Fermion production in time-dependent fields

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The exact fermion propagator in a classical time-dependent gauge field is derived by solving the equation of motion for the Dirac Green's functions. From the retarded propagator obtained in this way the momentum spectrum for the produced fermion pairs is calculated. Different approximations and the exact solution for the propagator and the momentum spectrum are presented.

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I. INTRODUCTION

Particle production in classical bosonic fields has been a topic of continuing interest in quantum electrodynamics (QED) and quantum chromodynamics (QCD). It is relevant for the physics of the early Universe [1] as well as of ultrarelativistic heavy-ion collisions and the quark-gluon plasma [2] (QGP). A lot of effort is made to study the QGP's production and equilibration [3] in nuclear collision experiments at the Relativistic Heavy-Ion Collider (RHIC) at the Brookhaven National Laboratory (BNL) and the Large Hadron Collider (LHC) under construction at CERN. The existence of such a state of matter is predicted by lattice QCD calculations at high temperatures [4].

At ultrarelativistic energies, the two nuclei are highly Lorentz contracted. When they pass through each other, a chromoelectric field is formed due to the exchange of soft gluons [5]. This is a natural extension of the color flux-tube model or the string model which are widely applied to high-energy pp, e^+e^-, and pA collisions [6]. Many other recent publications, e.g., [7–10], are based on the hypothesis that the initial state in heavy-ion collisions is dominated by gluons which on account of the large occupation number can be treated as a classical background field.

The larger the occupation number $\langle c_k^\dagger c_k\rangle$ of the bosonic sector of a physical system, the better it can be described by a classical field. Here c_k and $c_k^\dagger$ are the bosonic field annihilation and creation operators for particles of momentum k. Especially if the occupation number is much larger than one, the commutator of the creation and annihilation operator and hence quantum effects can be neglected: $[c_k^\dagger, c_k]=1 \ll \langle c_k^\dagger c_k\rangle$. The field operators can be approximated by complex numbers; i.e., they are treated classically.

For gluons in a heavy-ion collision at RHIC with $\sqrt{s}=130$ GeV the initial occupation number for gluons of average transverse momentum $|\vec{k}_T|\approx 1$ GeV in the center of the collisions is roughly equal to 1.5 [11]. Although this number is not much larger than unity, the classical field as the expectation value of the gauge field still constitutes the main contribution as compared to the fluctuations of the gauge field. The occupation number for lower transverse momenta is yet higher and the classical concept is an even better approximation. For larger transverse momenta the occupation number is smaller and thus the quantum fluctuations are more important there. As the above value for the occupation number at the average transverse momentum is larger than one, keeping only classical bosons is justified as a first approximation. One can investigate quantum fluctuations in a subsequent step.

The high-occupation number bosonic fields are of the order $A\sim g^{-1}$. Thus processes with multiple couplings to the classical field are not parametrically suppressed by powers of the coupling constant g. Without an additional scale, they have to be taken into account to all orders. Under the prerequisite of weak coupling, the most important quantum processes involve only terms of the classical action which are of second order in the quantum fields. These are the fermion and the antifermion fields as well as the field of the bosonic quantum fluctuations. The coefficient of the second order terms for a given field constitutes the inverse of the corresponding two-point Green's function. Inversion for selected boundary conditions yields the particle's propagator. The propagators contain all the information on two-(quantum-)particle reactions in the presence of classical fields to all orders in the coupling constant g. These reactions are scattering off the classical field or particle production by vacuum polarization.

In the following let us consider particle production. In quantum electrodynamics this means the production of electron-positron pairs. Analogously, in quantum chromodynamics quark-antiquark pairs can be produced. However, due to the non-linearity of the field tensor pairs of gluonic quantum fluctuations are produced, too. In fields of the magnitude $A\sim g^{-1}$ the production of both kinds of pairs is equally parametrically favored. This paper only deals with the production of fermions and antifermions. It is possible that in a given situation the bosonic sector is covered by the concept of a classical field sufficiently well. Corrections to the high-momentum sector could be perturbatively accessible [12].

No concept of a classical field exists for the fermionic sector. There the occupation number has always to be less than unity due to the Pauli principle. A purely perturbative treatment could only describe the high-momentum sector. The soft part would not be treated consistently.

An alternative to the perturbative approach for particle production is Schwinger's constant-field method [13] which is an exact one-loop non-perturbative approach. This method can also be understood as semiclassical tunneling across the mass gap [14]. However, this scheme is based on the as-

sumption of a slowly varying classical field. If the field changes too rapidly in space or time, the production of fermions is again not described properly. A different concept is needed that is independent of energy or time scales, respectively. Such a concept is especially important, if the time scale for a process is to be determined by a self-consistent calculation. If an approach is applied for such an investigation, which relies already on an assumption about the time scale, the result is likely to be misleading. For example, if the decay time scale for a classical field is to be calculated based on particle production, perturbative concepts are likely to lead to times which are too short, while the Schwinger method tends to predict a development of the system that is too slow.

For a concise treatment other methods are necessary. Exact results are desirable but hard to obtain. As mentioned before, neglecting bosonic quantum fluctuations the behavior of the fermions is governed by their two-point functions in the classic background field. It can be obtained by solving the equation of motion for the Dirac Green's function $G(x,y)$ exactly:

$$[i\gamma\cdot\partial(x)+\gamma\cdot A(x)-m]G(x,y)=\delta^{(4)}(x-y). \tag{1}$$

There are other ways to derive the full propagator, for instance by resumming all terms of the perturbative series or by adding up a set of (at all times complete) wave-function solutions of the Dirac equation.

In arbitrary fields a few general approximations are known. Neglecting the field in the equation of motion leads to the free Green's function $G^0(x-y)$. The standard perturbative series is a sum of terms containing powers of the background field A between free Green's functions. The asymptotic behavior of the free Green's function determines that of the approximated full Green's function. Another approach which applies in an arbitrary field is the static approximation $G^S(x,y)$. It is obtained by neglecting the spatial part of the covariant derivative in the differential equation. The remaining ordinary differential equation can be solved by direct integration. Yet another approach can be found in [15].

The following investigation concerns the case where the classical field depends arbitrarily on one rectilinear coordinate $A=A(n\cdot x)$. The equation of motion for the propagators shall be solved directly. If such solutions are investigated, care has to be taken: the result could be any Green's function which is not necessarily a propagator. If a propagator has been obtained, the imposed boundary conditions determine whether the result is the retarded or the Feynman propagator or one of their related singular functions.

A solution in a field depending on one rectilinear coordinate can also be seen as an approximation for the case where the strongest dependence is on this rectilinear coordinate and the dependence on all the others is much weaker.

It will be explored whether an approximation scheme can be found that is independent of assumptions on time and/or energy scales over a large range of parameters. The investigated approximations are the Born, the weak-field, the strong-field, and the Abelian approximation. The weak-field approach is an expansion in powers of the gauge field A based on the free propagator and valid for $A\ll\omega$. ω stands for the on-shell energy of the described particles. The strong-field approximation is justified for $\omega\ll A$ and consists of an expansion in powers of the on-shell energy. For the Abelian approach the commutators of the elements of the Clifford and the charge algebra are neglected.

Section II includes the exact solution of the equation of motion for the Dirac Green's function and gives several approximations to the full solution. Section III contains the application of the previous findings to the problem of particle production and the comparison of the different schemes to the exact result. In Sec. IV the contents of the paper are summarized.

Throughout the paper the metric is $g^{\mu\nu}=\mathrm{diag}(1,-1,-1,-1)$, angular momenta are measured in units of $\hbar$, and velocities in fractions of the speed of light c. From hereon, the coupling constant is included in the classical field: $gA^{\mu}_{old}=A^{\mu}_{new}$.

II. DETERMINATION OF THE PROPAGATOR

Let us consider homogeneous solutions $G_H(x,y)$ of Dirac's equation (1). In the special class of fields which only depend on one rectilinear coordinate $n\cdot x$ this equation can be Fourier transformed (three dimensionally) into an ordinary differential equation:

$$\left[i(\gamma\cdot n)\frac{d}{d(n\cdot x)}+\gamma\cdot\kappa+\gamma\cdot A(n\cdot x)-m\right]G_H(n\cdot x,n\cdot y,\kappa)=0, \tag{2}$$

with the conserved three-dimensional momentum coordinate $\kappa=k-n\partial(k\cdot x)/\partial(n\cdot x)$ orthogonal to n and where k stands for the four-momentum. As a further *ansatz* the matrix function $G_H(n\cdot x,n\cdot y,\kappa)$ is to be a functional of another matrix function $g_H(n\cdot x,n\cdot y,\kappa)$ with a special property for the derivative

$$\frac{d}{d(n\cdot x)}G_H[g_H(n\cdot x,n\cdot y,\kappa)]=\left[\frac{d}{d(n\cdot x)}g_H(n\cdot x,n\cdot y,\kappa)\right]G_H[g_H(n\cdot x,n\cdot y,\kappa)], \tag{3}$$

which looks like the derivative of an exponential function, but is not quite due to the matrix structure. Provided a function $g_H(n\cdot x,n\cdot y,\kappa)$ exists which satisfies Eq. (3), the form of the functional $G_H[g_H(n\cdot x,n\cdot y,\kappa)]$ can be determined. Exploiting the above property leads to a factorization in Eq. (2):

$$\left[i(\gamma\cdot n)\frac{d}{d(n\cdot x)}g_H(n\cdot x,n\cdot y,\kappa)+\gamma\cdot\kappa+\gamma\cdot A(n\cdot x)-m\right]\times G_H[g_H(n\cdot x,n\cdot y,\kappa)]=0. \tag{4}$$

Given the existence of a solution G_H other than the trivial solution, its matrix structure can be inverted. Multiplication with the inverse of the solution from the right then yields a differential equation for the function g_H,

$$i(\gamma\cdot n)\frac{d}{d(n\cdot x)}g_H(n\cdot x,n\cdot y,\kappa)+\gamma\cdot\kappa+\gamma\cdot A(n\cdot x)-m=0. \tag{5}$$

This ordinary differential equation can be solved by direct integration where the initial condition $g_H(n\cdot x=n\cdot y, n\cdot y,\kappa)=0$ is chosen,

$$g_H(n\cdot x,n\cdot y,\kappa)=i\frac{\gamma\cdot n}{n^2}\int_{n\cdot y}^{n\cdot x}d(n\cdot\xi)[\gamma\cdot\kappa+\gamma\cdot A(n\cdot\xi)-m]. \tag{6}$$

Here it is necessary to require $n^2\neq 0$. Otherwise, the matrix $\gamma\cdot n$ does not posses an inverse because of $\det\{\gamma\cdot n\}=(n^2)^2$. For cases with $n^2=0$ a different treatment is necessary.

