Classical confinement of test particles in higher-dimensional models: Stability criteria and a new energy condition

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We review the circumstances under which test particles can be localized around a spacetime section Σ_0 smoothly contained within a codimension-1 embedding space M. If such a confinement is possible, Σ_0 is said to be totally geodesic. Using three different methods, we derive a stability condition for trapped test particles in terms of intrinsic geometrical quantities on Σ_0 and M; namely, confined paths are stable against perturbations if the gravitational stress-energy density on M is larger than that on Σ_0, as measured by an observed travelling along the unperturbed trajectory. We confirm our general result explicitly in two different cases: the warped-product metric ansatz for $(n+1)$-dimensional Einstein spaces, and a known solution of the 5-dimensional vacuum field equation embedding certain 4-dimensional cosmologies. We conclude by defining a *confinement energy condition* that can be used to classify geometries incorporating totally geodesic submanifolds, such as those found in thick braneworld and other 5-dimensional scenarios.

I. INTRODUCTION

The past half-decade has seen a notable upswing in interest in noncompact higher-dimensional theories of physics. Most of this attention can be attributed to recent advances in string theory, which have postulated that we are living on a (3+1)-dimensional hypersurface embedded within some higher-dimensional manifold. Such "braneworld" scenarios have been extensively analyzed in the literature, and have been used to address issues such as the hierarchy problem of particle physics [1–4], as well as the idea that the post-inflationary epoch of our Universe was preceded by the collision of D3-branes [5,6]. In all fairness, it should be mentioned that the current flurry of interest in braneworld scenarios has been preceded by numerous other models making use of large or infinite extra dimensions [7–12].

In some braneworld scenarios, the idea of noncompact extra dimensions is made more palatable by postulating that the particles and fields of the standard model are confined to the brane universe. If we adopt the most conservative point of view, the notion of confinement is a prerequisite for any reasonable theory with noncompact extra dimensions; without such an assumption, the fact that we so not commonly see objects flying off in unseen directions becomes a thorny issue. In the context of a particular string theory-inspired model put forth by Horava and Witten [13,14], lower-dimensional confinement is a natural consequence of the idea that standard model degrees of freedom are associated with open strings that have endpoints residing on a Dp-brane. Conversely, since gravitational degrees of freedom are associated with closed strings, the graviton in such models is assumed to propagate both in the bulk and on the brane. Phenomenological 5-dimensional realizations based on this idea model the brane as a 4-dimensional domain wall or defect [3,4]. The discontinuity in the 5-geometry about the defect forces the graviton ground state to be sharply localized on the brane, which allows for the recovery of standard Newtonian gravitation in the low-energy limit. This kind of localization extends to other types of fields, thus representing a sort of concretization of the confinement mechanism envisioned in the original string model. In addition, if the matter localized on the brane satisfies the appropriate energy conditions and the $\mathbb{Z}_2$ symmetry is obeyed, one can show that test particles can be gravitationally confined to a small region about the defect [15]. This acts as a classical confinement mechanism.

A natural generalization of models involving thin geometric defects is scenarios involving thick, smooth domain walls [16,17]. There are a couple of reasons to consider such models: First, there is a natural minimum length in string theory given by the string scale, so the idea of an infinitely thin geometric feature is somewhat suspect even in a phenomenological model. Second, one would like to see these braneworld scenarios resulting from some genuine solutions of supergravity, which are *a priori* smooth and differentiable manifolds. The question is: what becomes of the confinement paradigm in bulk manifolds devoid of thin domain walls? For test particles in scenarios with one extra dimension, the answer is well known: if the brane has vanishing extrinsic curvature, geodesics may be naturally hypersurface-confined without the invocation of external nongravitational forces. A surface with zero extrinsic curvature is sometimes called totally geodesic.[1] But what of the stability of the trajectories confined on these surfaces? That is, if one perturbs a confined trajectory slightly off of a totally geodesic submanifold, will it naturally return or not? In other words, under which conditions is a totally geodesic hypersurface gravitationally attractive? For obvious reasons, such questions are of direct relevance to any serious attempt to classically describe our universe as a smoothly-embedded hypersurface on which we

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[1] An alternative name for a totally geodesic submanifold is "geodesically complete."

are gravitationally trapped. It is possible that this classical stability issue is irrelevant at the quantum level—perhaps because stable particle confinement can be guaranteed by other means—but for the purposes of this study we will assume that the classical formalism is applicable.

In this paper, we propose to address these issues n-dimensional totally geodesic submanifolds smoothly embedded in a space of one higher dimension, with either timelike or spacelike signature. We will utilize quite general methods that will ensure our results apply to any geometry and choice of coordinates in the bulk or on the submanifold. In Sec. II, we describe our geometric construction. In Sec. III, we review the covariant splitting of test particle equations of motion developed in [15,18,19] and use it to derive the zero-extrinsic curvature condition for totally geodesic submanifolds. Then, we find the stability condition for the confined trajectories, which is that the double contraction of the particle's velocity with the Ricci tensor of the bulk is greater than the double contraction with the Ricci tensor of the submanifold. In more physical terms, the stability of trapped particles demands that the locally measured gravitational density of the bulk is bigger than the density of the effective lower-dimensional matter living on the brane.[2] We briefly discuss the nature of the latter, emphasizing that the stress-energy content of the submanifold—as perceived by an observer ignorant of an extra dimension—is made up from contributions from the ''real'' higher-dimensional matter as well as the bulk Weyl tensor. For good measure, we derive the stability condition using two additional methods: the geodesic deviation equation in Sec. IV A, and the Raychaudhuri equation in Sec. IV B. We confirm the correctness of our general result for the special case of the warped-product metric ansatz in Sec. V A, and consider a simple 5-dimensional model of the solar neighborhood. In Sec. V B, we show that our stability condition is also correct in the Liu-Mashoon-Wesson solution [20,21] of the 5-dimensional vacuum field equations. Section VI summarizes our work and presents the *confinement energy condition*, which ensures that all timelike trajectories on a totally geodesic submanifold in a given bulk geometry will be stable. This energy condition can be used to classify solutions of the thick braneworld on other 5-dimensional scenarios.

