General relativistic interaction of massless fields in cylindrical waves

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(Received 21 May 2003; published 21 November 2003)

In this paper the problem of finding exact solutions of the combined system of the Einstein-Maxwell-Weyl equations for cylindrical waves is reduced to the solution of two complete singular integral equations in the complex plane of the auxiliary analytical parameter. In the case of the nonsingular symmetry axis the problem further simplifies and requires solving the only integral equation, the expressions for the Ernst potentials on the symmetry axis then defining the group transformations of internal symmetries during the solution generation process. A large class of exact solutions for neutrino electrovacuum is obtained, and the Cauchy problem related to some particular initial data is considered.

DOI: 10.1103/PhysRevD.68.104022 PACS number(s): 04.20.Jb

I. INTRODUCTION

It is well known that the concept of the neutrino as a highly penetrating uncharged particle of spin 1/2 was introduced in order to satisfy the energy conservation laws during the β decay [1,2]. In the experiments performed up to now, neutrinos have manifested themselves as the left-hand polarized particles and antineutrinos as the right-hand polarized particles. One may expect a strong neutrino radiation in some astrophysical situations, for instance, during the neutronization of matter inside a supercompressed neutron star, or in young shells of supernovas where the accelerated protons produce, while colliding, cosmic pion showers, and only neutrinos can escape through the dense shell.

In view of the clear asymmetry of matter and antimatter in the Universe, one can expect that the flows of neutrinos possessing a certain helicity may have observational manifestations. When the electromagnetic and gravitational waves pass through a region of strong coherent neutrino radiation of a given helicity, they may change their polarization. This effect is described in Ref. [3].

For the description of the neutrino, Lee, Yang, Landau, and Salam proposed in 1957 to use the Weyl equations of the two-component massless spinor field [4–6]. The energy-momentum tensor of a neutrino field does not have the property of energodominance, similar to the case of a classical electron described by the Dirac equations. That is why the interaction of a spinor field not subjected to the procedure of second quantization with the classical gravitational and electromagnetic fields has several peculiar features. For example, in the framework of general relativity there does not exist in such fields the effect of focusing the convergent normal congruence of isotropic geodesics. The gravitational fields generated by neutrinos of right-hand and left-hand helicity differ from each other.

The collisions of neutrino plane waves of some particular type in general relativity were studied by Griffiths [7]. In his monograph he gave a thorough review of the most interesting results regarding collisions of plane gravitational and electromagnetic waves. Henneaux [8] studied applications of the Weyl equations to neutrinos in the cosmological context. An exact solution for cylindrical gravitational waves was obtained by Piran, Safier, and Katz [9].

In the present paper an effort is made to maximally simplify the mathematical investigation of the full Einstein-Maxwell-Weyl system in the case of cylindrical waves. We succeeded in reducing this problem to a unique linear integral equation with a single kernel, the solutions of which permit us to calculate the metric coefficients and components of the spinor and electromagnetic fields via quadratures. In the absence of a neutrino field the equation obtained reduces to the singular integral equation derived by one of the authors two decades ago for the case of the electrovacuum [10]. We obtain a large family of exact solutions for the neutrino electrovacuum using the data on the symmetry axis, and give some illustrative examples for which the corresponding Cauchy problems are considered.

It is worth mentioning that a theory of singular equations similar to the one considered in the present paper was developed earlier mainly in applications to aerodynamics and theory of elasticity [11–13]. We mention also the classical Keldysh-Sedov result for boundary problems with several cuts [14], and that a detailed discussion of the theory of singular equations can be found in the book [15].

II. SIMPLIFICATION OF A CLOSED SYSTEM FOR FREE MASSLESS FIELDS IN THE CASE OF CYLINDRICAL WAVES

Exact solutions of the combined system of Einstein-Maxwell-Weyl equations admitting a group of motions with two commuting Killing vectors are of interest from the point of view of applications which include cylindrical and colliding waves, as well as stationary fields with axial symmetry. On the other hand, the discovery of the integrability of this system [16,3] provides the possibility of its far-reaching mathematical analysis. In the case of the axisymmetric electrovacuum problem, the first fundamental results for the linear matrix formulation of the field equations were obtained by Kinnersley [17]. The idea of the equivalence of all integrable systems to the boundary Riemann–Hilbert problem [18] was realized, in application to the electrovacuum, in the pioneering papers by Hauser and Ernst [19], who were able in particular to prove the Geroch conjecture for the electrovacuum fields [20]. In contradistinction to the matrix Riemann-Hilbert problem, in the book [10] a single linear integral equation for the electrovacuum was obtained which permitted construction of the solutions of the Ernst equations [21] corresponding to given potentials on the symmetry axis both in the stationary axisymmetric case [22] and in the case of cylindrical waves [10]. This equation also made it possible to construct solutions involving analytically extended parameter sets, e.g., the extended multisoliton solutions [23], or new exact solutions of astrophysical interest [24]. A solution describing the exterior field of a magnetized, rotating, massive deformed source [25] was used in [26] for the analysis of the disk accretion onto a neutron star within the framework of general relativity.

A detailed formulation of the Einstein-Maxwell-Weyl system can be found in the books [3,10]. Here we shall restrict ourselves to giving only the formulas essential for achieving our main goal—the derivation of the new integral equation.

In the case of cylindrical waves, the cyclical coordinates (the ones on the orbits of Killing vectors) are the angle φ and coordinate z along the symmetry axis. The line element can then be written in the form

$$ds^2=-\theta^2 du\,dv+g_{AB}(dx^A+g^A_u du+g^A_v dv)\times(dx^B+g^B_u du+g^B_v dv), \tag{1}$$

where the unknown functions θ, g_{AB}, g^A_u, g^A_v, g^B_u, g^B_v depend only on the coordinates u and v. The indices A,B take the values 1,2 and stand for the cyclical coordinates φ,z. The coordinates $u=t+\rho$ and $v=t-\rho$ have the meaning of the advanced and retarded times, respectively. The metric (1) admits a liberty in the choice of the coordinates u,v: $u\rightarrow U(u)$, $v\rightarrow V(v)$, and also in the choice of cyclical coordinates $x^A\rightarrow x^A+f^A(u,v)$. The energy-momentum tensor of spinor fields violates the Frobenius conditions of the existence of two-dimensional surfaces orthorgonal to the orbits of Killing vectors. Therefore, in the presence of massless spinor fields the metric (1) cannot be reduced to the Papapetrou form with only one nondiagonal term. Henceforth the system of units is used in which the speed of light in vacuum and the gravitational constant are used as scale units.

It follows from Einstein's equations that $\sqrt{|g_{AB}|}$ as a function of u,v satisfies the d'Alambert equation. In what follows we shall set $\sqrt{|g_{AB}|}=\rho$, thus fixing the choice of the coordinates u,v.[1]

The neutrino fields are described by the two-component spinors (ϕ,ψ) and satisfy the Weyl equations. An analysis of the respective components of the Einstein and Weyl equations yields

$$\phi\phi^*=\frac{\phi_1(u)}{8\pi\rho\theta},\qquad \psi\psi^*=\frac{\phi_2(v)}{8\pi\rho\theta}, \tag{2}$$

where $\phi_1(u)$ and $\phi_2(v)$ are arbitrary functions corresponding to the incident and expanding (from the axis) cylindrical neutrino waves.

The electromagnetic field is described by a four-potential with the nonzero components A_θ $(\theta=1,2)$.