In general, the argument function g_H does not commute with itself at different points $n\cdot x$. This is not (only) due to non-Abelian charges which might be included in the vector field A but to the non-commutative nature of the elements of the Clifford algebra. Thus the solution of Eq. (3) is not an exponential function but a path-ordered exponential:

$$G_H[g_H(n\cdot x,n\cdot y,\kappa)]=\mathcal{P}\exp\{g_H(n\cdot x,n\cdot y,\kappa)\}. \tag{7}$$

A sufficient but not necessary condition for its existence is that the norm of the integrand in Eq. (6) is bounded. It has to be noted that the invariance of an integral under the simultaneous exchange of the integration boundaries and the inversion of the sign cannot be used in Eq. (6) because the path ordering would be reversed.

If a more general initial condition had been chosen in Eq. (6) the additional addend $g_H(n\cdot y,n\cdot y,\kappa)$ would have lacked an ordering parameter necessary for the path ordering. Hence, it could only be treated by always setting it to the right-hand side (rhs) of the remaining path-ordered exponential. This would have led to an extra factor $\times\exp\{g_H(n\cdot y, n\cdot y,\vec{k})\}$. As here a homogeneous differential equation is investigated this factor does not lead to independent solutions.

Hereafter one has only to distinguish between the cases $n^2>0$ and $n^2<0$, because every field $A=A(n\cdot x)$ can be transformed into a field $A=A(n'\cdot x)$ with $\mathrm{sgn}(n^2)=\mathrm{sgn}(n'^2)$ by a Lorentz transformation. Overall factors in front of the normal vectors can be absorbed in a redefinition of the vector potential A. So, for the sake of simplicity it suffices to investigate one special case per class of fields. This is going to be done for the cases of $n^\mu=(1,0,0,0)$ and $n^\mu=(0,0,0,1)$. In situations where $n^2=0$, rotations in three-space can turn any normal vector n into $n^\mu=(1,0,0,-1)/\sqrt{2}$.

A. Time-like coordinates

For a purely time-dependent field, the solution for $g_H(x_0,y_0,\vec{k})$ in Eq. (6) is given by

$$g_H(x_0,y_0,\vec{k})=i\gamma^0\int_{y_0}^{x_0}d\xi_0[\gamma^j k_j+\gamma\cdot A(\xi_0)-m], \tag{8}$$

with $j\in\{1,2,3\}$. Constructing the matrix function $G_H(x_0,y_0,\vec{k})$ by setting Eq. (8) into Eq. (7) leads to

$$G_H(x_0,y_0,\vec{k})=\mathcal{P}\exp\left\{i\gamma^0\int_{y_0}^{x_0}d\xi_0[\gamma^j k_j+\gamma\cdot A(\xi_0)-m]\right\}. \tag{9}$$

In the following, various approximations are studied in order to learn more about the above solution.

1. Weak-field approximation

It is useful to investigate the case of a vanishing gauge field $A=0$. One then sees that the argument $g_H^0(x_0-y_0,\vec{k})$ now commutes with itself at different space-time points. The path-ordered exponential can now be replaced by an exponential function. The exponential function of matrices can be recast into exponential functions of scalar arguments multiplied with matrices:

$$G_H^0(x_0-y_0,\vec{k})=\gamma^0\frac{\gamma^0\omega+\gamma^j k_j-m}{2\omega}e^{+i\omega(x_0-y_0)} + \gamma^0\frac{\gamma^0\omega-\gamma^j k_j+m}{2\omega}e^{-i\omega(x_0-y_0)}, \tag{10}$$

with $\omega=\sqrt{|\vec{k}|^2+m^2}$.

Standard perturbation theory for small gauge fields $A\ll\omega$ which can be interpreted as an ultraviolet approximation is obtained by expanding the exact solution in powers of A. Prior to this, it has to be rewritten in order to include all powers of the momenta and the mass with every factor of the field. The path-ordered exponential can be expressed as

$$G_H(x_0,y_0,\vec{k})=\lim_{N\to\infty}\mathcal{P}\prod_{n=0}^{N-1}\{1+i\gamma^0\Delta\xi_0^{(n)} \times[\gamma^j k_j+\gamma\cdot A(\xi_0^{(n)})-m]\}. \tag{11}$$

The interval $[x_0,y_0]$ is decomposed into N disjoint pieces with the lengths $\Delta\xi_0^{(n)}$ which need not be equal and each with an inner point $\xi_0^{(n)}$. These are arranged according to $x_0=\xi_0<\xi_1<\cdots<\xi_N<\xi_{N+1}=y_0$ for $x_0<y_0$ or $x_0=\xi_0>\xi_1>\cdots>\xi_N>\xi_{N+1}=y_0$ for $x_0>y_0$. $\mathcal{P}$ indicates that the factors are ordered with respect to the index ν where the term with the lowest index is put furthest to the left. The expression can now be sorted with respect to powers of the field A,

$$G_H(x_0,y_0,\vec{k}) = \lim_{N\to\infty} \sum_{l=0}^{N} \sum_{n_1=0}^{N-1} \sum_{n_2=n_1+1}^{N-1} \cdots \sum_{n_l=n_{l-1}+1}^{N-1} \prod_{L=0}^{n_1-1} (1+i\gamma^0\Delta\xi_0^{(L)}[\gamma^j k_j - m]) \times [i\gamma^0\gamma\cdot A(\xi_0^{(n_1)})\Delta\xi_0^{(n_1)}]$$
$$\times \prod_{L=n_1+1}^{n_2-1} (1+i\gamma^0\Delta\xi_0^{(L)}[\gamma^j k_j - m]) \times [i\gamma^0\gamma\cdot A(\xi_0^{(n_2)})\Delta\xi_0^{(n_2)}] \times \cdots$$
$$\times \prod_{L=n_{l-1}+1}^{n_l-1} (1+i\gamma^0\Delta\xi_0^{(L)}[\gamma^j k_j - m]) \times [i\gamma^0\gamma\cdot A(\xi_0^{(n_{l-1})})\Delta\xi_0^{(n_{l-1})}] \prod_{L=n_l+1}^{N-1} (1+i\gamma^0\Delta\xi_0^{(L)}[\gamma^j k_j - m]). \tag{12}$$

For $l=0$ there are no further sums over n_i. Sums and products are not taken into account if the starting index is greater than the ending index. In the limit $N\to\infty$ the outer sum over the powers l of the gauge field A becomes an infinite sum, the intermediate sums turn into integrals over simplices, and the products give path-ordered exponentials. In fact, their arguments commute at every point, thus the path ordering can be dropped here

$$G_H(x_0,y_0,\vec{k}) = \sum_{l=0}^{\infty} \int_{y_0}^{x_0} d\xi_1 \int_{y_0}^{\xi_1} d\xi_2 \cdots \int_{y_0}^{\xi_{l-1}} d\xi_l \exp\{i\gamma^0[\gamma^j k_j - m](x_0-\xi_1)\} \times [i\gamma^0\gamma\cdot A(\xi_1)] \exp\{i\gamma^0[\gamma^j k_j - m](\xi_1-\xi_2)\}$$
$$\times [i\gamma^0\gamma\cdot A(\xi_2)] \times \cdots \times \exp\{i\gamma^0[\gamma^j k_j - m](\xi_{l-1}-\xi_l)\} \times [i\gamma^0\gamma\cdot A(\xi_l)] \exp\{i\gamma^0[\gamma^j k_j - m](\xi_l - y_0)\}. \tag{13}$$

This expression is a uniformly and absolutely converging series representation for a path-ordered exponential,

$$G_H(x_0,y_0,\vec{k}) = G_H^0(x_0-y_0,\vec{k}) \mathcal{P} \exp\left\{ \int_{y_0}^{x_0} d\xi_0 G_H^0(y_0-\xi_0,\vec{k}) \times [i\gamma^0\gamma\cdot A(\xi_0)] G_H^0(\xi_0-y_0,\vec{k}) \right\}. \tag{14}$$

The above derivation is a special case of a more general identity for the type of path-ordered exponentials encountered here (see the Appendix). The expansion of this formula in powers of A yields

$$G_H(x_0,y_0,\vec{k}) = G_H^0(x_0-y_0,\vec{k}) + \int_{y_0}^{x_0} d\xi_0 G_H^0(x_0-\xi_0,\vec{k}) \times [i\gamma^0\gamma\cdot A(\xi_0)] G_H^0(\xi_0-y_0,\vec{k})$$
$$+ \int_{y_0}^{x_0} d\xi_0 \int_{y_0}^{\xi_0} d\eta_0 G_H^0(x_0-\xi_0,\vec{k}) \times [i\gamma^0\gamma\cdot A(\xi_0)] G_H^0(\xi_0-\eta_0,\vec{k}) \times [i\gamma^0\gamma\cdot A(\eta_0)] G_H^0(\eta_0-y_0,\vec{k}) + \cdots. \tag{15}$$

Up to now, only the solution $G_H(x_0,y_0,\vec{k})$ of the homogeneous Dirac equation in the mixed representation has been investigated. According to the equation of motion (1) the inhomogeneous solution $iG(x_0,y_0,\vec{k})\gamma^0$ must jump by one at $x_0=y_0$. The retarded propagator $G_R(x_0,y_0,\vec{k})$ vanishes for negative time differences $x_0-y_0<0$. Due to the previous requirement on the argument $g_H(x_0=y_0,y_0,\vec{k})=0$ one has for the homogeneous solution $G_H(x_0=y_0,y_0,\vec{k})=1$. Hence, the following condition has to be fulfilled in order to relate the latter and the retarded propagator:

$$iG_R(x_0,y_0,\vec{k})\gamma^0 = \theta(x_0-y_0) G_H(x_0,y_0,\vec{k}). \tag{16}$$

It should be noted that if the case of a different, more general coordinate with $n^2>0$ should have been investigated at this point, the additional requirement $n_0>0$ would be needed here in order to ensure that really the *retarded* propagator is obtained. However, this can always be achieved by a redefinition of the functional form of the vector potential.