Conventions. Uppercase Latin indices run from 0 to n, while lowercase Greek indices run from 0 to $n-1$. Higher-dimensional curvature tensors are distinguished from their lower-dimensional counterparts by hats. Higher and lower dimensional covariant differentiation operators are denoted by ∇_A or ∇_α, respectively. A center dot indicates the scalar product between higher-dimensional vector fields; i.e., $u \cdot v \equiv u^A v_A$.

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[2] Roughly speaking, the gravitational density of a given matter-energy distribution differs from the ordinary density by terms involving the pressure. For example, according to an observer comoving with a $(n+1)$-dimensional perfect fluid, the gravitational density—as we define it below—is $[(n-2)\rho+np]/(n-1)$. It is important because it appears naturally in the Raychaudhuri equation, as we will see in Sec. IV B.

II. GEOMETRIC CONSTRUCTION

We will be concerned with an $(n+1)$-dimensional manifold (M, g_{AB}) on which we place a coordinate system $x \equiv \{x^A\}$. Sometimes, we will refer to M as the ''bulk manifold.'' In our working, we will allow for two possibilities: either there is one timelike and n spacelike directions tangent to M, *or* there are two timelike and $(n-1)$ spacelike directions tangent to M. Hence, the signature of g_{AB} is

$$\text{sig}(g_{AB}) = (- + \cdots + \varepsilon), \qquad (1)$$

where $\varepsilon = \pm 1$. We introduce a scalar function

$$\ell = \ell(x), \qquad (2)$$

which defines our foliation of the higher-dimensional manifold with the hypersurfaces given by $\ell = \text{const}$, denoted by Σ_ℓ. If there is only one timelike direction tangent to M, we assume that the vector field n^A normal to Σ_ℓ is spacelike. If there are two timelike directions, we take the unit normal to be timelike. In either case, the submanifold tangent to a given Σ_ℓ hypersurface contains one timelike and $(n-1)$ spacelike directions; that is, each Σ_ℓ hypersurface corresponds to an n-dimensional Lorentzian spacetime. The normal vector to the Σ_ℓ slicing is given by

$$n_A = \varepsilon \Phi \partial_A \ell, \quad n \cdot n = \varepsilon. \qquad (3)$$

The scalar Φ which normalizes n^A is known as the lapse function. We define the projection tensor as

$$h_{AB} = g_{AB} - \varepsilon n_A n_B. \qquad (4)$$

This tensor is symmetric ($h_{AB} = h_{BA}$) and orthogonal to n_A. We place an n-dimensional coordinate system on each of the Σ_ℓ hypersurfaces $y \equiv \{y^\alpha\}$. The n holonomic basis vectors

$$e_\alpha^A = \frac{\partial x^A}{\partial y^\alpha}, \quad n \cdot e_\alpha = 0 \qquad (5)$$

are by definition tangent to the Σ_ℓ hypersurfaces and orthogonal to n^A. It is easy to see that e_α^A behaves as a vector under coordinate transformations on M [$\phi : x \to \bar{x}(x)$] and a one-form under coordinate transformations on Σ_ℓ [$\psi : y \to \bar{y}(y)$]. We can use these basis vectors to project higher-dimensional objects onto Σ_ℓ hypersurfaces. For example, for an arbitrary one-form on M we have

$$T_\alpha = e_\alpha^A T_A = e_\alpha \cdot T. \qquad (6)$$

Here T_α is said to be the projection of T_A onto Σ_ℓ. Clearly T_α behaves as a scalar under ϕ and a one-form under ψ. The induced metric on the Σ_ℓ hypersurfaces is given by

$$h_{\alpha\beta} = e_\alpha^A e_\beta^B g_{AB} = e_\alpha^A e_\beta^B h_{AB}. \qquad (7)$$

Just like g_{AB}, the induced metric has an inverse:

$$h^{\alpha\gamma} h_{\gamma\beta} = \delta^\alpha_\beta. \qquad (8)$$

The induced metric and its inverse can be used to raise and lower the indices of tensors tangent to Σ_ℓ, and change the position of the spacetime index of the e_α^A basis vectors. This implies

$$e_A^\alpha e_\beta^A = \delta^\alpha{}_\beta. \tag{9}$$

Also note that since h_{AB} is entirely orthogonal to n^A, we can express it as

$$h_{AB} = h_{\alpha\beta} e_A^\alpha e_B^\beta. \tag{10}$$

At this juncture, it is convenient to introduce our definition of the extrinsic curvature $K_{\alpha\beta}$ of the Σ_ℓ hypersurfaces:

$$K_{\alpha\beta} = e_\alpha^A e_\beta^B \nabla_A n_B = \tfrac{1}{2} e_\alpha^A e_\beta^B \hat{\pounds}_n h_{AB}. \tag{11}$$

Note that the extrinsic curvature is symmetric ($K_{\alpha\beta} = K_{\beta\alpha}$). It may be thought of as the derivative of the induced metric in the normal direction. This n-tensor will appear often in what follows.

We will also require an expression that relates the higher-dimensional covariant derivative of $(n+1)$-tensors to the lower-dimensional covariant derivative of the corresponding n-tensors. We have that the n-dimensional Christoffel symbols are given by

$$\Gamma^\alpha_{\beta\gamma} = e_\gamma^B e_A^\alpha \nabla_B e_\beta^A. \tag{12}$$

This allows us to deduce that for one-forms, the following relation holds:

$$\nabla_\beta T_\alpha = e_\beta^B e_\alpha^A \nabla_B (h_A{}^C T_C), \tag{13}$$

where ∇_B is the covariant derivative on M defined with respect to g_{AB} and ∇_β is the covariant derivative on Σ_ℓ defined with respect to $h_{\alpha\beta}$. The generalization to tensors of higher rank is obvious. It is not difficult to confirm that this definition of ∇_α satisfies all the usual requirements imposed on the covariant derivative operator.

