Inverse approach to Einstein's equations for nonconducting fluids

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We show that a flow (timelike congruence) in any type B_1 warped product spacetime is uniquely and algorithmically determined by the condition of zero flux. (Though restricted, these spaces include many cases of interest.) The flow is written out explicitly for canonical representations of the spacetimes. With the flow determined, we explore an inverse approach to Einstein's equations where a phenomenological fluid interpretation of a spacetime follows directly from the metric irrespective of the choice of coordinates. This approach is pursued for fluids with anisotropic pressure and shear viscosity. In certain degenerate cases this interpretation is shown to be generically not unique. The framework developed allows the study of exact solutions in any frame without transformations. We provide a number of examples, in various coordinates, including spacetimes with and without unique interpretations. The results and algorithmic procedure developed are implemented as a computer algebra program called GRSOURCE.

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I. INTRODUCTION

The usual procedure for finding exact solutions of Einstein's equations involves writing down a phenomenological energy-momentum tensor, often a perfect fluid, in a set of coordinates, frequently comoving, so that the field equations can be integrated, often with the aid of simplifying assumptions [1,2]. In view of the difficulty of solving Einstein's equations, inverse problems are important [3]. An inverse problem of interest (not involving the classification problem for the Ricci tensor [4] explicitly) may be stated thus. Given a spacetime $(\mathcal{M},\mathbf{g})$ with manifold $\mathcal{M}$ and Lorentzian metric $\mathbf{g}$, what, if any, fluid flow could generate $(\mathcal{M},\mathbf{g})$ via Einstein's equations? This question is explored here for the case of type B_1 warped product spacetimes (see below). The problem is of interest since, as is shown, the flow (velocity field) is uniquely determined subject to a zero flux condition. With the flow determined, physical parameters, subject to a fluid decomposition that includes anisotropic pressure and shear viscosity, can be extracted directly from the metric irrespective of coordinates. The framework developed in this paper allows the study of exact solutions in any frame without transformations. Surprisingly, little work has been done in noncomoving frames [5]. The invariant procedure developed is algorithmic and suited to computer algebra projects such as the Interactive Geometric Database [6] where, given a metric, one might like to know if it is necessarily, for example, a perfect fluid, or on the contrary, incompatible with a perfect fluid. The paper is organized as follows. In Sec. II, we explore the zero flux condition and in Sec. III we derive explicit forms of the velocity field for canonical coordinate types. In Sec. IV we explore the phenomenology of the non-conducting fluid source showing that degenerate cases exist. Examples that illustrate the power of the results obtained are given in Sec. V. Section VI is a summary.

II. ZERO FLUX

We consider warped product spacetimes of class B_1 [7,8]. These can be written in the form

$$ds^2_{\mathcal{M}}=ds^2_{\Sigma_1}(x^1,x^2)+C(x^\alpha)ds^2_{\Sigma_2}(x^3,x^4), \tag{1}$$

where $C(x^\alpha)=r(x^1,x^2)^2w(x^3,x^4)^2$, signature$(\Sigma_1)=0$ and signature$(\Sigma_2)=2\epsilon$ $(\epsilon=\pm1)$. Although very special, these spaces include many of interest, for example, *all* spherical, plane, and hyperbolic spacetimes. We write

$$ds^2_{\Sigma_1}=a(dx^1)^2+2b\,dx^1dx^2+c(dx^2)^2, \tag{2}$$

with a, b and c functions of (x^1,x^2) only. A congruence of unit timelike vectors (a "flow" in what follows) $u^\alpha=(u^1,u^2,0,0)$ have an associated unit normal field n^α (in the tangent space of Σ_1) satisfying $n_\alpha u^\alpha=0, n_\alpha n^\alpha=1$ [9]. It follows that $n_\alpha=\psi(x^1,x^2)(u^2,-u^1,0,0)$ (ψ a normalization factor). The timelike condition on u^α is

$$-1=a(u^1)^2+2bu^1u^2+c(u^2)^2 \tag{3}$$

and $(\mathcal{M},\mathbf{g})$ is time orientated by the restriction

$$u^1>0. \tag{4}$$

The condition

$$G_\alpha^\beta u^\alpha n_\beta=0, \tag{5}$$

where G_α^β is the Einstein tensor, can be written as

$$Au^1u^2-B(u^1)^2+C(u^2)^2=0, \tag{6}$$

where

$$A\equiv G_1^1-G_2^2,\quad B\equiv G_1^2,\quad C\equiv G_2^1. \tag{7}$$

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Condition (5) is the defining criterion for the flows considered here. In general, A, B and C are functions of x^1 through x^4, assumed not to vanish simultaneously. Equations (3), (4) and (6) determine the flow uniquely. The flow need not exist. For example, if $A=C=0$ and $B\neq 0$ then there exists no such flow. The assumption of "comoving coordinates" ($u^2=0$ in the present notation) imposes restrictions on the coordinatization of Σ_1, but it is already clear from Eq. (6) that the existence of such a flow requires $B=0$.