All results obtained for the homogeneous solution of Dirac's equation in the present mixed representation are linked directly to the Green's function $G_R(x_0,y_0,\vec{k})$ by Eq. (16). After putting Eq. (15) into the previous expression, the Heaviside function can be multiplied to every free homogeneous solution $G_H^0(\zeta_0,\vec{k})$,

$$iG_R(x_0,y_0,\vec{k})\gamma^0 = \theta(x_0-y_0) G_H^0(x_0-y_0,\vec{k}) + \int_{y_0}^{x_0} d\xi_0 \theta(x_0-\xi_0) G_H^0(x_0-\xi_0,\vec{k}) \times [i\gamma^0\gamma\cdot A(\xi_0)] \theta(\xi_0-y_0) G_H^0(\xi_0-y_0,\vec{k})$$
$$+ \int_{y_0}^{x_0} d\xi_0 \int_{y_0}^{\xi_0} d\eta_0 \theta(x_0-\xi_0) G_H^0(x_0-\xi_0,\vec{k}) \times [i\gamma^0\gamma\cdot A(\xi_0)] \theta(\xi_0-\eta_0) G_H^0(\xi_0-\eta_0,\vec{k}) \times [i\gamma^0\gamma\cdot A(\eta_0)] \theta(\eta_0-y_0) G_H^0(\eta_0-y_0,\vec{k}) + \cdots. \tag{17}$$

This is possible due to the idempotency of the Heaviside function and the fact that $\theta(x_0-\xi_0)\,\theta(\xi_0-y_0)=\theta(x_0-y_0)$ if $\xi_0\in[x_0,y_0]$. Subsequently, in accordance with Eq. (16), the result can be reexpressed in terms of free Greens's functions

$$\begin{aligned} iG_R(x_0,y_0,\vec{k}) &= iG_R^0(x_0-y_0,\vec{k})+\int_{x_0}^{y_0} d\xi_0 iG_R^0(x_0-\xi_0,\vec{k}) \\ &\times[i\gamma\cdot A(\xi_0)]iG_R^0(\xi_0-y_0,\vec{k}) \\ &+\int_{x_0}^{y_0} d\xi_0\int_{\xi_0}^{y_0} d\eta_0 iG_R^0(x_0-\xi_0,\vec{k}) \\ &\times[i\gamma\cdot A(\xi_0)]iG_R^0(\xi_0-\eta_0,\vec{k}) \\ &\times[i\gamma\cdot A(\eta_0)]iG_R^0(\eta_0-y_0,\vec{k})+\cdots. \end{aligned} \tag{18}$$

Note that in the literature slightly different definitions exist for the propagator which account for the various occurrences of the imaginary unit *i*. The full retarded propagator $G_R(x_0,y_0,\vec{k})$ inherits the asymptotic behavior of the free retarded propagator $G_R^0(x_0-y_0,\vec{k})$ by virtue of the above formula (18).

The full Feynman, i.e. time-ordered propagator, cannot be expressed as a path-ordered exponential because it is defined with mixed boundary conditions: for the positive energy components at $x_0\to-\infty$ and for the negative energy components at $x_0\to+\infty$. This can also be seen from the free Feynman propagator

$$\begin{aligned} iG_F^0(x_0-y_0,\vec{k}) &= \theta(x_0-y_0)\frac{\gamma^0\omega-\gamma^j k_j+m}{2\omega}e^{-i\omega(x_0-y_0)} \\ &+\theta(y_0-x_0)\frac{\gamma^0\omega+\gamma^j k_j-m}{2\omega}e^{+i\omega(x_0-y_0)} \end{aligned} \tag{19}$$

which is a singular object in this and every mixed representation. That can be understood by looking at Fig. 1. Thus it is impossible to take its logarithm and express it as an exponential function. This is why the Feynman propagator cannot be equal to a path-ordered exponential of the form (9).

Figure 1 shows the contour integrations in the complex k_0 plane which have to be carried out in order to determine the contributions from the different poles of the corresponding propagator in momentum representation to that propagator in the mixed representation. Every pole included inside a contour results in an additive contribution to the free propagator proportional to one of the matrices $\gamma\cdot k\pm m$. On shell, i.e. for $k^2=m^2$, these are singular. The circles in Fig. 1 belong to the retarded (black) and the advanced (white) propagators. It is important to note that either none of the poles is included in a contour or both. This means that if one of these two propagators is non-zero, the two singular matrices occur in a nontrivial linear combination, which yields an invertible matrix

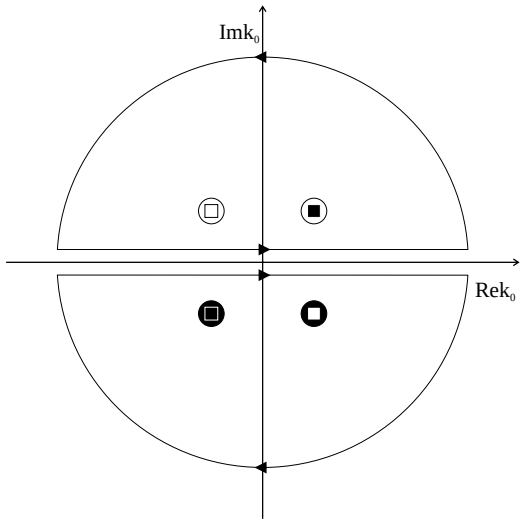

FIG. 1. Contour integration in the complex k_0 plane for the determination of the Green's function with the correct asymptotic behavior. The circles indicate the pairs of positions to which the poles are moved off the real axis by virtue of the corresponding ϵ prescription for the retarded (black) and the advanced (white) propagator. The squares show the position of the poles for the Feynman (white) and the reverse Feynman (black) propagators. For the retarded and the advanced propagators two poles or no pole is inside a given contour. For the Feynman and reverse Feynman propagators exactly one pole is always inside any contour.

structure. This is different for the Feynman propagator and its relatives. There only one pole at a time is included in a contour. Thus these propagators are always non-invertible.

2. Strong-field approximation

The previous expansion which is appropriate for weak fields $A(t)\ll\omega$ could be interpreted as a ultraviolet approximation. An infrared expansion requires a strong field $A(t)\gg\omega$. It can be obtained by applying the resummation formula of the Appendix in a different way. Resumming all scattering processes with the field for each power of the momentum, one obtains

$$\begin{aligned} &G_H(x_0,y_0,\vec{k}) \\ &=\mathcal{P}\exp\left\{i\int_{y_0}^{x_0} d\xi_0\,\gamma^0\gamma\cdot A(\xi_0)\right\} \\ &\times\mathcal{P}\exp\left[\int_{y_0}^{x_0} d\xi_0\,\mathcal{P}\exp\left\{i\int_{\xi_0}^{y_0} d\theta_0\,\gamma^0\gamma\cdot A(\theta_0)\right\}\right. \\ &\left.\times\{i\gamma^0[\gamma^j k_j-m]\}\mathcal{P}\exp\left\{i\int_{y_0}^{\xi_0} d\theta_0\,\gamma^0\gamma\cdot A(\theta_0)\right\}\right]. \end{aligned} \tag{20}$$

Now, one could start to expand the outer path-ordered exponential in powers of the momentum term:

$$G_H(x_0,y_0,\vec{k})=\mathcal{P}\exp\left\{ i\int_{y_0}^{x_0} d\xi_0\,\gamma^0\gamma\cdot A(\xi_0)\right\}+\int_{y_0}^{x_0} d\xi_0\,\mathcal{P}\exp\left\{ i\int_{\xi_0}^{y_0} d\theta_0\,\gamma^0\gamma\cdot A(\theta_0)\right\}\{i\gamma^0[\gamma^j k_j-m]\}$$
$$\times\mathcal{P}\exp\left\{ i\int_{x_0}^{\xi_0} d\theta_0\,\gamma^0\gamma\cdot A(\theta_0)\right\}+\int_{y_0}^{x_0} d\xi_0\int_{y_0}^{\xi_0} d\eta_0\,\mathcal{P}\exp\left\{ i\int_{\xi_0}^{y_0} d\theta_0\,\gamma^0\gamma\cdot A(\theta_0)\right\}$$
$$\times\{i\gamma^0[\gamma^j k_j-m]\}\mathcal{P}\exp\left\{ i\int_{\eta_0}^{\xi_0} d\theta_0\,\gamma^0\gamma\cdot A(\theta_0)\right\}\{i\gamma^0[\gamma^j k_j-m]\}\mathcal{P}\exp\left\{ i\int_{x_0}^{\eta_0} d\theta_0\,\gamma^0\gamma\cdot A(\theta_0)\right\}+\cdots. \tag{21}$$

This corresponds to an expansion in powers of the on-shell energy ω which can be understood by noting that $(\gamma^0[\gamma^j k_j-m])^2=\omega^2$. The weak-field approximation is based on the investigation of a given number of the—otherwise freely propagating—particle with the field which could be termed ''accelerations.'' The strong-field approach comes up to an expansion in powers of what could be called the ''inertia'' because ω equals the (asymptotic) relativistic mass. In the lowest order of the strong-field approximation the propagation of a particle without relativistic mass is governed only by the field. The higher order terms accord for deviations due to non-vanishing ω.

3. Abelian approximation

All of the above approximations in form of an expansion with respect to some part of the exponent are based on Eq. (A4). This is different for the Abelian approximation scheme(s), i.e. a commutative approximation with respect to the Clifford and the charge group algebra. The lowest order $G_H^{A_0}(x_0,y_0,\vec{k})$ of the Abelian approximation is given by omitting the path-ordering in Eq. (9),

$$G_H^{A_0}(x_0,y_0,\vec{k})=\exp\left\{ i\gamma^0\int_{y_0}^{x_0} d\xi_0[\gamma^j k_j+\gamma\cdot A(\xi_0)-m]\right\}. \tag{22}$$

Higher order approximations are not given by additive terms but by splitting Eq. (22) at an ordered set of points,

$$G_H^{A_N}(x_0,y_0,\vec{k})=\mathcal{P}\prod_{\nu=0}^{N}\exp\left\{ i\gamma^0\int_{\xi_{\nu+1}}^{\xi_\nu} d\xi_0 \times[\gamma^j k_j+\gamma\cdot A(\xi_0)-m]\right\}, \tag{23}$$

with $x_0=\xi_0<\xi_1<\cdots<\xi_N<\xi_{N+1}=y_0$ for $x_0<y_0$ or $x_0=\xi_0>\xi_1>\cdots>\xi_N>\xi_{N+1}=y_0$ for $x_0>y_0$. $\mathcal{P}$ denotes that the factors are ordered with respect to the index ν with the lowest index furthest to the left. The choice of the intermediate points ξ_ν is not unique, but in the limit of infinitely small intervals the result always becomes exact,

$$\begin{aligned}
&\lim_{N\to\infty} G_H^{A_N}(x_0,y_0,\vec{k})\\
&=\lim_{N\to\infty}\prod_{\nu=0}^{N}\exp\left\{ i\gamma^0\int_{\xi_{\nu+1}}^{\xi_\nu} d\xi_0[\gamma^j k_j+\gamma\cdot A(\xi_0)-m]\right\}\\
&=\lim_{N\to\infty}\prod_{\nu=0}^{N}\left\{ 1+i\gamma^0(\xi_\nu-\xi_{\nu+1})[\gamma^j k_j+\gamma\cdot A(\xi_0)-m]\right.\\
&\quad\left.+\mathcal{O}\left(\frac{(x_0-y_0)^2}{N^2}\right)\right\}\\
&=\mathcal{P}\exp\left\{ i\gamma^0\int_{y_0}^{x_0} d\xi_0[\gamma^j k_j+\gamma\cdot A(\xi_0)-m]\right\}\\
&=G_H(x_0,y_0,\vec{k}).
\end{aligned} \tag{24}$$

