

Spacetime noncommutativity and antisymmetric tensor dynamics in the early UniverseElisabetta Di Grezia,^{2,1,*} Giampiero Esposito,^{1,2,†} Agostino Funel,^{2,‡} Gianpiero Mangano,^{1,2,§} and Gennaro Miele^{2,1,||}¹*Istituto Nazionale di Fisica Nucleare, Sezione di Napoli, Complesso Universitario di Monte S. Angelo,
Via Cintia, Edificio N', 80126 Napoli, Italy*²*Dipartimento di Scienze Fisiche, Complesso Universitario di Monte S. Angelo, Via Cintia, Edificio N', 80126 Napoli, Italy*

(Received 15 May 2003; published 25 November 2003)

This paper investigates the possible cosmological implications of the presence of an antisymmetric tensor field θ related to a lack of commutativity of spacetime coordinates at the Planck era. For this purpose, θ is promoted to a dynamical variable, inspired by tensor formalism. By working to quadratic order in θ , we study the field equations in a Bianchi type I Universe in two models: an antisymmetric tensor plus scalar field coupled to gravity, or a cosmological constant and a free massless antisymmetric tensor. In the first scenario, numerical integration shows that, in the very early Universe, the effects of the antisymmetric tensor can prevail on the scalar field, while at late times the former approaches zero and the latter drives the isotropization of the Universe. In the second model, an approximate solution is obtained of a nonlinear ordinary differential equation which shows how the mean Hubble parameter and the difference between longitudinal and orthogonal Hubble parameter evolve in the early Universe.

DOI: 10.1103/PhysRevD.68.105012

PACS number(s): 03.70.+k, 04.60.Ds, 98.80.Cq

I. INTRODUCTION

Spacetime noncommutativity is one of the key new hints which follow from recent developments in quantum field theory. It has been recently realized, in particular, that a consequence of string theory [1,2] is that the structure of spacetime becomes noncommutative [3], which can be described loosely as an analog of a quantum phase space, in terms of the algebra generated by noncommuting coordinates $[x^\mu, x^\nu] = i\theta^{\mu\nu}$ with $\theta^{\mu\nu}$ an antisymmetric tensor.¹ The idea behind spacetime noncommutativity is very much inspired by quantum mechanics. A quantum spacetime is defined by replacing canonical variables with self-adjoint operators which obey Heisenberg-like commutation relations $[x^\mu, x^\nu] = i\theta^{\mu\nu}$, and can be viewed as the smearing out of a classical manifold, with the notion of a point replaced with that of a *Planck cell*. It was von Neumann who first attempted to rigorously describe such a quantum “space” and he called this study “pointless geometry,” referring to the fact that the notion of a point in a quantum phase space is meaningless because of the Heisenberg uncertainty principle of quantum mechanics. This led to the theory of von Neumann algebras and was essentially the birth of *noncommutative geometry*, referring to the study of topological spaces whose commutative C^* algebras of functions are replaced by noncommutative algebras [3–8]. The idea of noncommutative geometry was revived in the 1980s by Connes [5] and others, who generalized the notion of a differential structure to the noncommutative setting, i.e. to arbitrary C^* algebras. A theory on a noncommutative space replaces the noncommutativity

of operators associated with spacetime coordinates with a deformation of the algebra of functions defined on spacetime. In this context classical general relativity would break down at the Planck scale because spacetime would no longer be described by a differentiable manifold, and at these length scales quantum gravitational fluctuations become large and cannot be ignored. We stress, however, that the form of noncommutative geometry we are interested in is not directly related to current string theories (see below).

In the past few years several authors [9–11], including some of the present authors [12–16], have considered the possible effects of noncommutative geometry and Planck scale physics in cosmology. In particular, it has been shown that deformation of spacetime and/or phase space algebras may lead to several interesting features in the power spectrum of primordial perturbations produced during the inflationary era [17,18,19,20]. In all these investigations, however, $\theta^{\mu\nu}$ has been taken to be constant, or with an *a priori* modeled time evolution. In view of general covariance one may expect that $\theta^{\mu\nu}$ should be rather considered as a dynamical tensor, coupled to gravity and possibly affecting the cosmological evolution of the early Universe. This is actually a crucial point which deserves a thorough treatment, here summarized by relying in part upon Ref. [16]. On the one hand, it is true that, if one looks at the interplay between string theory and noncommutative geometry, one has to consider a constant B field and hence a constant $\theta^{\mu\nu}$ [21]. On the other hand, the commutator of the x^μ is a tensor, whose transformation under boosts yields nonvanishing spacetime components. Moreover, possible violations of unitarity are taken care of by imposing the conditions [16]

$$\theta^{\mu\nu}\theta_{\mu\nu} > 0, \quad \varepsilon_{\mu\nu\rho\sigma}\theta^{\mu\nu}\theta^{\rho\sigma} = 0.$$

From the point of view of general formalism, it is therefore legitimate to address the question of whether a broader noncommutative picture can be consistently built. This implies a departure from current models which only exploit a constant

*Electronic address: digrezia@na.infn.it

†Electronic address: giampiero.esposito@na.infn.it

‡Electronic address: funel@fe.infn.it

§Electronic address: mangano@na.infn.it

||Electronic address: gennaro.miele@na.infn.it

¹Hereafter we use the natural units $\hbar = c = 1$.

$\theta^{\mu\nu}$, and suggests starting from a nonlocal action functional with $*$ product of fields in the presence of nonvanishing spacetime curvature [16]. We therefore assume, hereafter, that the background geometry remains a classical pseudo-Riemannian geometry endowed with a Levi-Civita connection ∇ , while the $*$ product of scalar fields at the same spacetime point is defined by [16]

$$\begin{aligned} \varphi(x) * \psi(x) &\equiv \sum_{k=0}^{\infty} \frac{1}{k!} (i/2)^k \theta^{\mu_1 \nu_1}(x) \dots \theta^{\mu_k \nu_k}(x) \\ &\times (\nabla_{\mu_1} \dots \nabla_{\mu_k} \varphi)(\nabla_{\nu_1} \dots \nabla_{\nu_k} \psi). \end{aligned} \quad (1.1)$$

Similarly, having treated classically the geometry, we assume for tensor fields that

$$F_{\lambda_1 \dots \lambda_s} * F^{\lambda_1 \dots \lambda_s} \equiv g^{\lambda_1 r_1} \dots g^{\lambda_s r_s} F_{\lambda_1 \dots \lambda_s} * F_{r_1 \dots r_s}, \quad (1.2)$$

where

$$\begin{aligned} F_{\lambda_1 \dots \lambda_s} * F_{r_1 \dots r_s} &\equiv \sum_{k=0}^{\infty} \frac{1}{k!} (i/2)^k \theta^{\mu_1 \nu_1}(x) \dots \theta^{\mu_k \nu_k}(x) \\ &\times (\nabla_{\mu_1} \dots \nabla_{\mu_k} F_{\lambda_1 \dots \lambda_s}) \\ &\times (\nabla_{\nu_1} \dots \nabla_{\nu_k} F_{r_1 \dots r_s}). \end{aligned} \quad (1.3)$$

As is stressed in Ref. [16], the occurrence of covariant derivatives in our definitions (1.1) and (1.3) spoils associativity of the $*$ product. However, noncommutative effects are already present at quadratic order in $\theta^{\mu\nu}$, and our $\theta^{\mu\nu}$ will be taken to be sufficiently small so that higher order terms in the action functional are negligible.

