1)},e^{(2)}]=if_{12}^3e^{(3)}\delta(k-k')$, where $f_{12}^3=-f_{21}^3=1$ are the only nonvanishing structure constants. By taking the *coherent state* expectation value of the Heisenberg equations of motion for appropriate products of equal-time coherence group generators, it is possible to obtain the evolution equations for the subtracted n-point functions (or connected functions) in the same manner as described in Sec. V. In Appendix B, the evolution equations for the two-point function are derived in the Hartree approximation using the RY method and are shown to be equivalent to those obtained using the LvN approach.

VIII. CONCLUSION

In this paper we have used the LvN formalism, a canonical method, to study the nonequilibrium evolution of

equal-time, connected n-point functions for a symmetric ϕ^4 field theory. The usefulness and simplicity of the LvN method in obtaining perturbative, non-Gaussian corrections to the n-point correlation functions were first illustrated using the quantum mechanical anharmonic oscillator model. The formalism was then applied to a ϕ^4 field theory and used to obtain the evolution equations for the connected two-point function in the Hartree approximation, for nonequilibrium evolution as well as for the thermal equilibrium case. We also used the LvN formalism to go beyond the Gaussian approximation and obtain expressions for the two-point functions correct to $O(\lambda^2)$ after calculating the improved vacuum state. Expressions for four-point and higher n-point functions can be obtained similarly after some straightforward but tedious algebra. The corrections to the Gaussian approximation were found to appear first at $O(\lambda^2)$. The nonequilibrium evolution equations beyond the leading order approximation were then obtained by taking the vacuum expectation value of the Heisenberg equations of motion for appropriate products of field operators. This provides an alternative and nonperturbative approach to investigate systematically the nonequilibrium evolution of quantum fields, and yielded an infinite hierarchy of coupled equations for the connected n-point correlators which were truncated at the four-point level. This involved ignoring the contribution of connected n-point functions ($n \geqslant 6$) in the evolution and resulted in a closed set of equations involving connected two-point and four-point functions. Since we restricted our investigation to a symmetric ϕ^4 theory with vanishing vacuum expectation value, all connected odd n-point functions vanish and have no effect on the evolution. We have also established a connection between the canonical approach used in this paper and other methods [24,36] developed in the literature for deriving nonequilibrium evolution equations for equal-time correlation functions. It would be straightforward to generalize this technique for studying the nonequilibrium evolution of spontaneously broken field theories. Symmetry breaking would result in the generation of linear and cubic terms in the potential, as a consequence of which even connected odd n-point functions would contribute to the dynamical evolution.

The inadequacy of the Hartree approximation in studying the approach to thermalization has been extensively discussed in the literature. The Hartree approximation neglects scattering effects and therefore cannot account for long-time thermalization of the system. An elegant interpretation of this aspect has been put forward by Wetterich [36] and is based on the realization that the Hartree solution corresponds to a fixed point of the theory (i.e., configurations for which $\partial_t Z = 0$). The presence of an infinite number of conserved quantities (fixed points) prevents the system from escaping from these nonequilibrium fixed points and approaching the thermal equilibrium fixed point unless the initial values of all these conserved quantities coincide exactly with the ones corresponding to a thermal distribution. To take into account the effect of scattering, which would ultimately lead to thermalization of the system, requires going beyond the Hartree approximation. An approach based on the loop expansion of the 2PI effective action [13] in powers of $\hbar$ [or $1/N$ for an O(N) symmetric field theory] holds more promise in this context. In this paper we have discussed two different canonical approaches (perturbative and nonperturbative) of going beyond the Hartree approximation. The perturbative LvN approach developed in this paper provides an elegant method for obtaining corrections to the n-point functions in terms of powers of the coupling constant λ. The effect of scattering, which first appears at $O(\lambda^2)$, is manifest through the first and second order coefficients whose dynamical equations were derived. This method provides a straightforward method for obtaining non-Gaussian corrections to the domain size in systems undergoing a quenched second order phase transition. By making use of the exponentially growing solution of the soft modes of the theory, it is possible to obtain expressions for the first and second order coefficients, which in turn leads to expressions for the two-point functions to various orders in λ. The Fourier transform of the two-point functions would then yield the domain size, which includes corrections due to non-Gaussian effects [46]. Moreover, the LvN method provides an analytical, albeit perturbative, method for studying the role of interactions and the effect of scattering on thermalization of the system. This is possible by obtaining perturbative corrections to the two-point functions to various powers of λ and comparing the resulting expression with the thermal two-point correlation functions. We have also discussed in detail an alternative, nonperturbative canonical approach based on the Heisenberg formalism to study the nonequilibrium evolution of the ϕ^4 field theory. This nonperturbative Heisenberg formalism requires incorporating the effect of connected four-point function in the nonequilibrium evolution equations. The efficacy of truncating the hierarchy at the four-point level in ensuring thermalization needs to be addressed. Recent work [23] for classical field theories has been inconclusive because of the prohibitively long time required for equilibration and also due to the fact that the numerical evolution becomes unstable long before the system thermalizes. We believe that the canonical approaches developed in this paper provide a useful alternative method for studying nonequilibrium dynamics in field theory.