To estimate the error for an interval width of $y_0-x_0=2\Delta$, compare the lowest order result to the first order where the interval is divided into two halves exactly,

$$\begin{aligned}
\Delta G_H^A&=G_H^{A_0}(0,2\Delta,\vec{k})-G_H^{A_1}(0,2\Delta,\vec{k})\\
&=G_H^{A_0}(0,2\Delta,\vec{k})\\
&\quad-G_H^{A_0}(0,\Delta,\vec{k})G_H^{A_0}(\Delta,2\Delta,\vec{k}).
\end{aligned} \tag{25}$$

With the help of the Baker-Campbell-Hausdorff formula

$$\begin{aligned}
\Delta G_H^A&=\exp\{g_H(0,2\Delta,\vec{k})\}-\exp\{g_H(0,2\Delta,\vec{k})\\
&\quad+[g_H(0,\Delta,\vec{k}),g_H(\Delta,2\Delta,\vec{k})]+\mathcal{O}(\Delta^4)\}\\
&=-[g_H(0,\Delta,\vec{k}),g_H(\Delta,2\Delta,\vec{k})]+\mathcal{O}(\Delta^4)\\
&=-[g_H(0,\Delta,\vec{k}),g_H(0,2\Delta,\vec{k})]+\mathcal{O}(\Delta^4)\\
&=-[g_H(0,\Delta,\vec{k}),dg_H(0,\Delta,\vec{k})/d\Delta]\Delta+\mathcal{O}(\Delta^4).
\end{aligned} \tag{26}$$

The first occurrence of $\mathcal{O}(\Delta^4)$ results from a Taylor expansion of secondary and higher commutators. Thus, in leading order of the width of the interval Δ, the error is proportional to the commutator of the exponent g_H and its first derivative at an intermediate point of the interval. For a con-

stant integrand $dg_H(0,\Delta,\vec{k})/d\Delta$, i.e. for a constant gauge field A, the Abelian approximation is exact. Higher order terms are required for fields that lead to a commutator $[g_H(0,\Delta,\vec{k}),dg_H(0,\Delta,\vec{k})/d\Delta]\Delta$ not small against $G_H^{A_0}(0,2\Delta,\vec{k})$. (This comparison must be based on the definition of an adequate norm.)

This result can be compared to the error estimate for the standard form of expressing a path-ordered exponential as a product of linear factors [see Eq. (11)]. For that representation one finds

$$\begin{aligned}\Delta G_H &= 1+2g_H(0,\Delta,\vec{k})\\ &\quad -[1+g_H(0,\Delta/2,\vec{k})][1+g_H(0,3\Delta/2,\vec{k})]+\mathcal{O}(\Delta^2)\\ &= [2g_H(0,\Delta,\vec{k})-g_H(0,\Delta/2,\vec{k})-g_H(0,3\Delta/2,\vec{k})]\\ &\quad -g_H(0,\Delta/2,\vec{k})g_H(0,3\Delta/2,\vec{k})+\mathcal{O}(\Delta^2)\\ &= -g_H(0,\Delta,\vec{k})^2+[g_H(0,\Delta,\vec{k}),dg_H(0,\Delta,\vec{k})/d\Delta]\Delta/2\\ &\quad +\mathcal{O}(\Delta^2). \end{aligned} \tag{27}$$

Contrary to ΔG_H^A, ΔG_H does not become zero for a constant gauge field. In leading order it depends on the actual value of the exponent g_H. Thus, its convergence becomes slow not only for rapid changes of the gauge field but also for large values of the field and/or large energies. Even the free propagator then needs many terms to be approximated sufficiently well.

B. Space-like coordinates

The general solution scheme for a classical field, depending on an arbitrary rectilinear coordinate $n\cdot x$ leading to Eq. (7) with the argument (6), always yields a Green's function whose boundary conditions are given on a plane normal to n. Boundary conditions for propagators are given on surfaces with time-like normal vectors $n^2>0$. Hence, for a field only depending on the x_3 coordinate $[n^\mu=(0,0,0,1)]$, only a Green's function, but not a propagator, is given by Eqs. (7) and (6).

This can also be seen directly. The solution for the argument $g_H(x_3,y_3;k_0,\vec{k}_T)$ according to Eq. (6) is given by

$$\begin{aligned} &g_H(x_3,y_3;k_0,\vec{k}_T)\\ &\quad = -i\gamma^3\int_{y_3}^{x_3}d\xi_3[\gamma^0k_0+\gamma^Jk_J+\gamma\cdot A(\xi_3)-m], \end{aligned} \tag{28}$$

with an implicit sum over $J\in\{1,2\}$. Repeating the steps that led to the free homogeneous solution of the Dirac equation in the case $n^2>0$ in Eq. (10) yields

$$\begin{aligned} &G_H^0(x_3-y_3;k_0,\vec{k}_T)\\ &\quad = \gamma^3\frac{-\gamma^3\sqrt{(k_0)^2-m_T^2}+\gamma^0k_0+\gamma^Jk_J-m}{2\sqrt{(k_0)^2-m_T^2}}\\ &\qquad \times e^{-i\sqrt{(k_0)^2-m_T^2}(x_3-y_3)}\\ &\qquad +\gamma^3\frac{-\gamma^3\sqrt{(k_0)^2-m_T^2}-\gamma^0k_0-\gamma^Jk_J+m}{2\sqrt{(k_0)^2-m_T^2}}\\ &\qquad \times e^{+i\sqrt{(k_0)^2-m_T^2}(x_3-y_3)}, \end{aligned} \tag{29}$$

with the transverse mass $m_T=\sqrt{|\vec{k}_T|^2+m^2}$. This expression, multiplied with $\theta(x_3-y_3)$ in order to obtain a Green's function from the homogeneous solution, is not proportional to the free retarded propagator in this mixed representation.

C. Light-like coordinates

As mentioned before, the present way to derive a homogeneous solution cannot be followed if the four-vector n is light-like, because in that case $\gamma\cdot n$ has no inverse. However, for light-like coordinates there is a different approach that leads to a solution for G_H. In the case where the normal vector is $n^\mu=(1,0,0,-1)/\sqrt{2}$ Eq. (2) becomes

$$\begin{aligned} &\left\{i\gamma_-\frac{d}{dx_-}+\gamma_+[k_-+A_-(x_-)]-\vec{\gamma}_T\cdot[\vec{k}_T+\vec{A}_T(x_-)]\right.\\ &\qquad \left.+\gamma_-A_+(x_-)-m\right\}G_H(x_-,y_-;k_-,\vec{k}_T)=0, \end{aligned} \tag{30}$$

with $v_\pm=[v_0\pm v_3]/\sqrt{2}$ and $v_\pm=v^\mp$ where $v\in\{\gamma,x,k,A(x_-)\}$. Noting that $\gamma_+\gamma_-/2$ and $\gamma_-\gamma_+/2$ are two projection operators which project into disjoint subspaces of the Clifford algebra and satisfy the completeness relation $\gamma_+\gamma_-+\gamma_-\gamma_+=2$ the matrix function $G_H(x_-,y_-;k_-,\vec{k}_T)$ can be split into $2G_H(x_-,y_-;k_-,\vec{k}_T)=\gamma_+G_-(x_-,y_-;k_-,\vec{k}_T)+\gamma_-G_+(x_-,y_-;k_-,\vec{k}_T)$ with $G_\pm(x_-,y_-;k_-,\vec{k}_T)=\gamma_\pm G_H(x_-,y_-;k_-,\vec{k}_T)$. The argument will be suppressed in the following but until the end of this section the above mixed representation is addressed. Using this decomposition in Eq. (30) leads to

$$\begin{aligned} &i\frac{\gamma_-\gamma_+}{2}\frac{d}{dx_-}G_-+\frac{\gamma_+\gamma_-}{2}(k_-+A_-)G_+\\ &\qquad -[\vec{\gamma}_T\cdot(\vec{k}_T+\vec{A}_T)+m]G_H+\frac{\gamma_-\gamma_+}{2}A_+G_-=0. \end{aligned} \tag{31}$$

Use has been made of the idempotency of the projectors $(\gamma_\pm\gamma_\mp/2)^2=\gamma_\pm\gamma_\mp/2$ and their projection properties $(\gamma_\pm\gamma_\mp/2)\gamma_\mp=0$ and $(\gamma_\pm\gamma_\mp/2)\gamma_\pm=\gamma_\pm$. From here, two equations can be obtained with the help of the projection operators

$$i\frac{d}{dx_-}G_- + A_+ G_- - \frac{1}{2}[\vec{\gamma}_T\cdot(\vec{k}_T+\vec{A}_T)+m]\gamma_- G_+ = 0$$

$$(k_- + A_-)G_+ - \frac{1}{2}[\vec{\gamma}_T\cdot(\vec{k}_T+\vec{A}_T)+m]\gamma_+ G_- = 0. \tag{32}$$

The second equation is purely algebraic and can be used to replace G_+ in the first,

$$i\frac{d}{dx_-}G_- + \frac{1}{2}[\vec{\gamma}_T\cdot(\vec{k}_T+\vec{A}_T)+m](k_- + A_-)^{-1} \times[\vec{\gamma}_T\cdot(\vec{k}_T+\vec{A}_T)-m]G_- = 0. \tag{33}$$

When postulating a connection between the matrix function G_- and another g_- in direct analogy to Eq. (3) the resulting differential equation is given by

$$i\frac{d}{dx_-}g_- = -\frac{1}{2}[\vec{\gamma}_T\cdot(\vec{k}_T+\vec{A}_T)+m](k_- + A_-)^{-1} \times[\vec{\gamma}_T\cdot(\vec{k}_T+\vec{A}_T)-m] - A_+ . \tag{34}$$

The equation can be solved by direct integration. As already argued before, in general the functional $G_-[g_-]$ is given by the path-ordered exponential of its argument g_-. In the present situation already the absence of non-Abelian charges turns it into an ordinary exponential because then it only contains the neutral element for multiplication of the Clifford algebra. The other component of the matrix function is given by the second of the equations (32). Finally a homogeneous solution of the differential equation (30) has the form

$$\begin{aligned} G_H(x_-,y_-;k_-,\vec{k}_T) &= \frac{1}{2}\Bigg(\gamma_+ - \frac{1}{2}\gamma_-\gamma_+[k_- + A_-(x_-)]^{-1} \\ &\quad\times\{\vec{\gamma}_T\cdot[\vec{k}_T+\vec{A}_T(x_-)]-m\}\Bigg) \\ &\quad\times\mathcal{P}\exp\Bigg[i\int_{y_-}^{x_-}d\xi_-\Bigg(\frac{1}{2}\{\vec{\gamma}_T\cdot[\vec{k}_T+\vec{A}_T(\xi_-)]+m\} \\ &\quad\times[k_- + A_-(\xi_-)]^{-1}\{\vec{\gamma}_T\cdot[\vec{k}_T+\vec{A}_T(\xi_-)]-m\} \\ &\quad + A_+(\xi_-)\Bigg)\Bigg]. \end{aligned} \tag{35}$$

If one tries to construct a propagator with the help of this homogeneous solution it can only be retarded or advanced in the light-like coordinate x_-. Alternative approaches can be found in [20]. In the next section the production of fermion-antifermion pairs is described based on the results for the fermion propagator in a field that depends on a time-like coordinate.