III. U^α FOR CANONICAL REPRESENTATIONS OF Σ_1

There are four distinct canonical types of coordinates (not specific coordinates) that can be used to represent Σ_1. In what follows we write out u^α in each of these cases. We do not write out quantities that follow algorithmically from u^α.

A. Kruskal-Szekeres ($a=c=0$)

The distinguishing characteristic here is

$$A=0. \tag{8}$$

We can always choose $b<0$ so that

$$u^2=-\frac{1}{2bu^1}>0, \tag{9}$$

and so it follows from Eq. (6) that

$$u^1=\sqrt[4]{\frac{C}{4Bb^2}}. \tag{10}$$

B. Diagonal ($b=0$)

Here $a<0$, $c>0$ and

$$Bc=Ca. \tag{11}$$

With

$$\mathcal{D}\equiv A^2a+4B^2c \tag{12}$$

it follows that $\mathcal{D}a>0$,

$$u^1=\sqrt{\frac{1}{2}\left(\frac{-1}{a}+\sqrt{\frac{A^2}{\mathcal{D}a}}\right)}, \tag{13}$$

and

$$(u^2)^2=\frac{1}{2c}\left(-1+\sqrt{\frac{A^2a^2}{\mathcal{D}a}}\right). \tag{14}$$

The requirement of "comoving" coordinates is $u^2=0 \Leftrightarrow B=C=0$.

C. Bondi ($c=0$ or $a=0$)

It is sufficient to consider the case $c=0$. Now $b\neq 0$ and for $b>0$ x^1 is an "advanced" time and for $b<0$ x^1 is a "retarded" time. Now

$$Ab=Ca. \tag{15}$$

With

$$\mathcal{E}\equiv Aa^2+4Bab \tag{16}$$

it follows that $\mathrm{sgn}(\mathcal{E})=\mathrm{sgn}(A)$,

$$u^1=\sqrt[4]{\frac{A}{\mathcal{E}}}, \tag{17}$$

and

$$u^2=-\frac{1}{2b}\left(\sqrt[4]{\frac{\mathcal{E}}{A}}\right)\left(1+a\sqrt{\frac{A}{\mathcal{E}}}\right). \tag{18}$$

The condition of comoving coordinates $u^2=0$ requires $a<0$ and $B=0$. Conversely, if $B=0$ then $u^2=0$ for $a<0$ and $u^2=-\sqrt{a}/b$ for $a>0$.

D. $a\neq 0$, $b\neq 0$ and $c\neq 0$

With

$$\mathcal{R}\equiv A^2c-2AbC+2CBc+2C^2a \tag{19}$$

and

$$\mathcal{S}\equiv 2AbcB-2AbCa+B^2c^2-4BCb^2+2BcCa+C^2a^2+caA^2, \tag{20}$$

it follows that

$$(u^1)^2=\frac{-\mathcal{R}+\mathrm{sgn}(b)\sqrt{\mathcal{R}^2-4\mathcal{S}C^2}}{2\mathcal{S}} \tag{21}$$

and

$$u^2=\frac{-bu^1+\mathrm{sgn}(b)\sqrt{(bu^1)^2-c(a(u^1)^2+1)}}{c}. \tag{22}$$

The condition of comoving coordinates $u^2=0$ requires the same conditions as in the previous case.

IV. PHENOMENOLOGY

The discussion in this section requires no specification of coordinates on Σ_1.

A. Decomposition

Whereas condition (5) determines the flow on Σ_1, it is of interest to know when the flow reduces to that of a perfect fluid [isotropic pressure (including bulk stress) and zero shear stress], or conversely, to be able to state that a given spacetime cannot be compatible with a perfect fluid source. The familiar case is entirely obvious. If the Einstein tensor is diagonal with three equal components, then the metric is consistent with a perfect fluid in comoving coordinates. Such a circumstance is, however, a property of the spacetime and the coordinates in which it is exhibited, more general cases are explored in this section. With u^α known, it is possible to reconstruct phenomenological parameters associated with a decomposition of the energy-momentum tensor directly from the metric. This decomposition is not always unique.

