# Supersymmetric solutions of minimal gauged supergravity in five dimensions

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All purely bosonic supersymmetric solutions of minimal gauged supergravity in five dimensions are classified. The solutions fall into two classes depending on whether the Killing vector constructed from the Killing spinor is timelike or null. When it is timelike, the solutions are determined by a four-dimensional Kähler base manifold, up to an antiholomorphic function, are necessarily not static, and generically preserve 1/2 of the supersymmetry. When it is null we provide a precise prescription for constructing the solutions and we show that they generically preserve 1/4 of the supersymmetry. We show that five-dimensional anti-de Sitter space ($AdS_5$) is the unique maximally supersymmetric configuration. The formalism is used to construct some new solutions, including a nonsingular deformation of $AdS_5$, which can be uplifted to obtain new solutions of type IIB supergravity.

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## I. INTRODUCTION

There has been recent progress in classifying supersymmetric bosonic solutions in supergravity theories [1,2] (for older work using techniques specific to four dimensions, see [3]). Such a classification is desirable as it may allow one to find new kinds of solutions that have been hitherto missed by the usual procedure of starting with an inspired ansatz. In turn this could elucidate interesting new phenomena in string or M theory. In addition such a classification allows one to precisely characterize supersymmetric geometries of interest, which is important when explicit solutions are not available.

The basic strategy is to assume the existence of a Killing spinor, that is, assume a solution preserves at least one supersymmetry, and then consider the differential forms that can be constructed as bilinears from the spinor. These satisfy a number of algebraic and differential conditions which can be used to determine the form of the metric and other bosonic fields. Geometrically, the Killing spinor, or equivalently the differential forms, defines a preferred $G$ structure and the differential conditions restrict its intrinsic torsion.[1]

The analysis of the most general bosonic supersymmetric solutions of $D=11$ supergravity was initiated in [2]. It was shown that the solutions always have a Killing vector constructed as a bilinear from the Killing spinor and that it is either timelike or null. A detailed analysis was undertaken for the timelike case and it was shown that an $SU(5)$ structure plays a central role in determining the local form of the most general bosonic supersymmetric configuration. A similar analysis for the null case, which has yet to be carried out, would then complete this classification of the most general supersymmetric geometries of $D=11$ supergravity. A finer classification would be to carry out a similar analysis assuming that there is more than one Killing spinor and some indications of how this might be tackled were discussed in [2]. Of course, a fully complete classification of $D=11$ supersymmetric geometries would require classifying the explicit form of the solutions within the various classes, but this seems well beyond reach at present.

While more progress on $D=11$ or 10 supergravity is possible, it seems a daunting challenge to carry through the program of [2] in full. Thus, it is of interest to analyze simpler supergravity theories. In the cases where the theory arises via dimensional reduction from a higher dimensional supergravity theory, the analysis can be viewed as classifying a restricted class of solutions of the higher dimensional theory. In [1] minimal supergravity in $D=5$ was analyzed, which arises, for example, as a truncation of the dimensional reduction of $D=11$ supergravity on a six torus. As in $D=11$ supergravity, the general supersymmetric solutions of the $D=5$ theory have either a timelike or a null Killing vector that is constructed from the Killing spinor. In the timelike case there is an $SU(2)$ structure. More precisely, it was shown that working in a neighborhood in which the Killing vector is timelike, the $D=5$ geometry is completely determined by a hyper-Kähler base manifold, orthogonal to the orbits of the Killing vector, along with a function and a connection one-form defined on the base that satisfy a pair of simple differential equations. A similar analysis for the null case revealed that the most general solution was a plane fronted wave determined by three harmonic functions. Although the null case has an $\mathbb{R}^3$ structure, this did not play an important role in the analysis. In addition, it was shown that the generic solutions for both the timelike and null case preserve 1/2 supersymmetry, but they can also be maximally supersymmetric. A further analysis determined the explicit form of the most general maximally supersymmetric solutions.

Here we shall analyze minimal gauged supergravity in $D=5$. This theory arises as a consistent truncation of the dimensional reduction of type IIB supergravity on a five sphere [6,7]. The gauged theory has the same field content as the ungauged theory, and given their similarity it is not surprising that some of the analysis parallels that of [1]. However, it is interesting that there are some important differences. Once again there are two classes of supersymmetric solutions, the timelike class and the null class. In the timelike

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[1]The utility of $G$ structures in analyzing supersymmetric solutions of supergravity was discussed earlier in [4,5].

 

case, we show that the base manifold of the $D=5$ geometry orthogonal to the orbits of the Killing vector is now a Kähler manifold with a $U(2)$ structure, and the solutions generically preserve 1/2 of the supersymmetry. However, in contrast to the ungauged case, the whole of the geometry is determined by the base space up to an antiholomorphic function on the base. This formalism thus provides a very powerful method for the generation of new solutions. It is also interesting to highlight that we show that all of the solutions in the timelike case are necessarily not static. There are certainly static supersymmetric solutions in the timelike case, but the static Killing vector of such solutions never arises as a Killing spinor bilinear. Thus our analysis casts familiar static supersymmetric solutions in a novel way, which is suggestive that our formalism may lead to new kinds of solutions.

When the Killing vector is null, we show that the five-dimensional solution is again fixed up to three functions, as in the ungauged case. However, unlike the ungauged case, these functions are no longer harmonic, but rather satisfy more complicated elliptic differential equations on $\mathbb{R}^3$. Moreover, these solutions generically preserve only 1/4 of the supersymmetry rather than 1/2.

By examining the integrability conditions for the Killing spinor equation it is simple to show that five-dimensional anti–de Sitter space ($AdS_5$) is the unique solution preserving all supersymmetry. This is in contrast to the ungauged case where there is a rich class of maximally supersymmetric solutions.

By using this formalism, we construct some new solutions of five dimensional gauged supergravity. As in the ungauged case, we find that many of the new solutions have closed timelike curves. More specifically, we find a family of solutions corresponding to deformations of $AdS_5$, in which the deformation depends on a holomorphic function on a Kähler manifold equipped with the Bergmann metric. In the special case that the holomorphic function is constant, we find a regular deformation of $AdS_5$ with, for a range of parameters, no closed timelike curves. We also find a 1-parameter family of solutions for which the geometry corresponds to a certain double analytic continuation of the coset space $T^{pq}$. All of these solutions can be lifted on a five sphere to obtain solutions of type IIB theory using the formulas in [6,7].

The plan of this paper is as follows. In Sec. II we examine the structure of the minimal five dimensional gauged supergravity, and describe the algebraic and differential constraints which bilinears constructed out of the Killing spinor must satisfy. In Sec. III, we present a classification of the solutions when the Killing vector constructed from the Killing spinor is timelike. We show how the solutions are completely fixed up to an arbitrary Kähler 4-manifold together with an antiholomorphic function, and we present some new solutions. In Sec. IV, we examine solutions for which the Killing vector is null; again, we find a simple prescription for constructing solutions in this case. In Sec. V we investigate maximally supersymmetric solutions. In Sec. VI we present our conclusions.

## II. $D=5$ GAUGED SUPERGRAVITY

The bosonic action for minimal gauged supergravity in five dimensions is [8]

$$S=\frac{1}{4\pi G}\int\left(-\frac{1}{4}({}^5R-\chi^2)*1-\frac{1}{2}F\wedge *F-\frac{2}{3\sqrt{3}}F\wedge F\wedge A\right), \tag{1}$$

where $F=dA$ is a $U(1)$ field strength and $\chi\neq 0$ is a real constant. We will adopt the same conventions as [1], including a mostly minus signature for the metric. The bosonic equations of motion are

$$\begin{aligned}{}^5R_{\alpha\beta}+2F_{\alpha\gamma}F^{\gamma}_{\ \beta}-\tfrac{1}{3}g_{\alpha\beta}(F^2+\chi^2)&=0\\ d*F+\frac{2}{\sqrt{3}}F\wedge F&=0\end{aligned} \tag{2}$$

where $F^2\equiv F_{\alpha\beta}F^{\alpha\beta}$. A bosonic solution to the equations of motion is supersymmetric if it admits a supercovariantly constant spinor obeying

$$\left[D_\alpha+\frac{1}{4\sqrt{3}}(\gamma_\alpha^{\ \beta\gamma}-4\delta^\beta_\alpha\gamma^\gamma)F_{\beta\gamma}\right]\epsilon^a-\chi\epsilon^{ab}\left(\frac{1}{4\sqrt{3}}\gamma_\alpha-\frac{1}{2}A_\alpha\right)\epsilon^b=0 \tag{3}$$

where $\epsilon^a$ is a symplectic commuting Majorana spinor. We shall call such spinors Killing spinors. Our strategy for determining the most general bosonic supersymmetric solutions[2] is to analyze the differential forms that can be constructed from Killing spinors. We first investigate algebraic properties of these forms, and then their differential properties.

