Rotating black hole solution in a generalized topological 3D gravity with torsion

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A *first order* version of topological massive gravity is achieved by liberating its *translational* gauge degrees of freedom. In three dimensions, our Lagrangian consists of Chern-Simons (CS) terms for curvature and torsion inducing an effective cosmological constant dynamically, whereas a ''mixed'' CS term is substituting for the topological related Einstein-Cartan action. Anti–de Sitter and rotating black hole configurations are exact vacuum solutions. They also apply to a large class of Yang-Mills-type generalizations including ''exotic'' terms exclusively permitted in 3D. The reason for this can be partially traced back to a new strong/weak *duality* of the translational and rotational dynamical degrees of freedom.

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I. INTRODUCTION

Since the first three-dimensional model of Staruszkiewicz [1], the 3D topological model of gravity of Deser, Jackiw, and Templeton (DJT) [2–5] has aroused quite a lot of theoretical interest. In order to get a nontrivial vacuum theory, DJT added the Chern-Simons (CS) term for the Riemannian curvature to the 3D Hilbert-Einstein Lagrangian, regarded as the high-temperature limit (for Euclidean signature) or dimensional reduction of the four-dimensional Hilbert-Einstein Lagrangian. The topological term is supposed to come from the θ vacuum of four-dimensional physics. The nice, intrinsic feature of the DJT model is that the CS term induces a mass for the ''graviton'' without breaking infinitesimal gauge invariance or invoking the Higgs mechanism. The discovery of anti–de Sitter (AdS) [6] and black hole solutions [7] has added further attraction to 3D gravitational models as a ''laboratory'' to study geometric, dynamical, and statistical properties [8].

From a gauge theoretical point of view, however, it appears much more natural to formulate such a dimensionally reduced gravitational theory in a Riemann-Cartan (RC) space (time) with torsion [9] and thereby to go over to what is conventionally called a *first order formalism*. This has successfully been applied to simple supergravity, cf. [10,11], where torsion enters as an auxiliary field simplifying the local supersymmetric transformations [12]. In order to take proper care of the translational aspect, we construct a Poincaré gauge model in 3D, cf. Refs. [13,6,14,9].

Since the Einstein-Cartan (EC) action is nondynamical in vacuum, as reiterated in Sec. II, we construct in Sec. III a topological model based on the translational [15], rotational, and a new ''mixed'' Chern-Simons term. It is an intrinsic feature of our model that a cosmological term is *induced dynamically*. The vacuum field equations are similar to those of Ref. [13] inasmuch as they provide the constrictions of *constant* axial torsion and RC curvature. These turn out to be equivalent to a covariant Klein-Gordon equation together with the de Donder condition for the triads. As shown in Sec. III A, the three propagating vacuum modes are massive excitations with *quadratic* mass roots, in contradistinction to the problematic cubic eigenvalue equation of the DJT model with explicit mass term [16]. The known anti–de Sitter and rotating black hole solution of the 3D Einstein equations with effective cosmological constant are ''prolongated'' in Sec. IV to our purely topological model simply by deforming the Riemannian connection to the one with constant axial torsion.

To what extent can these exact solutions ''probe'' a more general dynamics? In Sec. V, an attempt is made to solve the general 3D Poincaré field equations by a *duality rotation* of the three rotational field momenta to those corresponding to the three translations. Although a complete integrability [17] as in 2D cannot be expected here, in RC space (times) with constant axial torsion the field momenta exhibit an intriguing S (strong/weak) duality with respect to the contortional coupling constant. Encouraged by this new finding, specific Yang-Mills-type extensions of our topological model are considered in Sec. VI. It is a new and special feature of 3D that a cubic term in the torsion and a mixed torsion/curvature term are permitted as rather *exotic* three-forms. Nevertheless, the previous prolongation of the AdS and black hole configurations to the solutions with constant axial torsion apply also to the Yang-Mills case, provided the contortional constant satisfies a quartic algebraic equation. Some generalizations and problems are briefly discussed in Sec. VII.

Our notation and the ''burden'' of useful geometrical identities are collected in the first three appendices, whereas in Appendix D the DJT model is *recovered* from our first order topological model by constraining, via a Lagrange multiplier, the torsion to vanish.

II. NONDYNAMICAL EINSTEIN THEORY IN 3D

The celebrated Poincaré group $P(n,\mathbf{R}) := R^n \otimes \mathrm{SO}(1,n-1)$ of particle physics in n dimensions is the semidirect product of the *translation* group R^n and the *Lorentz group* $\mathrm{SO}(1,n-1)$. It is particular for 3D, i.e., $n=3$, that the Poincaré group consists of three translation and three rotation generators, which invites some dynamical intertwining of the basic variables, the anholonomic coframe ϑ^α, and

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the connection one-form $\Gamma^{\alpha\beta}$, or "spin" connection $\Gamma^{\star}_{\alpha} := \frac{1}{2}\eta_{\alpha\beta\gamma}\Gamma^{\beta\gamma}$, when written in terms of the Lie dual, cf. Appendices A and B for details.