Another important goal to be pursued is a fully consistent study of primordial perturbations, which should take into account the $\theta^{\mu\nu}$ perturbations as well, generalizing the formalism of gauge-invariant perturbations [22] to anisotropic background metrics.

As a first step in this program, in this paper we discuss the possible dynamical evolution of a background, time-dependent antisymmetric tensor [23–38] in two possible scenarios: a free massless $\theta^{\mu\nu}$ in presence of a cosmological constant, the latter being introduced as the easiest way to trigger an inflationary dynamics, and a more general scenario where the noncommutative field $\theta^{\mu\nu}$ is coupled to a scalar inflaton. As mentioned the presence of $\theta^{\mu\nu}$ breaks the isotropy of the Universe, and hence only spatial homogeneity is preserved, leading in turn to a dependence of $\theta^{\mu\nu}$ on time only. In this framework the appropriate geometry for the Universe is therefore a Bianchi type I model. In Sec. II we obtain a nonlinear system of the background equations in the presence of inflaton plus a coupling term between inflation and $\theta^{\mu\nu}$, and the energy-momentum tensor. In Sec. III we consider the simpler cosmological term model which however can be worked out analytically, at least for those initial conditions which are of some interest and may lead to an early stage where the energy-momentum tensor is dominated

by $\theta^{\mu\nu}$. Concluding remarks and open problems are presented in Sec. IV, while relevant details are given in the Appendix.

II. ANTISYMMETRIC TENSOR PLUS SCALAR FIELD COUPLED TO GRAVITY

In order to describe a field dynamics which might lead to anisotropy in the early Universe, the appropriate model is a Bianchi type I Universe (as we stated before) if one wants to preserve the spatial homogeneity. In this case the line element can be written as

$$ds^2 = dt^2 - \sum_{i=1}^3 a_i^2(t) (dx^i)^2, \quad (2.1)$$

and correspondingly the nonvanishing connection coefficients are (no summation over i is here meant)

$$\Gamma_{ij}^0 = \delta_{ij} \dot{a}_i a_i, \quad \Gamma^i_{0i} = \frac{\dot{a}_i}{a_i}, \quad \forall i, j = 1, 2, 3, \quad (2.2)$$

and for the Ricci tensor one has

$$R_0^0 = - \sum_{i=1}^3 \frac{\ddot{a}_i}{a_i}, \quad R_i^j = - \delta_i^j \left(\frac{\ddot{a}_i}{a_i} + \frac{\dot{a}_i}{a_i} \sum_{k \neq i} \frac{\dot{a}_k}{a_k} \right). \quad (2.3)$$

We consider a model in which there are both the antisymmetric tensor responsible of noncommutativity of spacetime and a minimally coupled massive scalar field which drives the inflation. The corresponding nonlocal action reads

$$\begin{aligned} S = \int d^4x \sqrt{-g} &\left[- \frac{R}{16\pi G} + \frac{1}{12} H_{\mu\nu\sigma} * H^{\mu\nu\sigma} + \frac{1}{2} \varphi_{;\mu} * \varphi^{;\mu} \right. \\ &\left. - \frac{m^2}{2} \varphi * \varphi - \frac{\lambda}{2} (\varphi * \varphi) * (\theta_{\mu\nu} * \theta^{\mu\nu}) \right], \end{aligned} \quad (2.4)$$

where the part involving λ is here introduced to mimic a “time-dependent” mass term for $\theta^{\mu\nu}$, and

$$H_{\mu\nu\sigma} \equiv \nabla_\mu \theta_{\nu\sigma} + \nabla_\nu \theta_{\sigma\mu} + \nabla_\sigma \theta_{\mu\nu} \quad (2.5)$$

is the field strength associated to the antisymmetric tensor $\theta_{\mu\nu} = -\theta_{\nu\mu}$ (hereafter, Greek indices run from 0 through 3, whereas Latin indices run from 1 through 3). It should be noticed that the kinetic term for $\theta_{\mu\nu}$ is inspired by a generalization of the Maxwell theory [34] and that in Eq. (2.5) only the effects of partial derivatives survive.² Now, by virtue of the definitions (1.1)–(1.3), one finds

$$H_{\mu\nu\sigma} * H^{\mu\nu\sigma} = H_{\mu\nu\sigma} H^{\mu\nu\sigma} + \mathcal{O}(\theta^3), \quad (2.6)$$

$$\varphi * \varphi = \varphi^2 - \frac{1}{8} \theta^{\mu\nu} \theta^{\rho\sigma} (\nabla_\mu \nabla_\rho \varphi)(\nabla_\nu \nabla_\sigma \varphi) + \mathcal{O}(\theta^3), \quad (2.7)$$

$$(\varphi * \varphi) * (\theta_{\mu\nu} * \theta^{\mu\nu}) = \varphi^2 \theta_{\mu\nu} \theta^{\mu\nu} + \mathcal{O}(\theta^3). \quad (2.8)$$

²Our notation agrees with the one used, for example, in Ref. [35].

Thus, to second order in $\theta^{\mu\nu}$, which is relevant since $\theta^{\mu\nu}$ is taken to be sufficiently small, only kinetic and potential term for the scalar field contribute, but with vanishing coefficient, since the former changes by the amount [16] [integration by parts yields also a third term which however vanishes if $\varphi = \varphi(t)$ only]

$$\delta S_K = \frac{1}{32} \int d^4x \sqrt{-g} \theta^{\mu\nu} \theta^{\rho\sigma} (\nabla_\rho \nabla_\sigma \varphi) ([\nabla_\mu, \nabla_\nu] \nabla_\sigma \nabla^\tau \varphi), \quad (2.9)$$

and the latter changes by the amount [16]

$$\delta S_m = \frac{m^2}{32} \int d^4x \sqrt{-g} \theta^{\mu\nu} \theta^{\rho\sigma} R^\tau{}_{\sigma\mu\nu} (\partial_\tau \varphi) (\partial_\rho \varphi). \quad (2.10)$$

Since φ depends only on the time variable, both Eqs. (2.9) and (2.10) vanish in our Bianchi type I background. Thus, to quadratic order in $\theta^{\mu\nu}$, we end up with the local action functional

$$S = \int d^4x \sqrt{-g} \left[-\frac{R}{16\pi G} + \frac{1}{12} H_{\mu\nu\sigma} H^{\mu\nu\sigma} + \frac{1}{2} \varphi_{;\mu} \varphi^{;\mu} - V(\varphi) - \frac{\lambda}{2} \varphi^2 \theta_{\mu\nu} \theta^{\mu\nu} \right], \quad (2.11)$$

where $V(\varphi) \equiv (m^2/2)\varphi^2$ hereafter. At this stage, the resulting energy-momentum tensor is given by

$$T_\alpha^\beta = \delta_\alpha^\beta \left[-\frac{1}{12} H_{\mu\nu\sigma} H^{\mu\nu\sigma} - \frac{1}{2} \varphi_{;\mu} \varphi^{;\mu} + V(\varphi) + \frac{\lambda}{2} \varphi^2 \theta_{\mu\nu} \theta^{\mu\nu} \right] + \frac{1}{2} H_{\alpha\mu\nu} H^{\beta\mu\nu} + \varphi_{;\alpha} \varphi^{;\beta} - 2\lambda \varphi^2 \theta_{\alpha\nu} \theta^{\beta\nu}. \quad (2.12)$$