From a single commuting spinor $\epsilon^a$ we can construct a scalar $f$, a 1-form $V$ and three 2-forms $\Phi^{ab}\equiv\Phi^{(ab)}$:

$$\begin{aligned}f\epsilon^{ab}&=\overline{\epsilon^a}\epsilon^b\\ V_\alpha\epsilon^{ab}&=\overline{\epsilon^a}\gamma_\alpha\epsilon^b\\ \Phi^{ab}_{\alpha\beta}&=\overline{\epsilon^a}\gamma_{\alpha\beta}\epsilon^b,\end{aligned} \tag{4}$$

$f$ and $V$ are real, but $\Phi^{11}$ and $\Phi^{22}$ are complex conjugate and $\Phi^{12}$ is imaginary. It is convenient to work with three real two-forms defined by

$$\begin{aligned}\Phi^{(11)}&=X^{(1)}+iX^{(2)}\\ \Phi^{(22)}&=X^{(1)}-iX^{(2)}\end{aligned}$$

[2]Note that there are spacetimes admitting a Killing spinor that do not satisfy the equations of motion. These can be viewed as solutions of the field equations with additional sources, and supersymmetry imposes conditions on these sources. It is straightforward to determine the conditions, but for simplicity of presentation, we will restrict ourselves to solutions of the field equations without sources.

$$\Phi^{(12)} = -iX^{(3)}. \tag{5}$$

It will be useful to record some of the identities which can be obtained from various Fierz identities.

We first note that

$$V_\alpha V^\alpha = f^2 \tag{6}$$

which implies that $V$ is timelike, null or zero. The final possibility can be eliminated using the arguments in [9]. Now $f$ either vanishes everywhere or it is nonvanishing at a point $p$. In the former "null case," the Killing vector $V$ is a globally defined null Killing vector. In the latter "timelike case" there is a neighborhood of $p$ in which $f$ is nonvanishing and for which $V$ is timelike. We will work in such a neighborhood for this case, and then find the full solution by analytic continuation. In later sections we will analyze the timelike and null cases separately.

We also have

$$X^{(i)} \wedge X^{(j)} = -2\delta_{ij} f * V, \tag{7}$$

$$i_V X^{(i)} = 0, \tag{8}$$

$$i_V * X^{(i)} = -fX^{(i)}, \tag{9}$$

$$X^{(i)}_{\gamma\alpha} X^{(j)\gamma}_{\beta} = \delta_{ij}(f^2 \eta_{\alpha\beta} - V_\alpha V_\beta) + \epsilon_{ijk} f X^{(k)}_{\alpha\beta} \tag{10}$$

where $\epsilon_{123} = +1$ and, for a vector $Y$ and $p$-form $A$, $(i_Y A)_{\alpha_1, \ldots, \alpha_{p-1}} \equiv Y^\beta A_{\beta\alpha_1, \ldots, \alpha_{p-1}}$. Finally, it is useful to record

$$V_\alpha \gamma^\alpha \epsilon^a = f\epsilon^a, \tag{11}$$

and

$$\Phi^{ab}_{\alpha\beta} \gamma^{\alpha\beta} \epsilon^c = 8 f \epsilon^{c(a} \epsilon^{b)}. \tag{12}$$

We now turn to the differential conditions that can be obtained by assuming that $\epsilon$ is a Killing spinor. We differentiate $f$, $V$, $\Phi$ in turn and use Eq. (3). Starting with $f$ we find

$$df = -\frac{2}{\sqrt{3}} i_V F. \tag{13}$$

Taking the exterior derivative and using the Bianchi identity for $F$ then gives

$$\mathcal{L}_V F = 0, \tag{14}$$

where $\mathcal{L}$ denotes the Lie derivative. Next, differentiating $V$ gives

$$D_\alpha V_\beta = \frac{2}{\sqrt{3}} F_{\alpha\beta} f + \frac{1}{2\sqrt{3}} \epsilon_{\alpha\beta\gamma\delta\epsilon} F^{\gamma\delta} V^\epsilon + \frac{\chi}{2\sqrt{3}} (X^{(1)})_{\alpha\beta}, \tag{15}$$

which implies $D_{(\alpha} V_{\beta)} = 0$ and hence $V$ is a Killing vector. Combining this with Eq. (14) implies that $V$ is the generator of a symmetry of the full solution $(g, F)$. Note that Eq. (15) implies

$$dV = \frac{4}{\sqrt{3}} fF + \frac{2}{\sqrt{3}} *(F \wedge V) + \frac{\chi}{\sqrt{3}} X^{(1)}. \tag{16}$$

Finally, differentiating $X^{(i)}$ gives

$$D_\alpha X^{(i)}_{\beta\gamma} = -\frac{1}{\sqrt{3}} [2F_\alpha{}^\delta (*X^{(i)})_{\delta\beta\gamma} - 2F_{[\beta}{}^\delta (*X^{(i)})_{\gamma]\alpha\delta} + \eta_{\alpha[\beta} F^{\delta\epsilon} (*X^{(i)})_{\gamma]\delta\epsilon}] + \chi \epsilon^{1ij} \left[ A_\alpha X_{\beta\gamma}{}^{(j)} + \frac{1}{2\sqrt{3}} (*X^{(j)})_{\alpha\beta\gamma} \right] - \frac{\chi}{\sqrt{3}} \delta^{1i} \eta_{\alpha[\beta} V_{\gamma]}. \tag{17}$$

Note that Eq. (17) implies that

$$dX^{(i)} = \chi \epsilon^{1ij} \left( A \wedge X^{(j)} + \sqrt{\frac{3}{2}} * X^{(j)} \right) \tag{18}$$

so $dX^{(1)} = 0$ but $X^{(2)}$ and $X^{(3)}$ are not closed. In particular, this implies that

$$\mathcal{L}_V X^{(i)} = \chi \epsilon^{1ij} \left( i_V A - \sqrt{\frac{3}{2}} f \right) X^{(j)}. \tag{19}$$

It is useful to consider the effect of gauge transformations $A \to A + d\Lambda$. In particular, the Killing spinor equation is left invariant under the transformation

$$\epsilon^1 \to \cos\left(\frac{\chi\Lambda}{2}\right)\epsilon^1 - \sin\left(\frac{\chi\Lambda}{2}\right)\epsilon^2$$

$$\epsilon^2 \to \cos\left(\frac{\chi\Lambda}{2}\right)\epsilon^2 + \sin\left(\frac{\chi\Lambda}{2}\right)\epsilon^1. \tag{20}$$

Under these transformations, $f \to f$, $V \to V$ and $X^1 \to X^1$, but $X^2 + iX^3 \to e^{-i\chi\Lambda}(X^2 + iX^3)$. We shall choose to work in a gauge in which

$$i_V A = \sqrt{\frac{3}{2}} f \tag{21}$$

and so $\mathcal{L}_V A = 0$ and also $\mathcal{L}_V X^{(i)} = 0$.

To make further progress we will examine separately the case in which the Killing vector is timelike and the case in which it is null in the two following sections.

## III. THE TIMELIKE CASE

### A. The general solution

In this section we shall consider solutions in a neighborhood in which $f$ is nonzero and hence $V$ is a timelike Killing vector field. Eq. (10) implies that the 2-forms $X^{(i)}$ are all nonvanishing. Introduce coordinates such that $V = \partial/\partial t$. The metric can then be written locally as

$$ds^2 = f^2(dt + \omega)^2 - f^{-1} h_{mn} dx^m dx^n \tag{22}$$

where $f$, $\omega$ and $h$ depend only on $x^m$ and not on $t$, and we have assumed, essentially with no loss of generality, $f>0$. The metric $f^{-1}h_{mn}$ is obtained by projecting the full metric perpendicular to the orbits of $V$. The Riemannian 4-manifold with coordinates $x^m$ and metric $h$ will be referred to as the base space $B$.

Define

$$e^0=f(dt+\omega) \tag{23}$$

and if $\eta$ defines a positive orientation on $B$ then we define $e^0\wedge\eta$ to define a positive orientation for the $D=5$ metric. The two form $d\omega$ only has components tangent to the base space and can therefore be split into self-dual and anti-self-dual parts with respect to the metric $h_{mn}$:

$$f d\omega=G^+ + G^- \tag{24}$$

where the factor of $f$ is included for convenience.