As is well known, in 3D the Einstein-Cartan Lagrangian three-form

$$V_{\rm EC} = -\frac{1}{2\ell}R^{\alpha\beta}\wedge\eta_{\alpha\beta} = -\frac{1}{\ell}\vartheta^{\alpha}\wedge R^{\star}_{\alpha} = -\frac{(-1)^s}{\ell}\eta^{\alpha}\wedge {}^{*}R^{\star}_{\alpha}, \tag{2.1}$$

where ℓ denotes a fundamental length and $R^{\star}_{\alpha}$ the Lie dual (B4) of the Riemann-Cartan curvature, does not provide us with a model with dynamical degrees of freedom. Its variation with respect to the coframe ϑ^{α} and the connection $\Gamma^{\star}_{\alpha}$ yields the field equations

$$R^{\star}_{\alpha} = \ell\,\Sigma_{\alpha}, \qquad T_{\alpha} = (-1)^s 2\ell\,\tau^{\star}_{\alpha}, \tag{2.2}$$

where $\Sigma_{\alpha} := \delta L_{\rm mat}/\delta\vartheta^{\alpha}$ and $\tau^{\star}_{\alpha} := \frac{1}{2}\eta_{\alpha\beta\gamma}\tau^{\beta\gamma} = ((-1)^s/2)\,\delta L_{\rm mat}/\delta\Gamma^{\star\alpha}$ are the matter current two-forms of energy-momentum and spin, respectively. Thus in *vacuum*, the RC curvature and the torsion $T^{\alpha} := D\vartheta^{\alpha}$ are zero and Eq. (2.2) becomes trivial.

III. CHERN-SIMONS GRAVITY IN 3D

In 3D the gravitational Lagrangian has to be a three-form. Although the translational, (Lorentz-) rotational, and the mixed CS three-forms of our Appendix C are not all gauge invariant, they are viable candidates. Allowing for arbitrary "vacuum angles" $\theta_{\rm T}$, $\theta_{\rm L}$, and $\theta_{\rm TL}$, the most general purely *topological* Lagrangian in 3D then takes the form

$$V_{\infty} = \theta_{\rm T}C_{\rm T} + \theta_{\rm L}C_{\rm L} + \theta_{\rm TL}C_{\rm TL}. \tag{3.1}$$

This topological 3D model partially depends on the *translational* CS term $C_{\rm T}$, cf. [15], which is of decisive importance in order not to fall back to a trivial model. The new mixed topological term $C_{\rm TL}$ is simulating to some extent Einstein's theory in three dimensions. Following Witten [18], this can be traced back to the partially topological nature of the EC Lagrangian in 3D:

$$\vartheta^{\alpha}\wedge R^{\star}_{\alpha} = d(\Gamma^{\star}_{\alpha}\wedge\vartheta^{\alpha}) + \ell\,C_{\rm TL} \tag{3.2}$$

Varying Eq. (3.1) with respect to ϑ^{α} and $\Gamma^{\star\alpha}$ and employing the results of Appendix C yield the topological field equations

$$-\theta_{\rm TL}R^{\star}_{\alpha} - \frac{\theta_{\rm T}}{\ell}T_{\alpha} = \ell\Sigma_{\alpha} \tag{3.3}$$

and

$$\theta_{\rm TL}T_{\alpha} - \frac{\theta_{\rm T}}{2\ell}\eta_{\alpha} - \theta_{\rm L}\ell R^{\star}_{\alpha} = \ell\,\tau^{\star}_{\alpha}. \tag{3.4}$$

These field equations are of first order, similar to those of Mielke and Baekler [13]. Here, however, we have dismissed the traditional Einstein-Hilbert or EC Lagrangian, and substituted it by the new mixed Chern-Simons-type term $C_{\rm TL}$, in order to focus first on a completely topological theory.

Let us exclude a singular case and assume that $A =: 2(\theta^2_{\rm TL} + \theta_{\rm T}\theta_{\rm L}) \neq 0$. Then, combining the vacuum field equations (3.4) and (3.3) yields for the torsion and the RC curvature the constrictions

$$T_{\alpha} = \frac{2\kappa}{\ell}\eta_{\alpha}, \quad R^{\star}_{\alpha} = \frac{\rho}{\ell^2}\eta_{\alpha}, \tag{3.5}$$

where the contortional constant is $\kappa = (\theta_{\rm TL}\theta_{\rm T})/2A$ and $\rho = -\theta^2_T/A$. Since $\eta_{\alpha} = (1/2)\eta_{\alpha\beta\gamma}\vartheta^{\beta}\wedge\vartheta^{\gamma}$, both torsion and curvature carry in 3D only *one* independent irreducible component, i.e., the *axial* torsion and the *scalar* curvature, respectively. Thus, in vacuum, we end up with a space of constant RC curvature with a curvature radius of the order of the fundamental length ℓ. Let us remark that the (Lorentz-) rotational CS is of secondary importance here, it is only needed to reproduce, after imposing the dynamical constraint $D\vartheta^{\alpha} = 0$ of vanishing torsion, the Cotton tensor of the DJT model, cf. Appendix D or Sec. 8 of Ref. [6]. Crucial, however, is the translational CS action providing also a nontrivial planar dynamics [19].