By using the expressions (2.11), (2.12) the resulting equations of motion are

$$\nabla^\mu H_{\mu\nu\sigma} + 2\lambda \varphi^2 \theta_{\nu\sigma} = 0, \quad (2.13)$$

$$\nabla_\mu \nabla^\mu \varphi + \lambda \varphi \theta_{\mu\nu} \theta^{\mu\nu} + \frac{\delta V}{\delta \varphi} = 0, \quad (2.14)$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}. \quad (2.15)$$

To be consistent with the spacetime homogeneity ansatz we assume that all fields are depending on time only. In this case the above equations read (hereafter $H_i \equiv \dot{a}_i/a_i$)

$$\lambda \varphi^2 \theta_{0i} = 0, \quad (2.16)$$

$$\ddot{\theta}_{ij} + \left(\sum_{k=1}^3 H_k \right) \dot{\theta}_{ij} - 2(H_i + H_j) \dot{\theta}_{ij} + 2\lambda \varphi^2 \theta_{ij} = 0, \quad (2.17)$$

$$\ddot{\varphi} + \left(\sum_{k=1}^3 H_k \right) \dot{\varphi} + \lambda \varphi \theta^{\mu\nu} \theta_{\mu\nu} + \frac{\delta V}{\delta \varphi} = 0, \quad (2.18)$$

$$\sum_{k=1}^3 (\dot{\theta}_{ik} \dot{\theta}_{jk} - 2\lambda \varphi^2 \theta_{ik} \theta_{jk}) = 0, \quad \forall i \neq j \quad (2.19)$$

$$\sum_i \frac{\ddot{a}_i}{a_i} = 8\pi G \left(V(\varphi) - \dot{\varphi}^2 - \frac{\lambda}{2} \varphi^2 \theta_{\mu\nu} \theta^{\mu\nu} \right), \quad (2.20)$$

$$\frac{\ddot{a}_i}{a_i} + \frac{\dot{a}_i}{a_i} \sum_{k \neq i} \frac{\dot{a}_k}{a_k} = 8\pi G \left(\frac{1}{6} H_{\mu\nu\sigma} H^{\mu\nu\sigma} + V(\varphi) - \frac{\lambda}{2} \varphi^2 \theta_{\mu\nu} \theta^{\mu\nu} - \frac{1}{2} H_{i\nu\sigma} H^{i\nu\sigma} + 2\lambda \varphi^2 \theta_{i\nu} \theta^{i\nu} \right). \quad (2.21)$$

Of course, Eqs. (2.16) and (2.17) result from Eq. (2.13), whereas Eq. (2.18) is obtained from Eq. (2.14). The remaining equations are the Einstein equations where, in particular, Eq. (2.19) provides a Bianchi type I Universe.

Using Eq. (2.16) one easily gets $\theta^{0i} = \theta_{0i} = 0$. By virtue of Eq. (2.19) one can show that the only possible solution has only one nonvanishing component of θ_{ij} , e.g. θ_{12} . Moreover, since isotropy is broken and the residual invariance is $SO(2)$, it is rather natural to choose $a_1 = a_2 \equiv a_\perp$, $a_3 \equiv a_L$, with corresponding Hubble parameters $H_\perp \equiv \dot{a}_\perp/a_\perp$, $H_L \equiv \dot{a}_L/a_L$. In this case the equations become

$$\dot{\theta}_{12} + (H_L - 2H_\perp) \theta_{12} + 2\lambda \varphi^2 \theta_{12} = 0, \quad (2.22)$$

$$\ddot{\varphi} + (H_L + 2H_\perp) \dot{\varphi} + \lambda \frac{\varphi \theta_{12}^2}{a_\perp^4} + \frac{\delta V}{\delta \varphi} = 0, \quad (2.23)$$

$$H_\perp^2 + 2H_\perp H_L = 8\pi G \left(\frac{1}{2} \dot{\varphi}^2 + V(\varphi) + \frac{1}{2} \frac{\dot{\theta}_{12}^2}{a_\perp^4} + \lambda \frac{\varphi^2 \theta_{12}^2}{a_\perp^4} \right), \quad (2.24)$$

$$\dot{H}_\perp + H_\perp (H_L + 2H_\perp) = 8\pi G \left(V(\varphi) + \lambda \frac{\varphi^2 \theta_{12}^2}{a_\perp^4} \right), \quad (2.25)$$

$$\dot{H}_L + H_L (H_L + 2H_\perp) = 8\pi G \left(\frac{\dot{\theta}_{12}^2}{a_\perp^4} + V(\varphi) - \lambda \frac{\varphi^2 \theta_{12}^2}{a_\perp^4} \right). \quad (2.26)$$

Let us define the mass parameter $\mu \equiv m_{Pl}/\sqrt{8\pi} = 1/\sqrt{8\pi G}$. In terms of this quantity we can write a dimensionless system of differential equations. By defining

$$x \equiv \varphi/\mu, \quad y \equiv \theta_{12}/(\mu a_\perp^2), \quad \tilde{H}_\perp \equiv H_\perp/\mu,$$

$$\tilde{H}_L \equiv H_L/\mu, \quad \tilde{V}(x) \equiv V(\mu x)/\mu^4,$$

and using the dimensionless time $\tau \equiv t\mu$ we get

$$y'' + (\tilde{H}_L + 2\tilde{H}_\perp)y' + 2(\lambda x^2 - 2\tilde{H}_\perp^2 + \tilde{V}(x) + \lambda x^2 y^2)y = 0, \quad (2.27)$$

$$x'' + (\tilde{H}_L + 2\tilde{H}_\perp)x' + \lambda x y^2 + \frac{\delta \tilde{V}}{\delta x} = 0, \quad (2.28)$$

$$\tilde{H}_\perp^2 + 2\tilde{H}_\perp \tilde{H}_L = \frac{x'^2}{2} + \tilde{V}(x) + \frac{1}{2}(y' + 2\tilde{H}_\perp y)^2 + \lambda x^2 y^2, \quad (2.29)$$

$$\tilde{H}_\perp' + \tilde{H}_\perp(\tilde{H}_L + 2\tilde{H}_\perp) = \tilde{V}(x) + \lambda x^2 y^2, \quad (2.30)$$

$$\tilde{H}_L' + \tilde{H}_L(\tilde{H}_L + 2\tilde{H}_\perp) = (y' + 2\tilde{H}_\perp y)^2 + \tilde{V}(x) - \lambda x^2 y^2, \quad (2.31)$$

where the “prime” denotes the derivative with respect to τ . The truly independent equations are given by Eqs. (2.27), (2.28), (2.30) jointly with

$$\begin{aligned} & (\tilde{H}_L + 2\tilde{H}_\perp) \\ &= \frac{1}{2\tilde{H}_\perp} \left(\frac{x'^2}{2} + \tilde{V}(x) + \frac{1}{2}(y' + 2\tilde{H}_\perp y)^2 + \lambda x^2 y^2 + 3\tilde{H}_\perp^2 \right). \end{aligned} \quad (2.32)$$

On recalling the e -folding definition in the orthogonal direction, i.e. $N_\perp(\tau) \equiv \log(a_\perp(\tau)/a_\perp(\tau_i))$, one can easily prove that

$$\frac{d}{d\tau} = \tilde{H}_\perp \frac{d}{dN_\perp}, \quad (2.33)$$

$$\frac{d^2}{d\tau^2} = \tilde{H}_\perp^2 \frac{d^2}{dN_\perp^2} + \tilde{H}_\perp' \frac{d}{dN_\perp}. \quad (2.34)$$