A phenomenological fluid interpretation of Eq. (5), by way of Einstein's equations, follows from the well known Eckart relation [10] which gives

$$8\pi\kappa n^\alpha(\nabla_\alpha T + T u^\beta \nabla_\beta u_\alpha) = 0 \tag{23}$$

where κ is the thermal conductivity and T the temperature profile. The flows considered here can therefore be considered non-conducting ($\kappa=0$) setting aside contrived functions T. The energy-momentum tensor is decomposed here in the form [11]

$$T^\alpha_\beta = \rho u^\alpha u_\beta + p_1 n^\alpha n_\beta + p_2 \delta^\alpha_\beta + p_2(u^\alpha u_\beta - n^\alpha n_\beta) - 2\eta\sigma^\alpha_\beta, \tag{24}$$

where σ^α_β is the shear associated with u^α and η is the phenomenological shear viscosity. It follows from Eq. (1) that $G^3_3 = G^4_4$ for the spacetimes considered in this paper. With $G^3_3 = G^4_4$ it follows from Eq. (24) that either $\sigma^3_3 = \sigma^4_4$ or we must set $\eta \equiv 0$. Note that Eq. (24) distinguishes the shear stress from an anisotropic pressure. These are sometimes combined. For example, in the comoving frame in a spherically symmetric spacetime, it follows that $T^\alpha_\beta = \mathrm{diag}(\rho, P1, P2, P2)$ where $P1 = p_1 - 2\eta\sigma^2_2$ and $P2 = p_2 - 2\eta\sigma^3_3$. Such a combination is not, in general, possible outside the comoving frame. Since anisotropic pressures do not arise solely due to shear stresses (e.g., in a static spherically symmetric spacetime, $\sigma^\beta_\alpha = 0$ but $p_1 \neq p_2$ in general), the full decomposition (24) is used here.

B. Systems of equations

We set up systems of equations to be solved simultaneously for the functions (ρ, p_1, p_2, η) in terms of scalars that follow algorithmically from the metric. It is not possible to build more than three independent equations in an attempt to solve for (ρ, p_1, p_2, η) since, with Eq. (5), there are only three independent scalars that can be constructed from the set $(G^\beta_\alpha, u^\alpha, n_\alpha)$ [7].

1. General spacetimes

To proceed in a manifestly invariant way we construct scalars from the set $(G^\beta_\alpha, u^\alpha, n_\alpha)$ that are linear in G^β_α. These are $G^\beta_\alpha u^\alpha n_\beta$ [used in condition (5)], $G^\beta_\alpha u^\alpha u_\beta$, $G^\beta_\alpha n^\alpha n_\beta$ and $G^\alpha_\alpha \equiv G$. With Eq. (24) it follows that

$$G = 8\pi(-\rho + p_1 + 2p_2), \tag{25}$$

and

$$G^\beta_\alpha u^\alpha u_\beta \equiv G1 = 8\pi\rho. \tag{26}$$

In all cases we take Eq. (26) as the definition of ρ. Further,

$$G^\beta_\alpha n^\alpha n_\beta \equiv G2 = 8\pi(p_1 - 2\eta\Delta) \tag{27}$$

where

$$\Delta \equiv \sigma^\beta_\alpha n^\alpha n_\beta. \tag{28}$$

Rearrangement of Eqs. (25), (26) and (27) gives

$$48\pi\eta\Delta = G + G1 - 3G2 + 16\pi P \tag{29}$$

where $P \equiv p_1 - p_2$. If $\sigma^\beta_\alpha = 0$ or $\eta \equiv 0$ or $P \equiv 0$ then Eqs. (25), (26) and (27) form a complete set of equations. More generally, however, if we attempt to solve for the complete set of parameters (ρ, p_1, p_2, η) four equations are needed [12]. For the class of spacetimes considered here, it was shown in [7] that higher-order invariants are not independent of the linear ones. Therefore, use of higher-order invariants will not break the degeneracy but merely generate new syzygies (algebraic identities amongst invariantly defined quantities).