A. Massive gravitons

In order to exhibit the propagating degrees of freedom of our model, we form the gauge-covariant d'Alembertian of the coframe ϑ^{α}. In general, for a p form in n dimensions [20] this operator is given by $\Box := (-1)^{pn+s}[{}^{*}D{}^{*}D + (-1)^n D{}^{*}D{}^{*}]$. By employing the relation $\eta_{\alpha} = {}^{*}\vartheta_{\alpha}$ for the η basis, iteratively the algebraic relations (3.5) for the torsion and RC curvature, as well as the first Bianchi identity (C1), we find for the 3D coframe

$$[\Box + (-1)^s m^2]\vartheta^{\alpha} \cong 0 \tag{3.6}$$

and

$$D{}^{*}\vartheta^{\alpha} \cong 0 \tag{3.7}$$

where $m = 2\kappa/\ell = \theta_{\rm TL}\theta_{\rm T}/A\ell$ is the mass of the graviton, which is *real* for Lorentzian signature $s = 1$.

The Klein-Gordon (KG)-type equation (3.6) and the subsidiary condition (3.7) for the coframe are both *exact* consequences of our field equations. This is completely analogous to the three-dimensional topological gauge model for the spin-1 (or covector) field $A = A_i dx^i$ in which both the mass shell and the Lorentz condition $d{}^{*}A = 0$ are not imposed separately, but rather follow from the underlying gauge field equation [21]. The subsidiary condition (3.7) corresponds to the gauge-covariant Hilbert–de Donder or *transversality condition* for the coframe.

Therefore, in the count of the vacuum degrees of freedom of our topological gauge model with torsion, nowhere a linearization procedure [22] is needed: In 3D, the coframe $\vartheta^{\alpha} := e_i^{\alpha}dx^i$ has $3\times 3 = 9$ components. In view of the transformation formula $\vartheta'^{\alpha} = \Lambda_{\beta}{}^{\alpha}(x)\vartheta^{\beta}$, we have to subtract three gauge degrees of freedom due to local rotations $\Lambda_{\beta}{}^{\alpha}(x)$ or SO$(s,3-s)$-gauge transformations. The transversality condition (3.7), for vacuum, amounts to three further constraints such that three propagating vacuum modes remain. They consist of the two spin-2 degrees of freedom of a *massive* (non-ghost) *graviton* and one massive scalar mode, exactly as in the DJT model [2,3]. For $\Lambda_{\rm eff}\neq 0$, however, these modes "live," due to Eq. (4.2), on a background of *constant Riemannian* curvature, cf. [23]. Another virtue of our first order CS formulation is the quadratic mass roots in the effective KG equation (3.6) for the coframe. This is in contradistinction to the third order DJT model with explicit Pauli-Fierz mass term [16], where the cubic eigenvalue equation for the mass spectrum signals difficulties in the physical interpretation of complex roots or ghosts.

The Klein-Gordon-type reformulation of our topological model will likely facilitate its BRST quantization, cf. [24], without encountering a conformal "anomaly" in the renormalization procedure [25].

IV. PROLONGATION OF BLACK HOLE AND ANTI–de SITTER SOLUTIONS

In order to study vacuum solutions, let us consider the decomposition $\Gamma_\alpha^\star=\Gamma_\alpha^{\{\}\star}-K_\alpha^\star$ of the connection into Riemannian and contortional pieces. This implies the identity

$$R_\alpha^\star\equiv R_\alpha^{\{\}\star}-DK_\alpha^\star+\tfrac{1}{2}\eta_{\alpha\beta\gamma}K^{\star\beta}\wedge K^{\star\gamma} \tag{4.1}$$

for the RC curvature. Then from the relation $K_\alpha^\star=-\frac{1}{2}{}^*T_\alpha=-(-1)^s(\kappa/\ell)\vartheta_\alpha$ for the contortion and the definition $T^\alpha:=D\vartheta^\alpha$, it can be inferred that the Riemannian part $R_\alpha^{\{\}\star}$ of the curvature is also constant:

$$R_\alpha^{\{\}\star}=-\Lambda_{\rm eff}\eta_\alpha,$$

$$\Lambda_{\rm eff}=-\{\rho-[1+2(-1)^s]\kappa^2\}/\ell^2. \tag{4.2}$$

In principle, we can have a nonzero effective cosmological constant even for $\kappa=0$, i.e., in a purely Riemannian spacetime. Alternatively, for $\rho=0$, i.e., in the limit of vanishing RC curvature, there exists a nontrivial "parallelizing" torsion, resembling the "squashed" seven-sphere construction of Englert *et al.* [26] in higher dimensions.