By using N_\perp as the evolution parameter and defining $z \equiv \tilde{H}_\perp^2$ we find

$$\begin{aligned} & z \frac{d^2 y}{dN_\perp^2} + [\tilde{V}(x) + \lambda x^2 y^2] \frac{dy}{dN_\perp} + 2[\lambda x^2 - 2z + \tilde{V}(x) + \lambda x^2 y^2] y \\ &= 0, \end{aligned} \quad (2.35)$$

$$z \frac{d^2 x}{dN_\perp^2} + [\tilde{V}(x) + \lambda x^2 y^2] \frac{dx}{dN_\perp} + \lambda x y^2 + \frac{\delta \tilde{V}}{\delta x} = 0, \quad (2.36)$$

$$\frac{dz}{dN_\perp} = \tilde{V}(x) + \lambda x^2 y^2 - \frac{1}{2} z \left[\left(\frac{dx}{dN_\perp} \right)^2 + \left(\frac{dy}{dN_\perp} + 2y \right)^2 + 6 \right]. \quad (2.37)$$

In Figs. 1–4 we show, on choosing different initial conditions, that a range of N_\perp exists in which the θ_{12} component of the θ field dominates on the φ field; of course, this results from a special choice of initial conditions (see following section for a more thorough discussion of underlying issues). Moreover, H_L dominates as well, i.e., there is anisotropy.

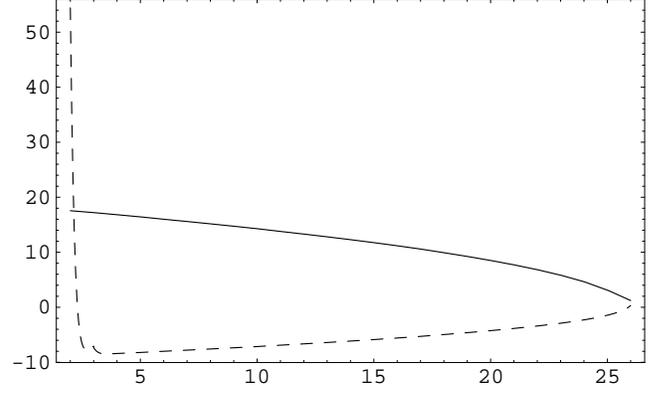


FIG. 1. The x axis corresponds to the e -folding parameter N_\perp and on the y axis we plot H_L and H_\perp with dashed and continuous line, respectively; we take $\tilde{V}(x) = qx^2$, with initial conditions: $H_\perp^2 = 10, \theta_{12} = 100, d\theta_{12}/dN_\perp = 0, \varphi = 10, d\varphi/dN_\perp = 0$, and coupling constants $\lambda = 10^{-4}, q = 10$.

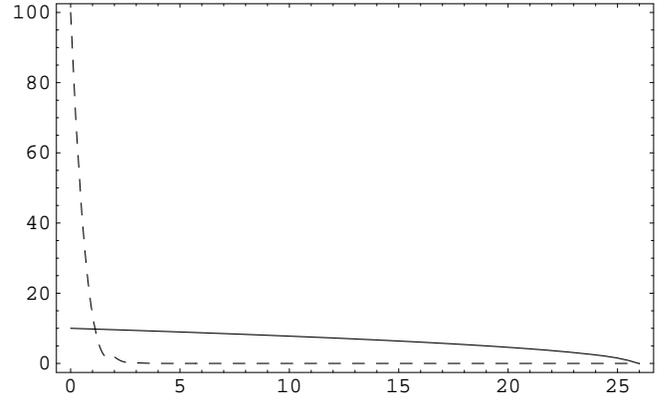


FIG. 2. The x axis corresponds to the e -folding parameter N_\perp , and on the y axis we plot θ_{12} and φ with dashed and continuous line, respectively, with same initial conditions of the previous figure.

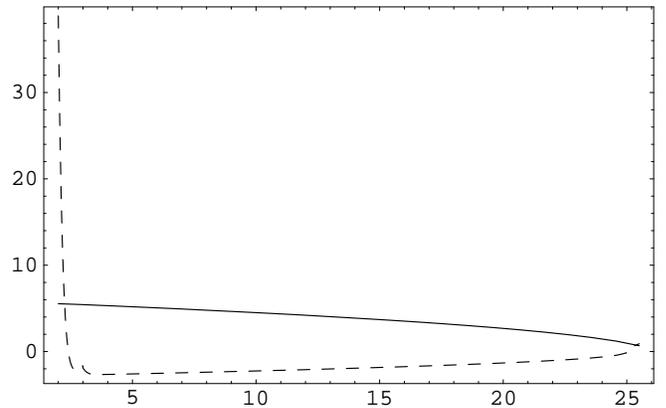


FIG. 3. We plot H_L and H_\perp as functions of the e -folding parameter N_\perp , with $\tilde{V}(x) = qx^2$, and initial conditions: $H_\perp^2 = 10, \theta_{12} = 100, d\theta_{12}/dN_\perp = 1, \varphi = 10, d\varphi/dN_\perp = 0$, and coupling constants $\lambda = -10^{-5}, q = 1$.

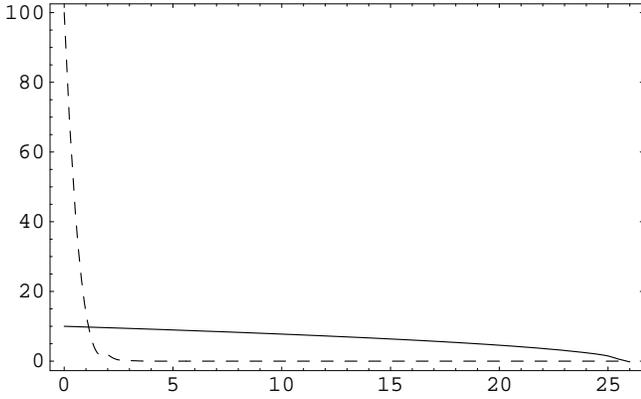


FIG. 4. The fields θ_{12} and φ are plotted as functions of the e -folding parameter N_{\perp} .

Outside this range, θ_{12} approaches zero and φ dominates driving the isotropization, which can be seen in the figures where H_L and H_{\perp} reach the same constant value.

III. COSMOLOGICAL CONSTANT AND MASSLESS ANTISYMMETRIC TENSOR

While the equations in the previous section can only be solved by means of numerical methods, one can envisage a simpler scenario which embodies the main features but allows for an analytic approach. For this purpose we consider a Lagrangian density for the antisymmetric tensor $\theta_{\mu\nu}$ and gravity including a cosmological constant term [with $H_{\mu\nu\rho}$ defined in Eq. (2.5)]

$$\begin{aligned} \mathcal{L} &= \sqrt{-g} \left[-\frac{1}{16\pi G} (R + 2\Lambda) + \frac{1}{12} H_{\mu\nu\rho} * H^{\mu\nu\rho} \right] \\ &= \sqrt{-g} \left[-\frac{1}{16\pi G} (R + 2\Lambda) + \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} \right] \\ &\quad + \mathcal{O}(\theta^3). \end{aligned} \quad (3.1)$$

Thus, on working to quadratic order in $\theta^{\mu\nu}$, the resulting action is invariant under the gauge transformations

$$\theta_{\mu\nu} \rightarrow \theta_{\mu\nu} + \partial_{\mu}\chi_{\nu} - \partial_{\nu}\chi_{\mu}, \quad (3.2)$$

and the $\theta_{\mu\nu}$ dynamics can be also recast in a different form by introducing the pseudoscalar Kalb-Ramond field χ via the duality transformation $\partial_{\mu}\chi = \epsilon_{\mu\nu\rho\sigma} H^{\nu\rho\sigma}$, with $\epsilon_{\mu\nu\rho\sigma}$ the volume four-form. Actually, since we are assuming homogeneity and a purely time-dependent θ field this implies a time-independent χ field with a linear dependence on spatial coordinates [see Eq. (3.15)].