2. Restricted spacetimes

From Eq. (24) and Einstein's equations it follows that for the spacetimes $(\mathcal{M}, \mathbf{g})$ considered here

$$\tilde{G}^1_2\tilde{G}^2_1 - (\tilde{G}^1_1 - \tilde{G}^3_3)(\tilde{G}^2_2 - \tilde{G}^3_3) \equiv \widetilde{G5} = (8\pi)^2 P(\rho + p_2), \tag{30}$$

where $\tilde{G}^\alpha_\beta = G^\alpha_\beta + 16\pi\eta\sigma^\alpha_\beta$. Although Eq. (30) holds in every $(\mathcal{M}, \mathbf{g})$ without specific coordinates specified on Σ_1, it is not manifestly invariant. Use of Eq. (30) merely generates further (restricted) syzygies.

C. Linear cases

1. $\sigma^\beta_\alpha = 0$

If $\sigma^\beta_\alpha = 0$ and $\rho + p_2 \neq 0$ then Eq. (30) with $P = 0$ gives the Walker's pressure isotropy condition [13]

$$G^1_2 G^2_1 = (G^1_1 - G^3_3)(G^2_2 - G^3_3) \tag{31}$$

which is here a necessary and sufficient condition for a perfect fluid. A manifestly invariant condition follows from Eqs. (25), (26) and (27) which give

$$p_1 = \frac{G2}{8\pi}, \tag{32}$$

and

$$p_2 = \frac{G + G1 - G2}{16\pi}. \tag{33}$$

Clearly

$$G + G1 = 3G2 \tag{34}$$

is a necessary and sufficient condition for a perfect fluid including the exceptional case $G + 3G1 = G2$ not covered by Eq. (31). Equation (34) is not sensitive to the presence of a cosmological constant term in the Einstein field equations and plays a central role in what follows [14].

In all of what now follows up to Sec. V we assume $\sigma^\beta_\alpha \neq 0$.

2. $\eta\equiv 0$

The decomposition (24) is consistent with some spacetimes if and only if $\eta\equiv 0$. Some examples of this are shown in Sec. V. If $\eta\equiv 0$ then Eqs. (32), (33) and (34) hold as in the previous case.

3. $p_1=p_2\equiv p$, $\Delta\neq 0$

Equations (25), (26) and (27) now give

$$p=\frac{G+G1}{24\pi}\tag{35}$$

and

$$\eta=\frac{G+G1-3G2}{48\pi\Delta}\tag{36}$$

so that Eq. (34) is once again a necessary and sufficient condition for a perfect fluid. The case $\Delta=0$ is equivalent to the case $\sigma_\alpha^\beta=0$.

4. $p_1\neq p_2$, $\Delta\neq 0$

Equations (25), (26) and (27) now give

$$p1=\frac{G2}{8\pi}+2\eta\Delta\tag{37}$$

and

$$p2=\frac{G+G1-G2}{16\pi}-\eta\Delta,\tag{38}$$

where η is arbitrary. If we set $\eta\equiv 0$ then the condition (34) is a necessary and sufficient condition for a perfect fluid. For other choices of η the fluid is imperfect.

D. Nonuniqueness of the source

In the last case above, it is not possible to solve for a unique set (ρ,p_1,p_2,η) as only three invariants are independent for the type of spacetimes considered in this paper. Another way to see this directly is to observe that substitution of the expressions for ρ, p_1 and p_2 as given by Eqs. (26), (37) and (38) into the energy-momentum tensor (24) and multiplication by 8π reproduces the Einstein tensor [15]. This of course all derives from the fact that in a canonical frame there are at most three independent components of the Einstein tensor for the spacetimes considered. As we show in Sec. V, the application of the framework to some given spacetimes known to represent perfect fluid solutions but where $\Delta\neq 0$ shows that there are other imperfect fluid sources possible. This is always the case where the perfect fluid condition (34) holds and $\Delta\neq 0$. For $\eta\equiv 0$ the fluid is perfect, for other choices of η the fluid is imperfect. For example, the Lemaître-Tolman-Bondi metric ("dust") is given [16,2] by

$$ds^2_{\mathcal{M}}=-(dt)^2+\frac{[R'(t,r)]^2(dr)^2}{1+f(r)}+R(t,r)^2d\Omega^2,\tag{39}$$

along with the constraints

$$\dot{R}(t,r)=\sqrt{2\frac{m(r)}{R(t,r)}+f(r)},\tag{40}$$

$$\ddot{R}(t,r)=-\frac{m(r)}{R^2(t,r)},\tag{41}$$

$$\ddot{R}'(t,r)=-\frac{m'(r)}{R^2(t,r)}+2\frac{m(r)R'(t,r)}{R^3(t,r)},\tag{42}$$