From EXCALC/REDUCE calculations [27], we know that $\Lambda_{\rm eff}:=\theta_{\rm T}^2[(9+2(-1)^s)\theta_{\rm TL}^2+8\theta_{\rm T}\theta_{\rm L}]/(2A\ell)^2$ is the effective cosmological constant, which is induced by the topological terms in our gauge Lagrangian (3.1). Inasmuch as the three-dimensional "image" of a cosmological term of either sign is already *induced* by the Chern-Simons terms in the Lagrangian, one can disregard a "bare" cosmological term and still be able to simulate cosmological models in 3D. In our topological model, however, the translational CS term proportional to $\theta_{\rm T}$ is indispensable for obtaining a nontrivial result.

In 3D with Lorentzian signature $s=1$, the three-dimensional Einstein theory (2.1) with vanishing torsion and effective cosmological term $\Lambda_{\rm eff}$ has the AdS metric

$$ds^2=-(1-\Lambda_{\rm eff}r^2)dt^2+(1-\Lambda_{\rm eff}r^2)^{-1}dr^2+r^2d\phi^2 \tag{4.3}$$

as an exact solution. For the topological gauge model of Ref. [13], this was first recognized by Baekler *et al.* [6].

From Eq. (4.3), by appropriate identifications, the vacuum solution

$$ds^2=-N^2(r)dt^2+N^{-2}(r)dr^2+r^2[d\phi+N^\phi(r)dt]^2. \tag{4.4}$$

can be obtained [7,28], where the lapse squared and shift are given by

$$N^2(r)=-M-\Lambda_{\rm eff}r^2+\frac{J^2}{4r^2},\quad N^\phi(r)=-\frac{J}{2r^2}, \tag{4.5}$$

respectively. Observe that the shift is proportional to the angular momentum J of the solution, which allows for $J^2\leqslant M^2$ to interpret this configuration as a *rotating black hole* with mass M, cf. [29]. Obviously, for $M=-1$ and $J=0$ it reduces to Eq. (4.3).

We have checked by EXCALC that the rotating black hole solution (4.4) has constant Riemannian curvature (4.2) and is therefore nowhere singular. Then the construction of configurations with constant axial torsion and RC curvature (3.5) rest essentially on a prolongation $\Gamma_\alpha^{\{\}\star}\to\Gamma_\alpha^\star=\Gamma_\alpha^{\{\}\star}-K_\alpha^\star$ of the Riemannian to a RC connection and an inversion of Eq. (4.1), cf. [30]. Provided the effective cosmological constant $\Lambda_{\rm eff}$ is related to κ and ρ via (4.2), the black hole configuration (4.4) with (4.5) is also an exact solution of our topological gauge model (3.1).

V. *S* DUALITY IN 3D

The *general* form of the Poincaré gauge field equations (5.8.10) and (5.8.11) of Ref. [9] follows from Noether's theorem. In 3D they can be converted into

$$DH_\alpha-E_\alpha=\Sigma_\alpha \tag{5.1}$$

and

$$DH_\alpha^\star-\tfrac{1}{2}\eta_{\alpha\beta}\wedge H^\beta=\tau_\alpha^\star, \tag{5.2}$$

where the one-forms

$$H_\alpha:=-\frac{\partial V}{\partial T^\alpha},\qquad H_\alpha^\star:=-\frac{(-1)^s}{2}\frac{\partial V}{\partial R^{\star\alpha}} \tag{5.3}$$

are the translational and rotational field momenta, respectively, and

$$E_\alpha:=\frac{\partial V}{\partial\vartheta^\alpha}=e_\alpha\rfloor V+(e_\alpha\rfloor T^\beta)\wedge H_\beta+2(-1)^s(e_\alpha\rfloor R^{\star\beta})\wedge H_\beta^\star \tag{5.4}$$

is the canonical energy-momentum two-form of the gravitational gauge fields.

In four dimensions, the symmetry of duality rotations has a long history. Since 1925 it was known to Rainich [31] and developed further in the context *geometrodynamics* by Misner and Wheeler [32], cf. [20]. More recently, Motonen and Olive [33,34] noted that then also a duality of the *strong-weak* coupling regime of gauge fields is generated, the so-called *S duality*. [For Chern-Simons (super) gravity, some aspects have also been discussed in Refs. [35,36].]