Finally, we note that $\{y, \ell\}$ defines an alternative coordinate system to x on M. The appropriate diffeomorphism is

$$dx^A = e_\alpha^A dy^\alpha + \ell^A d\ell, \tag{14}$$

where

$$\ell^A = \left(\frac{\partial x^A}{\partial \ell}\right)_{y^\alpha = \text{const}} \tag{15}$$

is the vector tangent to lines of constant y^α. We can always decompose higher dimensional vectors into the sum of a part tangent to Σ_ℓ and a part normal to Σ_ℓ: For ℓ^A we write

$$\ell^A = N^\alpha e_\alpha^A + \Phi n^A. \tag{16}$$

This is consistent with $\ell^A \partial_A \ell = 1$, which is required by the definition of ℓ^A, and the definition of n^A. The n-vector N^α is the shift vector, which describes how the y^α coordinate system changes as one moves from a given Σ_ℓ hypersurface to

another. Using our formulas for dx^A and ℓ^A, we can write the higher dimensional line element as

$$ds_{(M)}^2 = g_{AB} dx^A dx^B$$
$$= h_{\alpha\beta}(dy^\alpha + N^\alpha d\ell)(dy^\beta + N^\beta d\ell) + \varepsilon \Phi^2 d\ell^2. \tag{17}$$

This reduces to $ds_{(\Sigma_\ell)}^2 = h_{\alpha\beta} dy^\alpha dy^\beta$ if $d\ell = 0$. It is also possible to express the extrinsic curvature in terms of Φ and N^α:

$$K_{\alpha\beta} = \frac{1}{2\Phi}(\partial_\ell - \pounds_N) h_{\alpha\beta}, \tag{18}$$

where $\pounds_N$ is the Lie derivative in the direction of the shift vector.

In this paper, we will be primarily concerned with the Gaussian-normal coordinate gauge that has been termed canonical by some authors [22]. This is defined by the following choices of foliation parameters:

$$\Phi = 1, \quad N^\alpha = 0. \tag{19}$$

Obviously, this choice will result in significant simplification of many of the preceding and following formulas.

III. CONFINEMENT OF TEST PARTICLES

The equations of motion for a test particle traveling through M are taken to be

$$u^A \nabla_A u^B = \mathcal{F}^B, \quad u \cdot u = \kappa, \quad u^A = \frac{dx^A}{d\lambda}, \tag{20}$$

where $\kappa = -1, 0, +1$ to allow for massive, null and tachyonic particles respectively, λ is an affine parameter, and $\mathcal{F}$ is some nongravitational force per unit mass. One can decompose these equations into relations involving the particle's velocity tangent to the Σ_ℓ foliation $u^\alpha = e^\alpha \cdot u$ and parallel to the normal direction $u_n = n \cdot u$. This was first done in [18] for a 5-dimensional model with a spacelike extra dimension and pure geodesic motion, then generalized to accelerated trajectories and an extra dimension of arbitrary signature in [15], and further adapted to arbitrary dimension and refined notation in [19]. Here, we will merely adopt the final results, which are

$$u^\alpha \nabla_\alpha u^\beta = \varepsilon u_n [u_n \partial^\beta \ln \Phi - 2 K^{\alpha\beta} u_\alpha - \Phi^{-1}(\partial_\ell - \pounds_N) u^\beta]$$
$$+ \mathcal{F}^\beta, \tag{21a}$$

$$\dot{u}_n = K_{\alpha\beta} u^\alpha u^\beta - u_n u^\alpha \partial_\alpha \ln \Phi + \mathcal{F}_n, \tag{21b}$$

$$\kappa = h_{\alpha\beta} u^\alpha u^\beta + \varepsilon u_n^2, \tag{21c}$$

where we have defined $\mathcal{F}^\alpha \equiv e^\alpha \cdot \mathcal{F}$, $\mathcal{F}_n \equiv n \cdot \mathcal{F}$, and an overdot indicates $d/d\lambda$. We can express both u^α and u_n in terms of the foliation parameters:

$$u^\alpha = \dot{y}^\alpha + \dot{\ell} N^\alpha, \quad u_n = \varepsilon \Phi \dot{\ell}. \tag{22}$$

This form of the equations of motion has the virtue of being written entirely in terms of tensorial quantities on Σ_ℓ, which makes it invariant under n-dimensional coordinate transformations. Also note that one of the equations (21a)–(21c) is redundant; for example, if one contracts Eq. (21a) with u_β and makes use of Eq. (21c), one can recover Eq. (21b).

Now, if a test particle is confined to a given Σ_ℓ hypersurface, its ℓ coordinate must obviously be constant. This implies $u_n \equiv 0$, which by Eq. (21b) yields

$$0 = K_{\alpha\beta} u^\alpha u^\beta + \mathcal{F}_n. \tag{23}$$

In other words, if the normal force per unit mass equals $-K_{\alpha\beta} u^\alpha u^\beta$, then the particle can be hypersurface-confined. Since this quantity is quadratic in the particle's n-velocity, we can identify it as the generalized centripetal acceleration in curved space. Indeed, in [18] it was shown that when a particle is confined to the world tube of a circle $\mathbb{R} \times S$ embedded in 3-dimensional Minkowski space, $\mathcal{F}_n$ reduces to the familiar v^2/r from elementary mechanics.

Now, if one member Σ_0 of the Σ_ℓ foliation satisfies $K_{\alpha\beta} = 0$, then it is obvious that no external centripetal force $\mathcal{F}^A$ is required to ensure confinement. Indeed, when the extrinsic curvature vanishes one solution of the freely-falling equations of motion is

$$\dot{y}^\alpha \nabla_\alpha \dot{y}^\beta = 0, \quad \ell = 0, \tag{24}$$

i.e., the geodesics of Σ_0 are also geodesics of M. As mentioned in Sec. I, surfaces with this property are termed totally geodesic and they represent equilibrium surfaces for freely-falling test particles. We want to know how to tell if such surfaces represent stable or unstable equilibria.

