Generating G_2 cosmologies with a perfect fluid in dilaton gravity

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We present a method for generating exact diagonal cosmological solutions with two spacelike commuting Killing vectors (G_2 cosmologies) in dilaton gravity coupled to a radiation perfect fluid and with a cosmological potential of a special type. The method is based on the symmetry group of the system of G_2 field equations. Several new classes of explicit exact inhomogeneous perfect fluid scalar-tensor cosmologies are presented.

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I. INTRODUCTION

The generalized scalar-tensor theories of gravity are considered as the most natural generalization of general relativity. Their importance for current physics is related to string theory, which, in its low energy limit, predicts the existence of a scalar partner of the tensor graviton. A large amount of research has been devoted to dilaton cosmology [1–21] (and references therein).

The interest in studying inhomogeneous (and anisotropic) cosmological models is motivated by the following causes. As is well known, the present Universe is not exactly spatially homogeneous, not even at large scales. Although homogeneous models are good approximations of the present Universe, there is no reason to assume that such a regular expansion is suitable for a description of the early Universe. A theoretical explanation of the formation of large scale structures in the Universe also necessitates inhomogeneous models. Contrary to the general belief, it was shown that the existence of large inhomogeneities in the Universe does not necessarily lead to an observable effect left over in the spectrum of the cosmic microwave background (CMB) [22–25]. The existence of homogeneous but highly anisotropic cosmological models whose CMB is exactly isotropic was also demonstrated [26]. In addition, the inhomogeneous cosmological solutions allow us to investigate a number of long standing questions regarding the occurrence of singularities, the behavior of spacetime in the vicinity of a singularity, and the possibility of our universe arising from generic initial data.

In light of the above reasons, the study of inhomogeneous cosmological models is necessary and even imperative. The ideal case is to find general classes of inhomogeneous cosmological solutions of the field equation without any symmetry. However, this seems to be a hopeless task due to the complexity of the field equations. That is why we are forced to assume some simplifications in order to solve the field equations. Usually inhomogeneous models with two spacelike commuting Killing vectors (the so called G_2 cosmologies) are considered. Even for these simple cosmological models, few exact perfect fluid solutions are known in general relativity. The first such class of exact solutions was found by Wainwright and Goode [27]. Other classes were later given in [28–31]. All solutions were obtained by assuming the separation of variables of the metric components.

With regard to the scalar-tensor theories, there are no known exact inhomogeneous perfect fluid G_2 cosmological solutions. The reason is that the scalar-tensor equations are more complex than the Einstein ones and include arbitrary functions of the dilaton field. That is why finding exact perfect fluid solutions which hold for all scalar-tensor theories is unrealistic in the general case. However, for some special equations of state it is possible to find exact solutions that hold for all scalar-tensor theories. In [32], methods for generating scalar-tensor stiff perfect fluid cosmologies were developed and some explicit solutions were presented in [32], [33], and [34].

Another equation of state that is realistic and allows us to solve the field equations for all scalar-tensor theories (with a special form of the dilaton potential) is $\rho=3p$. It is the purpose of this paper to present a method for generating inhomogeneous perfect fluid diagonal G_2 cosmologies with equation of state $\rho=3p$ in scalar-tensor theories. As an illustration and important consequence of the method, new classes of exact inhomogeneous perfect fluid G_2 cosmological solutions are also presented for all scalar-tensor theories.

II. SOLUTION GENERATING

The general form of the extended gravitational action in scalar-tensor theories is

$$S=\frac{1}{16\pi G_*}\int d^4x\sqrt{-\tilde{g}}[F(\Phi)\tilde{R}-Z(\Phi)\tilde{g}^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi -2U(\Phi)]+S_m[\Psi_m;\tilde{g}_{\mu\nu}]. \tag{1}$$

Here, G_* is the bare gravitational constant, and $\tilde{R}$ is the Ricci scalar curvature with respect to the spacetime metric $\tilde{g}_{\mu\nu}$. The dynamics of the scalar field Φ depends on the functions $F(\Phi)$, $Z(\Phi)$, and $U(\Phi)$. In order for the gravitons to carry positive energy the function $F(\Phi)$ must be positive. The non-negativity of the energy of the dilaton field requires that $2F(\Phi)Z(\Phi)+3[dF(\Phi)/d\Phi]^2\geqslant 0$. The action of matter depends on the material fields Ψ_m and the spacetime metric $\tilde{g}_{\mu\nu}$. It should be noted that the stringy generated scalar-tensor theories, in general, admit a direct interaction between the matter fields and the dilaton in the Jordan (string) frame [3]. Here we consider the phenomenological case when the matter action does not involve the dilaton field in order for the weak equivalence principle to be satisfied.