Following Kinnersley [17], we shall be raising and lowering indices by means of the Levi-Cività symbols ε^{AB} and ε_{AB}: $\varepsilon_{11}=\varepsilon_{22}=0$, $\varepsilon_{12}=-\varepsilon_{21}=1$, $\varepsilon^{AB}=(\varepsilon_{AB})^{-1}$. Let $g_{AB}\equiv f_{AB}$. Then from the Maxwell equations it follows that

$$(\rho^{-1}f^C_A A_{C,u})_{,v}+(\rho^{-1}f^C_A A_{C,v})_{,u}=0. \tag{3}$$

Hence, there exist potentials B_A such that

$$\rho^{-1}f^C_A A_{C,u}=B_{A,u},\qquad \rho^{-1}f^C_A A_{C,v}=-B_{A,v}. \tag{4}$$

Denote the complex combinations A_C+iB_C as Φ_C and define the gradients ∇ and $\tilde{\nabla}$ as $(\partial/\partial u,\partial/\partial v)$, $(\partial/\partial u,-\partial/\partial v)$, respectively. We obtain from Eq. (4)

$$\rho^{-1}f^C_A\nabla A_C=\tilde{\nabla}B_A \Rightarrow \rho\nabla\Phi_A=if^C_A\tilde{\nabla}\Phi_C. \tag{5}$$

Similarly, from Einstein's equations in projections onto the orbits of Killing vectors follows the existence of the complex potentials H^B_A:

$$\nabla H^B_A=\nabla(f^B_A+W\delta^B_A)+\Phi^{B*}\nabla\Phi_A+\frac{i}{\rho}f^C_A(\tilde{\nabla}(f^B_C+W\delta^B_C)+\Phi^{B*}\tilde{\nabla}\Phi_C), \tag{6}$$

where $W\equiv\int\phi_1(u)du-\int\phi_2(v)dv$.

From Eq. (6) we get

$$\rho\nabla H^B_A=if^C_A\tilde{\nabla}H^B_C. \tag{7}$$

The potentials H^B_A, Φ^B in the absence of neutrino fields were introduced by Kinnersley [17], together with the potentials L^A and K:

$$\nabla L^B=2\Phi^{C*}\nabla H^B_C,\quad \nabla K=2\Phi^{C*}\nabla\Phi_C. \tag{8}$$

Following Hauser and Ernst [19], let us introduce the 3×3 matrix H^b_a $(a,b=1,2,3)$ via the definitions

$$H^b_a\equiv H^B_A\ \ (a=A,b=B),\quad H^3_a=\Phi_A\ \ (a=A),$$
$$H^b_3=L^B\ \ (b=B),\quad H^3_3=K. \tag{9}$$

The matrix $H=(H^b_a)$ introduced in this way, as follows from Eqs. (5)–(7), satisfies the matrix equation

$$2i(u-v)\frac{\partial^2 H}{\partial u\partial v}=\frac{\partial H}{\partial u}\frac{\partial H}{\partial v}-\frac{\partial H}{\partial v}\frac{\partial H}{\partial u}. \tag{10}$$

[1]In the case of colliding plane waves, $\sqrt{|g_{AB}|}$ should be set equal to t [7].

By construction, the trace of H is equal to

$$\mathrm{tr}H \equiv H^a_a = 2W - i(u+v). \tag{11}$$

This expression follows from Eqs. (6),(7) in which one has to put $B=A$ and take into account that $f^A_A = f_{AB}\varepsilon^{BA} = 0$.

The derivatives of H with respect to the coordinates u,v, as can be readily seen from Eqs. (5),(7), satisfy the equations

$$(I-M_1)\frac{\partial H}{\partial u} = 0, \quad (I-M_2)\frac{\partial H}{\partial v} = 0, \tag{12}$$

the matrices M_1 and M_2 being defined by the formulas

$$M_1 = \begin{pmatrix} i f^C_A & 0 \\ 2\Phi^{C*} & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} -i f^C_A & 0 \\ 2\Phi^{C*} & 0 \end{pmatrix}. \tag{13}$$

The components H^2_1 and H^3_1 of the matrix H are related to the generalized Ernst potentials $\mathcal{E},\Phi$ as $H^2_1 = -\mathcal{E}$, $H^3_1 = \Phi$, and they satisfy the following self-consistent system of two differential equations first reported in [16]:

$$\begin{aligned}
&(\mathrm{Re}\mathcal{E} - \Phi\Phi^*)\left\{\frac{\partial^2}{\partial t^2}\begin{pmatrix}\mathcal{E}\\ \Phi\end{pmatrix} - \frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho\frac{\partial}{\partial\rho}\begin{pmatrix}\mathcal{E}\\ \Phi\end{pmatrix}\right]\right.\\
&\quad\left. + \frac{i}{\rho}\left[\frac{\partial\tilde{W}}{\partial t}\frac{\partial}{\partial t}\begin{pmatrix}\mathcal{E}\\ \Phi\end{pmatrix} - \frac{\partial\tilde{W}}{\partial\rho}\frac{\partial}{\partial\rho}\begin{pmatrix}\mathcal{E}\\ \Phi\end{pmatrix}\right]\right\}\\
&\quad = \frac{\partial}{\partial t}\begin{pmatrix}\mathcal{E}\\ \Phi\end{pmatrix}\left(\frac{\partial\mathcal{E}}{\partial t} - 2\Phi^*\frac{\partial\Phi}{\partial t}\right) - \frac{\partial}{\partial\rho}\begin{pmatrix}\mathcal{E}\\ \Phi\end{pmatrix}\left(\frac{\partial\mathcal{E}}{\partial\rho} - 2\Phi^*\frac{\partial\Phi}{\partial\rho}\right),\\
&\tilde{W} \equiv \int \phi_1(u)du + \int \phi_2(v)dv.
\end{aligned} \tag{14}$$

The two sides of the vector equation (14) should be equated by components. We mention that in the characteristic coordinates (u,v) which can be more advantageous for use in some cases this system assumes the form

$$\begin{aligned}
&2(\mathrm{Re}\mathcal{E} - \Phi\Phi^*)\left[\frac{\partial^2}{\partial u\partial v}\begin{pmatrix}\mathcal{E}\\ \Phi\end{pmatrix} - \frac{1}{u-v}\left(\frac{1}{2} - i\frac{\partial\tilde{W}}{\partial v}\right)\frac{\partial}{\partial u}\begin{pmatrix}\mathcal{E}\\ \Phi\end{pmatrix}\right.\\
&\quad\left. + \frac{1}{u-v}\left(\frac{1}{2} + i\frac{\partial\tilde{W}}{\partial u}\right)\frac{\partial}{\partial v}\begin{pmatrix}\mathcal{E}\\ \Phi\end{pmatrix}\right]\\
&= \frac{\partial}{\partial u}\begin{pmatrix}\mathcal{E}\\ \Phi\end{pmatrix}\left(\frac{\partial\mathcal{E}}{\partial v} - 2\Phi^*\frac{\partial\Phi}{\partial v}\right)\\
&\quad + \frac{\partial}{\partial v}\begin{pmatrix}\mathcal{E}\\ \Phi\end{pmatrix}\left(\frac{\partial\mathcal{E}}{\partial u} - 2\Phi^*\frac{\partial\Phi}{\partial u}\right).
\end{aligned} \tag{15}$$

Equations (14) form a closed system for the determination of the potentials $\mathcal{E}$ and Φ, but for our purposes the formulation of the problem in terms of the matrix equation (10) is more advantageous. This is due to the fact that the matrix formulation (10) can be written in the form of the zero-curvature condition for the overdetermined matrix system first found by Kinnersley [17] and then rewritten in a more elegant form by Hauser and Ernst [19]:

$$\frac{\partial F}{\partial u} = \frac{i}{2(s-u)}\frac{\partial H}{\partial u}F, \quad \frac{\partial F}{\partial v} = \frac{i}{2(s-v)}\frac{\partial H}{\partial v}F. \tag{16}$$

Here the generating matrix F depends on the coordinates u,v and on the auxiliary analytical parameter s. As follows from Eq. (12), the derivatives of F with respect to u and v satisfy the conditions

$$(I-M_1)\frac{\partial F}{\partial u} = 0, \quad (I-M_2)\frac{\partial F}{\partial v} = 0. \tag{17}$$

In the complex plane of s, the matrix F has two branching points: $s=u$ and $s=v$. It follows from Eq. (16), taking into account Eqs. (11) and (2), that $|F| \equiv \det F$ satisfies the system of equations

$$\frac{\partial}{\partial u}\ln|F| = \frac{i}{2(s-u)}\frac{\partial}{\partial u}\mathrm{tr}H, \quad \frac{\partial}{\partial v}\ln|F| = \frac{i}{2(s-v)}\frac{\partial}{\partial v}\mathrm{tr}H, \tag{18}$$

whence

$$\begin{aligned}
|F| = \sqrt{\frac{(s-u_0)(s-v_0)}{(s-u)(s-v)}}\exp\left(i\int_{u_0}^{u}\frac{\phi_1(u)du}{s-u}\right.\\
\left. + i\int_{v}^{v_0}\frac{\phi_2(v)dv}{s-v}\right) \equiv \frac{1}{\lambda}
\end{aligned} \tag{19}$$

(u_0 and v_0 are the integration constants).