Equation (8) implies that the 2-forms $X^{(i)}$ can be regarded as 2-forms on the base space and Eq. (9) implies that they are anti-self-dual:

$$*_4 X^{(i)}=-X^{(i)}, \tag{25}$$

where $*_4$ denotes the Hodge dual associated with the metric $h_{mn}$. Equation (10) can be written

$$X^{(i)}{}_m{}^p X^{(j)}{}_p{}^n=-\delta^{ij}\delta_m{}^n+\epsilon_{ijk}X^{(k)}{}_m{}^n \tag{26}$$

where indices $m,n,\ldots$ have been raised with $h^{mn}$, the inverse of $h_{mn}$. This equation shows that the $X^{(i)}$'s satisfy the algebra of imaginary unit quaternions.

To proceed, we use Eqs. (13) and (16) to solve for the gauge field strength $F$. This gives

$$F=\sqrt{\frac{3}{2}}de^0-\frac{1}{\sqrt{3}}G^+-\frac{\chi}{2f}X^{(1)}. \tag{27}$$

It is convenient to write

$$H=\sqrt{\frac{3}{2}}de^0-\frac{1}{\sqrt{3}}G^+ \tag{28}$$

so that $F=H-(\chi/2f)X^{(1)}$. Substituting this into Eq. (17) we find that

$$\begin{aligned}D_\alpha X^{(i)}_{\beta\gamma}=&-\frac{1}{\sqrt{3}}[2H_\alpha{}^\delta(*X^{(i)})_{\delta\beta\gamma}-2H_{[\beta}{}^\delta(*X^{(i)})_{\gamma]\alpha\delta}\\&+\eta_{\alpha[\beta}H^{\delta\epsilon}(*X^{(i)})_{\gamma]\delta\epsilon}]\\&+\chi\epsilon^{1ij}\left(A_\alpha-\sqrt{\frac{3}{2f}}V_\alpha\right)X^{(j)}_{\beta\gamma}.\end{aligned} \tag{29}$$

We also find that

$$\begin{aligned}\nabla_m X^{(1)}_{np}&=0\\ \nabla_m X^{(2)}_{np}&=P_m X^{(3)}_{np}\\ \nabla_m X^{(3)}_{np}&=-P_m X^{(2)}_{np}\end{aligned} \tag{30}$$

where $\nabla$ is the Levi-Civita connection on $B$ with respect to $h$ and we have introduced

$$P_m=\chi\left(A_m-\sqrt{\frac{3}{2}}f\omega_m\right). \tag{31}$$

Recall that $X^{(1)}$ is gauge invariant. From Eqs. (26) and (30) we conclude that the base space is Kähler, with Kähler form $X^{(1)}$. Thus the base space has a $U(2)$ structure.

One might be tempted to conclude that the additional presence of $X^{(2)}$ and $X^{(3)}$ satisfying Eq. (26) implies that the manifold actually has an $SU(2)$ structure. However, this is not the case since $X^{(2)}$ and $X^{(3)}$ are not gauge invariant. To obtain some further insight, note that we can invert Eq. (30) to solve for $P$:

$$P_m=\tfrac{1}{8}(X^{(3)np}\nabla_m X^{(2)}_{np}-X^{(2)np}\nabla_m X^{(3)}_{np}) \tag{32}$$

from which we deduce that

$$dP=\mathfrak{R} \tag{33}$$

where $\mathfrak{R}$ is the Ricci form of the base space $B$ defined by

$$\mathfrak{R}_{mn}=\tfrac{1}{2}X^{(1)pq}R_{pqmn} \tag{34}$$

and $R_{pqmn}$ denotes the Riemann curvature tensor of $B$ equipped with metric $h$. Now on any Kähler four-manifold, with anti-self-dual Kähler two-form $X^{(1)}$ and Ricci form $\mathfrak{R}$, there is always a section of the canonical bundle, $X^{(2)}+iX^{(3)}$, with anti-self-dual two-forms $X^{(2)},X^{(3)}$, satisfying Eq. (26), and $(\nabla+iP)(X^{(2)}+iX^{(3)})=0$. But this is equivalent to the last two equations in Eq. (30). (Note that shifting $P$ by a gradient of a function on the Kähler manifold shifts $X^{(2)}+iX^{(3)}$ by a phase which precisely corresponds to the time-independent gauge transformations of $X^{(2)}+iX^{(3)}$.)

Thus the content of Eqs. (26), (30) and (31) is simply that the base $B$ is Kähler and that the base determines $A_m-(\sqrt{3}/2)f\omega_m$ (up to a gradient of a time independent function). In fact, as we now show, all of the five-dimensional geometry is determined in terms of the geometry of the Kähler base space $B$, up to an antiholomorphic function on the base. To see this we first substitute Eq. (31) into Eq. (33) to get

$$-\frac{1}{\sqrt{3}}G^+{}_{mn}-\frac{\chi}{2f}X^{(1)}_{mn}=\frac{1}{\chi}\mathfrak{R}_{mn}. \tag{35}$$

Upon contracting Eq. (35) with $X^{(1)mn}$, and using $\mathfrak{R}_{mn}X^{(1)mn}=R$, we obtain

$$f=-\frac{2\chi^2}{R} \tag{36}$$

where $R$ is the Ricci scalar curvature of $B$. In particular, we see $B$ cannot be hyper-Kähler, as we must have $R\neq 0$. Substituting back into Eq. (35) we find that

$$G^{+}{}_{mn} = -\sqrt{\frac{3}{\chi}}\left(\mathfrak{R}_{mn} - \frac{1}{4}RX^{(1)}_{mn}\right). \tag{37}$$

Now the Bianchi identity $dF=0$ is satisfied since

$$dG^{+} = \frac{\sqrt{3}\chi}{2f^2}df\wedge X^{(1)} \tag{38}$$

which is implied by Eq. (35). The gauge field equation implies that

$$\nabla_m\nabla^m f^{-1} = \frac{2}{9}(G^{+})^{mn}(G^{+})_{mn} + \frac{\chi}{2\sqrt{3}f}(G^{-})_{mn}(X^1)^{mn} - \frac{2\chi^2}{3f^2}. \tag{39}$$

If we write

$$G^{-} = \lambda^i X^{(i)}, \tag{40}$$

for some functions $\lambda^i$, we see that Eq. (39) fixes $\lambda^1$ in terms of the base space geometry via

$$\lambda^1 = \sqrt{\frac{3}{\chi R}}\left(\frac{1}{2}\nabla^m\nabla_m R + \frac{2}{3}\mathfrak{R}_{mn}\mathfrak{R}^{mn} - \frac{1}{3}R^2\right). \tag{41}$$

Next we note that Eq. (24) implies that $R(G^{+}+G^{-})$ is closed. Hence, on taking the exterior derivative and using Eq. (30) we find that

$$T + [d(R\lambda^2) - R\lambda^3 P]\wedge X^{(2)} + [d(R\lambda^3) + R\lambda^2 P]\wedge X^{(3)} = 0, \tag{42}$$

where

$$T \equiv \sqrt{\frac{3}{\chi}}\left(-dR\wedge\mathfrak{R} + d\left[\frac{1}{2}\nabla_m\nabla^m R + \frac{2}{3}\mathfrak{R}_{mn}\mathfrak{R}^{mn} - \frac{1}{12}R^2\right]\wedge X^{(1)}\right) \tag{43}$$

is determined by the geometry of the base. In particular, $\lambda^2 = \lambda^3 = 0$ is only possible if $T=0$. On defining

$$\Theta_m = (X^2)_m{}^n(*_4 T)_n \tag{44}$$

and adopting complex coordinates $z^j, z^{\bar{j}}$ on $B$ with respect to $X^1$, Eq. (42) simplifies to

$$\Theta_j = -(\partial_j - iP_j)[R(\lambda^2 - i\lambda^3)] \tag{45}$$

which fixes $\lambda^2 - i\lambda^3$ up to an arbitrary antiholomorphic function. In summary, we have determined $f$ and $G^{\pm}$ in terms of the Kähler base up to an antiholomorphic function; then, up to a time independent gradient, $\omega$ is determined by Eq. (24), and then $A_m$ by $P_m$. This state of affairs should be contrasted with the ungauged case [1], where $f$ and $\omega$ satisfied a pair of differential equations on a hyper-Kähler base.