To answer this, we can attempt to linearize the equations of motion about Σ_0; that is, we consider the motion of a test particle very close to the equilibrium hypersurface. To simplify matters, we will adopt the canonical gauge (19) discussed above. Then, it is straightforward to derive expressions for $\partial_\ell h_{\alpha\beta}$ and $\partial_\ell K_{\alpha\beta}$ [19]:

$$\partial_\ell h_{\alpha\beta} = 2 K_{\alpha\beta}, \tag{25a}$$

$$\partial_\ell K_{\alpha\beta} = K_\alpha{}^\mu K_{\mu\beta} - E_{\alpha\beta}, \tag{25b}$$

where $E_{\alpha\beta} \equiv e_\alpha^A e_\beta^B n^C n^D \hat{R}_{ACBD}$. This n-tensor can be related back to more familiar quantities by suitable manipulation of the Gauss-Codazzi relations:

$$\hat{R}_{AB} e_\alpha^A e_\beta^B = R_{\alpha\beta} + \varepsilon [E_{\alpha\beta} + K_\alpha{}^\mu (K_{\beta\mu} - K h_{\beta\mu})]. \tag{26}$$

Now, without loss of generality we can suppose that Σ_0 corresponds to $\ell = 0$; from this, it follows that $|\ell|$ is "small" in our approximation.[3] If we let ${}^0 h_{\alpha\beta}$ denote the induced metric on Σ_0, ${}^0 R_{\alpha\beta}$ the Ricci tensor on Σ_0, etc.; then we have

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[3]Technically, we require that ℓ be small compared to the characteristic size of the components of the curvature tensors on Σ_0 and M in order to justify dropping the $\mathcal{O}(\ell^2)$ terms in Eq. (27). In practical terms, this means ℓ should be much less than the radii of curvature of both manifolds.

$$h_{\alpha\beta} = {}^0 h_{\alpha\beta} + \mathcal{O}(\ell^2), \tag{27a}$$

$$K_{\alpha\beta} = \varepsilon ({}^0 R_{\alpha\beta} - {}^0 \hat{R}_{AB} e_\alpha^A e_\beta^B) \ell + \mathcal{O}(\ell^2). \tag{27b}$$

Furthermore, we suppose that the n-velocity of our particle to be approximately tangent to a geodesic on Σ_0:

$$u^\alpha = U^\alpha + \delta u^\alpha, \quad U^\alpha \nabla_\alpha U^\beta = 0. \tag{28}$$

Here, δu^α is considered to be a small quantity; that is, we are really considering perturbations of the confined trajectory tangent to $U^A = U^\alpha e_\alpha^A$. (We will comment on the validity of this assumption below.) Then, to lowest order in small quantities, Eq. (21b) reduces to

$$\ddot{\ell} = ({}^0 R_{\alpha\beta} - {}^0 \hat{R}_{AB} e_\alpha^A e_\beta^B) U^\alpha U^\beta \ell + \cdots. \tag{29}$$

If Σ_0 is a stable equilibrium for test particles, we require that $\ddot{\ell}/\ell < 0$. This condition translates into the following condition for the confinement of test particles on Σ_0:

$$({}^0 R_{\alpha\beta} - {}^0 \hat{R}_{AB} e_\alpha^A e_\beta^B) U^\alpha U^\beta < 0. \tag{30}$$

In order to interpret this result, we define the following quantities:

$$\rho_g^{(n+1)} \equiv {}^0 \hat{R}_{AB} e_\alpha^A e_\beta^B U^\alpha U^\beta, \tag{31a}$$

$$\rho_g^{(n)} \equiv {}^0 R_{\alpha\beta} U^\alpha U^\beta. \tag{31b}$$

Here, $\rho_g^{(n+1)}$ is our definition of the local gravitational density of higher-dimensional—or real—matter as measured by an observer freely-falling along Σ_0. It is guaranteed to be positive if the strong energy condition is satisfied in the bulk, or at least in the vicinity of the totally geodesic surface. Note that it is possible to define the gravitational density with a different normalization constant to obtain a "nicer" expression in the perfect fluid case. That is, for a $(3+1)$-dimensional perfect fluid, an observer comoving with the fluid will measure $\rho_g^{(4)} = \frac{1}{2}(\rho + 3p)$ using our definition. One might be motivated to give a different definition of $\rho_g^{(4)}$ that does not involve the $\frac{1}{2}$ prefactor, but such a choice would unnecessarily complicate the following discussion with spurious dimension-dependant terms.

Now, it is clear that $\rho_g^{(n)}$ is the lower-dimensional cousin of $\rho_g^{(n+1)}$, but the precise interpretation is a little more subtle. Imagine an observer living on Σ_0 that is entirely ignorant of the ℓ direction. Assuming that this observer can measure the local n-geometry and believes in the Einstein field equations, he will interpret the Einstein n-tensor of Σ_0 as being proportional to some effective stress-energy tensor. In other words, he will conclude that his local n-geometry is determined by an effective distribution of matter energy. From this, it follows that $\rho_g^{(n)}$ is the gravitational density of the effective n-dimensional matter.

A natural question is: how is the lower-dimensional effective matter related to the "real" higher-dimensional distribu-

tion? It is not hard to derive the following expression for the stress-energy tensor on Σ_0 from the Gauss-Codazzi relations:

$$G^{\alpha\beta} = -\varepsilon E^{\alpha\beta} + \hat{G}^{AB} e_A^\alpha e_B^\beta - \left[\hat{G}_{AB} h^{AB} - \left(\frac{n-2}{n-1} \right) \mathrm{Tr}(\hat{G}) \right] h^{\alpha\beta},$$
(32)

where all quantities are understood to be evaluated on Σ_0 and we have made note of $K_{\alpha\beta} = 0$. The last two terms on the right are explicitly related to the higher-dimensional stress-energy tensor, but the first can be rewritten using

$$E_{\alpha\beta} = e_\alpha^A e_\beta^B n^C n^D \hat{C}_{ACBD} + \frac{1}{n(n-1)} \varepsilon \hat{R} h_{\alpha\beta}$$

$$+ \frac{1}{n-1} (\hat{R}_{AB} n^A n^B h_{\alpha\beta} + \varepsilon \hat{R}_{AB} e_\alpha^A e_\beta^B). \quad (33)$$

The first term on the right shows how $E_{\alpha\beta}$ is in part determined by the Weyl $(n+1)$-tensor $\hat{C}_{ABCD}$ on M. This curvature tensor is related to the geometrical or gravitational degrees of freedom of the bulk, which are not directly fixed by the $(n+1)$-dimensional field equations. This implies that the Einstein n-tensor $G_{\alpha\beta}$ on M is not entirely determined by the distribution of higher-dimensional stress-energy—there is a purely geometric contribution from the appropriate projection of $\hat{C}_{ABCD}$. We call this contribution the *induced* [23] or *shadow* [24] matter stress-energy tensor because it represents a source of the lower-dimensional Einstein equation that cannot be unambiguously attributed to any "real" higher-dimensional fields. It follows that the n-dimensional effective gravitational density contains contributions from both real and induced matter.