The function $|F|$ is single valued in the plane with two cuts: one from u_0 to u, and the other from v to v_0. When $s\to\infty$, F has the asymptotics $F \approx I + (i/2s)H$, I being the unit matrix.

The simplest solution of the system (14) is $\mathring{\mathcal{E}} = 1$, $\mathring{\Phi} = 0$, for which the corresponding components of the metric g_{AB} are $g_{11} = 1$, $g_{12} = W$, $g_{22} = \rho^2 + W^2$. In this case, using the definition (6) of the potential H^A_B, we obtain the expressions for the nonzero components of the matrix $\mathring{H}$:

$$\begin{aligned}
&\mathring{H}^1_1 = 2W - i(u+v), \quad \mathring{H}^2_1 = -1,\\
&\mathring{H}^1_2 = \rho^2 + W^2 - iW(u+v)\\
&\qquad + 2i\left(\int \phi_1(u)u\,du - \int \phi_2(v)v\,dv\right).
\end{aligned} \tag{20}$$

The components of the corresponding generating matrix $\mathring{F}^{-1}$ can be found from the overdetermined system (16), the result being (cf. the analogous expression for $\mathring{F}^{-1}$ in the book [10])

$$\mathring{F}^{-1}=\begin{pmatrix} -\frac{i}{2s}Q+\frac{\lambda}{2} & \frac{i}{2s} & 0\\ -Q+is\lambda & 1 & 0\\ 0 & 0 & 1\end{pmatrix},\quad Q\equiv W-\frac{i}{2}(u+v)+is. \tag{21}$$

For integrable systems there exist continuous groups of internal symmetries which transform one solution into another. Along the one-parameter subgroup (the orbit) with parameter σ the following equation holds [19,10]:

$$\frac{dF(\mu,\sigma)}{d\sigma}=\frac{1}{2\pi i}\int_{\mathcal{L}}\frac{F(\nu,\sigma)\Gamma(\nu)F^{-1}(\nu,\sigma)d\nu}{\nu-\mu}F(\mu). \tag{22}$$

Here $\mathcal{L}$ is the contour bounding the simply connected region D in the plane of the analytical parameter ν. Outside the region D, the function $F(\nu)$ is supposed to be analytic. The matrices $\{\Gamma(\nu)\}$ form an infinite-parameter algebra and depend only on the analytical parameter ν. All the singularities of the matrix $\Gamma(\nu)$ lie off the region D. The integration of Eq. (22) along the orbit (the operation of exponentiation of the algebra) leads to the equation [19,10]

$$\int_{\mathcal{L}}\frac{F(\nu)e^{\sigma\Gamma(\nu)}\mathring{F}^{-1}(\nu)d\nu}{\nu-\mu}=0,\quad \mu\in D. \tag{23}$$

It follows from Eq. (23) that the function $F(\nu)\exp[\sigma\Gamma(\nu)]\mathring{F}^{-1}(\nu)$ is analytic outside the region D. The generating matrix $F(\nu)$ corresponds to a new solution into which the seed solution is transformed by means of a shift along the orbit of the group $\exp[\sigma\Gamma(\nu)]$. We mention that the result (23) can also be obtained with the aid of the "dressing" method of Zakharov and Shabat [18].

III. CANONICAL INTEGRAL EQUATIONS OF NEUTRINO ELECTROVACUUM FOR CYLINDRICAL WAVES

Let us consider now the particular case of Eqs. (16) for the neutrino electrovacuum where $\nu\equiv s$. By deforming the closed contour $\mathcal{L}$ to two cuts along the real axis, one from v to v_0 (unclosed contour $\mathcal{L}_1$), and the other from u_0 to u (unclosed contour $\mathcal{L}_2$), we obtain that the matrix $F(s)\exp[\sigma\Gamma(s)]\mathring{F}^{-1}(s)$ is continuous on these cuts.

From the papers of Kinnersley [17] and of Hauser and Ernst [19] one may draw the conclusion that the elements of the infinite-parameter algebra $\{\Gamma(s)\}$ can be represented as products of an arbitrary Hermitian matrix on the anti-Hermitian matrix with nonzero components $\Omega_{12}=-\Omega_{21}=1$, $\Omega_{33}=-i/4s$. The exponent of the matrix $\Gamma(s)$ can be calculated by means of the Lagrange-Sylvester formula, setting the components $(^1_2)$,$(^1_3)$ to zero, and the component $(^3_3)$ to 1. Then we shall obtain the general form of the matrix $\exp[\sigma\Gamma(s)]$ [10]

$$\exp[\sigma\Gamma(s)]=\begin{pmatrix} a & \frac{a(\gamma-i\alpha^*\alpha)}{2s} & \frac{i\alpha a}{2s}\\ 0 & 1/a^* & 0\\ 0 & -2\alpha^* & 1\end{pmatrix}. \tag{24}$$

Here α^*,a^* are understood as complex conjugations $(\alpha(s^*))^*,(a(s^*))^*$; moreover, $\gamma(s)=(\gamma(s^*))^*$.

Let us take the solution (19)–(21) as the seed one, and the expression (24) as the shift along the group of internal symmetries. Then the conditions of continuity of the matrix $F(s)\exp[\sigma\Gamma(s)]\mathring{F}^{-1}(s)$ on the cuts $\mathcal{L}_1$ and $\mathcal{L}_2$ give the following equations:

$$\frac{i}{2s}[F_a^1]e(s)+[F_a^2]=0,\qquad \frac{i}{2s}[F_a^1]f(s)+[F_a^3]=0, \tag{25}$$

$$\left[\lambda\left(F_a^2-\frac{i}{2s}F_a^1\tilde{e}(s)-2\tilde{f}(s)F_a^3\right)\right]=0,\qquad a=1,2,3, \tag{26}$$

where we have introduced

$$e(s)\equiv aa^*(1-i\gamma+\alpha^*\alpha),$$

$$\tilde{e}(s)\equiv aa^*(1+i\gamma+\alpha^*\alpha),$$

$$f(s)\equiv\alpha a,\quad \tilde{f}(s)\equiv\alpha^*a^*. \tag{27}$$

In Eqs. (25),(26), $[F_a^b]$ denote the jumps of the functions F_a^b when q tends to the point q_0 on the cut from above and from below:

$$[F_a^b]_{1,2}=\lim_{\epsilon\to+0}(F_a^b(q_0+i\epsilon)-F_a^b(q_0-i\epsilon)). \tag{28}$$

The matrix F as a function of the parameter q is analytic off the cuts $\mathcal{L}_1$ and $\mathcal{L}_2$, hence it can be represented in the form of the Cauchy integrals

$$F(q)=I+\frac{1}{2\pi i}\left(\int_{\mathcal{L}_1}\frac{[F]_1ds}{s-q}+\int_{\mathcal{L}_2}\frac{[F]_2ds}{s-q}\right). \tag{29}$$

Taking into account that $F(s)\approx I+(i/2s)H$ for $s\to\infty$, it follows from Eq. (29) that H is expressible in terms of the jumps $[F]$ as

$$H=\frac{1}{\pi}\left(\int_{\mathcal{L}_1}[F]_1ds+\int_{\mathcal{L}_2}[F]_2ds\right). \tag{30}$$

From the expressions (30) and conditions (25) we obtain

$$\mathcal{E}=H_{11}=\frac{i}{\pi}\left(\int_{\mathcal{L}_1}\frac{[F_1^1]_1}{2s}e(s)ds+\int_{\mathcal{L}_2}\frac{[F_1^1]_2}{2s}e(s)ds\right),$$

$$\Phi=H_1^3=\frac{i}{\pi}\left(\int_{\mathcal{L}_1}\frac{[F_1^1]_1}{2s}f(s)ds+\int_{\mathcal{L}_2}\frac{[F_1^1]_2}{2s}f(s)ds\right). \tag{31}$$

Therefore, we have reduced the problem of finding exact solutions of the nonlinear system (14) to determination of the jumps of the function F_1^1 on the cuts $\mathcal{L}_1$ and $\mathcal{L}_2$. Below we shall show that these jumps can be found from the conditions (26).