Consequently, let us assume here that the 3D field momenta one-forms are duality rotated to each other via

$$H^\star_\alpha = \ell H_\alpha + \frac{\vartheta_\alpha}{\ell}, \tag{5.5}$$

as is the case for the special models considered before, cf. Eq. (12.2) of Ref. [6]. (For convenience, two possible arbitrary constants are absorbed in the still to be sought-for field momenta.) Inserting this ansatz into Eq. (5.2) yields the following condition for the translational field momenta:

$$\frac{T_\alpha}{\ell} + \ell E_\alpha - \frac{1}{2}\eta_{\alpha\beta}\wedge H^\beta = \tau^\star_\alpha - \ell \Sigma_\alpha . \tag{5.6}$$

After multiplying (5.6) by ϑ^α and using the algebraic identity $\vartheta^\alpha \wedge \eta_{\alpha\beta} = -2{}^*\vartheta_\beta$, we get the explicit condition

$${}^*H_\alpha = \frac{T_\alpha}{\ell} + \ell(E_\alpha + \Sigma_\alpha) - \tau^\star_\alpha . \tag{5.7}$$

This converts Eq. (5.1) into a first order field equation for the translational momenta, but, due to the intertwining ansatz (5.5), coupled to the material spin,

$$\ell D H_\alpha + \frac{T_\alpha}{\ell} - {}^*H_\alpha = \tau^\star_\alpha . \tag{5.8}$$

A complete integrability [17], as in the case of 2D Poincaré gauge models, is not available here. Nevertheless, a vacuum solution of Eq. (5.8) is

$$H_\alpha = \frac{2\kappa}{\ell^2(1-2\kappa)}\vartheta_\alpha \quad \Leftrightarrow \quad T_\alpha = \frac{2\kappa}{\ell}\eta_\alpha , \tag{5.9}$$

provided the RC spaces have constant axial torsion, similarly as in the case of Eq. (3.5) of CS gravity.

Due to Eq. (5.5), we obtain

$$H^\star_\alpha = \frac{1}{\ell(1-2\kappa)}\vartheta_\alpha = \frac{\ell}{2\kappa} H_\alpha , \tag{5.10}$$

which can be viewed as a kind of S (strong/weak) duality inverting the coupling constant $\kappa \to 1/\kappa$. Here in 3D the novel feature that this occurs for the intertwining mapping (5.10) between the translational/rotational pair of field momenta arises. In particular, in the limit $\kappa \to 0$ of weak axial torsion coupling, the translational field momenta (5.9) will vanish, whereas the rotational momenta $H^\star_\alpha = \vartheta_\alpha/\ell$ will become unity, similarly as in the EC action.

VI. NEW YANG-MILLS TERMS IN 3D GRAVITY

For a concrete generalization of our topological model (3.1), we consider here possible additional terms which may enter the 3D gravitational Lagrangian, for instance, as counter terms after renormalization, cf. [37]. They are the EC Lagrangian, a bare cosmological term $\Lambda\eta$, as well as the following Yang-Mills-type three-forms:

$$V_{\rm YM} = \chi V_{\rm EC} - \frac{\Lambda}{\ell^3}\eta - \frac{a}{2\ell} T^\alpha \wedge {}^*T_\alpha - \frac{b\ell}{2} R^{\star\alpha}\wedge {}^*R^\star_\alpha - c T^\alpha \wedge {}^*R^\star_\alpha + \frac{\Lambda_{\rm T}}{3!}\eta_{\alpha\beta\gamma}{}^*T^\alpha \wedge {}^*T^\beta \wedge {}^*T^\gamma , \tag{6.1}$$

or the corresponding decomposition in irreducible pieces. The constant χ, weighing the EC Lagrangian, as well as a, b, c, Λ, and $\Lambda_{\rm T}$ are *dimensionless* coupling constants. The mixed torsion-curvature term as well as the cubic term in the dual torsion (or contortion, see the Appendix) do not arise in four dimensions, they are pertinent to 3D. However, these *new* exotic terms seem to be devoid of any topological interpretation.

Using ${}^*T^\alpha \wedge {}^*T^\beta \wedge {}^*T^\gamma = -(-1)^s T^\alpha \wedge {}^*({}^*T^\beta \wedge {}^*T^\gamma)$ for the cubic term, they provide the following field momenta:

$$H_\alpha = \frac{a}{\ell}{}^*T_\alpha + c\,{}^*R^\star_\alpha + (-1)^s \frac{\Lambda_{\rm T}}{2}\eta_{\alpha\beta\gamma}{}^*({}^*T^\beta \wedge {}^*T^\gamma), \tag{6.2}$$

$$H^\star_\alpha = \frac{(-1)^s}{2}\left[\frac{\chi}{\ell}\vartheta_\alpha + c\,{}^*T_\alpha + b\ell\,{}^*R^\star_\alpha\right], \tag{6.3}$$

and

$$E_\alpha = -\frac{\chi}{\ell}R^\star_\alpha - \frac{\Lambda}{\ell^3}\eta_\alpha . \tag{6.4}$$

Let us now demonstrate that the black hole solution (4.4) with constant axial torsion and RC curvature (3.5) is also a solution of this Yang-Mills system, albeit new algebraic constraints on the coupling constants.