The final point we wish to discuss in this section has to do with the validity of our approximations. Recall that above, in order to derive Eq. (29), we assumed that δu^α was a small quantity. We now want to describe under which circumstances this hypothesis holds by substituting the expansion $u^\alpha = U^\alpha + \delta u^\alpha$ in Eq. (21b) in the canonical gauge with $\mathcal{F} = 0$. For this purpose it is useful to assume that $U^\alpha = U^\alpha(y(\lambda))$, or equivalently $\partial_\ell U^\alpha = 0$. Coupled with Eq. (27a), this implies that $U^\alpha \nabla_\alpha U^\beta = \mathcal{O}(\ell^2)$. Under such circumstances, we obtain

$$\frac{d}{d\lambda} \delta u^\alpha = {}^0 E^{\alpha\beta} (U_\beta - \delta u_\beta) \frac{d}{d\lambda} \ell^2 - \Gamma_{\beta\gamma}^\alpha \delta u^\beta \delta u^\gamma - \delta u^\beta \nabla_\beta U^\alpha,$$
(34)

where we have made use of $K_{\alpha\beta} \approx - {}^0 E_{\alpha\beta} \times \ell$. It is clear that in order to have δu^α be consistently "small," we must have that the only term on the right inhomogeneous in δu^α is negligible. Now, since ${}^0 E_{\alpha\beta} = \varepsilon ({}^0 \hat{R}_{ABC} e_\alpha^A e_\beta^B - {}^0 R_{\alpha\beta})$ we see that its components are of the order of the inverse squares of the curvature lengths of M and Σ_0. Then, for our approximations to be valid, we need that $d(\ell^2)/d\lambda$ be small compared to the characteristic curvature of M or Σ_0, whichever is smaller. This is a sensible intuitive bound—if either the higher- or lower-dimensional manifolds are highly curved we expect that true confinement will be difficult to achieve. So,

in addition to demanding that ℓ is small in order to justify $K_{\alpha\beta} \approx - {}^0 E_{\alpha\beta} \times \ell$, we also need the extra dimensional velocity to be relatively tiny.

To summarize, we have seen that any n-surface Σ_0 embedded in M is an equilibrium position for freely-falling test particles if it has vanishing extrinsic curvature. Every geodesic on Σ_0 is automatically a geodesic of M, hence the hypersurface is called totally geodesic. Furthermore, if a given trajectory confined to Σ_0 is perturbed off of the equilibrium surface, the acceleration of the test particle is towards $\ell = 0$ provided that

(i) the gravitational density of the higher-dimensional matter is greater than the gravitational density of the effective lower-dimensional matter on Σ_0, as measured by an observer travelling on the unperturbed trajectory; and,

(ii) both ℓ and $d(\ell^2)/d\lambda$ are small compared with the characteristic curvature scales of M and Σ_0.

Notice that these comments apply to a particular trajectory only.

IV. TWO ALTERNATIVE DERIVATIONS OF THE STABILITY CONDITION

The stability condition for freely-falling test particle trajectories derived above depended on our decomposition of the higher-dimensional equation of motion (21) and the canonical gauge for the foliation parameters (19). It is possible to derive the same condition using two different methods that relax one or both of these assumptions, which is what we do in this section.

A. From the geodesic deviation equation

Consider a freely-falling test particle on Σ_0 that has an $(n+1)$-velocity $U^A = e_\alpha^A U^\alpha$ at a particular instant of time. As before, we assume that Σ_0 is totally geodesic, which means that the test particle will remain confined on the submanifold for all future times in the absence of non-gravitational influences. Now, consider an additional test particle that is separated from the first object by a vector $\xi^A = \ell n^A$. Here, ℓ is the proper distance separating the two particles and is assumed to be small. Then, the equation of geodesic deviation says that

$$a^A = -R^A{}_{BCD} U^B \xi^C U^D, \quad (35)$$

where a^A is the acceleration of ξ^A, defined as

$$a^A = (U \cdot \nabla)^2 \xi^A. \quad (36)$$

Now, consider $n \cdot a$. Making use of $\xi^A = \ell n^A$, we find the following expression for this scalar product:

$$n \cdot a = \varepsilon \ddot{\ell} - \ell h^{AB} (U^C \nabla_C n_A)(U^D \nabla_D n_B). \quad (37)$$

Here, we have used $g_{AB} = h_{AB} + \varepsilon n_A n_B$, $n^A U^B \nabla_B n_A = 0$, and $\ddot{\ell} = (U \cdot \nabla)^2 \ell$. Because U^A is parallel to Σ_0, the second term on the right reduces to $- \ell K_{\alpha\beta} K^{\alpha\gamma} U^\beta U_\gamma = 0$. Hence, we have

$$\ddot{\ell} = -\varepsilon R_{ABCD} n^A U^B n^C U^D \ell = -\varepsilon E_{\alpha\beta} U^\alpha U^\beta \ell. \quad (38)$$

If we now use Eq. (26) to substitute for $E_{\alpha\beta}$, we immediately recover our previous result (29). Hence, the test particle located just off Σ_0 will be accelerated towards $\ell = 0$ if the stability condition from the previous section $\rho_g^{(n+1)} > \rho_g^{(n)}$ holds. Notice that we did not assume the canonical gauge for this derivation.