Let $\{F\}$ denote the sum of limits of the function $F(q)$ when the point q tends to the point q_0 on a given cut from above and from below:

$$\{F\}=\lim_{\epsilon\to+0}(F(q_0+i\epsilon)+F(q_0-i\epsilon)). \tag{32}$$

Then from conditions (26) it follows that on each cut the following relation holds:

$$[\lambda]\{L\}+\{\lambda\}[L]=0,\quad L\equiv sF_1^2-\frac{i}{2}\tilde{e}(s)F_1^1-2s\tilde{f}(s)F_1^3. \tag{33}$$

On the cut $\mathcal{L}_1$, as follows from representations of the functions F_a^b in terms of the Cauchy integrals (29), we have

$$\{L\}_1=-\frac{1}{\pi}\left(\fint_{\mathcal{L}_1}\frac{[F_1^1]_1K_{11}(s,q)ds}{2s(s-q)}+\int_{\mathcal{L}_2}\frac{[F_1^1]_2K_{12}(s,q)ds}{2s(s-q)}\right)-\tilde{e}(q)i,$$

$$K_{ij}(s,q)\equiv qe(s)+s\tilde{e}(q)-2qf(s)\tilde{f}(q),\qquad q\in\mathcal{L}_i,\quad s\in\mathcal{L}_j, \tag{34}$$

and

$$[L]_1=-\frac{i}{2}[F_1^1]_1(e(q)+\tilde{e}(q)-2f(q)\tilde{f}(q)),\qquad q\in\mathcal{L}_1. \tag{35}$$

The analogous formulas are also readily obtainable for the cut $\mathcal{L}_2$. The expressions for one-half the sum and one-half the difference of the values of λ on different sides of the cut $\mathcal{L}_1$ (or $\mathcal{L}_2$) have the form

$$[\lambda]=\lambda_+(1+\exp(2\pi\phi_1)),\quad \{\lambda\}=\lambda_+(1-\exp(2\pi\phi_1)), \tag{36}$$

where λ_+ is the limiting value of the function λ on the upper side of the cut. From Eq. (33), taking into account Eqs. (34)–(36), we obtain a system of two linear integral equations for the determination of jumps of the function F_1^1 on the cuts $\mathcal{L}_1$ and $\mathcal{L}_2$:

$$\frac{1}{\pi}\left(\fint_{\mathcal{L}_1}\frac{\chi_1(s)K_{11}(s,q)ds}{s-q}+\int_{\mathcal{L}_2}\frac{\chi_2(s)K_{12}(s,q)ds}{s-q}\right)-i\tanh[\pi\phi_1(q)]K_{11}(q,q)\chi_1(q)=\tilde{e}_1(q),\quad q\in\mathcal{L}_1,$$

$$\frac{1}{\pi}\left(\int_{\mathcal{L}_1}\frac{\chi_1(s)K_{21}(s,q)ds}{s-q}+\fint_{\mathcal{L}_2}\frac{\chi_2(s)K_{22}(s,q)ds}{s-q}\right)-i\tanh[\pi\phi_2(q)]K_{21}(q,q)\chi_2(q)=\tilde{e}_2(q),\quad q\in\mathcal{L}_2, \tag{37}$$

χ_1 and χ_2 denoting the jumps of the function $iF_1^1/2s$ on the cuts $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively.

After finding a solution of the system of linear integral equations (37), the corresponding solution for the Ernst potentials can be obtained via quadratures:

$$\mathcal{E}=\frac{1}{\pi}\left(\int_{\mathcal{L}_1}\chi_1(s)e_1(s)ds+\int_{\mathcal{L}_2}\chi_2(s)e_2(s)ds\right),$$

$$\Phi=\frac{1}{\pi}\left(\int_{\mathcal{L}_1}\chi_1(s)\phi_1(s)ds+\int_{\mathcal{L}_2}\chi_2(s)\phi_2(s)ds\right). \tag{38}$$

Let us consider now a particular case describing cylindrical waves falling on and reflecting from the symmetry axis. In this case we can assume without any loss of generality that

$$u_0=v_0=0,\quad \phi_1(s)=\phi_2(s)=\phi(s),$$

$$\chi_1(s)=\chi_2(s)=\chi(s),$$

$$K(s,q)=qe(s)+s\tilde{e}(q)-2qf(s)\tilde{f}(q). \tag{39}$$

The system (37) then converts into one elegant equation

$$\frac{1}{\pi}\fint_v^u\frac{\chi(s)K(s,q)ds}{s-q}-i\tanh[\pi\phi(q)]K(q,q)\chi(q)=\tilde{e}(q), \tag{40}$$

in which s,q belong to the interval (v,u).

We mention that the nonhomogeneous equation (40) is equivalent to the homogeneous equation

$$\frac{1}{\pi}\fint_v^u\frac{\chi(s)K'(s,q)ds}{s-q}-i\tanh[\pi\phi(q)]K'(q,q)\chi(q)=0,$$

$$K'(s,q)\equiv e(s)+\tilde{e}(q)-2f(s)\tilde{f}(q), \tag{41}$$

with the normalizing condition

$$\frac{1}{\pi}\int_v^u\chi(s)ds=1. \tag{42}$$

Equation (40) has the form of a classical complete singular equation whose numerous applications are known [13,15,27]. In the absence of neutrino waves this equation simplifies further, and reduces to the equation for cylindrical electromagnetic and gravitational waves analogous to the one obtained for the stationary axisymmetric case [10].

The expressions for the Ernst potentials assume the form

$$\begin{pmatrix}\mathcal{E}\\ \Phi\end{pmatrix}=\frac{1}{\pi}\int_v^u \chi(s)\begin{pmatrix}e(s)\\ f(s)\end{pmatrix}ds. \tag{43}$$

IV. PROOF OF GEROCH CONJECTURE FOR INTERACTING MASSLESS CYLINDRICAL WAVES

It was conjectured by Geroch [28] that all stationary axisymmetric vacuum spacetimes of general relativity can be obtained from Minkowski space by an appropriate transformation from the group of internal symmetries of the Einstein equations. Geroch's conjecture was proved by Hauser and Ernst in the case of electrovacuum fields [20]. In this section we shall prove it for a more general situation involving neutrino fields.

The proof consists in showing that from Eqs. (40),(43) it follows that on the symmetry axis the functions $e(t),\tilde{e}(t),f(t),\tilde{f}(t)$ defined in terms of the functions $a(s),\gamma(s),\alpha(s)$ determining a shift from an initial solution to the final solution along the orbit, have the meaning of the Ernst potentials $\mathcal{E},\mathcal{E}^*,\Phi,\Phi^*$ on the symmetry axis $\rho=0$. Thus, the Ernst functions on the symmetry axis will define the group transformation of internal symmetries which passes the initial seed solution into another solution with prescribed Ernst potentials on the symmetry axis.