III. FERMION-ANTIFERMION PAIR PRODUCTION

Here the results for the full propagator in an external field depending on one rectilinear time-like coordinate are applied to the problem of particle production due to vacuum polarization. First it is argued where such a propagator is of use in describing the physics of fermions in a heavy-ion collision. Second, a detailed comparison of the different approximation schemes with the full solution for a given model field is presented.

This calculation can be understood in a twofold way. On the one hand the field could be really an external field in the sense of the production of particles via vacuum polarization. It is determined by the dynamics of the physical system without taking the back reaction of the particle creation into account. This field is used to calculate how many particles would be produced in its presence. This approach is justified if the process of particle production constitutes merely a small perturbation. Whether this condition is fulfilled has to be checked afterwards.

On the other hand, the field could already be a self-consistent solution of a system of equations. For this solution for the classical field one would like to know how many particles were created in the process. In the present scenario such a set of equations would include the Yang-Mills equations with the expectation value for the current of produced fermions and antifermions $\langle J_\nu\rangle$ and an (initially present) external current J^ν_{ext}

$$\partial^\mu\mathcal{F}_{\mu\nu} - i[\mathcal{A}^\mu,\mathcal{F}_{\mu\nu}] = J^\nu_{ext} + \langle J^\nu\rangle, \tag{36}$$

with the gauge field $\mathcal{A}^\mu$ and the corresponding field tensor $\mathcal{F}_{\mu\nu}=\partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu - i[\mathcal{A}_\mu,\mathcal{A}_\nu]$ in the adjoint representation. The expectation value for the current can be obtained from the causal propagator [21]:

$$\langle J^\nu\rangle \sim \mathrm{tr}\{\gamma^\nu G_C(x,x)\}, \tag{37}$$

where the trace is only running over the matrices of the Clifford algebra and with the definition: $G_C(x,x) = \lim_{\epsilon\to 0}[G_C(x+n\epsilon,x)+G_C(x,x+n\epsilon)]/2$ with $n^2>0$. Higher-order radiative corrections are suppressed by powers of the coupling constant which are not compensated by powers of the classical field. The causal propagator can be reexpressed as a linear combination of the retarded, the advanced, and the on-shell propagator,

$$G_C(x,y) = \frac{1}{2}[G_R(x,y)+G_A(x,y)+G_S(x,y)]. \tag{38}$$

The advanced propagator can be obtained from the homogeneous solution $G_H(x_0,y_0,\vec{k})$ for the equation of motion for the Dirac Green's function by the relation

$$iG_A(x_0,y_0,\vec{k})\gamma^0 = -\theta(y_0-x_0)G_H(x_0,y_0,\vec{k}). \tag{39}$$

The on-shell propagator can be reexpressed in terms of the retarded and advanced one-particle scattering operators [16]

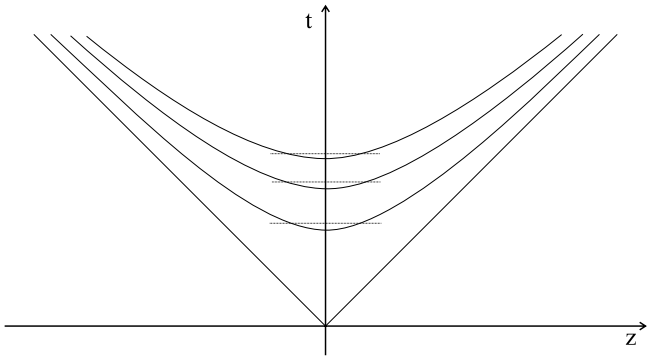

FIG. 2. Time dependence as an approximation to the situation found in the central region of a boost-invariant system. The proper time τ is constant on the hyperbolas.

$$G_S(x,y)=\int \frac{d^4k}{(2\pi)^4}2\pi\delta(k^2-m^2)G_R^0(q)\mathcal{T}_R(q,k)\times(\gamma\cdot k+m)\mathcal{T}_A(k,p)G_A^0(p). \quad (40)$$

The scattering operators are defined according to Eq. (43). In the present framework all required propagators are known as functionals of the classical gauge fields. Hence, Eq. (36) with Eq. (37) constitutes an integro-differential equation for the classical gauge field. Its solution would yield the form of the field. The expectation value for the produced fermions and antifermions could be calculated from this field.

Let us consider a model for the classical radiation-field in an ultrarelativistic heavy-ion collision. According to Bjorken [17], the mid-rapidity region in a heavy-ion collision is characterized by boost invariant quantities, i.e. boost invariant along the beam direction. Let us consider a central collision in a symmetric system in the center-of-mass frame. For absolute values of the longitudinal coordinate $|z|$ smaller than the kinematic time t in this frame of reference the dependence on proper time $\tau=\sqrt{t^2-z^2}$ is approximated well by a dependence on the kinematic time t (see Fig. 2). Most of the energy is deposited during $t<t_{in}$ close to the collision point at $z=0$ [7]. Hence, in good approximation, for $t>t_{in}$ and $|z|<t_{in}$, an in general proper-time τ dependent energy density can be reexpressed as an energy density depending on kinematic time t. Let the entire energy density be initially stored in an electric field of the form $E_\eta(\tau)$ (component along the hyperbolas). In the present approach, this fact is consistently approximated by a storage of the energy in $E_z(t)$. In temporal gauge ($A_0=0$) or even in Lorentz gauge this is equivalent to a gauge field $A_z(t)$. For this form of gauge field, the retarded propagator has been derived in the previous section.

In order to proceed, one needs to know how to describe particle production based on a given propagator. With the Fourier transform of the retarded propagator,

$$G_R(x,y)=\int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}e^{-iq\cdot x}e^{+ip\cdot y}G_R(q,p) \quad (41)$$

one can write the implicit definition of the corresponding one-particle scattering operator $\mathcal{T}$ in momentum space as

$$G_R(q,p)=(2\pi)^4\delta^{(4)}(q-p)G_R^0(p)+G_R^0(q)\times\mathcal{T}(q,p)\times G_R^0(p). \quad (42)$$

Explicitly, it is given by

$$\mathcal{T}(q,p)=\gamma\cdot A(q-p)+\int \frac{d^4k}{(2\pi)^4}\frac{d^4l}{(2\pi)^4}\times\gamma\cdot A(q-k)G_R(k,l)\gamma\cdot A(l-p). \quad (43)$$

In the following $\mathcal{T}$ always denotes the *retarded* one-particle scattering operator. The Born approximation, the expansion to lowest order in the fields, is given by

$$\mathcal{T}^B(q,p)=\gamma\cdot A(q-p). \quad (44)$$

This term is always contained in the scattering operator. The different approximation schemes discussed in the preceding section lead to differences in the remaining non-Bornian part in Eq. (43). In the presence of a purely time-dependent field the retarded one-particle scattering operator becomes

$$\mathcal{T}(q,p)=(2\pi)^3\delta^{(3)}(\vec{q}-\vec{p})\left[\gamma\cdot A(q_0-p_0)+\int dx_0dy_0e^{+iq_0x_0}e^{-ip_0y_0}\gamma\cdot A(x_0)G_R(x_0,y_0,\vec{p})\gamma\cdot A(y_0)\right]$$
$$=(2\pi)^3\delta^{(3)}(\vec{q}-\vec{p})\mathcal{T}(q_0,p_0)=(2\pi)^3\delta^{(3)}(\vec{q}-\vec{p})[\mathcal{T}^B(q_0,p_0)+\mathcal{T}^{NB}(q_0,p_0)], \quad (45)$$

with the non-Bornian part $\mathcal{T}^{NB}(q_0,p_0)$. Due to the occurrence of the Dirac δ distribution the conservation of the total three-momentum becomes obvious. In purely time-dependent fields the fermion-antifermion pairs are always produced in a *back-to-back* configuration. Terms of higher order in the gauge field—for instance in the weak-field expansion—can be obtained by replacing the full propagator in the scattering operator by terms from Eq. (18). Analogously, replacing the full propagator by various approximations leads to the corresponding approximations for the one-particle scattering operator.

From the retarded one-particle scattering operator $\mathcal{T}_R$ the expectation value of produced pairs can be obtained

$$\langle n\rangle = \int \frac{d^3q}{2(2\pi)^3 q_0}\frac{d^3p}{2(2\pi)^3 p_0}|\overline{u}(q)\mathcal{T}_R(q,-p)v(p)|^2, \tag{46}$$

where a summation over the spin degrees of freedom of the unit spinors $\overline{u}(q)$ and $v(p)$ is understood and where $p_0=\sqrt{|\vec{p}|^2+m^2}$ and $q_0=\sqrt{|\vec{q}|^2+m^2}$. For the special form of the scattering operator in spatially homogeneous purely time-dependent situations this simplifies to

$$\langle n\rangle = \frac{V}{4(2\pi)^3}\int \frac{d^3p}{p_0^2}|\overline{u}(p_0,-\vec{p})\mathcal{T}(p_0,-p_0)v(p_0,+\vec{p})|^2, \tag{47}$$

where use has been made of the relation $[\delta^{(3)}(\vec{p}-\vec{q})]^2 = V\delta^{(3)}(\vec{p}-\vec{q})/(2\pi)^3$. Carrying out the spin summation leads to

$$\frac{4(2\pi)^3}{V}\frac{d\langle n\rangle}{d^3p} = \mathrm{tr}\left\{\mathcal{T}(p_0,-p_0)\frac{\gamma^0 p_0+\gamma^j p_j - m}{p_0} \times \gamma^0\mathcal{T}^\dagger(p_0,-p_0)\gamma^0\frac{\gamma^0 p_0-\gamma^j p_j+m}{p_0}\right\}. \tag{48}$$

In order to gain some insight into the behavior of the differential expectation value (or momentum spectrum) and some information on the quality of the different approximations without having to solve the Yang-Mills equations (36) beforehand, the different formulas are going to be evaluated for a special choice of the field

$$A_\mu(t) = g_{3\mu}A_{in}e^{-t/t_0}\theta(t). \tag{49}$$

Many other forms could have been taken. This choice was also inspired by a numerical study [18] which indicates that the field decays in a similar fashion. In any case, the actual form of the classical field has to be determined in a self-consistent calculation.