Inserting ansatz (3.5) into Eqs. (6.2), (6.3), and (6.4) provides the simplification

$$H_\alpha = (-1)^s \frac{1}{\ell^2}[2a\kappa + c\rho + (-1)^s 4\Lambda_{\rm T}\kappa^2]\vartheta_\alpha , \tag{6.5}$$

$$H^\star_\alpha = \frac{1}{2\ell}[(-1)^s\chi + 2c\kappa + b\rho]\vartheta_\alpha , \tag{6.6}$$

and

$$E_\alpha = -\frac{1}{\ell^3}(\chi\rho + \Lambda)\eta_\alpha . \tag{6.7}$$

Employing $T_\alpha = D\vartheta_\alpha$ and again Eq. (3.5), the general 3D gauge field equations (5.1) and (5.2) yield

$$\ell^3\Sigma_\alpha = \{(-1)^s 2\kappa[2a\kappa + c\rho + (-1)^s 4\Lambda_{\rm T}\kappa^2] + \chi\rho + \Lambda\}\eta_\alpha , \tag{6.8}$$

and

$$\ell^2 \tau^*_\alpha = (-1)^s \{\kappa[\chi + (-1)^s 2c\kappa + (-1)^s b\rho] + 2a\kappa + c\rho + (-1)^s 4\Lambda_{\mathrm{T}}\kappa^2\}\eta_\alpha . \tag{6.9}$$

In vacuum, we obtain a system of algebraic equations, cubic in κ and linear in ρ, which can be solved by standard methods. Eliminating ρ from Eq. (6.9) provides us with

$$\rho = -(-1)^s \kappa \frac{\chi + 2a + (-1)^s 2\kappa(c + 2\Lambda_{\mathrm{T}})}{b\kappa + (-1)^s c}, \tag{6.10}$$

unless $\kappa = -(-1)^s c/b$, which, in turn, requires $(\chi + 2a)b = 2c(c + 2\Lambda_{\mathrm{T}})$ as constraint on the coupling constants. Substituting Eq. (6.10) into the vacuum equation (6.8) yields the *quartic* algebraic equation

$$[b\kappa + (-1)^s c][\Lambda + (-1)^s 4a\kappa^2 + 8\Lambda_{\mathrm{T}}\kappa^3] = (-1)^s \kappa[\chi + (-1)^s 2c\kappa][\chi + 2a + (-1)^s \times 2\kappa(c + 2\Lambda_{\mathrm{T}})]. \tag{6.11}$$

Let us concentrate again on a model for which the bare cosmological constant is zero, i.e., $\Lambda = 0$, and the effective constant gets ''induced'' dynamically. Then the quartic equation factorizes with $\kappa = 0$ as one solution. Since this also implies $\rho = 0$, we would end up with the trivial case of flat space (time). For $b\Lambda_{\mathrm{T}} \neq 0$, the cubic equation remains,

$$\kappa^3 + B\kappa^2 + C\kappa + \Delta = \left(\kappa + \frac{B}{3}\right)^3 + p\left(\kappa + \frac{B}{3}\right) + q = 0, \tag{6.12}$$

where $B = (ab - c^2)/2b\Lambda_{\mathrm{T}}$, $C = \chi(c + \Lambda_{\mathrm{T}})/2b\Lambda_{\mathrm{T}}$, $\Delta = \chi(\chi + 2a)/8b\Lambda_{\mathrm{T}}$, $p = C - B^2/3 = [3\chi(c + \Lambda_{\mathrm{T}}) - ab + c^2]/6b\Lambda_{\mathrm{T}}$, and $q = \Delta - BC/3 + 2B^3/27$. The formula of Cardano [38] provides the real solution

$$\kappa = -\frac{B}{3} + \left(\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)^{1/3} + \left(\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)^{1/3} \tag{6.13}$$

for the contortional constant, provided $q^2 + 4p^3/27 \geqslant 0$.

VII. DISCUSSION

As demonstrated explicitly, our purely CS version (3.1) of 3D topological gravity has the same dynamical degrees of freedom as the DJT model and admits rotating black hole and AdS solutions. These exact configurations with axial torsion solve also our Yang-Mills extensions in 3D, albeit some constraints on the weights or coupling constants of the individual torsion and/or curvature terms in Lagrangian (6.1). However, there the Cauchy formulation as well as the number of no-ghost dynamical degrees of freedom are not known and may again depend on the choice of the coupling constant.