To accomplish our purpose, let us perform the substitution

$$s=\frac{u+v}{2}+\frac{u-v}{2}\sigma, \qquad q=\frac{u+v}{2}+\frac{u-v}{2}\kappa \tag{44}$$

in Eq. (41). Then Eq. (41) takes the form

$$\frac{1}{\pi}\fint_{-1}^{1}\frac{\chi(t+\rho\sigma)K'(\sigma,\kappa)d\sigma}{\sigma-\kappa}-i\tanh[\pi\phi(t+\rho\kappa)]K'(\kappa,\kappa)\chi(t+\rho\kappa)=0,$$

$$K'(\sigma,\kappa)\equiv e(t+\rho\sigma)+\tilde{e}(t+\rho\kappa)-2f(t+\rho\sigma)\tilde{f}(t+\rho\kappa). \tag{45}$$

Denote $\lim_{\rho\to 0}\rho\chi(t+\rho\sigma)=\chi_0(\sigma)$. When $\rho\to 0$, the kernel $K'(\sigma,\kappa)\to K'(t,t)$.

Tending ρ to zero in Eq. (45) and canceling the common factor, we obtain the integral equation for $\chi_0(\sigma)$:

$$\frac{1}{\pi}\fint_{-1}^{1}\frac{\chi_0(\sigma)d\sigma}{\sigma-\kappa}-i\tanh[\pi\phi(t)]\chi_0(\kappa)=0, \qquad t=\frac{u+v}{2} \tag{46}$$

with the normalizing condition

$$\frac{1}{\pi}\int_{-1}^{1}\chi_0(\sigma)d\sigma=1, \tag{47}$$

which follows from Eq. (42).

Consider now the function

$$X(\kappa)=-\frac{\sqrt{\kappa^2-1}}{\pi}\exp\left(i\phi(t)\int_{-1}^{1}\frac{ds}{s-\kappa}\right)\int_{-1}^{1}\frac{\chi_0(\sigma)d\sigma}{\sigma-\kappa}. \tag{48}$$

This function, according to Eq. (47), tends to 1 when $\kappa\to\infty$. From Eq. (46) it follows that the jump of this function on the cut $(-1,1)$ is equal to zero. According to the Liouville theorem, an analytic function bounded in the extended region is constant; hence $\chi(\sigma)\equiv 1$ and

$$\int_{-1}^{1}\frac{\chi_0(\sigma)d\sigma}{\sigma-\kappa}=-\frac{\pi}{\sqrt{\kappa^2-1}}\exp\left(-i\phi(t)\int_{-1}^{1}\frac{ds}{s-\kappa}\right). \tag{49}$$

Using Sokhotsky's formula in the last equation, we obtain

$$\chi_0(\sigma)=\frac{1}{\sqrt{1-\sigma^2}}\left(\frac{1-\sigma}{1+\sigma}\right)^{i\phi(t)}. \tag{50}$$

From the formulas (43) then follows that on the symmetry axis the potentials $\mathcal{E}(t),\Phi(t)$ are equal to $e(t),f(t)$. Therefore, the Ernst potentials on the symmetry axis determine the transformation of the group of internal symmetries for the neutrino electrovacuum.

V. THE CAUCHY PROBLEM FOR CYLINDRICAL ELECTROVACUUM WAVES FORMING A CLASS OF EXACT SOLUTIONS

In this section we shall construct a class of exact solutions characterized by the rational structure of the Ernst potentials on the symmetry axis. Let

$$e(s)=\frac{E_l(s)}{Q_m(s)}, \quad f(s)=\frac{F_n(s)}{Q_m(s)}, \tag{51}$$

where $E_l,Q_m(s),F_n(s)$ are polynomials of degree l,m,n, respectively, which have no common factors in the form of polynomials of s.

Then, in the absence of the neutrino field, Eq. (41) for the unknown function

$$\chi(s)\equiv\mu(s)Q_m(s)/P(s) \tag{52}$$

reduces to the equation

$$\fint_v^u\frac{\mu(s)P(s,q)ds}{P(s)(s-q)}=0. \tag{53}$$

Here

$$P(s,q)\equiv E_l(s)Q_m^*(q)+E_l^*(q)Q_m(s)-2F_n(s)F_n^*(q),$$

$$P(s)\equiv P(s,s). \tag{54}$$

Let us show that the solution of Eq. (53) can be searched for in the form

$$\mu(s)=(A_0+A_1s+\cdots+A_ks^k)/\sqrt{(u-s)(s-v)}, \tag{55}$$

where k denotes the maximum degree of the polynomials $E_l(s), Q_m(s), F_n(s)$, i.e., $k=\max(l,m,n)$.

Indeed, Eq. (53) can be rewritten in the form

$$\frac{1}{\pi}\fint_v^u \frac{\mu(s)ds}{s-q} = M(q), \tag{56}$$

where

$$M(q) = -\int_v^u \left(E_l(s)\frac{Q_m^*(q)-Q_m^*(s)}{s-q} + Q_m(s)\frac{E_l^*(q)-E_l^*(s)}{s-q} - 2F_n(s)\frac{F_n^*(q)-F_n^*(s)}{s-q}\right)\frac{\mu(s)ds}{P(s)}. \tag{57}$$

It is easy to see that the expression $M(q)$ represents a polynomial in q of degree $k-1$.

Using the well-known inversion formulas for integrals of the type (56) (see, e.g., [27] or [15]), we obtain

$$\mu(s)\sqrt{(u-s)(s-v)} = -\frac{1}{\pi}\fint_v^u M(q)\frac{\sqrt{(u-q)(q-v)}}{s-q}dq + \text{const.} \tag{58}$$

Hence, if $M(q)$ is a polynomial of degree $k-1$, then $\mu(s)\sqrt{(u-s)(s-v)}$ is a polynomial of degree k.[2]

Turning back to Eq. (53), we observe that it can be rewritten in the form

$$[\sqrt{(q-u)(q-v)}R(q)] = 0, \quad R(q) \equiv \frac{1}{\pi}\int_v^u \frac{P(s,q)\mu(s)ds}{P(s)(s-q)}. \tag{59}$$

Here the analytic function $R(q)$ is defined everywhere off the cut. Taking into account Eq. (55), this function can be calculated explicitly with the aid of the residues theorem. To do this, it is advantageous to rewrite $R(q)$ in the form of an integral over the closed contour $\mathcal{L}$ running clockwise (the negative direction), which bounds the cut in the plane s from v to u:

$$R(q) = \frac{1}{2\pi i}\oint_{\mathcal{L}} \frac{P(s,q)(A_0+\cdots+A_k s^k)ds}{P(s)(s-q)\sqrt{(s-u)(s-v)}}. \tag{60}$$

Taking now a closed contour $\mathcal{L}_\infty$ in the vicinity of ∞ which is homotopic to the circumference and contains inside it the cut (v,u) and all the zeros of the polynomial $P(s)$, we get, after applying the residues theorem,

$$R(q) = \frac{A_0+\cdots+A_k q^k}{\sqrt{(q-u)(q-v)}} + \sum_{i=1}^{N}\frac{P(\xi_i,q)(A_0+\cdots+A_k\xi_i^k)}{P'(\xi_i)(\xi_i-q)\sqrt{(\xi_i-u)(\xi_i-v)}} - \frac{1}{2\pi i}\oint_{\mathcal{L}_\infty}\frac{P(s,q)(A_0+\cdots+A_k s^k)ds}{P(s)(s-q)\sqrt{(s-u)(s-v)}}. \tag{61}$$

To calculate the integral over the contour $\mathcal{L}_\infty$, it is sufficient to find the coefficient at the power $1/s$ in the Lorent expansion of the integrand in the vicinity of the point at infinity. After performing this calculation, we obtain

$$R(q) = \frac{A_0+\cdots+A_k q^k}{\sqrt{(q-u)(q-v)}} + \sum_{i=1}^{N}\frac{P(\xi_i,q)(A_0+\cdots+A_k\xi_i^k)}{P'(\xi_i)(\xi_i-q)\sqrt{(\xi_i-u)(\xi_i-v)}} - \sum_{p=0}^{k} A_p \operatorname{res}\left(\frac{P(s,q)s^p}{P(s)(s-q)\sqrt{(s-u)(s-v)}}\right)_{s=\infty}. \tag{62}$$