For this field, the one-particle scattering operator in Born approximation (44) is given by

$$\mathcal{T}^B(2\omega) = \frac{\gamma^3 A_{in}t_0}{1+2it_0\omega}. \tag{50}$$

The following sections show the various approximations of the remaining part of the retarded one-particle scattering operator in the field (49).

1. The weak-field approximation

The lowest-order weak-field term of the full propagator is given by the free propagator [Eq. (16) together with Eq. (10)]. Here, the non-Bornian part of the one-particle scattering operator is

$$\mathcal{T}^{UV} = -i\gamma^3\frac{\gamma^0\omega+\gamma^j k_j - m}{2\omega}\gamma^3\mathcal{T}_+^{UV} - i\gamma^3\frac{\gamma^0\omega-\gamma^j k_j+m}{2\omega}\gamma^3\mathcal{T}_-^{UV}, \tag{51}$$

with

$$\mathcal{T}_\pm^{UV} = A_{in}^2\int_0^\infty dx_0\int_0^{x_0}dy_0 e^{i\omega(x_0+y_0)}e^{\pm i\omega(x_0-y_0)}e^{-x_0/t_0}e^{-y_0/t_0} = \frac{(A_{in}t_0)^2/2}{[1-i\omega t_0][1-i(\omega\pm\omega)t_0]}. \tag{52}$$

2. The strong-field approximation

For any purely time-dependent field, the general expression for the homogeneous solution in the lowest-order strong-field approximation:

$$G_H^{IR}(x_0,y_0,\vec{k}) = \mathcal{P}\exp\left\{i\int_{y_0}^{x_0}d\xi_0\gamma^0\gamma\cdot A(\xi_0)\right\} \tag{53}$$

is not much simpler to evaluate than the exact solution. However, for a field of constant direction $A_\mu(t)=A_\mu\times f(t)$ the path ordering can be dropped. In the lowest-order strong-field approximation the non-Bornian part of the one-particle scattering operator is

$$\mathcal{T}^{IR} = -i\frac{\gamma^0-\gamma^3}{2}\mathcal{T}_+^{IR} - i\frac{\gamma^0+\gamma^3}{2}\mathcal{T}_-^{IR}, \tag{54}$$

with

$$\begin{aligned}\mathcal{T}_\pm^{IR} &= A_{in}^2\int_0^\infty dx_0\int_0^{x_0}dy_0 e^{i\omega(x_0+y_0)}e^{-x_0/t_0}e^{-y_0/t_0} \\ &\quad\times\exp\{\mp iA_{in}t_0[e^{-x_0/t_0}-e^{-y_0/t_0}]\} \\ &= (A_{in}t_0)^2\sum_{\mu=0}^\infty\frac{(\mp iA_{in}t_0)^\mu}{\mu!}\frac{1}{\mu+1-i\omega t_0} \\ &\quad\times\sum_{\nu=0}^\infty\frac{(\pm iA_{in}t_0)^\nu}{\nu!}\frac{1}{\nu+\mu+2-2i\omega t_0},\end{aligned} \tag{55}$$

where use has been made of the uniform convergence of the exponential series for bounded arguments. With formula 6.5.29 in [19]:

$$\gamma^*(a,z) = \frac{1}{\Gamma(a)}\sum_{n=0}^\infty\frac{(-z)^n}{(a+n)n!} \tag{56}$$

for a bounded norm of $A_{in}t_0$

$$\mathcal{T}_\pm^{IR} = (A_{in}t_0)^2\sum_{\mu=0}^\infty\frac{(\mp iA_{in}t_0)^\mu}{\mu!}\frac{1}{\mu+1-i\omega t_0} \times\gamma^*(\mu+2-2i\omega t_0,\mp iA_{in}t_0)\Gamma(\mu+2-2i\omega t_0). \tag{57}$$

Formula 6.5.4 in [19]

$$\gamma^*(a,z)=\frac{z^{-a}}{\Gamma(a)}\gamma(a,z) \tag{58}$$

leads to

$$\mathcal{T}_{\pm}^{IR}=-(\mp A_{in}t_0)^{2i\omega t_0}\sum_{\mu=1}^{\infty}\frac{1}{\mu!}\frac{\gamma(\mu+1-2i\omega t_0,\mp iA_{in}t_0)}{1-i\omega t_0/\mu}. \tag{59}$$

In the case of multiple charges, A_{in} can be decomposed according to $A_{in}=A_{in}^a T^a$ where the T^a are the generators of the corresponding algebra. Due to the requirement of unitarity these generators have to be Hermitian. This is also true for any linear combination of the generators with real coefficients. Thus every matrix $A_{in}^a T^a$ with real A_{in}^a can be diagonalized, yielding

$$A_{in}^a T^a=\sum_{n=1}^{N}\lambda_n|n\rangle\langle n|, \tag{60}$$

with the eigenvalues λ_n and the N orthonormal eigenvectors $|n\rangle$. The $|n\rangle\langle n|$ are projectors onto subspaces of different charges. Thus one gets

$$\mathcal{T}_{\pm}^{IR}=-\sum_{n=1}^{N}|n\rangle\langle n|(\mp\lambda_n t_0)^{2i\omega t_0}\times\sum_{\mu=1}^{\infty}\frac{1}{\mu!}\frac{\gamma(\mu+1-2i\omega t_0,\mp i\lambda_n t_0)}{1-i\omega t_0/\mu}. \tag{61}$$

3. The modified strong-field approximation

In the present situation the special form of the field allows for a variation of the strong-field approximation, where the component of the momentum parallel to the field—in this case k_3—is included in the exponent of the lowest-order expression $A_3\to A_3+k_3$:

$$G_H^{IR'}(x_0,y_0,\vec{k})=\mathcal{P}\exp\left\{i\int_{y_0}^{x_0}d\xi_0\gamma^0\gamma^3[k_3+A_3(\xi_0)]\right\}. \tag{62}$$

In the lowest-order modified strong-field approximation the additional part of the one-particle scattering operator beyond the Born approximation is

$$\mathcal{T}^{IR'}=-i\frac{\gamma^0-\gamma^3}{2}\mathcal{T}_{+}^{IR'}-i\frac{\gamma^0+\gamma^3}{2}\mathcal{T}_{-}^{IR'}, \tag{63}$$

with

$$\mathcal{T}_{\pm}^{IR'}=A_{in}^2\int_0^{\infty}dx_0\int_0^{x_0}dy_0 e^{i(\omega\pm k_3)(x_0+y_0)}e^{-x_0/t_0}e^{-y_0/t_0}\times\exp\{\mp iA_{in}t_0[e^{-x_0/t_0}-e^{-y_0/t_0}]\}. \tag{64}$$

Repeating the above steps leads to

$$\mathcal{T}_{\pm}^{IR'}=-\sum_{n=1}^{N}|n\rangle\langle n|(\mp\lambda_n t_0)^{2i\omega t_0}\times\sum_{\mu=1}^{\infty}\frac{1}{\mu!}\frac{\gamma(\mu+1-2i\omega t_0,\mp i\lambda_n t_0)}{1-i(\omega\pm k_3)t_0/\mu}. \tag{65}$$

4. The Abelian approximation

In the lowest-order Abelian approximation the interacting part of the retarded propagator is

$$\mathcal{T}^A=+i\mathcal{T}_{+}^A+i\mathcal{T}_{-}^A, \tag{66}$$

with

$$\mathcal{T}_{\pm}^A=A_{in}^2\int_0^{\infty}dx_0\int_0^{x_0}dy_0 e^{i\omega(x_0+y_0)}e^{\pm i\Omega(x_0-y_0)}\times e^{-x_0/t_0}e^{-y_0/t_0}\frac{\gamma^0\Omega\pm[\gamma^J k_J-\gamma^3 K_3+m]}{2\Omega}, \tag{67}$$

with the generalized energy $\Omega=\sqrt{m_T^2+K_3^2}$ and the generalized momentum $K_3=k_3+A_{in}t_0(e^{-x_0/t_0}-e^{-y_0/t_0})/(x_0-y_0)$.

Decomposition with respect to multiple charges leads to

$$\mathcal{T}_{\pm}^A=\sum_{n=1}^{N}|n\rangle\langle n|\lambda_n^2\int_0^{\infty}dx_0\int_0^{x_0}dy_0 e^{i\omega(x_0+y_0)}\times e^{\pm i\Omega_n(x_0-y_0)}e^{-x_0/t_0}e^{-y_0/t_0}\times\frac{\gamma^0\Omega_n\pm[\gamma^J k_J-\gamma^3(K_3)_n+m]}{2\Omega_n}, \tag{68}$$

with the generalized energy $\Omega_n=\sqrt{m_T^2+(K_3)_n^2}$ and the generalized momentum $(K_3)_n=k_3+\lambda_n t_0(e^{-x_0/t_0}-e^{-y_0/t_0})/(x_0-y_0)$ belonging to the respective eigenvalue λ_n. The modification to the longitudinal momentum is equal to the arithmetic average of the gauge field over the interval $[x_0,y_0]$. Hence the Abelian approximation can be interpreted as the description of the propagation of the fermions with their arithmetically averaged canonical momentum. In the weak-field expansion they are propagated with their asymptotic kinematic momentum. Higher orders in the Abelian approximation scheme make better approximations similar to a Fourier series. The particle is propagated with its canonical momentum averaged over every piece of the trajectory. The finer the partitioning of the path, the closer the average canonical momentum is to its actual value in a particular interval. In the weak-field perturbative approximation scheme the particle is always propagated with its asymptotic kinetic momentum. For higher orders it only interacts with the field more and more often.

For the strong-field approximation and the modified strong-field approximation the replacement of the path-ordered exponential by an exponential function is only possible due to the special form of the field. In the Abelian approximation scheme this exchange is possible in the presence of an arbitrary field.