ACKNOWLEDGMENTS

S.S. would like to thank L. Yaffe for helpful discussions. S.P.K. would like to thank D. N. Page for useful discussions and also appreciates the warm hospitality of the Theoretical Physics Institute, University of Alberta. The work of S.S. and F.C.K. was supported in part by the Natural Sciences and Engineering Research Council (NSERC), Canada. The work of S.P.K. was supported in part by the Korea Research Foundation under Grant No. KRF-2002-041-C00053 and also by the Korea Science and Engineering Foundation under Grant No. 1999-2-112-003-5.

APPENDIX A

In this appendix we write down the set of evolution equations for the connected n-point functions in configuration space by making use of the cluster expansion for the four-point and six-point functions. The cluster expansion of

n-point functions can be formally derived from the generating functional of the interacting field theory [37]. The difference between the ordinary and connected *n*-point correlators first appears at the four-point level. To simplify the notation, we represent the ϕ and π field operators by the numbers 1 and 2, respectively. So the correlation functions $g^{(2C)}_{\phi\phi}(x_1,x_2)$, $g^{(2C)}_{\pi\phi}(x_1,x_2)$, etc., are symbolically represented as $g^{(2C)}_{11}(1,2)$, $g^{(2C)}_{21}(1,2)$, respectively, where the numbers in parentheses refer to the subscript indices of the spatial coordinates.

The equations for the two-point functions in configuration space are easily obtained by using Eq. (151) and the appropriate cluster expansion of the four-point functions. They are

$$\dot{g}^{(2C)}_{11}(1,2)=g^{(2C)}_{12}(1,2)+g^{(2C)}_{21}(1,2),$$

$$\dot{g}^{(2C)}_{21}(1,2)=g^{(2C)}_{22}(1,2)+(\nabla^2_1-m^2)g^{(2C)}_{12}(1,1)-\frac{\lambda}{2}g^{(2C)}_{11}(1,1)g^{(2C)}_{11}(1,2)-\frac{\lambda}{6}g^{(4C)}_{1111}(1,1,1,2),$$

$$\dot{g}^{(2C)}_{22}(1,2)=(\nabla^2_1-m^2)g^{(2C)}_{12}(1,2)+(\nabla^2_2-m^2)g^{(2C)}_{21}(1,2)-\frac{\lambda}{2}(g^{(2C)}_{11}(1,1)g^{(2C)}_{12}(1,2)+g^{(2C)}_{11}(2,2)g^{(2C)}_{21}(1,2))-\frac{\lambda}{6}(g^{(4C)}_{1112}(1,1,1,2)+g^{(4C)}_{2111}(1,2,2,2)), \tag{A1}$$

where the last two equations in the above set have been obtained by making use of the following cluster expansion of the four-point functions:

$$g^{(4)}_{1111}(1,1,1,2)=g^{(4C)}_{1111}(1,1,1,2)+3g^{(2C)}_{11}(1,1)g^{(2C)}_{11}(1,2),$$

$$g^{(4)}_{1112}(1,1,1,2)=g^{(4C)}_{1112}(1,1,1,2)+3g^{(2C)}_{11}(1,1)g^{(2C)}_{12}(1,2),$$

$$g^{(4)}_{2111}(1,2,2,2)=g^{(4C)}_{2111}(1,1,1,2)+3g^{(2C)}_{11}(2,2)g^{(2C)}_{21}(1,2). \tag{A2}$$

We have used lower case letters to represent the correlation functions in configuration space and upper case letters to represent their Fourier transformed counterparts in momentum space. The *n*-point connected correlation functions appearing above and in the subsequent discussion are all normal ordered vacuum expectation values of products of operators.