The insertion of the expression for $R(q)$ into the jump condition (59) gives that the following polynomial of degree $k-1$ in q is equal to zero:

$$\sum_{p=0}^{k} A_p\left(\sum_{i=1}^{N}\frac{P(\xi_i,q)(A_0+\cdots+A_k\xi_i^k)}{P'(\xi_i)(\xi_i-q)\sqrt{(\xi_i-u)(\xi_i-v)}} - \operatorname{res}\left(\frac{P(s,q)s^p}{P(s)(s-q)\sqrt{(s-u)(s-v)}}\right)_{s=\infty}\right) = 0. \tag{63}$$

Collecting the coefficients of the same powers of q on the left of Eq. (63) and equating the resulting expressions to zero, we obtain a system of k homogeneous linear equations for the coefficients $A_0, A_1, \ldots, A_k$. The remaining equation is obtainable from the normalizing condition

$$\frac{1}{\pi}\int_v^u \frac{Q_m(s)\mu(s)ds}{P(s)} = 1, \tag{64}$$

the result being

$$\sum_{i=1}^{N}\frac{(A_0+A_1\xi_i+\cdots+A_k\xi_i^k)Q_m(\xi_i)}{P'(\xi_i)\sqrt{(\xi_i-u)(\xi_i-v)}} - \operatorname{res}\left(\frac{(A_0+\cdots+A_k s^k)Q_m(s)}{P(s)\sqrt{(s-u)(s-v)}}\right)_{s=\infty} = 1. \tag{65}$$

[2] $M(q)$ can be expanded in terms of the Chebyshev polynomials $U_0, \ldots, U_{k-1}$, and then one can get from Eq. (58) the expansion of $\mu(s)\sqrt{(u-s)(s-v)}$ in terms of the Chebyshev polynomials $T_0, T_1, \ldots, T_k$ [29].

With the aid of the coefficients A_0, A_k thus defined, the corresponding solution for $\mathcal{E}$ and Φ can be constructed via the formulas

$$\mathcal{E}=\sum_{i=1}^{N}\frac{A_0+A_1\xi_i+\cdots+A_k\xi_i^k}{P'(\xi_i)\sqrt{(\xi_i-u)(\xi_i-v)}}E_l(\xi_i) -\operatorname{res}\left(\frac{(A_0+\cdots+A_ks^k)E_l(s)}{P(s)\sqrt{(s-u)(s-v)}}\right)_{s=\infty},$$

$$\Phi=\sum_{i=1}^{N}\frac{A_0+A_1\xi_i+\cdots+A_k\xi_i^k}{P'(\xi_i)\sqrt{(\xi_i-u)(\xi_i-v)}}F_n(\xi_i) -\operatorname{res}\left(\frac{(A_0+\cdots+A_ks^k)F_n(s)}{P(s)\sqrt{(s-u)(s-v)}}\right)_{s=\infty}. \tag{66}$$

The formulas (66) fully solve the problem of reconstruction of the Ernst potentials from their values (51) on the symmetry axis.

Let us consider now two particular rational-axis-data solutions from the family obtained:

$$\text{(a)}\quad e(s)=\frac{s-a}{s+a^*},\quad f(s)=\frac{b}{s+a^*},$$

and

$$\text{(b)}\quad e(s)=\frac{s+i\alpha}{s+i\beta},\quad f(s)=\frac{b}{s+i\beta},$$

where a is an arbitrary complex constant, and b,α,β are arbitrary real parameters.

The case (a). Using the theory developed above, we obtain

$$\mathcal{E}=1-\frac{4\xi(a+a^*)}{\Delta},\quad \Phi=\frac{4b\xi}{\Delta},$$
$$\Delta\equiv 2(a+a^*)\xi+(a-a^*+2\xi)\sqrt{(\xi-u)(\xi-v)} -(a-a^*-2\xi)\sqrt{(\xi+u)(\xi+v)},$$
$$\xi\equiv\sqrt{\frac{a^2+a^{*2}}{2}+b^2}. \tag{67}$$

It is of interest to see to which Cauchy problem the solution (67) corresponds. Setting in Eq. (67) $t=0$, i.e., $u=-v=\rho$, we obtain

$$\mathcal{E}\bigg|_{t=0}=1-\frac{a+a^*}{\Delta_0},\quad \Phi\bigg|_{t=0}=\frac{b}{\Delta_0},\quad \Delta_0\equiv\frac{a+a^*}{2}+\sqrt{\xi^2-\rho^2}. \tag{68}$$

When $a+a^*>0$, the initial values (68) of the Ernst potentials are continuous. Calculating $(\partial\mathcal{E}/\partial t)|_{t=0}$ and $(\partial\Phi/\partial t)|_{t=0}$, we have

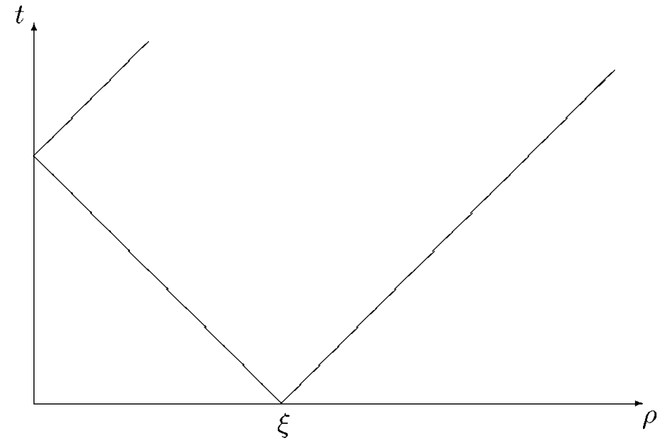

FIG. 1. The discontinuity lines of the electrovacuum wave solution [case (a)].

$$\frac{\partial\mathcal{E}}{\partial t}\bigg|_{t=0}=-\frac{a^2-a^{*2}}{2\Delta_0^2\sqrt{\xi^2-\rho^2}},\quad \frac{\partial\Phi}{\partial t}\bigg|_{t=0}=\frac{b(a-a^*)}{2\Delta_0^2\sqrt{\xi^2-\rho^2}}. \tag{69}$$

Therefore, in the initial Cauchy data (69) the discontinuity of derivatives is present. This discontinuity is represented by two parts: the one going to infinity, and the other first coming to the symmetry axis, then reflecting from it, and finally going away to infinity. In Fig. 1 the lines of discontinuity in the plane (ρ,t) as well as the regions of continuity of the solution are shown.

The case (b). This case corresponds to smooth initial Cauchy data since for $t=0$ the following formulas occur

$$\mathcal{E}\bigg|_{t=0}=1-\frac{\beta-\alpha}{\Delta_0},\qquad \Phi\bigg|_{t=0}=-\frac{ib}{\Delta_0},$$
$$\Delta_0\equiv\frac{\beta-\alpha}{2}+\sqrt{\rho^2+\xi^2},\quad \frac{\partial\mathcal{E}}{\partial t}\bigg|_{t=0}=-\frac{i(\beta^2-\alpha^2)}{\Delta_0^2},$$
$$\frac{\partial\Phi}{\partial t}\bigg|_{t=0}=-\frac{b(\alpha+\beta)}{\Delta_0^2}. \tag{70}$$

The solution of the Cauchy problem corresponding to the initial data (70) is given by the formulas

$$\mathcal{E}=1+\frac{4i\xi(\beta-\alpha)}{\Delta},\qquad \Phi=-\frac{4b\xi}{\Delta},$$
$$\Delta\equiv 2i\xi(\alpha-\beta)+(\alpha+\beta-2\xi)\sqrt{(i\xi-u)(i\xi-v)} -(\alpha+\beta+2\xi)\sqrt{(i\xi+u)(i\xi+v)},$$
$$\xi\equiv\sqrt{\alpha\beta-b^2}. \tag{71}$$

For $\beta>\alpha>0$ and $\alpha\beta>\beta^2$, this solution has no discontinuities and is smooth everywhere.