Using the results of the previous section in the particular case we are considering, the resulting equations of motion are

$$\nabla^{\mu} H_{\mu\nu\sigma} = 0, \quad (3.3)$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}(\theta) + \Lambda g_{\mu\nu}, \quad (3.4)$$

and the energy-momentum tensor reads

$$T_{\mu}^{\nu}(\theta) = \frac{1}{2} H_{\mu\lambda\rho} H^{\nu\lambda\rho} - \frac{\delta^{\nu}_{\mu}}{12} H_{\lambda\sigma\rho} H^{\lambda\sigma\rho}. \quad (3.5)$$

As in the previous section, we assume Bianchi type I geometry. The equations of motion (3.3) and (3.4) now read

$$\ddot{\theta}_{ij} + \left(\sum_{k=1}^3 H_k \right) \dot{\theta}_{ij} - 2(H_i + H_j) \dot{\theta}_{ij} = 0, \quad (3.6)$$

$$\sum_{k=1}^3 \dot{\theta}_{ik} \dot{\theta}_{jk} = 0, \quad \forall i \neq j \quad (3.7)$$

$$\sum_{i=1}^3 \frac{\ddot{a}_i}{a_i} = \Lambda, \quad (3.8)$$

$$\frac{\ddot{a}_i}{a_i} + \frac{\dot{a}_i}{a_i} \sum_{k \neq i} \frac{\dot{a}_k}{a_k} = \Lambda - 4\pi G \left(H_{i\nu\sigma} H^{i\nu\sigma} - \frac{1}{3} H_{\mu\nu\sigma} H^{\mu\nu\sigma} \right). \quad (3.9)$$

Unlike the previous section, Eq. (2.16) is now replaced by a gauge-fixing condition. In the following we will use the gauge $\theta_{0i} = 0$. A detailed analysis of the admissibility of this gauge fixing is reported in the Appendix.

The effect of Eqs. (3.6) and (3.7) is to restrict severely the possible choice of initial conditions for $\dot{\theta}_{ij}$. In particular either all $\dot{\theta}_{ij}(0)$ values vanish and thus θ_{ij} is a constant tensor, or at most only one component $\dot{\theta}_{ij}(0)$ is nonvanishing. Since the antisymmetric tensor enters the field equations only through its first- and second-order time derivatives the case of a constant θ tensor reduces to a pure Λ -term cosmology and is thus uninteresting for our analysis. We therefore consider the case $\dot{\theta}_{13}(0) = \dot{\theta}_{23}(0) = 0$, $\dot{\theta}_{12}(0) \neq 0$, and we eventually get

$$\ddot{\theta}_{12} + (H_L - 2H_{\perp}) \dot{\theta}_{12} = 0, \quad (3.10)$$

$$H_{\perp}^2 + 2H_{\perp}H_L = \Lambda + 4\pi G \frac{\dot{\theta}_{12}^2}{a_{\perp}^4}, \quad (3.11)$$

$$\dot{H}_{\perp} + H_{\perp}(H_L + 2H_{\perp}) = \Lambda, \quad (3.12)$$

$$\dot{H}_L + H_L(H_L + 2H_{\perp}) = \Lambda + 8\pi G \frac{\dot{\theta}_{12}^2}{a_{\perp}^4}. \quad (3.13)$$

Equation (3.10) determines the behavior of θ_{12} in terms of the scale factors

$$\dot{\theta}_{12} = \dot{\theta}_{12}(0) \frac{a_{\perp}^2}{a_L}, \quad (3.14)$$

where in Eq. (3.14), by virtue of the arbitrariness in the choice of $a_L(0)$ and $a_{\perp}(0)$, we have assumed, without loss

of generality, $a_{\perp}(0) = a_L(0) = 1$. Notice that this immediately gives the spacetime dependence of the Kalb-Ramond scalar field

$$\chi = \dot{\theta}_{12}(0)x^3 + \chi_0, \quad (3.15)$$

where, according to Eq. (2.1), x^3 denotes the longitudinal spatial coordinate. Equation (3.15) is actually consistent with spatial homogeneity, while it breaks isotropy.

The previous equations can be recast in a simpler and more useful form by introducing the variable

$$\xi \equiv \frac{1}{3} \log(a_{\perp}^2 a_L), \quad (3.16)$$

which represents the average e -folding. Hence one gets

$$H_{\perp}^2 + 2H_{\perp}H_L = \Lambda + \frac{c^2}{2a_L^2}, \quad (3.17)$$

$$(H'_{\perp} + 3H_{\perp})(H_L + 2H_{\perp}) = 3\Lambda, \quad (3.18)$$

$$(H'_L + 3H_L)(H_L + 2H_{\perp}) = 3\Lambda + 3\frac{c^2}{a_L^2}, \quad (3.19)$$

where $c^2 \equiv 8\pi G(\dot{\theta}_{12}(0))^2$ and $' \equiv d/d\xi$. It is also convenient to set

$$a_L(\xi) = \exp\left\{\xi + \frac{\Omega(\xi)}{2}\right\}, \quad (3.20)$$

and hence from Eq. (3.16)

$$a_{\perp}(\xi) = \exp\left\{\xi - \frac{\Omega(\xi)}{4}\right\}. \quad (3.21)$$

Furthermore, we introduce the mean Hubble parameter, $H \equiv (H_L + 2H_{\perp})/3$, and the asymmetry function $h \equiv H_L - H_{\perp}$. Since $a_L(0) = a_{\perp}(0) = 1$, Eqs. (3.20) and (3.21) yield $\Omega(0) = 0$. Thus one gets

$$\frac{1}{2}(H^2)' + 3H^2 = \Lambda + \frac{c^2}{3} \exp\{-2\xi - \Omega(\xi)\}, \quad (3.22)$$

$$h' + 3h = \frac{c^2}{H} \exp\{-2\xi - \Omega(\xi)\}, \quad (3.23)$$

$$H^2 - \frac{1}{9}h^2 = \frac{\Lambda}{3} + \frac{c^2}{6} \exp\{-2\xi - \Omega(\xi)\}, \quad (3.24)$$

where the last equation is the Hamiltonian constraint. Actually the functional dependence of h in terms of H and Ω can be already obtained by its very definition

$$h \equiv H \left(\frac{a'_L}{a_L} - \frac{a'_{\perp}}{a_{\perp}} \right) = \frac{3}{4} H \Omega', \quad (3.25)$$

showing that Ω' is directly related to the isotropy breaking. In the following we will consider an expanding Universe in both longitudinal and transverse directions, $H_L, H_{\perp} > 0$. In this case we note that $-2 \leq \Omega' < 4$. We will use this information later on. Notice that $\Omega' = 4$ would correspond to $H_{\perp} = 0$, which is forbidden by Eq. (3.17).