We remark that there are no solutions for which $V$ is hypersurface orthogonal; in other words there are no solutions with $d\omega=0$. To see this note that if $d\omega=0$ then $G^{+}=G^{-}=0$, and from Eq. (38) we find that $df=0$. On substituting this into Eq. (39) and using $G^{+}=G^{-}=0$ we obtain a contradiction. This would seem problematic, as it is known that many of the known solutions such as $AdS_5$ and certain types of nakedly singular black hole solutions can be written in coordinates in which the solution is static with respect to some timelike killing vector. This apparent contradiction is resolved by noting that this timelike killing vector is not the killing vector constructed from the Killing spinor. Hence, it is clear that the coordinates which arise naturally from the construction described here are not in fact the coordinates in which the known solutions can be written in a static form. This is, however, a minor inconvenience in recovering the known solutions, since as we shall see, the two coordinate systems are typically related by rather simple coordinate transformations. Moreover, it is clear that the formalism described above is particularly useful in generating new solutions.

We have obtained all of the constraints on the bosonic quantities $f$, $V$, and $X^{(i)}$ imposed by the Killing spinor equations and the equations of motion. It remains to check whether, conversely, the geometry we have found always admits Killing spinors. Now the Killing spinor equation can be rewritten in terms of $H$ as

$$\left[D_\alpha + \frac{1}{4\sqrt{3}}(\gamma_\alpha{}^{\beta\lambda} - \delta^\beta_\alpha\gamma^\lambda)H_{\beta\lambda} + \frac{\sqrt{3}\chi}{4f}(X^1)_{\alpha\lambda}\gamma^\lambda\right]\epsilon^a - \frac{\chi}{2}\epsilon^{ab}\left(\sqrt{\frac{3}{2}}\gamma_\alpha - A_\alpha\right)\epsilon^b = 0 \tag{46}$$

where we have used Eq. (12). We recall also from Eq. (11) that we need to impose $\gamma^0\epsilon^a = \epsilon^a$. Then, using Eq. (21), the $t$ component of Eq. (46) requires that $\partial\epsilon^a/\partial t = 0$, so that $\epsilon^a$ depends only on the $x^m$. Next we consider the $m$ component of Eq. (46); it is convenient to rescale

$$\epsilon^a = f^{1/2}\eta^a. \tag{47}$$

Using the fact that Eq. (12) implies

$$\eta^a = -\tfrac{1}{4}\epsilon^{ab}X^{(1)}_{pq}\hat{\gamma}^{pq}\eta^b \tag{48}$$

we obtain

$$\nabla_m\eta^a + \tfrac{1}{2}P_m\epsilon^{ab}\eta^b = 0 \tag{49}$$

where $\hat{\gamma}$ are rescaled gamma matrices satisfying the algebra

$$\hat{\gamma}_m\hat{\gamma}_n + \hat{\gamma}_n\hat{\gamma}_m = -2h_{mn} \tag{50}$$

and all spatial coordinate indices are raised with respect to $h^{mn}$, the inverse of $h_{mn}$. Since Eq. (49) always has a solution[3] on a Kähler manifold (see, e.g., Ref. [10]), we have shown that the geometry does indeed admit Killing spinors.

### B. Some examples

Using the techniques described in the previous section, it is possible to construct gauged supergravity solutions with timelike $V$. In the following, we shall denote an orthonormal basis of the Kähler base space $B$ by $\{e^1,e^2,e^3,e^4\}$ and take $e^1\wedge e^2\wedge e^3\wedge e^4$ to define a positive orientation with

$$X^1=e^1\wedge e^2-e^3\wedge e^4$$
$$X^2=e^1\wedge e^3+e^2\wedge e^4$$
$$X^3=e^1\wedge e^4-e^2\wedge e^3. \tag{51}$$

#### *1. Bergmann base space and deformations of $AdS_5$*

The simplest class of examples are those for which the base space $B$ is Einstein. From Eq. (37) we see that this is equivalent to $G^+=0$. Moreover, if $G^+=0$ then from Eq. (38) we obtain $df=0$, and without loss of generality we set $f=1$, and so $R=-2\chi^2$ and $\mathfrak{R}=-(\chi^2/2)X^1$. Hence, from Eq. (41) we find $\lambda^1=\chi/\sqrt{3}$ and we note that $\Theta=0$. Hence, locally Eq. (45) can be written as

$$\partial_j(\lambda^2-i\lambda^3)+\frac{\chi^2}{4}\partial_j K(\lambda^2-i\lambda^3)=0 \tag{52}$$

where $K$ is the Kähler potential of $B$, so

$$\lambda^2-i\lambda^3=e^{-\chi^2K/4}\mathcal{F}(\overline{z}) \tag{53}$$

where $\mathcal{F}(\overline{z})$ is an antiholomorphic function. Note that the field strength takes the simple form

$$F=\frac{\sqrt{3}}{2}(\lambda^2X^2+\lambda^3X^3). \tag{54}$$

A simple Einstein base is obtained by taking the base metric to be given by the Bergmann metric

$$ds^2=dr^2+\frac{3}{\chi^2}\sinh^2\left(\frac{\chi r}{2\sqrt{3}}\right)[(\sigma_1^L)^2+(\sigma_2^L)^2]$$
$$+\frac{3}{\chi^2}\sinh^2\left(\frac{\chi r}{2\sqrt{3}}\right)\cosh^2\left(\frac{\chi r}{2\sqrt{3}}\right)(\sigma_3^L)^2 \tag{55}$$

where $\sigma_L^i$ are right invariant one-forms on the three sphere and we use the same Euler angles and notation as in [1]. The $\chi^{(i)}$ are given by Eq. (51) if we choose the orthonormal basis

$$e^1=dr$$
$$e^2=\sqrt{\frac{3}{\chi}}\sinh\left(\frac{\chi r}{2\sqrt{3}}\right)\cosh\left(\frac{\chi r}{2\sqrt{3}}\right)\sigma_3^L$$
$$e^3=\sqrt{\frac{3}{\chi}}\sinh\left(\frac{\chi r}{2\sqrt{3}}\right)\sigma_1^L$$
$$e^4=\sqrt{\frac{3}{\chi}}\sinh\left(\frac{\chi r}{2\sqrt{3}}\right)\sigma_2^L. \tag{56}$$

More explicitly we have

$$X^1=\frac{3}{\chi^2}d\left[\sinh^2\left(\frac{\chi r}{2\sqrt{3}}\right)\sigma_3^L\right]$$
$$X^2=\frac{3}{\chi^2}\cosh^3\left(\frac{\chi r}{2\sqrt{3}}\right)d\left[\tanh^2\left(\frac{\chi r}{2\sqrt{3}}\right)\sigma_1^L\right]$$
$$X^3=\frac{3}{\chi^2}\cosh^3\left(\frac{\chi r}{2\sqrt{3}}\right)d\left[\tanh^2\left(\frac{\chi r}{2\sqrt{3}}\right)\sigma_2^L\right], \tag{57}$$

and $P=-\frac{3}{2}\sinh^2(\chi r/2\sqrt{3})\sigma_3^L$.

For solutions with $\mathcal{F}=0$ (and so $\lambda^2=\lambda^3=0$ and $F=0$) we find $\omega=(\sqrt{3}/\chi)\sinh^2(\chi r/2\sqrt{3})\sigma_3^L$. The five dimensional geometry can be written, after shifting the Euler angle $\phi\to\phi+(\chi/\sqrt{3})t$, as

$$ds^2=\cosh^2\left(\frac{\chi r}{2\sqrt{3}}\right)dt^2-dr^2-\frac{12}{\chi^2}\sinh^2\left(\frac{\chi r}{2\sqrt{3}}\right)d\Omega_3^2 \tag{58}$$

which is the simply the metric of $AdS_5$ with radius $2\sqrt{3}/\chi$.