B. From the Raychaudhuri equation

To establish the stability condition from the Raychaudhuri equation, we again consider a freely-falling test particle on Σ_0 with a trajectory γ tangent to $U^A = e_\alpha^A U^\alpha$. Consider some small $(n-1)$-dimensional region $\delta V_{n-1} \subset \Sigma_0$ orthogonal to γ at some given time λ_0. We extend δV_{n-1} a small distance of ℓ on either side of Σ_0 to define an n-dimensional region δV_n. Since U^A is tangent to Σ_0, our test particle's trajectory is also orthogonal to δV_n. Now consider a geodesic congruence centered about γ and threading every point within δV_n. At the moment of interest, we can take each member of the $(n+1)$-congruence to be parallel to γ. This means that the subset of the total congruence situated on Σ_0 is an n-dimensional geodesic congruence on the submanifold. To evolve the orthogonal regions forward in time, we imagine that each point in δV_n is glued to the associated geodesic, so the small region deforms in the same manner as the congruence. Since the congruence is instantaneously parallel at λ_0, we have that $\dot{\ell}(\lambda_0) = 0$.

Now in the canonical gauge, the volume of δV_n is related to the volume of δV_{n-1} by

$$\text{vol } \delta V_n = 2\ell \text{ vol } \delta V_{n-1}. \quad (39)$$

We can define expansion scalars for both the higher- and lower-dimensional congruences:

$$\theta_n = \frac{d}{d\lambda} \ln(\text{vol } \delta V_n) = \nabla_A U^A, \quad (40a)$$

$$\theta_{n-1} = \frac{d}{d\lambda} \ln(\text{vol } \delta V_{n-1}) = \nabla_\alpha U^\alpha. \quad (40b)$$

It is easy to see that at $\lambda = \lambda_0$ we have

$$\dot{\theta}_n = \ddot{\ell}/\ell + \dot{\theta}_{n-1}, \quad (41)$$

where we have made use of $\dot{\ell}(\lambda_0) = 0$. The Raychaudhuri equation applied to each congruence gives

$$\dot{\theta}_n = -\nabla^A U^B \nabla_B U_A - {}^0\hat{R}_{AB} U^A U^B, \quad (42a)$$

$$\dot{\theta}_{n-1} = -\nabla^\alpha U^\beta \nabla_\beta U_\alpha - {}^0 R_{\alpha\beta} U^\alpha U^\beta. \quad (42b)$$

A reasonably quick calculation reveals

$$\nabla^A U^B \nabla_B U_A = (h^{AC} + \varepsilon n^A n^C)(h^{BD} + \varepsilon n^B n^D) \nabla_C U_D \nabla_B U_A$$

$$= \nabla^\alpha U^\beta \nabla_\beta U_\alpha - 2\varepsilon K_{\alpha\beta} U^\beta e_A^\alpha n^B \nabla_B U^A$$

$$+ (U^\alpha \partial_\alpha \ln \Phi)^2$$

$$= \nabla^\alpha U^\beta \nabla_\beta U_\alpha. \quad (43)$$

In going from the third to fourth line we have made use of the fact that Σ_0 is totally geodesic ($K_{\alpha\beta} = 0$) and our gauge choice ($\Phi = 1$). Putting it all together, we get

$$\ddot{\ell} = \ell [\rho_g^{(n)} - \rho_g^{(n+1)}]. \quad (44)$$

The geodesics of the congruence will accelerate towards the equilibrium hypersurface if the quantity in the square brackets is negative. This yields the same stability condition as before: $\rho_g^{(n)} < \rho_g^{(n+1)}$.

V. EXAMPLES

We have now established the stability condition $\rho_g^{(n)} < \rho_g^{(n+1)}$ using three different methods in general geometric backgrounds. In this section, we give a few examples of specific manifolds containing totally geodesic hypersurfaces. For each case, we explicitly confirm that our stability condition for confined trajectories is correct.

A. Warped-product spaces

For our first example of test particle confinement, we consider the so-called warped product metric ansatz:

$$ds_{(M)}^2 = e^\Omega q_{\alpha\beta} dy^\alpha dy^\beta + \varepsilon d\ell^2, \quad (45)$$

where the warp factor $\Omega = \Omega(\ell)$ is independent of the n-dimensional y coordinates and the warp metric $q_{\alpha\beta} = q_{\alpha\beta}(y)$ is independent of the extra dimensional ℓ. The induced metric on Σ_ℓ hypersurfaces is $h_{\alpha\beta} = e^\Omega q_{\alpha\beta}$. We will assume that the bulk is an Einstein space satisfying

$$\hat{G}_{AB} = -\Lambda g_{AB}, \quad \hat{R}_{AB} = \frac{2\Lambda}{n-1} g_{AB}. \quad (46)$$

Then, solutions for the warp factor and warp metric are easily found [19]:

$$e^{\Omega/2} = A \begin{cases} \cos \omega \ell, & \varepsilon\Lambda > 0, \\ \cosh \omega \ell, & \varepsilon\Lambda < 0, \end{cases} \quad (47a)$$

$$R_{\alpha\beta} = \frac{2\Lambda A^2}{n} q_{\alpha\beta}, \quad (47b)$$

$$\omega^2 \equiv \frac{2|\Lambda|}{n(n-1)}, \quad (47c)$$

where A is a constant and $R_{\alpha\beta}$ is the Ricci tensor formed from either the induced or warp metrics (both n-tensors give the same result). Essentially, the above states that the warp metric can be taken to be *any* n-dimensional solution of the Einstein field equations sourced by a cosmological constant $\Lambda_n = A^2 \Lambda (n-2)/n$.