Equation (3.22) is a first-order inhomogeneous equation whose solution reads

$$H^2(\xi) = \frac{\Lambda}{3} + \left(H^2(0) - \frac{\Lambda}{3} \right) \exp(-6\xi) + \frac{2}{3} c^2 \exp(-6\xi) \int_0^{\xi} \exp(4\zeta - \Omega(\zeta)) d\zeta, \quad (3.26)$$

which, together with Eq. (3.25), gives the Hubble parameters in terms of the function $\Omega(\xi)$. The latter can be determined as the solution of a second-order differential equation, obtained by inserting Eq. (3.25) into the differential equation (3.23) and the Hamiltonian constraint (3.24). The latter operation yields

$$H^2 = \left(1 - \frac{\Omega'^2}{16} \right)^{-1} \left(\frac{\Lambda}{3} + \frac{c^2}{6} \exp(-2\xi - \Omega(\xi)) \right).$$

If the differential equation (3.22) is also exploited to express HH' , we eventually find

$$\Omega''(\xi) + 2\Omega'(\xi) \left(1 - \frac{\Omega'(\xi)^2}{16} \right) \frac{3\Lambda + c^2 \exp(-2\xi - \Omega(\xi))}{2\Lambda + c^2 \exp(-2\xi - \Omega(\xi))} - 8 \left(1 - \frac{\Omega'(\xi)^2}{16} \right) \frac{c^2 \exp(-2\xi - \Omega(\xi))}{2\Lambda + c^2 \exp(-2\xi - \Omega(\xi))} = 0, \quad (3.27)$$

with initial conditions $\Omega(0) = 0$, $\Omega'(0) = (4/3)h(0)/H(0)$.

This equation can be hardly solved analytically in the general case, but it reduces to much simpler forms in the two regimes, when either the cosmological constant or the $\theta_{\mu\nu}$ fields dominate the energy-momentum tensor. At very early times the latter is likely to be largely the dominant component. We can rewrite the parameter c^2 as $c^2 = m_{NC}^4/m_{Pl}^2$, with m_{NC} the scale where the classical picture of spacetime manifold breaks down. The order of magnitude of Λ can be instead constrained by the fact that it drives inflation in the late stages, and represents in the slow-roll approximation the potential of the inflaton field. This is severely bounded by the fact that it should account for the correct amplitude of primordial perturbations. For example, for a polynomial potential $V(\phi) = \lambda \phi^n/n$, the requirement of slow-roll dynamics and perturbation amplitudes of the order of 10^{-5} gives $\Lambda \leq 10^{-12} m_{Pl}^2$ [38]. As long as $m_{NC} \sim m_{Pl}$ the value of Λ is several orders of magnitude smaller than c^2 , so in this case the early dynamics is fully determined by $\theta_{\mu\nu}$. Smaller values of m_{NC} , such that $c^2 \sim \Lambda$, cannot be ruled out of course, but in this case Eq. (3.27) can only be solved numerically.

Hereafter we specialize to the case $m_{NC} \sim m_{Pl}$. At early times therefore Eq. (3.27) takes the simplified form

$$\Omega''(\xi) + 2(\Omega'(\xi) - 4) \left(1 - \frac{\Omega'(\xi)^2}{16} \right) = 0. \quad (3.28)$$

Since the $\theta_{\mu\nu}$ contribution is diluted with expansion as a_L^2 , this equation holds approximatively for values of ξ smaller than the value ξ_* such that $\Lambda \sim c^2/a_L^2(\xi_*)$, i.e.

$$\xi_* + \frac{\Omega(\xi_*)}{2} \sim \log \frac{c^2}{\Lambda}. \quad (3.29)$$

Later expansion is instead driven by Λ and hence, by neglecting the $\theta_{\mu\nu}$ contribution, we have

$$\Omega''(\xi) + 3\Omega'(\xi) \left(1 - \frac{\Omega'(\xi)^2}{16} \right) = 0. \quad (3.30)$$

This equation of course should give back the isotropization phase leading to a de Sitter phase.

We begin by studying the early time evolution. Before doing this it is worth discussing the values of initial conditions for Hubble parameters. From Eqs. (3.12) and (3.13) we see that $\theta_{\mu\nu}$ acts as source for a_L only, the evolution of a_\perp being expected to be much slower. In other words, *the anti-symmetric tensor drives the expansion of the longitudinal scale factor only*. The most natural choice at $\xi=0$ is therefore $H_L(0) \gg H_\perp(0)$, that is to say $\Omega'(0) = 4 - 12\epsilon$, with $\epsilon = H_\perp(0)/H_L(0) \ll 1$. In this case the solution of Eq. (3.28) is particularly simple. On defining $\lambda(\xi) \equiv \Omega(\xi) - 4\xi$, the latter reduces to

$$\lambda''(\xi) = \lambda'(\xi)^2 \left(1 + \frac{\lambda'(\xi)}{8} \right). \quad (3.31)$$

Upon considering $\log \lambda'(\xi) \equiv y(\xi)$, this equation is solved exactly by separation of variables and subsequent integration, and yields (C being an integration constant)

$$\xi + C = -\exp(-y) - \frac{y}{8} + \frac{1}{8} \log \left(1 + \frac{1}{8} \exp(y) \right). \quad (3.32)$$

On choosing the initial conditions

$$\lambda(0) = 0, \quad \lambda'(0) = -12\epsilon, \quad (3.33)$$

the approximate solution at small ϵ reads, with a very good accuracy,

$$\lambda(\xi) = -\frac{1}{3} \log \left(1 + 12\epsilon \left(1 - \frac{3}{2}\epsilon \right) \xi \right). \quad (3.34)$$

For example, we have checked numerically that this solution is accurate at the per thousand level up to $\xi=10^2$, if $\epsilon < 10^{-2}$. These ranges fully cover the early stage. In fact Eq.

(3.28) no longer holds at ξ_* , see Eq. (3.29), which, in view of the logarithmic behavior of $\lambda(\xi)$, for $\epsilon < 10^{-2}$ is approximatively fixed by

$$\xi_* \sim \frac{1}{3} \log \frac{c^2}{\Lambda}. \quad (3.35)$$

If $c^2/\Lambda \ll 10^{12}$ we get $\xi_* \ll 15$. On using Eq. (3.34) it is now possible to determine the evolution of the Hubble parameters as functions of ξ . From Eqs. (3.26) and (3.25) we get

$$H^2(\xi) \sim \exp(-6\xi) \left[H^2(0) + \frac{2}{3} c^2 \left(1 - \frac{3}{4}\epsilon \right) \xi \right], \quad (3.36)$$

$$h(\xi) \sim 3 \exp(-3\xi) H(0) [1 - 3\epsilon]. \quad (3.37)$$

Late-time evolution is ruled by Eq. (3.30). Since $\Omega'(\xi) < 4$ it therefore follows that asymptotically Ω reaches a positive constant value Ω_∞ , so that h vanishes in the large- ξ limit as expected. In particular, for large ξ Eq. (3.30) can be linearized, i.e.

$$\Omega''(\xi) \sim -3\Omega'(\xi), \quad (3.38)$$

and hence it is immediate to get the approximate solution

$$H^2(\xi) \sim \frac{\Lambda}{3} + \left(H^2(\xi_*) - \frac{\Lambda}{3} \right) \exp(-6(\xi - \xi_*)), \quad (3.39)$$

$$h(\xi) \sim h(\xi_*) \exp(-3(\xi - \xi_*)). \quad (3.40)$$

Note that the action functional studied in the present section or in Sec. II coincides with the low-energy limit of string theory with a constant dilaton, *to lowest order in the expansion of the nonlocal action functional with * product of fields*. Readers interested in string cosmology can be referred, for example, to the work in Refs. [39,40]. In particular, the work in Ref. [41] has studied string cosmology with a time-dependent antisymmetric tensor in a Bianchi type I Universe, but as is stressed in Sec. IV of Ref. [41], since a pure radiation plus dilaton solution has $\phi \rightarrow \text{const}$, a late-time isotropic radiation-dominated solution is a contracting Universe. By contrast, in our model, when the Universe reaches an isotropic stage it is still expanding.