In order to construct new solutions with $F\neq 0$ we exploit the fact that the Kähler potential is well known in complex coordinates (see, e.g., Ref. [11]). In particular if we introduce the complex coordinates

$$z^1=\tanh\left(\frac{\chi r}{2\sqrt{3}}\right)\cos\left(\frac{\theta}{2}\right)e^{(i/2)(\phi+\psi)}$$
$$z^2=\tanh\left(\frac{\chi r}{2\sqrt{3}}\right)\sin\left(\frac{\theta}{2}\right)e^{(i/2)(\phi-\psi)} \tag{59}$$

the Kähler potential is

$$K=-\frac{6}{\chi^2}\log(1-|z^1|^2-|z^2|^2). \tag{60}$$

Thus in the real coordinates, $K=(12/\chi^2)\log\cosh(\chi r/2\sqrt{3})$ and hence $\lambda^2-i\lambda^3=\cosh^{-3}(\chi r/2\sqrt{3})\mathcal{F}(\overline{z})$. If we write $\mathcal{F}\equiv\mathcal{F}_1-i\mathcal{F}_2$ then we find

[3]The term quadratic in gamma matrices in $\nabla_m\eta^a$ is of opposite sign to the usual convention for the supercovariant derivative, though this is consistent with the sign appearing in Eq. (50). Note also that one can use the spinorial construction of $X^{(2)},X^{(3)}$ to show that a Kähler manifold always satisfies Eq. (30), as claimed earlier.

$$d\omega = d\left[\sqrt{\frac{3}{\chi}}\sinh^2\left(\frac{\chi r}{2\sqrt{3}}\right)\sigma_3^L\right] + \mathcal{F}_1 d\left[\frac{3}{\chi^2}\tanh^2\left(\frac{\chi r}{2\sqrt{3}}\right)\sigma_1^L\right] + \mathcal{F}_2 d\left[\frac{3}{\chi^2}\tanh^2\left(\frac{\chi r}{2\sqrt{3}}\right)\sigma_2^L\right]. \tag{61}$$

It would be interesting to explore these deformation of $AdS_5$ in more detail. Let us just note here that if we consider the special case when $\mathcal{F}_i$ are constant, it is trivial to find the explicit form of $\omega$. Interestingly, this case seems to be a completely regular deformation of $AdS_5$. Moreover, by considering the norm of the left vector fields $\xi_1^L$ and $\xi_2^L$, we find that there are closed timelike curves, for sufficiently small $r$, when $\mathcal{F}_i^2 > \frac{4}{3}\chi^2$ and they appear to be absent otherwise.

#### *2. Base space is a product of two manifolds*

Let us now consider some examples in which the base manifold is a product $B = M_2 \times N_2$ where $M_2, N_2$ are two 2-manifolds. When the base space is itself not Einstein, then these solutions have $G^+ \neq 0$. In the first case, we take $B = H^2 \times H^2$ equipped with the metric

$$ds^2 = (dr^2 + \sinh^2 r d\theta^2) + \beta^2(d\rho^2 + \sinh^2\rho d\phi^2) \tag{62}$$

for $\beta$ constant. Note that setting the radius of the first factor to 1, as we have done, does not in fact result in any loss of generality in the resulting five dimensional geometries. Clearly this base is Einstein iff $\beta^2 = 1$. We take the orthonormal basis to be

$$e^1 = dr, \quad e^2 = \sinh r d\theta$$
$$e^3 = \beta d\rho, \quad e^4 = \beta \sinh \rho d\phi. \tag{63}$$

It is straightforward to show that for this solution

$$P = -\coth r e^2 + \beta^{-1}\coth\rho e^4$$
$$f = \frac{\chi^2\beta^2}{1+\beta^2}$$
$$G^+ = \frac{\sqrt{3}(\beta^2-1)}{2\chi\beta^2}(e^1\wedge e^2 + e^3\wedge e^4)$$
$$\lambda^1 = \frac{4}{\sqrt{3}\chi(1+\beta^2)}$$
$$\Theta = 0. \tag{64}$$

Since $\Theta = 0$ we can set $\lambda^2 = \lambda^3 = 0$, which we do for simplicity. We then find that

$$f\omega = \frac{1}{2\sqrt{3}\chi(1+\beta^2)}[\beta^{-2}(3\beta^2-1)(\beta^2+3)\cosh r d\theta + (\beta^2-3)(3\beta^2+1)\cosh\rho d\phi] \tag{65}$$

where

$$F = \frac{(\beta^2-1)}{4\chi(1+\beta^2)}[\beta^{-2}(3-\beta^2)\sinh r dr\wedge d\theta + (3\beta^2-1)\sinh\rho d\rho\wedge d\phi]. \tag{66}$$

After rescaling $t = [(1+\beta^2)/\chi^2\beta^2]t'$ we find

$$ds^2 = \left[dt' + \frac{1}{2\sqrt{3}\chi(1+\beta^2)}\{\beta^{-2}(3\beta^2-1)(\beta^2+3)\cosh r d\theta + (\beta^2-3)(3\beta^2+1)\cosh\rho d\phi\}\right]^2 - \frac{(1+\beta^2)}{\chi^2}\left[\frac{1}{\beta^2}(dr^2 + \sinh^2 r d\theta^2) + (d\rho^2 + \sinh^2\rho d\phi^2)\right]. \tag{67}$$

From these expressions we observe that the solution remains unchanged (up to a coordinate transformation) under the operation $\beta \to 1/\beta$.

There are two special cases to consider. First, when $\beta = 1$ we obtain the geometry

$$ds^2 = \left[d\hat{t} + \frac{2}{\sqrt{3}\chi}(\cosh r d\theta - \cosh\rho d\phi)\right]^2 - \frac{2}{\chi^2}[dr^2 + \sinh^2 r d\theta^2 + d\rho^2 + \sinh^2\rho d\phi^2] \tag{68}$$

with $F = 0$. This an Einstein metric, admitting a Killing spinor (it is not maximally symmetric and so it is not $AdS_5$). Second, if we take $\beta = 1/\sqrt{3}$ we obtain

$$ds^2 = \left[d\hat{t} - \frac{2}{\sqrt{3}\chi}\cosh\rho d\phi\right]^2 - \frac{4}{3\chi^2}[3(dr^2 + \sinh^2 r d\theta^2) + (d\rho^2 + \sinh^2\rho d\phi^2)] \tag{69}$$

with $F = -\chi^{-1}\sinh r dr\wedge d\theta$. This is the metric of $AdS_3 \times H^2$, and we recover the near horizon limit of the supersymmetric black string solution with hyperbolic transverse space [12].

Thus our general solution, Eqs. (67) and (66), is a one parameter family of supersymmetric solutions interpolating between the Einstein metric (68) and $AdS_3 \times H^2$. Note that for the entire family of solutions there are closed timelike curves in the neighborhood of $r = 0$ or $\rho = 0$ parallel to $\partial/\partial\theta$ or $\partial/\partial\phi$ respectively. Of course we know that the closed timelike curves can be eliminated for $AdS_3 \times H^2$ by going to the covering space, and it would be interesting to know if this happens for the entire family of solutions. Finally, we note that if we perform a double analytic continuation $\theta \to i\theta$, $\phi \to i\phi$, and periodically identify the time coordinate, we see from the discussion in, for example, [13], that the metric is that on the coset space $T^{p,q} = SU(2) \times SU(2)/U(1)_{p,q}$ with squashing parametrized by $\beta$ and $p$, $q$ are related via

$$\beta^{-2}(3\beta^2-1)(\beta^2+3)p - (\beta^2-3)(3\beta^2+1)q = 0. \tag{70}$$

The second class of solutions is obtained when we take the base space to be $B=H^2\times S^2$. In fact this solution can be obtained from the expressions given above on mapping $\rho\to i\rho$ (and restricting $0<\rho<\pi$) and $\beta\to -i\beta$. We thus find that the solution, with $\lambda^2=\lambda^3=0$, can be written

$$ds^2=\left[dt'+\frac{1}{2\sqrt{3}\chi(1-\beta^2)}\{\beta^{-2}(3\beta^2+1)(-\beta^2+3)\cosh r d\theta-(\beta^2+3)(-3\beta^2+1)\cos\rho d\phi\}\right]^2-\frac{(\beta^2-1)}{\chi^2}\left[\frac{1}{\beta^2}(dr^2+\sinh^2 r d\theta^2)+(d\rho^2+\sin^2\rho d\phi^2)\right] F=\frac{(\beta^2+1)}{4\chi(1-\beta^2)}[\beta^{-2}(3+\beta^2)\sinh r dr\wedge d\theta-(3\beta^2+1)\sin\rho d\rho\wedge d\phi]. \quad (71)$$

In contrast to the previous solution, it is clear that we must have $\beta>1$. Thus, in this case, it is not possible to choose $\beta$ in such a way as to obtain an Einstein metric. By considering the norm of the vector $\partial_\phi$ we see that the solutions have closed timelike curves. It is also interesting to note that for the special solution $\beta^2=3$ the metric becomes a direct product of a three space with $H^2$.