It would be interesting to implement the generalized solutions [39–43] of the DJT model in our topological CS with torsion, or even with nonmetricity [44,45]. In order to avoid factor 2 problems in the definition of the conserved quantities *M* and *J*, further insights in their group-theoretical interpretation via AdS Casimir operators [46,47] in 3D are desirable.

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APPENDIX A: THE DUAL BASIS FOR EXTERIOR FORMS IN 3D

The symbol $\wedge$ denotes the exterior product of forms, the symbol $\rfloor$ the interior product of a vector with a form, and $*$ the Hodge star (or left dual) operator, which maps a p-form into a $(3-p)$-form. It has the property that $**\Phi^{(p)} = (-1)^{p(3-p)+s}\Phi^{(p)}$, where p is the degree of the form Φ and s the signature of the metric.

The volume three-form is defined by

$$\eta := \frac{1}{3!}\eta_{\alpha\beta\gamma}\vartheta^\alpha \wedge \vartheta^\beta \wedge \vartheta^\gamma, \tag{A1}$$

where $\eta_{\alpha\beta\gamma} := \sqrt{|\det g_{\mu\nu}|}\epsilon_{\alpha\beta\gamma}$ and $\epsilon_{\alpha\beta\gamma}$ is the Levi-Civita symbol. Together with η, the following forms span a *dual basis* for the algebra of arbitrary p forms in 3D:

$$\eta_\alpha := e_\alpha \rfloor \eta = {}^*\vartheta_\alpha ,$$

$$\eta_{\alpha\beta} := e_\beta \rfloor \eta_\alpha = {}^*(\vartheta_\alpha \wedge \vartheta_\beta),$$

$$\eta_{\alpha\beta\gamma} := e_\gamma \rfloor \eta_{\alpha\beta}. \tag{A2}$$

We will call these forms the η basis of the three-dimensional space(time); for more details see the Appendix of Ref. [6].

APPENDIX B: LIE DUAL IN 3D

It is peculiar for 3D that the Poincaré group consists of three translation and three rotation generators. Then the Lie dual, that is, a duality operation with respect to the Lie-algebra indices, is mapping a vector into a bivector and vice versa; in particular for a bivector-valued p-form $\psi^{\alpha\beta} = -\psi^{\beta\alpha}$, the Lie dual is defined by

$$\psi^\star_\alpha := \tfrac{1}{2}\eta_{\alpha\beta\gamma}\psi^{\beta\gamma} \quad \Leftrightarrow \quad \psi^{\alpha\beta} = (-1)^s \eta^{\alpha\beta\gamma}\psi^\star_\gamma . \tag{B1}$$

In particular, we define

$$\Gamma^\star_\alpha := \tfrac{1}{2}\eta_{\alpha\beta\gamma}\Gamma^{\beta\gamma}, \quad K^\star_\alpha := \tfrac{1}{2}\eta_{\alpha\beta\gamma}K^{\beta\gamma} = -\tfrac{1}{2}{}^*T_\alpha , \tag{B2}$$

such that Cartan's structure equations for torsion and curvature get converted in

$$T^\alpha = d\vartheta^\alpha - (-1)^s \eta^{\alpha\beta} \wedge \Gamma^\star_\beta \tag{B3}$$

and

$$R^{\star}_{\alpha} := \frac{1}{2}\,\eta_{\alpha\beta\gamma}R^{\beta\gamma} = d\Gamma^{\star}_{\alpha} + \frac{(-1)^s}{2}\,\eta_{\alpha\beta\gamma}\Gamma^{\star\beta}\wedge\Gamma^{\star\gamma}. \quad \text{(B4)}$$

APPENDIX C: CHERN-SIMONS TERMS

Gauging the Poincaré group, translations, and (Lorentz) rotations gives rise two types of gauge potentials: the coframe ϑ^{γ} and the Lorentz-connection $\Gamma^{\star}_{\alpha}$ where the Lie dual of the connection has been used for a more condensed notation. Then the two Bianchi identities of Riemann-Cartan geometry can be rewritten as

$$DT^{\alpha} \equiv (-1)^s \eta^{\alpha\beta}\wedge R^{\star}_{\beta}, \quad \text{(C1)}$$

$$DR^{\star}_{\alpha} \equiv 0. \quad \text{(C2)}$$

The corresponding Chern-Simons three-forms [48] of gauge type $C = \mathrm{tr}\{A\wedge F\}$ are the translational Chern-Simons term

$$C_{\mathrm{T}} := \frac{1}{2\ell^2}\,\vartheta^{\alpha}\wedge T_{\alpha} = -\frac{(-1)^s}{\ell^2}\,\eta^{\alpha}\wedge K^{\star}_{\alpha} \quad \text{(C3)}$$

as well as the Lorentz-rotational one involving the curvature

$$C_{\mathrm{L}} := (-1)^s \Gamma^{\star\alpha}\wedge R^{\star}_{\alpha} - \frac{1}{3!}\,\eta_{\alpha\beta\gamma}\Gamma^{\star\alpha}\wedge\Gamma^{\star\beta}\wedge\Gamma^{\star\gamma}. \quad \text{(C4)}$$