Therefore, if the algebraic equation $P(s)=0$ has p real positive roots $\xi_1,\dots,\xi_p$, then the corresponding Cauchy problem will involve p initial discontinuities of the first derivatives. Similar to the case (a), each discontinuity decom-

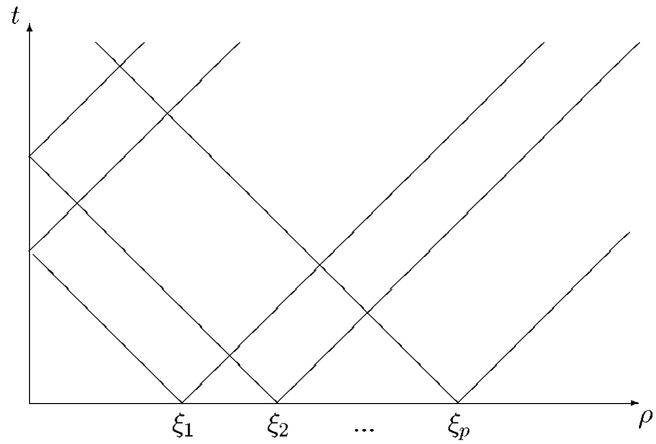

FIG. 2. The discontinuity lines emerging in the first derivatives of the Ernst potentials of the class C^1 solutions.

poses into two parts, the one going to infinity and the other running toward the symmetry axis with a subsequent reflection. In Fig. 2 the lines of discontinuity for the first derivatives of $\mathcal{E},\Phi$ in the plane (ρ,t), and the regions of continuity for solutions of the class C^1 are shown.

VI. CYLINDRICAL WAVES IN THE NEUTRINO ELECTROVACUUM

Consider now a solution of the integral equation (41) describing cylindrical waves in the neutrino electrovacuum, which corresponds to the rational functions $e(s),f(s)$ defined by formulas (51). According to Sec. III, $e(s)$ and $f(s)$ determining the transformation of internal symmetry have the meaning of the Ernst potentials on the symmetry axis.

Equation (41), after introducing the function $\mu(s)$ via the formula

$$\chi(s)=\frac{\mu(s)Q_m(s)}{P(s)}, \tag{72}$$

assumes the form

$$\frac{1}{\pi}\fint_v^u \frac{\mu(s)P(s,q)ds}{P(s)(s-q)}-i\tanh[\pi\phi(q)]\mu(s)=0, \tag{73}$$

where the polynomials $P(s,q),P(s)$ are defined by the formulas (54).

Let us show that the function $\mu(s)$ to be determined can be searched for in the form

$$\mu(s)=(A_0+\cdots+A_k s^k)\left[\frac{1}{\lambda}\right], \tag{74}$$

where

$$\lambda(s)\equiv\sqrt{(s-u)(s-v)}\exp\left(i\int_v^u \frac{\phi(\tau)d\tau}{\tau-s}\right),$$

$$\left[\frac{1}{\lambda(s)}\right]=\frac{2\cosh[\pi\phi(s)]}{\sqrt{(s-u)(s-v)}}\exp\left(-i\fint_v^u\frac{\phi(\tau)d\tau}{\tau-s}\right). \tag{75}$$

For this purpose, following Carleman (see Ref. [29]), we extract the singular part from the kernel (73), and denote the remaining part as $M(q)$:

$$\frac{1}{\pi}\fint_v^u\frac{\mu(s)ds}{s-q}-i\tanh[\pi\phi(q)]\mu(s)=M(q),$$

$$M(q)=-\int_v^u\frac{P(s,q)-P(s)}{P(s)(s-q)}\mu(s)ds. \tag{76}$$

The expression $M(q)$ is a polynomial in q of degree $k-1$.

Equation (76) is a singular integral equation of the Carleman type whose exact solution is known [29]. However, for our analysis it is convenient to use the Carleman-Tricomi ideas in a slightly different form than exposed in the original approach.

Note that Eq. (76) can be written down in the form

$$\begin{aligned}2[\lambda(q)\varphi(q)]&=\{\lambda(q)\}[\varphi(q)]+[\lambda(q)]\{\varphi(q)\}\\&=M(q)[\lambda(q)],\end{aligned} \tag{77}$$

where

$$\varphi(q)=\frac{1}{2\pi}\int_v^u\frac{\mu(s)ds}{s-q}\Rightarrow\quad\{\varphi(q)\}=\frac{1}{\pi}\fint_v^u\frac{\mu(s)ds}{s-q}, \tag{78}$$

whence

$$\lambda(q)\varphi(q)=\frac{1}{4\pi i}\int_v^u\frac{M(s)[\lambda(s)]ds}{s-q}+\frac{C_1}{2},\quad C_1=\text{const.} \tag{79}$$

From Eq. (79) follows that

$$\begin{aligned}2[\varphi(q)]=2i\mu(q)=&\left[\frac{1}{\lambda}\right]\left(\frac{1}{2\pi i}\fint_v^u\frac{M(s)[\lambda(s)]ds}{s-q}+C_1\right)\\&+\left\{\frac{1}{\lambda}\right\}\frac{M(q)[\lambda(q)]}{2}.\end{aligned} \tag{80}$$

The function $\lambda-q-C_2$ with

$$C_2\equiv-\frac{u+v}{2}+i\int_v^u\phi(\tau)d\tau \tag{81}$$

is equal to zero at infinity, and can be represented in the form of the Cauchy type integral

$$\lambda-q-C_2=\frac{1}{2\pi i}\int\frac{[\lambda]ds}{s-q}, \tag{82}$$

whence

$$\{\lambda\}=2q+2C_2+\frac{1}{\pi i}\int_v^u\frac{[\lambda(s)]ds}{s-q}. \tag{83}$$

Taking into account Eq. (83), the right-hand side of Eq. (80) assumes the form

$$\begin{aligned}2i\mu(q)=&\left[\frac{1}{\lambda}\right]\frac{1}{2\pi i}\int_v^u\frac{M(s)-M(q)}{s-q}[\lambda(s)]ds+\frac{M(q)}{2}\\&\times\left(\left\{\frac{1}{\lambda}\right\}[\lambda]+\left[\frac{1}{\lambda}\right]\{\lambda\}\right)+\left[\frac{1}{\lambda}\right](C_1-(q+C_2)M(q))\\=&\left[\frac{1}{\lambda}\right]\left(\frac{1}{2\pi i}\int_v^u\frac{M(s)-M(q)}{s-q}[\lambda(s)]ds\right.\\&\left.+C_1-(q+C_2)M(q)\right).\end{aligned} \tag{84}$$

In the above equation we have used the fact that

$$\left\{\frac{1}{\lambda}\right\}[\lambda]+\left[\frac{1}{\lambda}\right]\{\lambda\}=0. \tag{85}$$

From Eq. (84) one finally concludes that $\mu(s)$ is indeed representable in the form (74).

Let us now write down Eq. (73) in a form equivalent to Eq. (77):

$$\begin{aligned}&[\lambda(q)R(q)]=0,\\&R(q)\equiv\frac{1}{2\pi i}\int_v^u\frac{P(s,q)(A_0+\cdots+A_ks^k)}{P(s)(s-q)}\left[\frac{1}{\lambda}\right]ds\\&=\frac{1}{2\pi i}\oint_{\mathcal{L}}\frac{P(s,q)(A_0+\cdots+A_ks^k)ds}{P(s)(s-q)\lambda(s)}.\end{aligned} \tag{86}$$

In the formula (86), $\mathcal{L}$ is an arbitrary closed contour which bounds the cut (v,u) and should be passed over clockwise; it does not contain zeros of the function $P(s)$.

The function $R(q)$ can be found in explicit form with the aid of the residues theorem. Indeed, according to this theorem

$$\begin{aligned}&\frac{1}{2\pi i}\left(\int_{\mathcal{L}_\infty}+\int_{\mathcal{L}}\right)\frac{P(s,q)(A_0+\cdots+A_ks^k)ds}{P(s)(s-q)\lambda(s)}\\&=\frac{A_0+\cdots+A_kq^k}{\lambda(q)}+\sum_{a=1}^N\frac{P(\xi_a,q)(A_0+\cdots+A_k\xi_a^k)}{P'(\xi_a)(\xi_a-q)\lambda(\xi_a)},\end{aligned} \tag{87}$$

where $\mathcal{L}_\infty$ is a closed contour in the vicinity of the point at infinity.