The decomposition with respect to the charge projectors is also possible for the full non-Bornian part and the omnipresent Born part of the one-particle scattering operator $\mathcal{T}^B = \Sigma_{n=1}^N |n\rangle\langle n|\mathcal{T}_n^B$. Hence, the whole operator can always be decomposed with respect to the same projectors $\mathcal{T} = \Sigma_{n=1}^N |n\rangle\langle n|[\mathcal{T}_n^B + \mathcal{T}_n^{NB}]$. In the squared expression needed to calculate the expectation value the contributions belonging to the different projectors do not mix. They lead to a sum over the expectation values for the different charge subspaces

$$\mathrm{tr}_c\left\{\sum_{n'=1}^{N}|n'\rangle\langle n'|\mathcal{T}_{n'}\sum_{n''=1}^{N}|n''\rangle\langle n''|\mathcal{T}_{n''}^{*}\right\} = \sum_{n'=1}^{N}|\mathcal{T}_{n'}|^2\mathrm{tr}_c\{|n'\rangle\langle n'|\} = \sum_{n'=1}^{N}|\mathcal{T}_{n'}|^2,$$

where tr_c denotes the trace over the generators of the charge group. If the eigenvectors $|n'\rangle$ are normalized, the remaining trace is equal to unity. Due to these facts, it suffices to compare the contributions from the different approximation schemes for one of the sub-spaces.

It is always possible to measure all momenta, energies, and gauge field strengths in units of a scale parameter with the dimension of momentum. Then all lengths and times have to be given in units of inverse momenta. In the following, the eigenvalue belonging to the corresponding subspace is chosen as a scale parameter and is going to be called A_{in}/g again. The calculations are carried out assuming that all the energy of the system is included in one of the subspaces.

The expected energy density produced in a central heavy-ion collision at LHC (Pb-Pb at $\sqrt{s}=5.5$ TeV) is $\epsilon \approx 1000$ GeV/fm^3 [7–9]. For the strong coupling constant one expects $\alpha_s \approx 0.15$ [9]. If all the energy density was deposited in the field sector a rough estimate for the initial gauge field magnitude would be $A_{in} \approx \sqrt{g\sqrt{2\epsilon}} \approx 2$ GeV. For RHIC (Au-Au at $\sqrt{s}=200$ GeV) the typical coupling constant is around $\alpha_s \approx 0.33$ and the initial energy density $\epsilon \approx 50$ GeV/fm^3. This would lead to $A_{in} \approx 1$ GeV. With decay times in the range from 0.1 fm/c to 0.5 fm/c this leads to $A_{in}t_0$ between 0.5 and 5.0. Here only massless particles are investigated.

The expressions for the Born (50) and the weak-field approximation (51) can be evaluated straightforwardly. For the strong-field (54) and the modified strong-field approach (63) the few first terms of the infinite series representations suffice for obtaining an accurate result. The integrals for the Abelian approximation have to be treated with standard numerical methods. The exact solution requires the handling of path-ordered exponentials and subsequent integrations.

The general aspects of the exact solution for the momentum spectrum (48) are best seen in Figs. 3(a) and 4. As a function of the transverse momentum k_T it peaks once and shows no further relative extrema or other distinct structures. For increasing values of the parameter $A_{in}t_0$ [from Fig. 4(a) over Fig. 3(a) to Fig. 4(b)] the peak in the transverse momentum spectrum becomes more pronounced for a fixed value of the longitudinal momentum k_3. In other words it increases in height and decreases in width [see especially the different scale of the transverse momentum axis in Fig. 4(b)]. Actually, the differential expectation value is a function of the variable ωt_0. Hence the width of the transverse-momentum spectrum for massless particles at mid rapidity scales exactly inversely proportionally to t_0. The same holds still after ω and t_0 have been rescaled with A_{in}. For fields of a functional form analogous to that of the present special model field the peak height seems to be strictly monotonically decreasing with increasing longitudinal momentum, as is suggested in Fig. 3(a). Further, at zero transverse momentum no particles are produced; the fermions and antifermions are never produced with momenta along the direction of the field but preferentially with momenta perpendicular to the field.

A comparison of the different approaches shows that for large momenta all approximations and the exact solution tend towards the Born result. This is due to the form of the one-particle scattering operator (43). Together with the Born approximation all other graphs tend toward zero for higher particle energies. As shown in Fig. 3(b) the Born approach overestimates the exact value for low momenta but underestimates it for high momenta. The weak-field approximation is an improvement compared to the Born approach for most values of the transverse momentum. Looking at Fig. 3(c) the strong field and the modified strong field are generally closer to the exact result than the weak-field approximation. However, for more general forms of time-dependent fields the propagators in these schemes are not much simpler to deal with than the full one. The modified strong-field approximation even ceases to be available because the terms longitudinal and transverse might no longer be well defined with respect to the field. For all momenta the Abelian approximation scheme [see Fig. 3(d)] is closest to the exact values. The largest deviations are found for small energies and large values of the parameter $A_{in}t_0$ [compare Figs. 3(d), 4(a), and 4(b)]. The reason is that there the situation is maximally non-Abelian, i.e. there the condition, where the typical commutator of the exponent $g_H(x_0, y_0, \vec{k})$ at different points is negligible with respect to the typical propagator, is least well satisfied [see also Eq. (26)]. While for low values of $A_{in}t_0$ the Born approximation is reasonably good it is not appropriate for large values [see Fig. 4].

IV. SUMMARY

The exact homogeneous solutions for the Dirac equation in a gauge field depending on one rectilinear coordinate has been presented. An alternative way had to be taken for a dependence on a light-like coordinate. In the case where this coordinate was time-like, the retarded propagator has been constructed from the homogeneous solution. The analogous result for a space-like coordinate was seen to constitute a Dirac Green's function but not a propagator.

For the situation of a time-like coordinate various approximation schemes for the exact solution have been determined. Explicitly, these are the weak-field approximation, the strong-field approach, and the Abelian approximation.

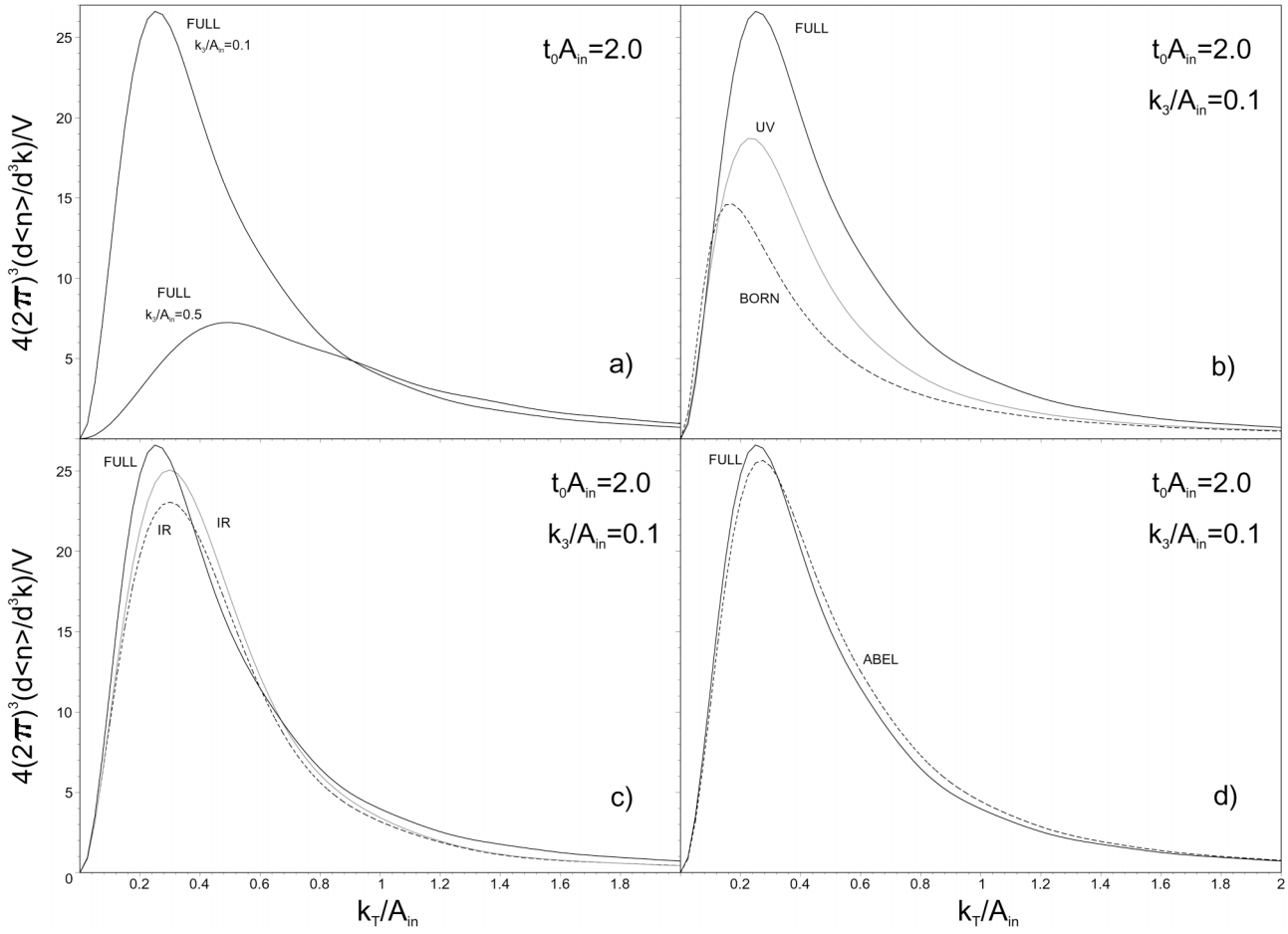

FIG. 3. Momentum spectrum of produced massless fermion-antifermion pairs versus transverse momentum and with the decay time $t_0A_{in}=2.0$: (a) The exact result for different values of the longitudinal momentum. In plots (b), (c), and (d) the longitudinal momentum is fixed at $k_3=0.1A_{in}$. (b) The exact result (solid) compared to the Born (dashed) and the weak-field (gray) approximation. (c) The exact result (solid) compared to the strong-field (dashed) and the modified strong-field (gray) approximation. (d) The exact result (solid) compared to the Abelian (dashed) approximation.