and

$$\dot{R}'(t,r)=\frac{2\,m'(r)R(r,t)-2\,m(r)R'(r,t)+f'(r)R^2(r,t)}{2R(r,t)\sqrt{[2\,m(r)+f(r)R(r,t)]R(r,t)}},\tag{43}$$

where $'\equiv\partial/\partial r$ and $\cdot\equiv\partial/\partial t$. Condition (34) holds, so the source is consistent with (but not necessarily) a perfect fluid, in fact simply dust (since $G2=0=G+G1$) with

$$\rho=\frac{m'(r)}{4\pi R^2(r,t)R'(r,t)}.\tag{44}$$

The metric (39) (with the given constraints) is also consistent with an imperfect fluid with $p_1=2\eta\Delta$, $p_2=-\eta\Delta$, η arbitrary and

$$\Delta=-\frac{-2m'(r,t)R(r,t)+6m(r)R'(r,t)-f'(r)R^2(r,t)+2R'(r,t)f(r)R(r,t)}{3R'(r,t)R^{3/2}(r,t)\sqrt{2\,m(r)+f(r)R(r,t)}}\tag{45}$$

as is easily verified. The fact that the Lemaître metric need not be considered as dust is not a new result [17]. Here, the degenerate case ($\Delta\neq 0$) is shown to be generic.

V. EXAMPLES

We provide here examples in various coordinate types in order to illustrate the results obtained.

A. Kruskal-Szekeres coordinates

Aside from the Kruskal-Szekeres metric, little use has been made of double null coordinates. The example used here is the Einstein–de Sitter universe [18]

$$ds^2_{\mathcal{M}}=\mathcal{C}^2(u+v)^4\left(-du\,dv+\frac{(u-v)^2}{4}d\Omega^2\right),\tag{46}$$

where $\mathcal{C}$ is a constant and $d\Omega^2$ is the metric of a unit sphere. It follows that $u^\alpha = 1/\mathcal{C}(u+v)^2(1,1,0,0)$, $\sigma_\alpha^\beta = 0$ and Eq. (34) holds so that the fluid is necessarily perfect (in fact simply dust since $G2 = 0 = G + G1$).

B. Bondi coordinates

The Bondi metric [19]

$$ds^2_{\mathcal{M}} = c^2(w,r)f(w,r)(dw)^2 \pm 2c(w,r)dwdr + r^2 d\Omega^2 \tag{47}$$

in advanced (+) or retarded (−) w has $A = C = 0, B \neq 0$ for $\partial c/\partial r = 0$ and $\partial f/\partial w \neq 0$ and so there is no non-conducting fluid source of Eq. (47) under these conditions. The Vaidya metric [20] (corresponding to a null flux) provides a familiar example. The metric [21]

$$ds^2_{\mathcal{M}} = -2H(u,r)(du)^2 - 2dudr + ur^{2n}((dx)^2 + (dy)^2), \tag{48}$$

where $H(u,r) = (r/u + kr^m u^{(2-m)/(m-1)})/2 > 0$ and $n = m(m-1)/2$ is necessarily comoving. Moreover, $\sigma_\alpha^\beta \neq 0$ and Eq. (34) holds, so the source is consistent with (but not necessarily) a perfect fluid. If we set $\eta \equiv 0$ then the fluid is perfect but for other choices of η the fluid is imperfect.

C. Comoving diagonal coordinates

The next examples are in diagonal coordinates and have $B = C = 0$ so that u^α is necessarily comoving.

1. $\sigma_\alpha^\beta = 0$

The Robertson-Walker metric

$$ds^2_{\mathcal{M}} = -(dt)^2 + \frac{a(t)^2(dr)^2}{1 - kr^2} + r^2 d\Omega^2 \tag{49}$$

gives $\sigma_\alpha^\beta = 0$ and Eq. (34) holds. It follows that the fluid is necessarily perfect [22]. Equations (26) and (27) reproduce Friedmann's equations. In contrast, the Kantowski-Sachs metric [23]

$$ds^2_{\mathcal{M}} = -(dt)^2 + \frac{a(t)^2(dr)^2}{1 - kr^2} + d\Omega^2 \tag{50}$$

gives $\sigma_\alpha^\beta = 0$ but Eq. (34) never holds. It follows that the fluid can never be perfect. For the general spherical static metric

$$ds^2_{\mathcal{M}} = -e^{2\Phi(r)}(dt)^2 + \frac{(dr)^2}{1 - 2m(r)/r} + r^2 d\Omega^2, \tag{51}$$