For a final example of a solution with product base space, we take the base to be $B=M_2\times\mathbb{R}^2$ with metric

$$ds^2=\frac{1}{r^2\left(\alpha r^2+\frac{\beta}{r^4}\right)z}dr^2+r^4\left(\alpha r^2+\frac{\beta}{r^4}\right)dz^2+dx^2+dy^2 \quad (72)$$

for positive constants $\alpha,\beta$; and we take an orthonormal basis

$$e^1=\frac{1}{r\sqrt{\alpha r^2+\frac{\beta}{r^4}}}dr,\quad e^2=r^2\sqrt{\alpha r^2+\frac{\beta}{r^4}}dz$$
$$e^3=dx,\quad e^4=dy. \quad (73)$$

This solution has

$$P=-3\alpha r^4 dz$$
$$f=\frac{\chi^2}{12\alpha r^2}$$
$$G^+=\frac{6\sqrt{3}\alpha r^2}{\chi}(e^1\wedge e^2+e^3\wedge e^4)$$
$$\lambda^1=\frac{6\sqrt{3}\alpha r^2}{\chi}$$
$$\Theta=0. \quad (74)$$

For simplicity we set $\lambda^2=\lambda^3=0$ and obtain

$$\omega=\frac{24\alpha^2\sqrt{3}r^6}{\chi^3}dz. \quad (75)$$

The solution has $F=-(\sqrt{3}/2)dt\wedge df$ and setting $z=(z'/\sqrt{\beta})+(\chi^3 t/24\sqrt{3}\alpha\beta)$ the metric simplifies to

$$ds^2=\frac{\chi^4}{144\alpha^2\beta}\left(\alpha r^2+\frac{\beta}{r^4}\right)dt^2-\frac{12\alpha}{\chi^2}\left(\alpha r^2+\frac{\beta}{r^4}\right)^{-1}dr^2-\frac{12\alpha r^2}{\chi^2}ds^2(\mathbb{R}^3). \quad (76)$$

This metric is a supersymmetric ‘‘topological black hole’’ [6] and it can be obtained from taking the infinite volume limit of the nakedly singular supersymmetric ‘‘black hole’’ solution to be discussed next.

### 3. Black hole solutions

In order to obtain black hole solutions we shall set the metric on the Kähler base manifold to be

$$ds^2=H^{-2}dr^2+\frac{r^2}{4}H^2(\sigma_3^L)^2+\frac{r^2}{4}[(\sigma_1^L)^2+(\sigma_2^L)^2] \quad (77)$$

with orthonormal basis

$$e^1=H^{-1}dr,\quad e^2=\frac{rH}{2}\sigma_3^L$$
$$e^3=\frac{r}{2}\sigma_1^L,\quad e^4=\frac{r}{2}\sigma_2^L \quad (78)$$

and we set

$$H=\sqrt{1+\frac{\chi^2}{12}r^2\left(1+\frac{\mu}{r^2}\right)^3}. \quad (79)$$

With this choice of $H$, $\Theta=0$. Once again, this allows us to set $\lambda^2=\lambda^3=0$ for simplicity. Moreover,

$$P=-\frac{\chi^2}{8r^2}(r^2+\mu)^2\sigma_3^L$$
$$\omega=\frac{\chi}{4\sqrt{3}r^4}(r^2+\mu)^3\sigma_3^L$$
$$f=\left(1+\frac{\mu}{r^2}\right)^{-1}$$
$$\lambda^1=\frac{\chi}{2\sqrt{3}r^4}(r^2+\mu)(2r^2-\mu)$$
$$G^+=-\frac{\sqrt{3}\chi\mu}{2r^4}(r^2+\mu)(e^1\wedge e^2+e^3\wedge e^4). \quad (80)$$

On setting $\phi=\phi'+(\chi/\sqrt{3})t$ the spacetime geometry simplifies to

$$ds^2=f^2\left(1+\frac{\chi^2}{12}r^2f^{-3}\right)dt^2-f^{-1}\left[\left(1+\frac{\chi^2}{12}r^2f^{-3}\right)^{-1}dr^2+r^2d\Omega_3^2\right] \tag{81}$$

with $F=-\sqrt{3}/2dt\wedge df$, where $d\Omega_3^2$ denotes the metric on $S^3$. These are the supersymmetric black holes, with naked singularities, first constructed in [14] (to get the same coordinates shift $r^2=R^2-\mu$). On taking the "infinite volume" limit, in which the 3-sphere blows-up to $\mathbb{R}^3$, we recover, up to a coordinate transformation, metric (76) [6]. Note that on holding $\mu$ constant and letting $\chi\to 0$, we obtain, as expected, the electrically charged static black hole solution of the ungauged theory.

We remark that all of these timelike solutions have $\Theta=0$, which is a strong restriction on the base. It would appear therefore that there is a rich structure of new solutions for which $\Theta\neq 0$. It would be interesting to see if the rotating black hole solutions examined in [15] lie within this class.

## IV. THE NULL CASE

### A. The general solution

In this section we shall find all solutions of minimal gauged $N=1$, $D=5$ supergravity for which the function $f$ introduced in Sec. II vanishes everywhere.

From Eq. (16) it can be seen that $V$ satisfies $V\wedge dV=0$ and is therefore hypersurface-orthogonal. Hence there exist functions $u$ and $H$ such that

$$V=H^{-1}du. \tag{82}$$

A second consequence of Eq. (15) is

$$V^\alpha D_\alpha V=0, \tag{83}$$

so $V$ is tangent to affinely parametrized geodesics in the surfaces of constant $u$. One can choose coordinates $(u,v,y^m)$, $m=1,2,3$, such that $v$ is the affine parameter along these geodesics, and hence

$$V=\frac{\partial}{\partial v}. \tag{84}$$

The metric must take the form

$$ds^2=H^{-1}(\mathcal{F}du^2+2dudv)-H^2\gamma_{mn}dy^mdy^n, \tag{85}$$

where the quantities $H$, $\mathcal{F}$, and $\gamma_{mn}$ depend on $u$ and $y^m$ only (because $V$ is Killing). It is particularly useful to introduce a null basis

$$e^+=V=H^{-1}du,\quad e^-=dv+\tfrac{1}{2}\mathcal{F}du,\quad e^i=H\hat{e}^i \tag{86}$$

satisfying

$$ds^2=2e^+e^--e^ie^i, \tag{87}$$

where $\hat{e}^i=\hat{e}^i{}_m dy^m$ is an orthonormal basis for the 3-manifold with $u$-dependent metric $\gamma_{mn}$; $\delta_{ij}\hat{e}^i\hat{e}^j=\gamma_{mn}dy^mdy^n$.

Equations (8) and (9) imply that $X^{(i)}$ can be written

$$X^{(i)}=e^+\wedge L^{(i)}, \tag{88}$$

where $L^{(i)}=L^{(i)}{}_m e^m$ satisfy $L^{(i)}{}_m L^{(j)}{}_n\delta^{mn}=\delta^{ij}$. In fact, by making a change of basis we can set $L^{(i)}=e^i$, so

$$X^{(i)}=e^+\wedge e^i=du\wedge\hat{e}^i. \tag{89}$$

We set $\epsilon_{+-123}=\eta$; $\eta^2=1$. Then Eq. (18) implies

$$du\wedge d\hat{e}^1=0$$

$$du\wedge\left[d\hat{e}^2-\chi\left(A\wedge\hat{e}^3+\eta\sqrt{\frac{3}{2}}H\hat{e}^1\wedge\hat{e}^2\right)\right]=0$$

$$du\wedge\left[d\hat{e}^3+\chi\left(A\wedge\hat{e}^2-\eta\sqrt{\frac{3}{2}}H\hat{e}^1\wedge\hat{e}^3\right)\right]=0. \tag{90}$$

Now define $\tilde{d}\hat{e}^i=\frac{1}{2}(\partial\hat{e}^i_m/\partial y^n-\partial\hat{e}^i_n/\partial y^m)dy^n\wedge dy^m$. Then Eq. (90) implies that

$$\tilde{d}\hat{e}^1=0$$

$$\tilde{d}\hat{e}^2-\chi\left(A\wedge\hat{e}^3+\eta\sqrt{\frac{3}{2}}H\hat{e}^1\wedge\hat{e}^2\right)=0$$

$$\tilde{d}\hat{e}^3+\chi\left(A\wedge\hat{e}^2-\eta\sqrt{\frac{3}{2}}H\hat{e}^1\wedge\hat{e}^3\right)=0. \tag{91}$$

Hence, in particular $(\hat{e}^2+i\hat{e}^3)\wedge\tilde{d}(\hat{e}^2+i\hat{e}^3)=0$, from which it follows that there exists a complex function $S(u,y)$ and real functions $x^2=x^2(u,y)$, $x^3=x^3(u,y)$ such that

$$(\hat{e}^2+i\hat{e}^3)_m=S\frac{\partial}{\partial y^m}(x^2+ix^3) \tag{92}$$

and hence $(\hat{e}^2+i\hat{e}^3)=Sd(x^2+ix^3)+\psi du$ for some complex function $\psi(u,y)$. Similarly, there exists a real function $x^1=x^1(u,y)$ such that $\hat{e}^1=dx^1+a^1du$ for some real function $a^1$. Hence, from this it is clear that we can change coordinates from $u,y^m$ to $u,x^m$. Moreover, we can make a gauge transformation of the form $A\to A+d\Lambda$ where $\Lambda=\Lambda(u,x)$ in order to set $X^2+iX^3\to Sdu\wedge(dx^2+idx^3)$ where $S$ is now a real function. Note that such a gauge transformation preserves the original gauge restriction (21) that $A_v=0$.