Now, for both of the cases shown in Eq. (47a), it is clear that the $\ell=0$ hypersurface Σ_0 is totally geodesic. If we consider some timelike geodesic confined to Σ_0, the higher- and lower-dimensional gravitational densities measured by an observer travelling along that geodesic are

$$\rho_g^{(n+1)} = -\frac{2\Lambda}{n-1}, \quad \rho_g^{(n)} = -\frac{2\Lambda}{n}, \qquad (48)$$

where we have used

$$-1 = U \cdot U = h_{\alpha\beta} U^\alpha U^\beta = A^2 q_{\alpha\beta} U^\alpha U^\beta. \qquad (49)$$

So, the condition that trajectories on Σ_0 be stable against perturbations in this case simply reads

$$\Lambda < 0, \qquad (50)$$

i.e., the bulk must have anti–de Sitter characteristics.[4]

This scenario is simple enough to verify our conclusion directly from the equations of motion. The particle Lagrangian can be taken as $L = \frac{1}{2}\dot{x} \cdot \dot{x}$, which leads directly to the following equation of motion:

$$\ddot{\ell} = \frac{\varepsilon}{2} \frac{d\Omega}{d\ell} h_{\alpha\beta} u^\alpha u^\beta. \qquad (51)$$

Now, assuming that our particle is very close to $\ell=0$, we can expand $d\Omega/d\ell$ to first order in ℓ and approximate $h_{\alpha\beta}u^\alpha u^\beta \sim -1$. In both of the relevant cases $\varepsilon\Lambda \lesssim 0$, we obtain

$$\ddot{\ell} = \frac{2\Lambda}{n(n-1)}\ell + \mathcal{O}(\ell^2) = (\mathrm{sgn}\,\Lambda)\omega^2\ell + \mathcal{O}(\ell^2). \quad (52)$$

The obvious stability condition from this expression is $\Lambda < 0$, which matches the above result precisely. Perhaps not surprisingly, the frequency of oscillation about $\ell=0$ is the same frequency found in the warp factor.

We finish by noting that this metric ansatz easily lends itself to toy models of spherically-symmetric astrophysical situations. For example, suppose that we believed that there was a—suitably tiny—cosmological constant Λ_4 permeating the immediate vicinity of the Sun. We could construct a 5-dimensional description by taking $q_{\alpha\beta}$ to be the 4-metric around a Λ_4-black hole:

$$q_{\alpha\beta}dy^\alpha dy^\beta = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2, \qquad (53a)$$

$$f(r) = 1 - \frac{2M}{r} - \frac{1}{3}\Lambda_4 r^2, \qquad (53b)$$

where $d\Omega_2^2$ is the metric on the unit 2-sphere. Using Eq. (47b), we see that Λ_4 is related to the 5-dimensional cosmological constant by

$$\Lambda_4 = \frac{1}{3} A^2 \Lambda. \qquad (54)$$

Interestingly, Λ_4 has the same sign as Λ. So in this scenario, test particle like comets, asteroids and planets will have a stable equilibrium about $\ell=0$ if $\Lambda_4 < 0$; that is, the vacuum energy has negative density. If Λ_4 is positive, as suggested by recent cosmological observations, the Σ_0 4-manifold will be gravitationally repulsive to test particles. Hence, if a particle at $\ell=0$ were to acquire a small extra dimensional velocity—perhaps by emitting gravitational radiation into the bulk—there would be no guarantee that it would return to our "native" spacetime.

B. The Liu-Mashhoon-Wesson metric

In [15], test particle trajectories in the following 5-geometry were considered:

$$ds_{(M)}^2 = -b^2(t,\ell)dt^2 + a^2(t,\ell)d\sigma_k^2 + d\ell^2, \qquad (55a)$$

$$a^2(t,\ell) \equiv (t^2+k)\ell^2 + \frac{\mathcal{K}}{t^2+k}, \qquad (55b)$$

$$b(t,\ell) \equiv \frac{[(t^2+k)^2\ell^2 - \mathcal{K}]}{(t^2+k)^{3/2}[(t^2+k)^2\ell^2 + \mathcal{K}]^{1/2}}, \qquad (55c)$$

$$d\sigma_k^2 \equiv \frac{dr^2}{1-kr^2} + r^2 d\Omega_2^2. \qquad (55d)$$

Here, $\mathcal{K}$ is an integration constant and $k=0,\pm 1$. This line element is a special case of one originally discovered by Liu and Mashhoon [20] and subsequently rediscovered by Liu and Wesson [21], and is a solution of the 5-dimensional vacuum field equations $\hat{R}_{AB}=0$. It is interesting because the line element on each of the Σ_ℓ hypersurfaces is of the cosmological Robertson-Walker form. However, this 5-geometry has recently been shown to be isometric the 5-dimensional topological Schwarzschild solution [19,25].

It is easy to confirm that $\ell=0$ is a totally geodesic 4-surface in the geometry (55) with line element:

$$ds_{(\Sigma_0)}^2 = \frac{\mathcal{K}}{t^2+k}\left[-\frac{dt^2}{(t^2+k)^2} + d\sigma_k^2\right], \qquad (56)$$

which can be shown to be isometric to a radiation-dominated cosmology. Note that in order to have a sensible solution, we need to ensure that

$$\frac{\mathcal{K}}{t^2+k} > 0 \qquad (57)$$

by judiciously choosing k and restricting the range of t. Now, the tangent vector to a comoving geodesic path on Σ_0 is

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[4]It is a misnomer to call the bulk anti–de Sitter space in this case, it merely satisfies the same Einstein equations as anti–de Sitter space.

$$U^\alpha \partial_\alpha = (t^2 + k) \sqrt{\frac{t^2 + k}{\mathcal{K}}} \partial_t. \tag{58}$$

The gravitational densities measured by such an observer are easily calculated from the basic definitions (31):

$$\rho_g^{(5)} = 0, \quad \rho_g^{(4)} = \frac{3(t^2 + k)^2}{\mathcal{K}}. \tag{59}$$

Since the bulk is devoid of matter in this case, the stability of the comoving trajectory on Σ_0 demands $\rho_g^{(4)} < 0$. In other words, the $\ell = 0$ hypersurface will be gravitationally attractive only if the strong energy condition is *violated* on Σ_0; i.e., if $\mathcal{K} < 0$. Notice that in order to have a negative value of the integration constant $\mathcal{K}$, the inequality (57) implies that $k = -1$ and we restrict $t \in (-1, 1)$.