The integral $\int_{\mathcal{L}_\infty}$ can also be calculated with the help of the residues theorem:

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\mathcal{L}_\infty}\frac{P(s,q)(A_0+\cdots+A_ks^k)ds}{P(s)(s-q)\lambda(s)}\\&=\sum_{p=0}^kA_p\mathrm{res}\left(\frac{P(s,q)s^p}{P(s)(s-q)\lambda(s)}\right)_{s=\infty}.\end{aligned} \tag{88}$$

In this formula, $\mathrm{res}_{s=\infty}f(s)$ is the coefficient at $1/s$ of the Lorent expansion of the function $f(s)$ in the vicinity of the point at infinity where it has a pole of finite order.

Substituting Eq. (88) into Eq. (87) we obtain the expression for $R(q)$ which in turn should be substituted into Eq. (86). Having performed this operation, we get

$$\begin{aligned}&\sum_{a=1}^N\frac{P(\xi_a,q)(A_0+\cdots+A_k\xi_a^k)}{P'(\xi_a)(\xi_a-q)\lambda(\xi_a)}\\&-\sum_{p=0}^kA_p\mathrm{res}\left(\frac{P(s,q)s^p}{P(s)(s-q)\lambda(s)}\right)_{s=\infty}=0.\end{aligned} \tag{89}$$

The polynomial $P(\xi_a,q)$ is divisible by ξ_a-q. Hence, the left side of Eq. (89) is a polynomial of order $k-1$ in q. Equating to zero each coefficient of this equation, we arrive at k homogeneous equations for the $k+1$ quantities $A_0,\ldots,A_k$. The remaining equation we obtain from the normalizing condition

$$\begin{aligned}&2i\sum_{a=1}^N\frac{(A_0+\cdots+A_k\xi_a^k)Q_m(\xi_a)}{P'(\xi_a)\lambda(\xi_a)}\\&-2i\,\mathrm{res}\left(\frac{(A_0+\cdots+A_ks^k)Q_m(s)}{P(s)\lambda(s)}\right)_{s=\infty}=1.\end{aligned} \tag{90}$$

In terms of the known coefficients $A_0,\ldots,A_k$ the Ernst potentials can be found via the formulas

$$\begin{aligned}\mathcal{E}=&2i\sum_{a=1}^N\frac{A_0+\cdots+A_k\xi_a^k}{P'(\xi_a)\lambda(\xi_a)}E_l(\xi_a)\\&-2i\,\mathrm{res}\left(\frac{A_0+\cdots+A_ks^k}{P(s)\lambda(s)}E_l(s)\right)_{s=\infty},\\\Phi=&2i\sum_{a=1}^N\frac{A_0+\cdots+A_k\xi_a^k}{P'(\xi_a)\lambda(\xi_a)}F_n(\xi_a)\\&-2i\,\mathrm{res}\left(\frac{A_0+\cdots+A_ks^k}{P(s)\lambda(s)}F_n(s)\right)_{s=\infty}.\end{aligned} \tag{91}$$

Therefore, the problem of finding the Ernst potentials from their data (51) on the symmetry axis for cylindrical waves in the neutrino electrovacuum is completely solved.

It is worthwhile pointing out that the case of exact solutions possessing the pseudoeuclidean asymptotics at $t\to\infty$ is defined by $l=m>n$. Therefore, $k=l$, and the system of linear algebraic equations (89),(90), as well as its analogue in the absence of the neutrino field (63),(65), assumes a simpler form:

$$\sum_{a=1}^{N} \frac{P(\xi_a,q)(A_0+\cdots+A_k\xi_a^k)}{P'(\xi_a)(\xi_a-q)\lambda(\xi_a)}=0, \tag{92}$$

and

$$2i\sum_{a=1}^{N} \frac{(A_0+\cdots+A_k\xi_a^k)Q_m(\xi_a)}{P'(\xi_a)\lambda(\xi_a)}-2iA_k a_0=1,$$

$$a_0\equiv \lim_{s\to\infty}\frac{s^kQ_m(s)}{P(s)}, \tag{93}$$

respectively. Formulas (91) also simplify:

$$\mathcal{E}d=2i\sum_{a=1}^{N} \frac{A_0+\cdots+A_k\xi_a^k}{P'(\xi_a)\lambda(\xi_a)}E_l(\xi_a)-2iA_kb_0,$$

$$\Phi=2i\sum_{a=1}^{N} \frac{A_0+\cdots+A_k\xi_a^k}{P'(\xi_a)\lambda(\xi_a)}F_n(\xi_a),$$

$$b_0\equiv \lim_{s\to\infty}\frac{s^kE_l(s)}{P(s)}. \tag{94}$$

Let us compare the rational-axis-data solutions for the neutrino electrovacuum obtained in this section and the electrovacuum solutions from the previous section. It is easy to notice that in the case $k=l=m>n$ the former solutions are obtainable from the latter by simply changing the square roots $\sqrt{(\xi_i-u)(\xi_i-v)}$ to the expressions $\sqrt{(\xi_i-u)(\xi_i-v)}\exp[i\int_v^u \phi(\tau)d\tau/(\tau-\xi_i)]$. Thus, the neutrino generalizations of two particular electrovacuum solutions from the previous section have the forms

$$\mathcal{E}=1-\frac{4\xi(a+a^*)}{\Delta},\quad \Phi=\frac{4b\xi}{\Delta},$$

$$\Delta\equiv 2(a+a^*)\xi+(a-a^*+2\xi)\sqrt{(\xi-u)(\xi-v)}$$
$$\times\exp\left(i\int_v^u\frac{\phi(\tau)d\tau}{\tau-\xi}\right)-(a-a^*-2\xi)$$
$$\times\sqrt{(\xi+u)(\xi+v)}\exp\left(i\int_v^u\frac{\phi(\tau)d\tau}{\tau+\xi}\right),$$

$$\xi\equiv\sqrt{\frac{a^2+a^{*2}}{2}+b^2} \tag{95}$$

[the analogue of case (a)] and

$$\mathcal{E}=1+\frac{4i\xi(\beta-\alpha)}{\Delta},\qquad \Phi=-\frac{4b\xi}{\Delta},$$

$$\Delta\equiv 2i\xi(\alpha-\beta)+(\alpha+\beta-2\xi)\sqrt{(i\xi-u)(i\xi-v)}$$
$$\times\exp\left(i\int_v^u\frac{\phi(\tau)d\tau}{\tau-i\xi}\right)-(\alpha+\beta+2\xi)$$
$$\times\sqrt{(i\xi+u)(i\xi+v)}\exp\left(i\int_v^u\frac{\phi(\tau)d\tau}{\tau+i\xi}\right), \tag{96}$$

$$\xi\equiv\sqrt{\alpha\beta-b^2}$$

[the analogue of case (b)].

VII. CONCLUSIONS

Therefore, we have been able to develop a general approach to the mathematical description of cylindrical waves starting from the self-consistent Einstein-Maxwell-Weyl system. We obtained the canonical integral equations, as well as the generalization of the Ernst equations in the presence of neutrino fields. The integral equations that govern the behavior of cylindrical waves in the general case are of classical type, similar to those found in the theory of elasticity and aerodynamics [29,13,14]. The method developed allowed us to construct a wide class of exact solutions defined by the rational data for the Ernst potentials on the symmetry axis. We solved the Cauchy problem for some initial data and showed in particular that the initial disruption disintegrates into two discontinuities, one of which propagates toward the axis of symmetry with subsequent reflection from it, while the other goes directly to infinity.

The nontrivial complementary problem of reconstruction of all the metric coefficients entering Eq. (1) from the generalized Ernst potentials, which represents a by far more difficult technical procedure than in the pure electrovacuum case, will be considered elsewhere.

ACKNOWLEDGMENTS

We are thankful to Alex Feinstein for valuable comments. This work has been partially supported by Projects 34222-E and 38495-E from CONACyT of Mexico. N.S. acknowledges financial support from SNI-CONACyT, Mexico.

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