whereas $\sigma_\alpha^\beta = 0$, Eq. (34) does not in general hold. The fluid has anisotropic pressure, the perfect fluid being a special case [24]. The metric [25]

$$ds^2_{\mathcal{M}} = -(dt)^2 + R(t)^2[(dr)^2 + sin(r)^2(dz)^2 + f(r)^2(d\phi)^2] \tag{52}$$

with the constraint $2R\ddot{R} + (\dot{R})^2 + 1 = 0$ where $\dot{} \equiv \partial/\partial t$ has $\sigma_\alpha^\beta = G_\phi^\phi = 0$. The condition $G_z^z = 0$ gives $f(r) = \cos(r + A)$ where A is constant. Condition (34) does not in general hold. Rather, $p_2 = 0$ but $p_1 = 0$ only for $A = 0$.

2. $\eta \equiv 0$

The metric [26]

$$ds^2_{\mathcal{M}} = S(t)^{-2m}C(x)^{-2m-2}[-(dt)^2 + (dx)^2] + S(t)C(x)^\alpha[T(t)^n(dy)^2 + T(t)^{-n}(dz)^2] \tag{53}$$

has $G_x^x = G_y^y = G_z^z$ and so is obviously a perfect fluid in comoving coordinates. However, $\sigma_y^y \neq \sigma_z^z$ and so the decomposition (24) holds only for $\eta = 0$. Condition (34) holds in agreement with the obvious.

D. Non-comoving diagonal coordinates

Few examples are available in non-comoving coordinates. From the pioneering work of McVittie and Wiltshire [5] we note for example, that their solution (6.12)

$$ds^2_{\mathcal{M}} = e^{2\beta(z)}[-(dt)^2 + (d\xi)^2 + \xi^2 d\Omega^2] \tag{54}$$

where $z = \epsilon(\xi^2 - t^2)/\xi_0^2$, and $\beta(z)$ is an undetermined function of z with $\beta_{zz} - \beta_z^2 \neq 0$, has $\sigma_\alpha^\beta = 0$ and the condition (34) holds so the source is necessarily a perfect fluid. Their solution (7.20)

$$ds^2_{\mathcal{M}} = \exp(Ae^{z/L} + Bz/l - 2\epsilon Lt)[-(dt)^2 + (d\omega)^2 + d\Omega^2], \tag{55}$$

where $z = \omega + \epsilon t$, $\epsilon = \pm 1$ and A, B and L are constants, has $\sigma_\alpha^\beta \neq 0$. Condition (34) holds so the source is consistent with a perfect fluid (if $\eta \equiv 0$). Their solutions (6.21) and (8.11) also have shear and Eq. (34) holds so the solutions are compatible with a perfect fluid source. As with the Davidson metric (48), McVittie and Wiltshire solutions (6.21), (7.20) and (8.11) are also consistent with an imperfect fluid with $\eta \neq 0$.

From the more recent work of Senovilla and Vera [5], for example, their solution (40)

$$ds^2_{\mathcal{M}} = -(dt)^2 + (dx)^2 + \frac{\cos^{1+2\nu}(\mu x)}{\cos h^{2\nu-1}(\mu t)}(dy)^2 + \frac{\cos^{1-2\nu}(\mu x)}{\cos h^{-2\nu-1}(\mu t)}(dz)^2 \tag{56}$$

has $G_y^y = G_z^z$ but $\sigma_y^y \neq \sigma_z^z$ and so the metric (56) is consistent with the decomposition (24) only for $\eta \equiv 0$ [the same holds for their solutions (38) and (41)]. Equation (34) holds and so the metric (56) necessarily represents a perfect fluid.

VI. SUMMARY

It has been shown that a flow (timelike congruence u^α) in any type B_1 warped product spacetime is uniquely and algorithmically determined by the condition of zero flux ($G^\beta_\alpha u^\alpha n_\beta = 0$). Explicit forms of u^α have been written out for canonical representations. With u^α known, a phenomenological interpretation of the spacetime ($\mathcal{M}$,**g**) in terms of a non-conducting fluid follows. The following cases are delineated

(i) If $\eta\Delta=0$ then the condition (34) is a necessary and sufficient condition for a perfect fluid.

(ii) If $p_1 \equiv p_2 \equiv p$ and $\Delta \neq 0$ then p is given by Eq. (35) and η is given by Eq. (36). The condition (34) is a necessary and sufficient condition for a perfect fluid.