However, the method we present here holds for the general case since we consider a radiation fluid with a traceless energy-momentum tensor.

It is much more convenient from a mathematical point of view to analyze the scalar-tensor theories with respect to the conformally related Einstein frame given by the metric

$$g_{\mu\nu}=F(\Phi)\tilde{g}_{\mu\nu}. \tag{2}$$

Further, let us introduce the scalar field φ (the so called dilaton) via the equation

$$\left(\frac{d\varphi}{d\Phi}\right)^2=\frac{3}{4}\left(\frac{d\ln(F(\Phi))}{d\Phi}\right)^2+\frac{Z(\Phi)}{2F(\Phi)} \tag{3}$$

and define

$$\mathcal{A}(\varphi)=F^{-1/2}(\Phi),\quad 2V(\varphi)=U(\Phi)F^{-2}(\Phi). \tag{4}$$

In the Einstein frame the action (1) takes the form

$$S=\frac{1}{16\pi G_*}\int d^4x\sqrt{-g}[R-2g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi-4V(\varphi)]$$
$$+S_m[\Psi_m;\mathcal{A}^2(\varphi)g_{\mu\nu}], \tag{5}$$

where R is the Ricci scalar curvature with respect to the Einstein metric $g_{\mu\nu}$.

The Einstein frame field equations are then

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=8\pi G_*T_{\mu\nu}+2\partial_\mu\varphi\partial_\nu\varphi$$
$$-g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi-2V(\varphi)g_{\mu\nu},$$
$$\nabla^\mu\nabla_\mu\varphi=-4\pi G_*\alpha(\varphi)T+\frac{dV(\varphi)}{d\varphi}, \tag{6}$$
$$\nabla_\mu T^\mu_\nu=\alpha(\varphi)T\partial_\nu\varphi.$$

Here $\alpha(\varphi)=d\ln[\mathcal{A}(\varphi)]/d\varphi$ and the Einstein frame energy-momentum tensor $T_{\mu\nu}$ is related to the Jordan frame one $\tilde{T}_{\mu\nu}$ via $T_{\mu\nu}=\mathcal{A}^2(\varphi)\tilde{T}_{\mu\nu}$. In the case of a perfect fluid one has

$$\rho=\mathcal{A}^4(\varphi)\tilde{\rho},$$
$$p=\mathcal{A}^4(\varphi)\tilde{p}, \tag{7}$$
$$u_\mu=\mathcal{A}^{-1}(\varphi)\tilde{u}_\mu.$$

In the present paper we consider spacetimes admitting two hypersurfaces and mutually orthogonal Killing vectors $K_1=\partial/\partial y$ and $K_2=\partial/\partial z$. We also require the dilaton field to satisfy

$$\mathcal{L}_{K_1}\varphi=\mathcal{L}_{K_2}\varphi=0 \tag{8}$$

where $\mathcal{L}_K$ is the Lie derivative along the Killing vector K.

The metric can be presented in the Einstein-Rosen form

$$ds^2=D(t,x)[-dt^2+dx^2]$$
$$+B(t,x)[C(t,x)dy^2+C^{-1}(t,x)dz^2] \tag{9}$$

and the fluid velocity is given by

$$u=D^{-1/2}\frac{\partial}{\partial t}. \tag{10}$$

In what follows we will consider a scalar potential of the form $V(\varphi)=\Lambda=\text{const}$ [i.e., $U(\Phi)=2\Lambda F^2(\Phi)$].

Under all these assumptions we obtain the following system of partial differential equations:

$$-\partial_t^2\ln D+\partial_x^2\ln D+\partial_t\ln D\partial_t\ln B-\partial_t^2\ln B-\frac{\partial_t^2B}{B}+\partial_x\ln D\partial_x\ln B$$
$$-(\partial_t\ln C)^2=8\pi G_*(\rho+3p)D+4(\partial_t\varphi)^2-4\Lambda D, \tag{11}$$

$$\partial_t^2\ln D-\partial_x^2\ln D+\partial_t\ln D\partial_t\ln B+\partial_x\ln D\partial_x\ln B-\partial_x^2\ln B-\frac{\partial_x^2B}{B}$$
$$-(\partial_x\ln C)^2=8\pi G_*(\rho-p)D+4(\partial_x\varphi)^2+4\Lambda D, \tag{12}$$