The evolution equations for the connected four-point functions in configuration space can be similarly derived by making use of the appropriate cluster expansion of the six-point functions. The five independent equations for the four-point function are

$$\dot{g}^{(4C)}_{1111}(1,2,3,4)=g^{(4C)}_{2111}(1,2,3,4)+g^{(4C)}_{1211}(1,2,3,4)+g^{(4C)}_{1121}(1,2,3,4)+g^{(4C)}_{1112}(1,2,3,4), \tag{A3}$$

$$\dot{g}^{(4C)}_{2111}(1,2,3,4)=g^{(4C)}_{2211}(1,2,3,4)+g^{(4C)}_{2121}(1,2,3,4)+g^{(4C)}_{2112}(1,2,3,4)+(\nabla^2_1-m^2)g^{(4C)}_{1111}(1,2,3,4)-\lambda g^{(2C)}_{11}(1,2)g^{(2C)}_{11}(1,3)g^{(2C)}_{11}(1,4)-\frac{\lambda}{2}(g^{(2C)}_{11}(1,1)g^{(4C)}_{1111}(1,2,3,4)+g^{(2C)}_{11}(1,2)g^{(4C)}_{1111}(1,1,3,4)+g^{(2C)}_{11}(1,3)g^{(4C)}_{1111}(1,1,2,4)+g^{(2C)}_{11}(1,4)g^{(4C)}_{1111}(1,1,2,3)), \tag{A4}$$

$$\dot{g}^{(4C)}_{2211}(1,2,3,4)=g^{(4C)}_{2221}(1,2,3,4)+g^{(4C)}_{2212}(1,2,3,4)+(\nabla^2_1-m^2)g^{(4C)}_{1211}(1,2,3,4)+(\nabla^2_2-m^2)g^{(4C)}_{2111}(1,2,3,4)-\lambda(g^{(2C)}_{12}(1,2)g^{(2C)}_{11}(1,3)g^{(2C)}_{11}(1,4)+g^{(2C)}_{21}(1,2)g^{(2C)}_{11}(2,3)g^{(2C)}_{11}(2,4))-\frac{\lambda}{2}(g^{(2C)}_{11}(1,1)g^{(4C)}_{1211}(1,2,3,4)+g^{(2C)}_{12}(1,2)g^{(4C)}_{1111}(1,1,3,4)+g^{(2C)}_{11}(1,3)g^{(4C)}_{1121}(1,1,2,4)+g^{(2C)}_{11}(1,4)g^{(4C)}_{1121}(1,1,2,3))-\frac{\lambda}{2}(g^{(2C)}_{21}(1,2)g^{(4C)}_{1111}(2,2,3,4)+g^{(2C)}_{11}(2,2)g^{(4C)}_{2111}(1,2,3,4)+g^{(2C)}_{11}(2,3)g^{(4C)}_{2111}(1,2,2,4)+g^{(2C)}_{11}(2,4)g^{(4C)}_{2111}(1,2,2,3)). \tag{A5}$$

Since the correlation functions are all normal ordered, the above equation is symmetric under the interchange of indices $x_1\leftrightarrow x_2$ and $x_3\leftrightarrow x_4$:

$$\dot{g}^{(4C)}_{2221}(1,2,3,4) = g^{(4C)}_{2222}(1,2,3,4) + (\nabla_1^2 - m^2) g^{(4C)}_{1221}(1,2,3,4) + (\nabla_2^2 - m^2) g^{(4C)}_{2121}(1,2,3,4) + (\nabla_3^2 - m^2) g^{(4C)}_{2211}(1,2,3,4)$$
$$- \lambda \big(g^{(2C)}_{12}(1,2) g^{(2C)}_{12}(1,3) g^{(2C)}_{11}(1,4) + g^{(2C)}_{21}(1,2) g^{(2C)}_{12}(2,3) g^{(2C)}_{11}(2,4)$$
$$+ g^{(2C)}_{21}(1,3) g^{(2C)}_{21}(2,3) g^{(2C)}_{11}(3,4) \big)$$
$$- \frac{\lambda}{2} \big(g^{(2C)}_{11}(1,1) g^{(4C)}_{1221}(1,2,3,4) + g^{(2C)}_{12}(1,2) g^{(4C)}_{1111}(1,1,3,4)$$
$$+ g^{(2C)}_{12}(1,3) g^{(4C)}_{1121}(1,1,2,4) + g^{(2C)}_{11}(1,4) g^{(4C)}_{1122}(1,1,2,3) \big)$$
$$- \frac{\lambda}{2} \big(g^{(2C)}_{11}(1,2) g^{(4C)}_{2121}(1,2,3,4) + g^{(2C)}_{21}(1,2) g^{(4C)}_{1121}(2,2,3,4)$$
$$+ g^{(2C)}_{12}(1,3) g^{(4C)}_{2111}(1,2,2,4) + g^{(2C)}_{11}(2,4) g^{(4C)}_{2112}(1,2,2,3) \big)$$
$$- \frac{\lambda}{2} \big(g^{(2C)}_{11}(3,3) g^{(4C)}_{2211}(1,2,3,4) + g^{(2C)}_{21}(1,3) g^{(4C)}_{2111}(2,3,3,4)$$
$$+ g^{(2C)}_{21}(1,3) g^{(4C)}_{2111}(1,3,3,4) + g^{(2C)}_{11}(3,4) g^{(4C)}_{2211}(1,2,3,3) \big). \tag{A6}$$

The above equation is symmetric under the exchange of coordinate indices $x_1 \leftrightarrow x_2 \leftrightarrow x_3$:

$$\dot{g}^{(4C)}_{2222}(1,2,3,4) = (\nabla_1^2 - m^2) g^{(4C)}_{1222}(1,2,3,4) + (\nabla_2^2 - m^2) g^{(4C)}_{2122}(1,2,3,4)$$
$$+ (\nabla_3^2 - m^2) g^{(4C)}_{2212}(1,2,3,4) + (\nabla_4^2 - m^2) g^{(4C)}_{2221}(1,2,3,4)$$
$$- \lambda \big(g^{(2C)}_{12}(1,2) g^{(2C)}_{12}(1,3) g^{(2C)}_{12}(1,4) + g^{(2C)}_{21}(1,2) g^{(2C)}_{12}(2,3) g^{(2C)}_{12}(2,4)$$
$$+ g^{(2C)}_{21}(1,3) g^{(2C)}_{21}(2,3) g^{(2C)}_{12}(3,4) + g^{(2C)}_{21}(1,4) g^{(2C)}_{21}(2,4) g^{(2C)}_{21}(3,4) \big)$$
$$- \frac{\lambda}{2} \big(g^{(2C)}_{11}(1,1) g^{(4C)}_{1222}(1,2,3,4) + g^{(2C)}_{12}(1,2) g^{(4C)}_{1122}(1,1,3,4)$$
$$+ g^{(2C)}_{12}(1,3) g^{(4C)}_{1122}(1,1,2,4) + g^{(2C)}_{12}(1,4) g^{(4C)}_{1122}(1,1,2,3) \big)$$
$$- \frac{\lambda}{2} \big(g^{(2C)}_{21}(1,2) g^{(4C)}_{1122}(2,2,3,4) + g^{(2C)}_{11}(2,2) g^{(4C)}_{2122}(1,2,3,4)$$
$$+ g^{(2C)}_{12}(2,3) g^{(4C)}_{2112}(1,2,2,4) + g^{(2C)}_{12}(2,4) g^{(4C)}_{2112}(1,2,2,3) \big)$$
$$- \frac{\lambda}{2} \big(g^{(2C)}_{21}(1,3) g^{(4C)}_{2112}(2,3,3,4) + g^{(2C)}_{21}(2,3) g^{(4C)}_{2112}(1,3,3,4)$$
$$+ g^{(2C)}_{11}(3,3) g^{(4C)}_{2212}(1,2,3,4) + g^{(2C)}_{12}(3,4) g^{(4C)}_{2211}(1,2,3,3) \big)$$
$$- \frac{\lambda}{2} \big(g^{(2C)}_{21}(1,4) g^{(4C)}_{2211}(2,3,4,4) + g^{(2C)}_{21}(2,4) g^{(4C)}_{2211}(1,3,4,4)$$
$$+ g^{(2C)}_{21}(3,4) g^{(4C)}_{2211}(1,2,4,4) + g^{(2C)}_{11}(4,4) g^{(4C)}_{2221}(1,2,3,4) \big). \tag{A7}$$