IV. CONCLUDING REMARKS

Motivated by cosmology and noncommutative geometry, we have investigated the effects of an antisymmetric tensor in a Bianchi type I early Universe. What we do only holds at an intermediate stage where departure from ordinary spacetime geometry can be appreciated, while lack of associativity of the resulting * product in curved spacetime is negligible [16]. Moreover, such a stage implies a departure from the models with constant $\theta^{\mu\nu}$ which are relevant for string theory [21]. With this understanding, our original results are as follows.

(i) In the first model, where the antisymmetric tensor θ

resulting from noncommutative geometry and a minimally coupled scalar field driving inflation both occur, we have found by numerical methods that a suitable range of the e -folding exists such that θ prevails on the scalar field and the longitudinal Hubble parameter prevails on H_{\perp} . This is the anisotropic era, but for larger values of the e -folding the antisymmetric tensor is damped and the scalar field dominates, leading in turn to isotropization. The very existence of the anisotropic era depends of course on the initial conditions chosen, but it appears interesting to have shown explicitly how the corresponding model can be built.

(ii) In the second model, again in a Bianchi type I Universe, gravity with a cosmological constant Λ is coupled to an antisymmetric tensor. The resulting nonlinear system of equations for the Hubble parameters H and h has been solved by first obtaining the differential equation (3.27) for the unknown function Ω , and then finding approximate solutions when the effects of $\theta_{\mu\nu}$ prevail upon Λ , or the other way around. An accurate analytic description of the process leading to an isotropic final state of the Universe has been therefore obtained.

An outstanding problem in sight is now the development of cosmological perturbations' theory [38] within such a framework, with the hope of obtaining quantitative information on the effect of θ on the formation of structure in the early Universe.

ACKNOWLEDGMENTS

We are indebted to Fedele Lizzi for enlightening conversations. The work of G. Esposito has been partially supported by PRIN 2002 "Sintesi"; the work of G. Mangano and G. Miele has been partially supported by COFIN 2002 "Fisica Astroparticellare."

APPENDIX

In this appendix we consider the admissibility of the gauge condition $\theta_{0i}=0$ from the point of view of constraint analysis, and we begin with the simplest model, with action functional (as in Secs. II and III, the starting point is a non-local action functional, which reduces to a local action by retaining only quadratic terms in $\theta^{\mu\nu}$)

$$S = \int \mathcal{L} \sqrt{-g} d^4x, \quad (\text{A1})$$

with Lagrangian density $\mathcal{L} \equiv \frac{1}{12} H^{\mu\nu\rho} H_{\mu\nu\rho}$ in a Bianchi type I background, and field strength given by cyclic permutations of $\nabla\theta$ terms, according to Eq. (2.5). Only the effects of partial derivatives survive in $H_{\mu\nu\rho}$, and hence we find, by virtue of Eq. (2.1),

$$\mathcal{L} = \frac{1}{12} H^{\mu\nu\rho} H_{\mu\nu\rho} = \frac{1}{4} g^{ij} g^{kl} \theta_{ik,0} \theta_{jl,0}, \quad (\text{A2})$$

since the assumption of spatial homogeneity implies that $\theta^{\mu\nu}$ can only depend on t . The term $g^{ij} \theta_{0i,0} \theta_{0j,0}$ is weighted with coefficient $3-3=0$ and hence does not occur in Eq. (A2). The Lagrangian L is obtained from \mathcal{L} by means of (recall that

the lapse function N is equal to 1 in a Bianchi type I background, and that h denotes the determinant of the induced three-metric)

$$L = \int \mathcal{L} \sqrt{h} d^3x, \quad (\text{A3})$$

and hence we find

$$\pi^{ij} \equiv \frac{\delta L}{\delta \theta_{ij,0}} = \frac{1}{2} \sqrt{h} g^{ir} g^{js} \theta_{rs,0}, \quad (\text{A4})$$

$$\pi^{0i} \equiv \frac{\delta L}{\delta \theta_{0i,0}} \approx 0. \quad (\text{A5})$$

Equation (A5) deserves some comments: since the Lagrangian is independent of $\theta_{0i,0}$, the momentum conjugate to θ_{0i} vanishes. More precisely, the π^{0i} can be seen as 3 primary constraints arising from the structure of the Lagrangian; as such, their vanishing is only a weak equation (≈ 0), because π^{0i} are well defined over the whole phase space of the theory, and only vanish on the constraint submanifold [42].

Our Lagrangian (A3) reads

$$L = \int \frac{1}{2} \pi^{ij} \theta_{ij,0} d^3x, \quad (\text{A6})$$

with corresponding canonical Hamiltonian

$$H_c \equiv \int \pi^{ij} \theta_{ij,0} d^3x - L = \int \frac{1}{2} \pi^{ij} \theta_{ij,0} d^3x. \quad (\text{A7})$$

By virtue of the primary constraints (A5), however, the Hessian matrix is singular, and the time evolution is only well defined when the effective Hamiltonian \hat{H} is considered. The latter is given by H_c plus a linear combination of primary constraints, i.e. [42]

$$\hat{H} = \int \left(\frac{1}{2} \pi^{ij} \theta_{ij,0} + \mu^i \pi_{0i} \right) d^3x. \quad (\text{A8})$$

Moreover, we are still free to impose supplementary (more frequently called "gauge") conditions, here chosen in the form

$$\theta_{0i} \approx 0. \quad (\text{A9})$$

By doing so, we choose to regard the gauge conditions as constraint equations, in much the same way as the Coulomb gauge can be treated as a constraint equation in Maxwell theory [43]. We are therefore working with an extended Hamiltonian

$$H_e \equiv \int \left(\frac{1}{2} \pi^{ij} \theta_{ij,0} + \mu^i \pi_{0i} + \lambda^i \theta_{0i} \right) d^3x, \quad (\text{A10})$$

where the Lagrange multipliers μ^i, λ^i can be evaluated by requiring preservation in time of the primary constraints π_{0i} and gauge constraints θ_{0i} . For this purpose, note first that Eq. (A4) yields

$$\theta_{ij,0} = \frac{2}{\sqrt{h}} \pi_{ij}, \quad (\text{A11})$$

and hence (with $\rho \equiv 2/\sqrt{h}$)

$$H_e = \int \left(\frac{\rho}{2} \pi^{ij} \pi_{ij} + \mu^i \pi_{0i} + \lambda^i \theta_{0i} \right) d^3x. \quad (\text{A12})$$

All our constraints are then trivially preserved, without giving rise to further constraints, because

$$\begin{aligned} \frac{d}{dt} \pi_{0i} \equiv \{ \pi_{0i}(\vec{x}, t), H_e \} &\approx \int \lambda^j(\vec{y}, t) \{ \pi_{0i}(\vec{x}, t), \theta_{0j}(\vec{y}, t) \} d^3y \\ &= -\lambda_i(\vec{x}, t), \end{aligned} \quad (\text{A13})$$

$$\begin{aligned} \frac{d}{dt} \theta_{0i} \equiv \{ \theta_{0i}(\vec{x}, t), H_e \} &\approx \int \mu^l(\vec{y}, t) \{ \theta_{0i}(\vec{x}, t), \pi_{0l}(\vec{y}, t) \} d^3y \\ &= \mu_i(\vec{x}, t). \end{aligned} \quad (\text{A14})$$

Note that, by virtue of our gauge constraints (A9), the set of constraints has been turned into the second class, a feature shared by all field theories after a gauge condition has been imposed [43,44].