Via the variational derivatives

$$\frac{\delta C_{\mathrm{T}}}{\delta\vartheta^{\alpha}} = \frac{1}{\ell^2}T_{\alpha} \quad \text{and} \quad \frac{\delta C_{\mathrm{T}}}{\delta\Gamma^{\star\alpha}} = \frac{(-1)^s}{\ell^2}\,\eta_{\alpha} \quad \text{(C5)}$$

as well as

$$\frac{\delta C_{\mathrm{L}}}{\delta\vartheta^{\alpha}} = 0 \quad \text{and} \quad \frac{\delta C_{\mathrm{L}}}{\delta\Gamma^{\star\alpha}} = (-1)^s 2R^{\star}_{\alpha}, \quad \text{(C6)}$$

these three-forms are uniquely related to the torsion T_{α}, the curvature $R^{\star}_{\alpha}$, and the cosmological term η_{α}, respectively, cf. [49].

In 3D with torsion, there exists another mixed topological term

$$C_{\mathrm{TL}} := \frac{1}{\ell}\left(\Gamma^{\star}_{\alpha}\wedge T^{\alpha} - \frac{(-1)^s}{2}\,\eta_{\alpha\beta\gamma}\Gamma^{\star\alpha}\wedge\Gamma^{\star\beta}\wedge\vartheta^{\gamma}\right). \quad \text{(C7)}$$

Its variations lead to

$$\frac{\delta C_{\mathrm{TL}}}{\delta\vartheta^{\alpha}} = \frac{1}{\ell}R^{\star}_{\alpha} \quad \text{and} \quad \frac{\delta C_{\mathrm{TL}}}{\delta\Gamma^{\star\alpha}} = \frac{1}{\ell}T_{\alpha}. \quad \text{(C8)}$$

APPENDIX D: DJT MODEL BY ENFORCING VANISHING TORSION

In order to extract the Riemannian content of our topological model, one cannot simply put torsion to zero, because $D\vartheta^{\alpha}=0$ is a *dynamical constraint*. Rather, we follow Refs. [13,6] and supplement our Lagrangian (3.1) with the constraint of vanishing torsion by means of a Lagrange multiplier

$$V_{\mathrm{DJT}} = V_{\infty} + \lambda_{\alpha}\wedge T^{\alpha}. \quad \text{(D1)}$$

By varying it with respect to the independent variables ϑ^{α}, $\Gamma^{\star}_{\alpha}$, the additional terms $D\lambda_{\alpha}$ and $-(\ell/2)\,\eta_{\alpha\beta\gamma}\vartheta^{[\beta}\wedge\lambda^{\gamma]}$ arise, respectively, in the field equations (3.3) and (3.4). The variation with respect to the Lagrange multiplier one-form λ_{α} provides the constraint $T^{\alpha}=0$ of vanishing torsion.

In order to resolve for λ_{α}, we employ the algebraic identity (A.1.26) of Ref. [9]. Then the first field equation reads

$$-\frac{\theta_{\mathrm{TL}}}{\ell}R^{\{\}\star}_{\alpha} - 2(-1)^s\theta^{*}_{L}C_{\alpha} = \Sigma_{\alpha} + 2D^{\{\}}(e^{\beta}\rfloor\tau_{\alpha\beta} - \tfrac{1}{4}\vartheta_{\alpha}e^{\gamma}\rfloor e^{\delta}\rfloor\tau_{\gamma\delta}). \quad \text{(D2)}$$

The one-form

$$C^{l} := e^{l}{}^{*}_{\alpha}[D^{\{\}}(e_{\beta}\rfloor R^{\{\}\alpha\beta} - \tfrac{1}{4}\vartheta^{\alpha}e^{\gamma}\rfloor e^{\delta}\rfloor R^{\{\}}_{\gamma\delta})] = C_{k}{}^{l}dx^{k} \quad \text{(D3)}$$

is associated with the symmetric Cotton tensor $C^{kl}=C^{lk}$. Therefore, the vacuum field equation (D2) is for $\theta_{\mathrm{TL}}=-1$ identical to that of the topological gauge model of gravity considered by Deser *et al.* [2]. Since the Cotton tensor vanishes for Eq. (4.4), the field equations (D2) are trivially satisfied.

Thus the constraint of vanishing torsion transforms the original system (3.3) and (3.4) of first order field equations in the variables $(\vartheta^{\alpha},\ \Gamma^{\star}_{\alpha})$ into a third order one with respect to the components g_{ij} of the metric. The Cotton tensor is mediating on the gravitational field side of Eq. (D2), between the canonical and the Belinfante-Rosenfeld symmetrized energy-momentum current of matter, cf. [50,51].

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