(iii) If $\Delta \neq 0$ then p_1 is given by Eq. (37) and p_2 is given by Eq. (38) and η is a freely specified function. This is a generic degenerate case. If condition (34) holds then the fluid is compatible with a perfect fluid source. If $\eta \equiv 0$ the fluid is perfect. For other choices of η the fluid is imperfect.

Furthermore, the derived covariant perfect fluid condition can be used to study exact solutions of Einstein's equations as well as deriving new families of solutions [27]. Examples, in various coordinates, including spacetimes with and without unique interpretations have been provided. The procedure developed has been implemented in a computer algebra program described in the Appendix.

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APPENDIX: GRSOURCE, A COMPUTER ALGEBRA PROGRAM

GRSOURCE [29] runs under the system GRTENSORII which in turn runs under the system Maple. The program is called with a spacetime metric name as an input. GRSOURCE calculates the velocity field and related physical quantities. Moreover, it performs an algorithmic analysis on the nature of the fluid. The results of the evaluated quantities are displayed, followed by a summary report on the analysis of the possible fluid sources for the input spacetime. An example follows:

> restart; We start a GRTENSORII and GRSOURCE session

> grtw( );

GRTENSORII *Version 1.80-pre2 (R6)*

Developed by Peter Musgrave, Denis Pollney and Kayll Lake

Copyright 1994–2003 by the authors.

Latest version available from: http://grtensor.org

GRSOURCE *Package Version 1.00*

Developed by Mustapha Ishak and Kayll Lake (c) 2002–2003

Help available via ?grsource

Usage: Load a metric and enter the command source(metricname);

> qload(kantosachs);

Default spacetime=kantosachs

For the kantosachs spacetime

Coordinates: x(up)

$$x^{a} = [r,\ \theta,\ \phi,\ t]$$

Signature=2

Line element: ds²

$$ds^2 = \frac{a(t)^2 dr^2}{1-Kr^2} + a(t)^2 d\theta^2 + \mathrm{a}(t)^2 \sin(\theta)^2 d\phi^2 - dt^2$$

Kantowski-Sachs metric, J. Math. Phys. **7**, 443 (1966)

> source(kantosachs);

A general velocity field will be generated automatically from the metric.

Please answer the following or enter exit anytime to stop the session.

Enter the timelike or null coordinate, for example, t;>t;

Enter the spacelike or null coordinate, for example, r;>r;

Enter the spacelike coordinate, for example, θ;>θ;

Enter the spacelike coordinate, for example, ϕ;>ϕ;

For the kantosachs spacetime

$$u^a = [0,\ 0,\ 0,\ 1]$$

$$uSq = -1$$

flux=All components are zero

ShearTensor$^a{}_b$=All components are zero

Expansion scalar: expsc

$$\Theta[u] = 3\frac{\frac{d}{dt}a(t)}{a(t)}$$

$\Delta =$ *All components are zero*

$$\rho = \frac{1}{8}\,\frac{3\left[\frac{d}{dt}a(t)\right]^2 + 1}{a(t)^2 \pi}$$

$$p1 = -\frac{1}{8}\,\frac{\left[\frac{d}{dt}a(t)\right]^2 + 2a(t)\left[\frac{d^2}{dt^2}a(t)\right] + 1}{a(t)^2 \pi}$$

$$p2 = -\frac{1}{8}\,\frac{\left[\frac{d}{dt}a(t)\right]^2 + 2a(t)\left[\frac{d^2}{dt^2}a(t)\right]}{a(t)^2 \pi}$$

$$PFCondition = 2\frac{1}{a(t)^2}$$

Report

η Δ=0 but PFCondition was not simplified to zero, the fluid has anisotropic pressure. Further simplifications can be applied to the objects calculated using the commands gralter( ) and grdisplay( ).*

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[1] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt, *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, Cambridge, 2003).

[2] A. Krasinski, *Inhomogeneous Cosmological Models* (Cambridge University Press, Cambridge, 1997).

[3] Any spacetime can be associated with an ''exact solution'' to Einstein's equations, but only those which are ''physically reasonable'' are worthy of consideration. Whereas the term exact solution has no precise definition, the concept is well understood [1,2]. The concept of physically reasonable is somewhat more slippery and it is the responsibility of the author of any new exact solution to justify its significance.

[4] See, for example [1]. For a classical perspective see J. Plebański and J. Stachel, J. Math. Phys. **9**, 269 (1968) and for a modern perspective see F.M. Paiva, M.J. Rebouas, G.S. Hall, and M.A.H. MacCallum, Class. Quantum Grav. **15**, 1031 (1998). The purpose of our approach is to derive an efficient algorithmic procedure (to be incorporated into a computer program) to invariantly extract physical information directly from a given spacetime. Whereas one could put the spacetime in canonical form, this is not part of the algorithmic procedure as no transformations are allowed here. We wish to extract the physical information without solving the eigenvalue problem for the Ricci tensor. This is a matter of efficiency.