This conclusion is odd enough to warrant direct verification from the higher-dimensional geodesic equation. The 5-dimensional Lagrangian for comoving trajectories is

$$L = \frac{1}{2} [-b^2(t, \ell) \dot{t}^2 + \dot{\ell}^2]. \tag{60}$$

We can obtain an equation for $\ddot{\ell}$ by extremizing the action, which yields

$$\ddot{\ell} = -\frac{1}{2} \dot{t}^2 \frac{\partial}{\partial \ell} b^2(t, \ell) = \left(\frac{3\dot{t}^2}{t^2 + k} \right) \ell + O(\ell^3). \tag{61}$$

We can use the solution for U^α above to approximate $\dot{t}^2 \approx (t^2 + k)^3 / \mathcal{K}$ and write

$$\ddot{\ell} = \frac{3(t^2 + k)^2}{\mathcal{K}} \ell = \rho_g^{(4)} \ell. \tag{62}$$

Here, we have omitted the higher-order terms from the equation of motion. This is what is expected from Eq. (29), and confirms to us that the comoving trajectory with $\ell = 0$ is stable only if $\rho_g^{(4)} < 0$.

VI. SUMMARY AND DISCUSSION: AN ENERGY CONDITION FOR HIGHER DIMENSIONS

In this paper, we have shown that confined particle trajectories on totally geodesic n-surfaces embedded in $(n + 1)$-dimensional bulk manifolds are stable against small perturbations if $\rho_g^{(n+1)} > \rho_g^{(n)}$. Here, $\rho_g^{(n+1)}$ and $\rho_g^{(n)}$ are the gravitational densities of M and Σ_0 as measured by observers travelling on the confined trajectories, respectively. We established this result using a covariant decomposition of the higher-dimensional equation of motion in Sec. III, the equation of geodesic deviation in Sec. IV A, and the Raychaudhuri equation in Sec. IV B. In Sec. V, we gave several concrete examples of our results involving warped product and Liu-Mashhoon-Wesson metrics.

We conclude by noting that the stability condition as formulated above is only applicable to particular geodesic paths on Σ_0. For some applications, one might want to ensure that *all* the timelike trajectories through some region of Σ_0 are stable against perturbations. It is not difficult to see how to generalize our previous results to satisfy this stronger demand. Consider the following definition:

The confinement energy condition. Let Σ_0 be an n-dimensional totally geodesic Lorentzian submanifold smoothly embedded in $(n + 1)$-dimensional bulk M. Also, let $^0 R_{\alpha\beta}$ be the Ricci tensor on Σ_0 and $^0 \hat{R}_{AB}$ be the Ricci tensor on M, both evaluated at a point $P \in \Sigma_0 \subset M$. The confinement energy condition at P is

$$(^0 \hat{R}_{AB} e_\alpha^A e_\beta^B - {}^0 R_{\alpha\beta}) U^\alpha U^\beta > 0, \tag{63}$$

where U^α is an arbitrary timelike vector tangent to Σ_0. If the confinement energy condition holds in some neighborhood $N[P] \subset \Sigma_0$ of P, then any test particle travelling along a timelike trajectory on Σ_0 passing through P will be stable against small perturbations while in $N[P]$.

There are obviously significant similarities between this and the familiar strong energy condition from 4-dimensional relativity, and we note that it can be used to place conditions on the densities and principle pressures associated with the Einstein tensors of M and Σ_0. It is clear that for the examples of Sec. V, the confinement energy condition is satisfied in the warped-product Einstein space situation if $\Lambda < 0$, and in the Liu-Mashhoon-Wesson metric if the 4-dimensional strong energy condition is false on Σ_0. We have no doubt that it would be interesting to apply this condition to other higher-dimensional situations with totally geodesic submanifolds, but this is the subject for future work.

[1] N. Arkani-Hamed, S. Dimopoulos, and G.R. Dvali, Phys. Lett. B **429**, 263 (1998).
[2] N. Arkani-Hamed, S. Dimopoulos, and G.R. Dvali, Phys. Rev. D **59**, 086004 (1999).
[3] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999).
[4] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999).
[5] J. Khoury, B.A. Ovrut, P.J. Steinhardt, and N. Turok, Phys. Rev. D **64**, 123522 (2001).
[6] M. Bucher, Phys. Lett. B **530**, 1 (2002).
[7] D.W. Joseph, Phys. Rev. **126**, 319 (1962).
[8] K. Akama, Lect. Notes Phys. **176**, 267 (1982).
[9] V.A. Rubakov and M.E. Shaposhnikov, Phys. Lett. **125B**, 136 (1983).
[10] M. Visser, Phys. Lett. **159B**, 22 (1985).
[11] G.W. Gibbons and D.L. Wiltshire, Nucl. Phys. **B287**, 717 (1987).
[12] P.S. Wesson and J. Ponce de Leon, J. Math. Phys. **33**, 3883 (1992).

[13] P. Horava and E. Witten, Nucl. Phys. **B460**, 506 (1996).

[14] P. Horava and E. Witten, Nucl. Phys. **B475**, 94 (1996).

[15] S.S. Seahra and P.S. Wesson, Class. Quantum Grav. **20**, 1321 (2003).

[16] O. DeWolfe, D.Z. Freedman, S.S. Gubser, and A. Karch, Phys. Rev. D **62**, 046008 (2000).

[17] C. Csaki, J. Erlich, T.J. Hollowood, and Y. Shirman, Nucl. Phys. **B581**, 309 (2000).

[18] S.S. Seahra, Phys. Rev. D **65**, 124004 (2002).

[19] S.S. Seahra, Ph.D. thesis, University of Waterloo, 2003.

[20] H. Liu and B. Mashhoon, Ann. Phys. (Leipzig) **4**, 565 (1995).

[21] H. Liu and P.S. Wesson, Astrophys. J. **562**, 1 (2001).

[22] B. Mashhoon, H. Liu, and P. Wesson, Phys. Lett. B **331**, 305 (1994).

[23] P.S. Wesson, *Space-Time-Matter* (World Scientific, Singapore, 1999).

[24] V. Frolov, M. Snajdr, and D. Stojkovic, Phys. Rev. D **68**, 044002 (2003).

[25] S.S. Seahra and P.S. Wesson, gr-qc/0309006.