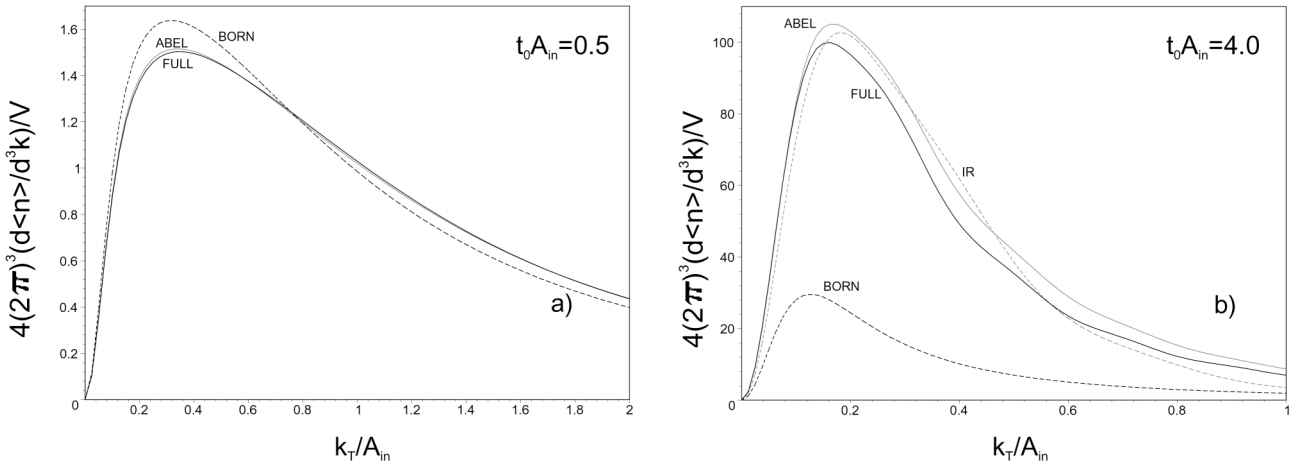

FIG. 4. Momentum spectrum (solid) of produced massless fermion-antifermion pairs versus transverse momentum compared to the Born (dashed) and the Abelian (gray) approximation for fixed longitudinal momentum $k_3=0.1A_{in}$ and for different values of the decay time: (a) $A_{in}t_0=0.5$ and (b) $A_{in}t_0=4.0$. In plot (b) the enhanced strong-field approximation (dashed gray) is shown, too.

Additionally, a larger variety of approximations can be obtained with the help of the general resummation formula.

Subsequently, the retarded fermion propagator and all the lowest orders of the various approximation schemes in the presence of a gauge field depending on one rectilinear timelike coordinate have been used to calculate the momentum spectrum of produced fermion-antifermion pairs. The resulting expressions are evaluated for a decaying model field and the results are mutually compared for parameters expected to be found in ultrarelativistic heavy-ion collisions. In this situation an additional modified strong-field approximation could be obtained.

In the present situation, the exact momentum spectrum is a singly peaked function of the transverse momentum with no further distinct structure. The quality of the approximations increases from the Born approach over the lowest-order weak field, strong field, and modified strong field, towards the Abelian approximation. It should be mentioned that in more general situations the strong field and the modified strong field approaches are not much simpler to evaluate than the full result. The model parameter is $A_{in}t_0$. It is the product of the initial magnitude of the gauge field A_{in} and the decay time scale of the field. For the smallest expected values the Born approximation is still acceptable. Nevertheless, the other schemes like the Abelian or the enhanced strong field are even better. For the highest values of the decay time only the latter come close to the exact result. Hence, in order to ensure the maximum possible independence from the scale parameter $A_{in}t_0$ without having to evaluate the exact solution it would be best to use the Abelian approximation for self-consistent calculations.

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APPENDIX: GENERAL RESUMMATION FORMULA

In general, a path-ordered exponential with an integrand depending on a single variable ξ_0 can be rewritten in the following way:

$$\begin{aligned}
&\mathcal{P}\exp\left\{\int_{y_0}^{x_0} d\xi_0[B(\xi_0)+C(\xi_0)]\right\}\\
&\quad=\sum_{l=0}^{\infty}\int_{y_0}^{x_0} d\xi_1\int_{y_0}^{\xi_1} d\xi_2\ldots\int_{y_0}^{\xi_{l-1}} d\xi_l\\
&\qquad\times\mathcal{P}\exp\left\{\int_{\xi_1}^{x_0} d\xi_0 B(\xi_0)\right\}C(\xi_1)\\
&\qquad\times\mathcal{P}\exp\left\{\int_{\xi_2}^{\xi_1} d\xi_0 B(\xi_0)\right\}C(\xi_2)\cdots\\
&\qquad\times\mathcal{P}\exp\left\{\int_{\xi_l}^{\xi_{l-1}} d\xi_0 B(\xi_0)\right\}C(\xi_l)\mathcal{P}\exp\left\{\int_{y_0}^{\xi_l} d\xi_0 B(\xi_0)\right\}.
\end{aligned}\tag{A1}$$

Making use of the group property valid for the present path-ordered exponentials,

$$\begin{aligned}
&\mathcal{P}\exp\left\{\int_{z_0}^{x_0} d\xi_0 B(\xi_0)\right\}\times\mathcal{P}\exp\left\{\int_{y_0}^{z_0} d\xi_0 B(\xi_0)\right\}\\
&\qquad=\mathcal{P}\exp\left\{\int_{y_0}^{x_0} d\xi_0 B(\xi_0)\right\}
\end{aligned}\tag{A2}$$

the above equation can be reexpressed as

$$\begin{aligned}
&\mathcal{P}\exp\left\{\int_{y_0}^{x_0} d\xi_0[B(\xi_0)+C(\xi_0)]\right\}\\
&\quad=\mathcal{P}\exp\left\{\int_{y_0}^{x_0} d\xi_0 B(\xi_0)\right\}\sum_{l=0}^{\infty}\int_{y_0}^{x_0} d\xi_1\int_{y_0}^{\xi_1} d\xi_2\cdots\int_{y_0}^{\xi_{l-1}} d\xi_l\mathcal{P}\exp\left\{\int_{x_0}^{y_0} d\xi_0 B(\xi_0)\right\}\\
&\qquad\times\mathcal{P}\exp\left\{\int_{\xi_1}^{x_0} d\xi_0 B(\xi_0)\right\}C(\xi_1)\mathcal{P}\exp\left\{\int_{y_0}^{\xi_1} d\xi_0 B(\xi_0)\right\}\mathcal{P}\exp\left\{\int_{x_0}^{y_0} d\xi_0 B(\xi_0)\right\}\mathcal{P}\exp\left\{\int_{\xi_2}^{x_0} d\xi_0 B(\xi_0)\right\}C(\xi_2)\\
&\qquad\times\mathcal{P}\exp\left\{\int_{y_0}^{\xi_2} d\xi_0 B(\xi_0)\right\}\times\cdots\times\mathcal{P}\exp\left\{\int_{x_0}^{y_0} d\xi_0 B(\xi_0)\right\}\mathcal{P}\exp\left\{\int_{\xi_l}^{x_0} d\xi_0 B(\xi_0)\right\}C(\xi_l)\mathcal{P}\exp\left\{\int_{y_0}^{\xi_l} d\xi_0 B(\xi_0)\right\}\\
&\quad=\mathcal{P}\exp\left\{\int_{y_0}^{x_0} d\xi_0 B(\xi_0)\right\}\mathcal{P}\exp\left[\mathcal{P}\exp\left\{\int_{x_0}^{y_0} dz_0 B(z_0)\right\}\int_{x_0}^{y_0} d\xi_0\mathcal{P}\exp\left\{\int_{\xi_0}^{x_0} dz_0 B(z_0)\right\}C(\xi_0)\mathcal{P}\exp\left\{\int_{y_0}^{\xi_0} dz_0 B(z_0)\right\}\right].
\end{aligned}\tag{A3}$$

Again, by virtue of the group property the most compact form is given by

$$\mathcal{P}\exp\left\{\int_{y_0}^{x_0} d\xi_0[B(\xi_0)+C(\xi_0)]\right\}=\mathcal{P}\exp\left\{\int_{y_0}^{x_0} d\xi_0 B(\xi_0)\right\}\mathcal{P}\exp\left[\int_{y_0}^{x_0} d\xi_0 \mathcal{P}\exp\left\{\int_{\xi_0}^{y_0} dz_0 B(z_0)\right\} C(\xi_0)\mathcal{P}\exp\left\{\int_{y_0}^{\xi_0} dz_0 B(z_0)\right\}\right]. \tag{A4}$$

Summarizing, more general resummations are possible in which the dominant quantity can be chosen arbitrarily. Further, the above steps can be repeated so as to resum the obtained quantity several times, e.g. after splitting $C(\xi_0)$ into a dominant part and a deviation.

The above result can also be obtained in another way. Write the Fourier transformed Dirac equation in a generic form, where the dependence on $\vec{k}$ will not be denoted in the following:

$$\left[\frac{d}{dx_0}-B(x_0)-C(x_0)\right]G_H(x_0,y_0)=0. \tag{A5}$$

With the ansatz

$$G_H(x_0,y_0)=U(x_0,y_0)\hat{G}_H(x_0,y_0) \tag{A6}$$

and the product rule for differentiation, one obtains

$$\left[\frac{d}{dx_0}U(x_0,y_0)\right]\hat{G}_H(x_0,y_0)+U(x_0,y_0)\left[\frac{d}{dx_0}\hat{G}_H(x_0,y_0)\right] -[B(x_0)+C(x_0)]U(x_0,y_0)\hat{G}_H(x_0,y_0)=0. \tag{A7}$$

When postulating that $U(x_0,y_0)$ satisfies the differential equation

$$\frac{d}{dx_0}U(x_0,y_0)-B(x)U(x_0,y_0)=0, \tag{A8}$$

the above expression reduces to

$$U(x_0,y_0)\left[\frac{d}{dx_0}\hat{G}_H(x_0,y_0)\right]-C(x_0)U(x_0,y_0)\hat{G}_H(x_0,y_0)=0. \tag{A9}$$

This formula can be transformed by multiplying with $U^{-1}(x_0,y_0)$ from the left

$$\left[\frac{d}{dx_0}\hat{G}_H(x_0,y_0)\right]-U^{-1}(x_0,y_0)C(x_0)U(x_0,y_0)\hat{G}_H(x_0,y_0) =0. \tag{A10}$$

As has been shown above, the solutions for the preceding equation and Eq. (A8) are given by

$$\hat{G}(x_0,y_0)=\mathcal{P}\exp\left\{\int_{y_0}^{x_0} d\xi_0 U^{-1}(\xi_0,y_0)C(\xi_0)U(\xi_0,y_0)\right\} \tag{A11}$$

and

$$U(x_0,y_0)=\mathcal{P}\exp\left\{\int_{y_0}^{x_0} d\xi_0 B(\xi_0)\right\}, \tag{A12}$$

respectively. Making use of the group property (A2), the last line of Eq. (A3) is reproduced. These derivations can be repeated for any variable $n\cdot x$. Only for $n^2=0$ the Dirac equation cannot be written in the generic form (A5).

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