[5] J.M. Senovilla and R. Vera, Class. Quantum Grav. **15**, 1737 (1998); G.C. McVittie and R.J. Wiltshire, Int. J. Theor. Phys. **16**, 121 (1977); P.C. Vaidya, Phys. Rev. **174**, 1615 (1968).

[6] Specifically, we have in mind algorithms as supplements to classification procedures in the grdb project. See M. Ishak and K. Lake, Class. Quantum Grav. **19**, 505 (2002) (see http://grdb.org).

[7] K. Santosuosso, D. Pollney, N. Pelavas, P. Musgrave, and K. Lake, Comput. Phys. Commun. **115**, 381 (1998).

[8] J. Carot and da Costa, Class. Quantum Grav. **10**, 461 (1993).

[9] We use signature +2 with $\alpha=1,\ldots,4$.

[10] C. Ecart, Phys. Rev. **58**, 919 (1940).

[11] Here we are interested in fluid flows alone and ignore contributions from a null flux, and from scalar and electromagnetic fields.

[12] Component expressions follow immediately from Eq. (24). For example, $G_3^3 = 8\pi(p_2 - 2\eta\sigma_3^3)$ could be used. These component expressions fail to solve for a unique set (ρ, p_1, p_2, η). Furthermore, the approach used in this section is manifestly invariant.

[13] A.G. Walker, Q. J. Math. **6**, 81 (1935).

[14] Equation (34) is linear in G_α^β. Alternate forms can be derived if one considers higher-order invariants built only from the Einstein tensor or the trace-free Ricci tensor $[S_\beta^\alpha \equiv R_\beta^\alpha - (R/4)\delta_\beta^\alpha]$. An example of such an equation is $3R2^2 - R1^3 = 0$ where $R1 \equiv \frac{1}{4}S_\beta^\alpha S_\alpha^\beta$ and $R2 \equiv -\frac{1}{8}S_\beta^\alpha S_\gamma^\beta S_\alpha^\gamma$. The velocity and the normal vector fields do not enter such a relation, however one needs to verify the existence of the fluid flow (as explained earlier, the flow need not exist).

[15] K. Lake, gr-qc/0209063.

[16] G. Lemaître, Ann. Soc. Sci. Bruxelles A **53**, 51 (1933). This is commonly called as the ''Tolman-Bondi'' metric (see [1] and [2]).

[17] R.A. Sussman and M. Ishak, Gen. Relativ. Gravit. **34**, 1589 (2002).

[18] See, for example, W.G. Laarakkers and E. Poisson, in *General Relativity and Relativistic Astrophysics*, edited by C.P. Buroyess and R.C. Myers, AIP Conf. Proc. No. 493 (AIP, Melville, NY, 1999), p. 156.

[19] H. Bondi, Proc. R. Soc. London **281**, 39 (1964).

[20] P.C. Vaidya, Proc. Indian Acad. Sci., Sect. A **33A**, 264 (1951).

[21] W. Davidson, Class. Quantum Grav. **5**, 147 (1988) (see [2] p. 245).

[22] Of course the metric (49) allows other interpretations if condition (5) is relaxed. See, for example, A.A. Coley and B.O.J. Tupper, Astrophys. J. **271**, 1 (1983).

[23] R. Kantowski and R.K. Sachs, J. Math. Phys. **7**, 443 (1966) (see [2] p. 12).

[24] Indeed, it is rather difficult to find isotropic solutions. See M.S.R. Delgaty and K. Lake, Comput. Phys. Commun. **115**, 395 (1998); K. Lake, Phys. Rev. D **67**, 104015 (2003).

[25] N.V. Mitskievič and Y.E. Senin, Acta Phys. Pol. B **12**, 541 (1981) (see [2] p. 247).

[26] J. Wainwright and S.W. Goode, Phys. Rev. D **22**, 1906 (1980) (see [2] p. 178).

[27] Lake [24].

[28] This is a package which runs within MAPLE. It is entirely distinct from packages distributed with MAPLE and must be obtained independently. The GRTENSORII software and documentation is distributed freely on the World Wide Web from the address http://grtensor.org

[29] GRSOURCE program is freely distributed from the authors.