Hence, the null basis can be simplified to

$$\begin{aligned} e^{+} &= V = H^{-1}du \\ e^{-} &= dv + \tfrac{1}{2}\mathcal{F}du \\ e^{1} &= H(dx^{1} + a^{1}du) \\ e^{2} &= H(Sdx^{2} + S^{-1}a^{2}du) \\ e^{3} &= H(Sdx^{3} + S^{-1}a^{3}du) \end{aligned} \tag{93}$$

for real functions $H(u,x^m)$, $S(u,x^m)$, $a^i(u,x^m)$, and $X^i = e^+ \wedge e^i$.

Equation (13) implies that $i_V F = 0$ and hence

$$F = F_{+i} e^{+} \wedge e^{i} + \tfrac{1}{2} F_{ij} e^{i} \wedge e^{j}. \tag{94}$$

To proceed, we use Eq. (16) to solve for the components $F_{ij}$; we find

$$\begin{aligned} F_{12} &= -\eta \sqrt{\frac{3}{2}} H^{-2} S^{-1} \nabla_3 H \\ F_{13} &= \eta \sqrt{\frac{3}{2}} H^{-2} S^{-1} \nabla_2 H \\ F_{23} &= \eta \left( \frac{\chi}{2} - \sqrt{\frac{3}{2}} H^{-2} \nabla_1 H \right) \end{aligned} \tag{95}$$

where $\nabla$ denotes the flat connection on $\mathbb{R}^3$, $\nabla_i \equiv \partial/\partial x^i$, and we set $a^1 = a_1$, $a^2 = a_2$ and $a^3 = a_3$. Next we consider the constraints implied by Eq. (17). After a long calculation we find[4] $\eta = -1$ together with

$$\nabla_1 S = -\chi \sqrt{\frac{3}{2}} H S \tag{96}$$

and we also find that the gauge field strength is

$$\begin{aligned} F = &\left( -\frac{\chi}{\sqrt{3}} H A_u + \frac{1}{2\sqrt{3}} S^{-2} H^{-2} [\nabla_2 (H^3 a_3) \right. \\ &\left. - \nabla_3 (H^3 a_2)] \right) du \wedge dx^1 - \frac{1}{2\sqrt{3}} H^{-2} [\nabla_1 (H^3 a_3) \\ &- \nabla_3 (H^3 a_1)] du \wedge dx^2 + \frac{1}{2\sqrt{3}} H^{-2} [\nabla_1 (H^3 a_2) \\ &- \nabla_2 (H^3 a_1)] du \wedge dx^3 + \sqrt{\frac{3}{2}} (\nabla_3 H dx^1 \wedge dx^2 - \nabla_2 H dx^1 \\ &\wedge dx^3) + \frac{1}{2} (\sqrt{3} \nabla_1 H - \chi H^2) S^2 dx^2 \wedge dx^3 \end{aligned} \tag{97}$$

and the gauge field potential is

$$A = A_u du + \frac{1}{\chi S} (\nabla_2 S dx^3 - \nabla_3 S dx^2). \tag{98}$$

We require that $F = dA$, which implies that

$$\begin{aligned} &\frac{1}{2\sqrt{3}} [\nabla_2 (H^3 a_3) - \nabla_3 (H^3 a_2)] \\ &\quad = -H^2 S^{\frac{4}{3}} \nabla_1 (S^{\frac{2}{3}} A_u) \\ &\frac{1}{2\sqrt{3}} [\nabla_3 (H^3 a_1) - \nabla_1 (H^3 a_3)] \\ &\quad = -H^2 \nabla_2 A_u - \frac{H^2}{\chi} \nabla_3 \left( S^{-1} \frac{\partial S}{\partial u} \right) \\ &\frac{1}{2\sqrt{3}} [\nabla_1 (H^3 a_2) - \nabla_2 (H^3 a_1)] \\ &\quad = -H^2 \nabla_3 A_u + \frac{H^2}{\chi} \nabla_2 \left( S^{-1} \frac{\partial S}{\partial u} \right) \end{aligned} \tag{99}$$

and

$$\begin{aligned} &S \nabla_1 \nabla_1 S - \tfrac{1}{3} (\nabla_1 S)^2 + S^{-1} (\nabla_2 \nabla_2 S + \nabla_3 \nabla_3 S) - S^{-2} [(\nabla_2 S)^2 \\ &\quad + (\nabla_3 S)^2] = 0 \end{aligned} \tag{100}$$

where we have made use of Eq. (96) in order to simplify these equations. Observe that Eq. (99) implies the following integrability condition:

$$\begin{aligned} &\frac{2H}{\chi} \left[ \nabla_3 H \nabla_2 \left( S^{-1} \frac{\partial S}{\partial u} \right) - \nabla_2 H \nabla_3 \left( S^{-1} \frac{\partial S}{\partial u} \right) \right] \\ &\quad = \nabla_1 [H^2 S^{4/3} \nabla_1 (S^{2/3} A_u)] + \nabla_2 (H^2 \nabla_2 A_u) \\ &\quad\quad + \nabla_3 (H^2 \nabla_3 A_u). \end{aligned} \tag{101}$$

In fact, it is straightforward to show that these constraints ensure that the Bianchi identity and the gauge field equations hold automatically. In addition, all but the $uu$ component of the Einstein equations also hold automatically. The $uu$ component fixes $\mathcal{F}$ in terms of the other fields.

Finally, it remains to substitute the bosonic constraints into the Killing spinor equation (3) and to check that the geometry does indeed admit Killing spinors. Recall from Eq. (11) that the Killing spinor is annihilated by $\gamma^+$,

$$\gamma^+ \epsilon^a = 0. \tag{102}$$

Then the $\alpha = -$ component of the Killing spinor equation implies that

$$\frac{\partial \epsilon^a}{\partial v} = 0, \tag{103}$$

so $\epsilon^a = \epsilon^a(u, x^1, x^2, x^3)$. Next we set $\alpha = +$; we find that

[4]The origin of this fixed orientation is that we chose a frame such that $X^{(i)} = e^+ \wedge e^i$, as in Eq. (89), rather than $X^{(i)} = -e^+ \wedge e^i$.

$$H\left(\frac{\partial\epsilon^a}{\partial u}-a^1\nabla_1\epsilon^a-S^{-2}a^2\nabla_2\epsilon^a-S^{-2}a^3\nabla_3\epsilon^a\right)$$
$$-\frac{\chi}{4\sqrt{3}}\gamma^-(\gamma^1\epsilon^a+\epsilon^{ab}\epsilon^b)+\frac{\chi A_+}{2}(\gamma^1\epsilon^a+\epsilon^{ab}\epsilon^b)=0. \tag{104}$$

Acting on Eq. (104) with $\gamma^+$ we find the algebraic constraint

$$\gamma^1\epsilon^a+\epsilon^{ab}\epsilon^b=0. \tag{105}$$

Next set $\alpha=1,2,3$; it is straightforward to show that these components of the Killing spinor equation imply that

$$\nabla_1\epsilon^a=\nabla_2\epsilon^a=\nabla_3\epsilon^a=0 \tag{106}$$

and substituting this back into Eq. (104) we also find

$$\frac{\partial\epsilon^a}{\partial u}=0. \tag{107}$$

Hence the Killing spinor equation implies that $\epsilon^a$ is constant and is constrained by Eqs. (102) and (105).

It is also useful to examine the effect on the solution of certain coordinate transformations. In particular, under the shift $v=v'+g(u,x)$ we note that the form of the solution remains the same, with $v$ replaced by $v'$, and $a_i$ and $\mathcal{F}$ replaced by

$$a_i'=a_i-H^{-3}\nabla_i g$$

$$\mathcal{F}'=\mathcal{F}+2\frac{\partial g}{\partial u}-2(a_1\nabla_1 g+S^{-2}[a_2\nabla_2 g+a_3\nabla_3 g])$$
$$+H^{-3}\{(\nabla_1 g)^2+S^{-2}[(\nabla_2 g)^2+(\nabla_3 g)^2]\} \tag{108}$$

hence we see that $H^3a$ is determined only up to a gradient.