For the model studied in Sec. III, the full Hamiltonian constraint (3.11), when expressed in integral form, is the sum of Eq. (A12) and of the gravitational contribution. The latter is obtained from spatial integration of

$$(16\pi G) G_{ijkl} \tilde{\pi}^{ij} \tilde{\pi}^{kl} + \frac{\sqrt{h} {}^{(3)}R}{16\pi G} + \Lambda \frac{\sqrt{h}}{8\pi G},$$

where $G_{ijkl} \equiv [1/(2\sqrt{h})](h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl})$ is the DeWitt supermetric on the space of Riemannian geometries on Σ [45], $\tilde{\pi}^{ij}$ is the momentum conjugate to the induced three-metric and ${}^{(3)}R$ is the three-dimensional scalar curvature (our sign for such a curvature is opposite to the one of Ref. [45]).

-
- [1] J. Polchinski, *String Theory. Vol. 1: An Introduction to the Bosonic String* (Cambridge University Press, Cambridge, England, 1998).
- [2] J. Polchinski, *String Theory. Vol. 2: Superstring Theory and Beyond* (Cambridge University Press, Cambridge, England, 1998).
- [3] A. Connes, M.R. Douglas, and A. Schwarz, *J. High Energy Phys.* **02**, 003 (1998).
- [4] A. Connes, *Noncommutative Geometry* (Academic, New York, 1994).
- [5] A. Connes, *C.R. Acad. Sci., Ser. I; Math.* **290**, 599 (1980).
- [6] G. Landi, *An Introduction to Noncommutative Spaces and their Geometries* (Springer, Berlin, 1997).
- [7] J. Madore, *An Introduction to Noncommutative Geometry and its Physical Applications* (Cambridge University Press, Cambridge, England, 1999).
- [8] J.M. Gracia-Bondía, J.C. Varilly, and H. Figueroa, *Elements of Noncommutative Geometry* (Birkhäuser, Boston, 2001).
- [9] A. Kempf, *Phys. Rev. D* **63**, 083514 (2001).
- [10] J. Martin and R. Brandenberger, *Phys. Rev. D* **63**, 123501 (2001).
- [11] A. Kempf and J.C. Niemeyer, *Phys. Rev. D* **64**, 103501 (2001).
- [12] F. Lizzi, G. Mangano, G. Miele, and G. Sparano, *Int. J. Mod. Phys. A* **11**, 2907 (1996).
- [13] F. Lizzi, G. Mangano, G. Miele, and G. Sparano, *Mod. Phys. Lett. A* **11**, 2561 (1996).
- [14] F. Lizzi, G. Mangano, G. Miele, and G. Sparano, *Phys. Rev. D* **55**, 6357 (1997).
- [15] F. Lizzi, G. Mangano, and G. Miele, *Mod. Phys. Lett. A* **16**, 1 (2001).
- [16] F. Lizzi, G. Mangano, G. Miele, and M. Peloso, *J. High Energy Phys.* **06**, 049 (2002).
- [17] F.C. Adams, J.R. Bond, K. Freese, J.A. Frieman, and A.V. Olinto, *Phys. Rev. D* **47**, 426 (1993).
- [18] J.D. Barrow and A.R. Liddle, *Phys. Rev. D* **47**, 5219 (1993).
- [19] J. Garcia-Bellido and A.R. Liddle, *Phys. Rev. D* **55**, 4603 (1997).
- [20] Q.G. Huang and M. Li, *J. High Energy Phys.* **06**, 014 (2003).
- [21] N. Seiberg and E. Witten, *J. High Energy Phys.* **09**, 032 (1999).
- [22] J.M. Bardeen, *Phys. Rev. D* **22**, 1882 (1980).
- [23] M.J. Duff, in *Supergravity 1981*, edited by S. Ferrara and J.G. Taylor (Cambridge University Press, Cambridge, England, 1982).
- [24] P.G.O. Freund and R.I. Nepomechie, "Unified Geometry of Antisymmetric Tensor Gauge Field and Gravity," National Science Foundation, 1981.
- [25] C. Pathinayake, A. Vilenkin, and B. Allen, *Phys. Rev. D* **37**, 2872 (1988).
- [26] R.K. Kaul, *Phys. Rev. D* **18**, 1127 (1978).
- [27] C.R. Hagen, *Phys. Rev. D* **19**, 2367 (1979).
- [28] E. Sezgin and P. van Nieuwenhuizen, *Phys. Rev. D* **22**, 301 (1980).
- [29] W. Mecklenburg and L. Mizrachi, *Phys. Rev. D* **29**, 1709 (1984).
- [30] S. Deser and E. Witten, *Nucl. Phys.* **B178**, 491 (1981).
- [31] A.A. Slavnov and S.A. Frolov, "Quantization of Interacting Antisymmetric Tensor Field," DFPD 13/87, 1987.
- [32] S.P. de Alwis, M.T. Grisaru, and L. Mezincescu, "Quantization and Unitarity in Antisymmetric Tensor Gauge Theories," BRX TH-235, 1988.
- [33] N.Yu. Obukhov, *Phys. Lett.* **109B**, 195 (1982).
- [34] Z. Tokuoka, *Phys. Lett.* **87A**, 215 (1982).
- [35] I. Bena, *Phys. Rev. D* **62**, 127901 (2000).
- [36] S. Deguchi, T. Mukai, and T. Nakajima, *Phys. Rev. D* **59**, 065003 (1999).
- [37] M. Abud, J.P. Ader, and L. Cappiello, *Nuovo Cimento A* **105**, 1507 (1992).
- [38] V.F. Mukhanov, H.A. Feldman, and R.H. Brandenberger, *Phys. Rep.* **215**, 203 (1992).

- [39] J.E. Lidsey, D. Wands, and E.J. Copeland, *Phys. Rep.* **337**, 343 (2000).
- [40] R. Easther, K.I. Maeda, and D. Wands, *Phys. Rev. D* **53**, 4247 (1996).
- [41] E.J. Copeland, A. Lahiri, and D. Wands, *Phys. Rev. D* **51**, 1569 (1995).
- [42] P.A.M. Dirac, *Lectures on Quantum Mechanics* (Dover, New York, 2001).
- [43] A. Hanson, T. Regge, and C. Teitelboim, *Constrained Hamiltonian Systems*, Contributi del Centro Linceo Interdisciplinare di Scienze Matematiche e loro Applicazioni, n. 22 (Accademia Nazionale dei Lincei, Roma, 1976).
- [44] G. Esposito, *Quantum Gravity, Quantum Cosmology and Lorentzian Geometries*, Lecture Notes in Physics Vol. m12 (Springer, Berlin, 1994).
- [45] B.S. DeWitt, *Phys. Rev.* **160**, 1113 (1967).