APPENDIX B

We now show the equivalence between the LvN method and the RY method by obtaining the evolution equations for the Fourier transform of the subtracted two-point correlations using the RY method in the Hartree approximation. The expectation value of the Hamiltonian (79) can be expressed in terms of the Fourier transforms $\Phi_{0i}(k,t)(i=1,2)$ of the coherent state expectation value of the fluctuation field $\phi_0(x,t) \equiv \langle \hat{\phi}_f \rangle_{\rm cs}$ and that of the conjugate momentum operator $\pi_0(x,t) \equiv \langle \hat{\pi}_f \rangle_{\rm cs}$, respectively. In order to apply the RY method, it is necessary to isolate from the expectation value of the Hamiltonian the parts that depend only on the Fourier transforms $\Phi_{01}(k,t) = \Phi_0(k,t)$ and $\Phi_{02}(k,t) = \Pi_0(k,t)$ of the coherent state expectation value of the field and its conjugate momentum operator. Using the relations $\langle \hat{\pi}_f^2(x) \rangle_{\rm cs} = \pi_0^2(x,t) + g_{22}(0,t)$ and $\langle \hat{\phi}_f^2(x) \rangle_{\rm cs} = \phi_0^2(x,t) + g_{11}(0,t)$ and

the Hartree factorization of the cubic and quartic terms specified in Eq. (94), the coherent state expectation value of the full Hamiltonian can be expressed as

$$\langle \hat{H}(t)\rangle_{\rm cs} = \int [dk]\overline{H}(\Phi_0(k,t),\Pi_0(k,t)) + \cdots, \quad \text{(B1)}$$

where the ellipsis corresponds to terms that are independent of $\Phi_{0i}(k,t)(i=1,2)$ and

$$\overline{H}(\Phi_0(k,t),\Pi_0(k,t)) = \frac{1}{2}\Pi_0^2(k,t) + \left[\omega_k^2 + \frac{\lambda}{2}\phi_{cl}^2(t) + \frac{\lambda}{2}g_{11}(0,t)\right]\frac{\Phi_0^2(k,t)}{2}. \quad \text{(B2)}$$

By noting that the underlying coherence group in this case is the Heisenberg group whose generators $e^{(1)},e^{(2)},e^{(3)} \equiv \hat{\pi}_k,\hat{\phi}_k,\hat{1}$ satisfy the commutation relations $[e^{(1)},e^{(2)}] = if_{12}^3 e^{(3)}\delta(k-k')$ where $f_{12}^3 = -f_{21}^3 = 1$ are the only nonvanishing structure constants, and by making use of the general form of the NLO evolution equations for the one-point and two-point functions, we get the following self-consistent set of equations to $O((\lambda)^2)$:

$$\frac{d\Phi_0(k,t)}{dt} = \Pi_0(k,t),$$

$$\frac{d\Pi_0(k,t)}{dt} = -\left[\omega_k^2 + \frac{\lambda}{2}\phi_{cl}^2(t) + \frac{\lambda}{2}g_{11}(0,t)\right]\Phi_0(k,t) + O((\lambda)^2),$$

$$\frac{dG_{11}(k,t)}{dt} = G_{12}(k,t) + G_{21}(k,t),$$

$$\frac{dG_{22}(k,t)}{dt} = -\left[\omega_k^2 + \frac{\lambda}{2}\phi_{cl}^2(t) + \frac{\lambda}{2}g_{11}(0,t)\right] \times [G_{12}(k,t) + G_{21}(k,t)] + O((\lambda)^2),$$

$$\frac{dG_{12}(k,t)}{dt} = G_{22}(k,t) - \left[\omega_k^2 + \frac{\lambda}{2}\phi_{cl}^2(t) + \frac{\lambda}{2}g_{11}(0,t)\right] \times G_{11}(k,t) + O((\lambda)^2), \quad \text{(B3)}$$