To summarize, it is possible to construct a null supersymmetric solution as follows. First choose $S(u,x)$ satisfying Eq. (100). Then use Eq. (96) to obtain $H$. Next find $A_u(u,x)$ satisfying Eq. (101). Given such an $A_u$ Eq. (99) can always be solved, at least locally, to give $H^3a$ up to a gradient; this gradient term can be removed by making a shift in $v$ as described above. Then the gauge potential is given by Eq. (98). Lastly, fix $\mathcal{F}$ by solving the $uu$ component of the Einstein equations. In this sense the solutions are determined by three functions $S$, $A_u$ and $\mathcal{F}$. The Killing spinors are constant and constrained by Eqs. (102) and (105). Note that these solutions are generically 1/4 supersymmetric, in contrast with the null solutions in the ungauged supergravity, which are generically 1/2 supersymmetric.

### B. Magnetic string solutions

To construct a solution to these equations, we take $S$ to be independent of $u$ and separable, $S=P(x^1)Q(x^2,x^3)$, so that from Eq. (100) we find that

$$(\nabla_2\nabla_2+\nabla_3\nabla_3)\log Q=-kQ^2 \tag{109}$$

and

$$P\ddot{P}-\tfrac{1}{3}(\dot{P})^2-k=0 \tag{110}$$

for constant $k$, where here $\cdot=d/dx^1$. We then have $H=-(2/\chi\sqrt{3})P^{-1}\dot{P}$. We set $A_u=a^1=a^2=a^3=0$ and seek solutions that also have $\mathcal{F}=0$. The metric and the gauge field strength are given by

$$ds^2=-\chi\sqrt{3}P(\dot{P})^{-1}du\,dv-\frac{4}{3\chi^2}P^{-2}(\dot{P})^2(dx^1)^2$$
$$-\frac{4}{3\chi^2}(\dot{P})^2ds^2(M_2) \tag{111}$$

and

$$F=-k\chi^{-1}dvol(M_2) \tag{112}$$

where $M_2$ is a 2-manifold with metric

$$ds^2(M_2)=Q^2[(dx^2)^2+(dx^3)^2]. \tag{113}$$

Because $Q$ satisfies Eq. (109), we see that $M_2$ has constant curvature and hence can be taken to be $\mathbb{R}^2$ if $k=0$, $S^2$ if $k>0$ (with radius $k^{-1/2}$), or $H^2$ if $k<0$ [with radius $(-k)^{-1/2}$]. Next we simplify the metric by defining $R=\dot{P}$, and we note that Eq. (110) implies that $R=\sqrt{\mu P^{2/3}-3k}$ for constant $\mu$ and also $R^2(R^2/3+k)^{-2}dR^2=P^{-2}\dot{P}^2(dx^1)^2$. Hence, on rescaling $\hat{v}=-9\chi\mu^{-3/2}v$ we obtain

$$ds^2=R^{1/2}\left(\frac{R}{3}+\frac{k}{R}\right)^{3/2}du\,d\hat{v}-\frac{4}{3\chi^2}\left(\frac{R}{3}+\frac{k}{R}\right)^{-2}dR^2$$
$$-\frac{4}{3\chi^2}R^2ds^2(M_2). \tag{114}$$

It is straightforward to show that all components of the Einstein equations are satisfied. These solutions are the black string solutions of [12,16]. For $k<0$ the solution has a horizon at $R^2=-3k$ and the near horizon limit gives $AdS_3\times H^2$, which we also found in the timelike class of solutions.

## V. INTEGRABILITY AND MAXIMAL SUPERSYMMETRY

The Killing spinor equation (3) implies the following integrability conditions on the Killing spinor:

$$\frac{1}{8}{}^5R_{\rho\mu\nu_1\nu_2}\gamma^{\nu_1\nu_2}\epsilon^a = -\frac{1}{4\sqrt{3}}(\gamma_{[\mu}{}^{\nu_1\nu_2}+4\gamma^{\nu_1}\delta^{\nu_2}_{[\mu})\nabla_{\rho]}F_{\nu_1\nu_2}\epsilon^a + \frac{1}{48}(-2F^2\gamma_{\mu\rho}-8F^2_{\nu[\rho}\gamma^{\nu}{}_{\mu]} + 12F_{\mu\nu_1}F_{\rho\nu_2}\gamma^{\nu_1\nu_2} + 8F_{\nu_1\nu_2}F_{\nu_3[\rho}\gamma_{\mu]}{}^{\nu_1\nu_2\nu_3})\epsilon^a + \frac{\chi}{24}(\gamma_{\rho\mu}{}^{\nu_1\nu_2}F_{\nu_1\nu_2}-4F_{\nu[\rho}\gamma_{\mu]}{}^{\nu} - 6F_{\rho\mu})\epsilon^{ab}\epsilon^b + \frac{\chi^2}{48}\gamma_{\rho\mu}\epsilon^a. \tag{115}$$

To obtain a geometry preserving maximal supersymmetry, we require that this integrability condition imposes no algebraic constraints on the Killing spinor. In particular, it is required that the terms which are zeroth, first and second order in the gamma matrices should vanish independently (after rewriting the terms cubic, quartic and quintic in gamma matrices in terms of quadratic, linear and zeroth order terms, respectively). Hence from the zeroth order term we immediately obtain $F=0$. The integrability condition then simplifies considerably to give

$${}^5R_{\rho\mu\nu_1\nu_2} = \frac{\chi^2}{12}(g_{\rho\nu_1}g_{\mu\nu_2}-g_{\rho\nu_2}g_{\mu\nu_1}), \tag{116}$$

which implies that that the five dimensional geometry must be $AdS_5$. This is in contrast to the case of the ungauged theory, for which it has been shown [1] that there is a rich structure of maximally supersymmetric solutions.

Note also that if we contract the integrability condition with $\gamma^\mu$ we get

$$0 = \left(R_{\rho\mu}+2F_{\rho\nu}F_{\mu}{}^{\nu}-\frac{1}{3}g_{\rho\mu}(F^2+\chi^2)\right)\gamma^\mu\epsilon - \frac{1}{\sqrt{3}}\left[*\left(d*F + \frac{2}{\sqrt{3}}F\wedge F\right)\right]^{\nu}(2g_{\nu\rho}-\gamma_{\rho\nu})\epsilon - \frac{1}{6\sqrt{3}}dF_{\nu_1\nu_2\nu_3}(\gamma_{\rho}^{\nu_1\nu_2\nu_3} - 6\delta^{\nu_1}_{\rho}\gamma^{\nu_2\nu_3})\epsilon. \tag{117}$$

Suppose we have a geometry admitting a Killing spinor and in addition the equation of motion and Bianchi identity for $F$ are satisfied. By following exactly the same argument presented in [1] we conclude that if the Killing spinor is timelike, then all of Einstein's equations are automatically satisfied while if it is null, only the $++$ component, in frame (87), might not be satisfied.

## VI. CONCLUSIONS

In this paper we have presented a classification of all supersymmetric solutions of minimal five-dimensional gauged supergravity. One of the interesting differences with the ungauged theory is that in the timelike case much more of the solution is fixed by the geometric structure of the base manifold. On the other hand, in the gauged case the base must be Kähler and not hyper-Kähler, whereas in the ungauged case the base must be hyper-Kähler. In the null case the solutions are still determined by three differential equations as in the ungauged case, but these equations are more complicated than those in the ungauged theory. In addition we have shown that the gauging generically reduces the proportion of supersymmetry preserved in the null case from 1/2 to 1/4. In the gauged theory, $AdS_5$ is the unique maximally supersymmetric solution, while there are a number of different possibilities in the ungauged case.

We have also presented some new solutions, that would be worth investigating further both in $D=5$ and in $D=10$ after uplifting with a five sphere. Many of the new solutions we have presented have closed timelike curves, as was also seen in the ungauged case, which provides additional evidence that they are commonplace amongst supersymmetric solutions. It would be interesting to see if they can be removed in our explicit solutions by going to a covering space either in five or ten dimensions. Moreover, all of the timelike solutions which we have examined correspond to Kähler geometries for which the tensor $T$ given by Eq. (43) vanishes. Clearly, there are many new solutions for which $T\neq 0$.

It may also be possible to use the generic form of the supersymmetric solutions to examine the geometry of black hole solutions. In [9], the constraints on ungauged solutions found in [1] were used to show that the near horizon geometry of all supersymmetric black holes is isometric to the near horizon geometry of the BMPV solutions; and from this a uniqueness theorem was proven. In contrast, it is known that the static asymptotically anti–de-Sitter black holes have no horizon, as they are nakedly singular. However, there does exist a class of rotating AdS black hole solutions which have horizons, and hence a similar investigation could be feasible.

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