where we have used $\overline{H}^{(11)} \equiv (1/2!)\partial^2\overline{H}/\partial\Phi_0^2(k,t) = (1/2) \times [\omega_k^2 + (\lambda/2)\phi_{cl}^2(t) + (\lambda/2)g_{11}(0,t)]$ and $\overline{H}^{(22)} \equiv (1/2!)\partial^2\overline{H}/\partial\Pi_0^2(k,t) = 1/2$. The set of equations (B3) are identical to the equations (107) obtained using the LvN formalism which clearly establishes the equivalence of the LvN and RY methods. The coherent state can be considered as the vacuum state of the theory with its expectation value providing the classical background. For a symmetric theory, the coherent state expectation value of the field and the conjugate momenta vanishes, and we recover the set of equations for the two-point functions (in the Hartree limit) obtained in Sec. V.

[1] L. Kofman, A. Linde, and A.A. Starobinsky, Phys. Rev. D **56**, 3258 (1997); A. Dolgov and D. Kirrilova, Sov. J. Nucl. Phys. **51**, 172 (1990); Y. Shtanov, J. Traschen, and R. Brandenberger, Phys. Rev. D **51**, 5438 (1995); M. Yoshimura, Prog. Theor. Phys. **94**, 873 (1995); S.Yu. Khlebnikov and I.I. Tkachev, Phys. Rev. Lett. **77**, 219 (1996); Phys. Lett. B **390**, 80 (1997); D.I. Kaiser, Phys. Rev. D **53**, 1776 (1996); D. Boyanovsky, H.J. de Vega, R. Holman, and J.F.J. Salgado, *ibid.* **54**, 7570 (1996).

[2] D. Boyanovsky, D.S. Lee, and A. Singh, Phys. Rev. D **48**, 800 (1993).

[3] T.W.B. Kibble, J. Phys. A **9**, 1387 (1976).

[4] W.H. Zurek, Nature (London) **317**, 505 (1985); Phys. Rep. **276**, 177 (1996); P. Laguna and W.H. Zurek, Phys. Rev. Lett. **78**, 2519 (1997); A. Yates and W.H. Zurek, *ibid.* **80**, 5477 (1998); N.D. Antunes, L.M.A. Bettencourt, and W.H. Zurek, *ibid.* **82**, 2824 (1999).

[5] A.J. Gill and R.J. Rivers, Phys. Rev. D **51**, 6949 (1995); G. Karra and R.J. Rivers, Phys. Lett. B **414**, 28 (1997); "The Densities, Correlations and Length Distributions of Vortices Produced at a Gaussian Quench," hep-ph/9604413; R.J. Rivers, Int. J. Theor. Phys. **39**, 1623 (2000); R.J. Rivers, E. Kavoussanaki, and G. Karra, Condens. Matter Phys. **3**, 133 (2000).

[6] G.J. Stephens, E. Calzetta, B.L. Hu, and S.A. Ramsey, Phys. Rev. D **59**, 045009 (1999); D. Ibaceta and E. Calzetta, Phys. Rev. E **60**, 2999 (1999).

[7] S. Digal and A.M. Srivastava, Phys. Rev. Lett. **76**, 583 (1996); see also S. Digal, S. Sengupta, and A.M. Srivastava, Phys. Rev. D **58**, 103510 (1998); **55**, 3824 (1997); **56**, 2035 (1997); S. Digal, R. Ray, S. Sengupta, and A.M. Srivastava, Phys. Rev. Lett. **84**, 826 (2000); R. Ray and S. Sengupta, Phys. Rev. D **65**, 063521 (2002).

[8] J. Borrill, T.W.B. Kibble, T. Vachaspati, and A. Vilenkin, Phys. Rev. D **52**, 1934 (1995).

[9] K. Rajagopal and F. Wilczek, Nucl. Phys. **B404**, 577 (1993); D. Boyanovsky, H.J. de Vega, and R. Holman, Phys. Rev. D **51**, 734 (1995); F. Cooper, Y. Kluger, E. Mottola, and J.P. Paz, *ibid.* **51**, 2377 (1995).

[10] M. Gleiser and R.O. Ramos, Phys. Rev. D **50**, 2441 (1994).

[11] D. Boyanovsky, H.J. de Vega, R. Holman, D.S. Lee, and A. Singh, Phys. Rev. D **51**, 4419 (1995).

[12] C. Greiner and B. Muller, Phys. Rev. D **55**, 1026 (1997).

[13] J. Berges and J. Cox, Phys. Lett. B **517**, 369 (2001); G. Aarts and J. Berges, Phys. Rev. D **64**, 105010 (2001); J. Berges, Nucl. Phys. **A699**, 847 (2002); G. Aarts, D. Ahrensmeier, R. Baier, J. Berges, and J. Serreau, Phys. Rev. D **66**, 045008

(2002); E. Calzetta and B.L. Hu, hep-ph/0205271.
[14] C. Wetterich, Phys. Rev. Lett. **78**, 3598 (1997); Phys. Lett. B **399**, 123 (1997); Phys. Rev. E **56**, 2687 (1997); L. Bettencourt and C. Wetterich, Phys. Lett. B **430**, 140 (1998); G.F. Bonini and C. Wetterich, Phys. Rev. D **60**, 105026 (1999).
[15] K.B. Blagoev, F. Cooper, J.F. Dawson, and B. Mihaila, Phys. Rev. D **64**, 125003 (2001); B. Mihaila, J.F. Dawson, and F. Cooper, *ibid.* **63**, 096003 (2001); F. Cooper, A. Khare, and H. Rose, Phys. Lett. B **515**, 463 (2001).
[16] O. Eboli, R. Jackiw, and So-Young Pi, Phys. Rev. D **37**, 3557 (1988); see also R. Jackiw, *Diverse Topics in Theoretical and Mathematical Physics* (World Scientific, Singapore, 1995).
[17] F. Cooper, S. Habib, Y. Kluger, E. Mottola, J.P. Paz, and P.R. Anderson, Phys. Rev. D **50**, 2848 (1994); F. Cooper, J. Dawson, S. Habib, Y. Kluger, D. Meredith, and H. Shepard, Physica D **83**, 74 (1995); F. Cooper, S. Habib, Y. Kluger, and E. Mottola, Phys. Rev. D **55**, 6471 (1997); B. Mihaila, T. Athan, F. Cooper, J. Dawson, and S. Habib, *ibid.* **62**, 125015 (2000).
[18] D. Boyanovsky, M. D'Attanasio, H.J. de Vega, R. Holman, and D.-S. Lee, Phys. Rev. D **52**, 6805 (1995); D. Boyanovsky, D. Cormier, H.J. de Vega, and R. Holman, *ibid.* **55**, 3373 (1997).
[19] J. Baacke and K. Heitmann, Phys. Rev. D **62**, 105022 (2000); J. Baacke and S. Michalski, *ibid.* **65**, 065019 (2002).
[20] M. Salle, J. Smit, and J.C. Vink, Phys. Rev. D **64**, 025016 (2001); Nucl. Phys. **B625**, 495 (2002).
[21] L.M.A. Bettencourt, K. Pao, and J.G. Sanderson, Phys. Rev. D **65**, 025015 (2002).
[22] M. Salle and J. Smit, Phys. Rev. D **67**, 116006 (2003).
[23] G. Aarts, G.F. Bonini, and C. Wetterich, Phys. Rev. D **63**, 025012 (2001); G. Aarts and J. Berges, Phys. Rev. Lett. **88**, 041603 (2002).
[24] A.V. Ryzhov and L.G. Yaffe, Phys. Rev. D **62**, 125003 (2000).
[25] J.M. Cornwall, R. Jackiw, and E. Tomboulis, Phys. Rev. D **10**, 2428 (1974).
[26] E. Calzetta and B.L. Hu, Phys. Rev. D **37**, 2878 (1988).
[27] R.H. Kraichnan, J. Fluid Mech. **5**, 497 (1959); J. Math. Phys. **2**, 124 (1961); **3**, 205 (1962).
[28] J. Schwinger, J. Math. Phys. **2**, 407 (1961); L.V. Keldysh, Zh. Éksp. Teor. Fiz. **47**, 1515 (1964) [Sov. Phys. JETP **20**, 1018 (1965)]; P.M. Bakshi and K. Mahanthappa, J. Math. Phys. **4**, 1 (1963); **4**, 12 (1963); K. Mahanthappa, Phys. Rev. **126**, 329 (1962); for review and references, see K. Chao, Z. Su, B. Hao, and L. Yu, Phys. Rep. **118**, 1 (1985); see also M. Le Bellac, *Thermal Field Theory* (Cambridge University Press, Cambridge, England, 1996).
[29] S.P. Kim and C.H. Lee, Phys. Rev. D **62**, 125020 (2000).
[30] H.R. Lewis, Jr. and W.B. Riesenfeld, J. Math. Phys. **10**, 1458 (1969).
[31] S.P. Kim, J. Korean Phys. Soc. **43**, 11 (2003).
[32] J.K. Kim and S.P. Kim, J. Phys. A **32**, 2711 (1999); S.P. Kim and D.N. Page, Phys. Rev. A **64**, 012104 (2001).
[33] S.P. Kim, S. Sengupta, and F.C. Khanna, Phys. Rev. D **64**, 105026 (2001).
[34] S.P. Kim and C.H. Lee, Phys. Rev. D **65**, 045013 (2002).
[35] G. Flores-Hidalgo and Rudnei O. Ramos, ''Nonequilibrium Dynamics of Coupled Quantum Systems,'' hep-th/0206022.
[36] Wetterich [14].
[37] S.J. Wang, W. Cassing, J.M. Hauser, A. Peter, and M.H. Thoma, Ann. Phys. (N.Y.) **242**, 235 (1995); J.M. Hauser, W. Cassing, A. Peter, and M.H. Thoma, Z. Phys. A **353**, 301 (1996); A. Peter, J.M. Hauser, M.H. Thoma, and W. Cassing, Z. Phys. C **71**, 515 (1996).
[38] R. Balian and M. Veneroni, Ann. Phys. (N.Y.) **164**, 334 (1985).
[39] S.P. Kim, A.E. Santana, and F.C. Khanna, Phys. Lett. A **272**, 46 (2000).
[40] I.A. Malkin, V.I. Man'ko, and D.A. Trifnov, Phys. Rev. D **2**, 1371 (1970); V.V. Dodonov and V.I. Man'ko, Phys. Rev. A **20**, 550 (1979).
[41] A.K. Rajagopal and J.T. Marshall, Phys. Rev. A **26**, 2977 (1982).
[42] W.E. Caswell, Ann. Phys. (N.Y.) **123**, 153 (1979); J. Killingbeck, J. Phys. A **14**, 1005 (1981); E.J. Austin and J. Killingbeck, *ibid.* **15**, L443 (1982).
[43] S.J. Chang, Phys. Rev. D **12**, 1071 (1975); **13**, 2778 (1976); P.M. Stevenson, *ibid.* **30**, 1712 (1984); **32**, 1389 (1985).
[44] D. Boyanovsky, D.-S. Lee, and A. Singh, Phys. Rev. D **48**, 800 (1993).
[45] D. Bak, S.P. Kim, S.K. Kim, K.-S. Soh, and J.H. Yee, J. Korean Phys. Soc. **37**, 168 (2000).
[46] S.P. Kim and F.C. Khanna, ''Non-Gaussian Effects on Domain Growth,'' hep-ph/0011115; S. Sengupta, S.P. Kim, and F.C. Khanna, in *Fundamental Interactions*, Proceedings of the Lake Louise Winter Institute, Alberta, Canada, 2001, edited by A. Astbury, B.A. Campbell, F.C. Khanna, and M.G. Vincter (World Scientific, Singapore, 2001).
[47] H. Kleinert, Phys. Lett. A **173**, 332 (1993).
[48] See S.P. Kim, Phys. Rev. D **46**, 3403 (1992), Appendix B.
[49] C.M. Bender and L.M.A. Bettencourt, Phys. Rev. D **54**, 7710 (1996).
[50] J. Baacke, K. Heitmann, and C. Patzold, Phys. Rev. D **55**, 2320 (1997).
[51] G. Aarts, G.F. Bonini, and C. Wetterich, Nucl. Phys. **B587**, 403 (2000).
[52] R. Balescu, *Equilibrium and Nonequilibrium Statistical Mechanics* (Wiley, New York, 1975).