$$\partial_t\ln B\partial_x\ln D+\partial_t\ln D\partial_x\ln B+\partial_t\ln B\partial_x\ln B$$
$$-2\frac{\partial_t\partial_xB}{B}-\partial_t\ln C\partial_x\ln C=4\partial_t\varphi\partial_x\varphi, \tag{13}$$

$$\frac{\partial_t^2B}{B}-\frac{\partial_x^2B}{B}=8\pi G_*(\rho-p)D+4\Lambda D, \tag{14}$$

$$\frac{1}{B}\partial_t(B\partial_t\ln C)-\frac{1}{B}\partial_x(B\partial_x\ln C)=0, \tag{15}$$

$$\frac{1}{B}\partial_t(B\partial_t\varphi)-\frac{1}{B}\partial_x(B\partial_x\varphi)=0. \tag{16}$$

The above system of partial differential equations (11)–(16) is invariant under the group of symmetries $\mathrm{Iso}(\mathcal{R}^2)$. Let us introduce

$$X=\begin{pmatrix}\ln C\\ 2\varphi\end{pmatrix}\in\mathcal{R}^2. \tag{17}$$

The explicit action of the group is given as follows:

$$X\to MX+\xi \tag{18}$$

where $M\in O(2)$ and $\xi\in\mathcal{R}^2$.

The group of symmetries can be used to generate new solutions from known ones, especially to generate solutions with a nontrivial dilaton field from pure general relativistic G_2 cosmologies.

The subgroup of translations corresponds to a constant shift of the dilaton field ($\varphi\to\varphi+\text{const}$)and to a constant rescaling of the metric function $C(C\to\text{const}\times C)$. That is why, without loss of generality we shall restrict ourselves to the subgroup $SO(2)\in\mathrm{Iso}(\mathcal{R}^2)$ consisting of the matrices

$$M=\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}. \tag{19}$$

The remaining discrete Z_2 subgroup corresponds to the transformation $C\to C^{-1}$ or $\varphi\to-\varphi$.

Let us consider an arbitrary solution of the G_2 Einstein equations with a radiation perfect fluid and a cosmological term:

$$ds_E^2=D_E(t,x)[-dt^2+dx^2] +B_E(t,x)[C_E(t,x)dy^2+C_E^{-1}(t,x)dz^2], \tag{20}$$

$$\rho_E=\rho_E(t,x), \tag{21}$$

$$u_E^\mu=u_E^\mu(t,x). \tag{22}$$

The $SO(2)$ transformation then generates a new Einstein frame scalar-tensor solution as follows:

$$ds^2=D_E(t,x)[-dt^2+dx^2] +B_E(t,x)[C(t,x)dy^2+C^{-1}(t,x)dz^2], \tag{23}$$

$$\rho=\rho_E, \tag{24}$$

$$u^\mu=u_E^\mu, \tag{25}$$

$$\varphi=-\frac{1}{2}\sin(\theta)\ln C_E, \tag{26}$$

where

$$\ln C=\cos(\theta)\ln C_E. \tag{27}$$

The Z_2 transformations can be used to restrict[1] the range of the parameter θ to $0\leqslant\theta\leqslant\pi$. Let us note that for the particular value $\theta=\pi/2$ we obtain plane symmetric solutions.

The Jordan frame solutions are given by

$$F[\Phi(t,x)]=\mathcal{A}^2[-\sin(\theta)\ln C_E(t,x)], \tag{28}$$

$$d\tilde{s}^2=F^{-1}(\Phi)ds^2, \tag{29}$$

$$\tilde{\rho}=F^2(\Phi)\rho_E, \tag{30}$$

$$\tilde{u}^\mu=F^{-1/2}(\Phi)u_E^\mu. \tag{31}$$

In the above considerations the metric of the (t,x) space was taken to be in an isotropic form. It should be noted (and it is easy to see) that the solution generating method is applicable for an arbitrary form of the (t,x)-space metric.

[1]When the coordinates y and z have the same topology we can restrict the range of θ to $0\leqslant\theta\leqslant\pi/2$ since the metric is invariant under the simultaneous transformations $C(t,x)\to C^{-1}(t,x)$ and $y\to z$.

III. EXAMPLES OF EXPLICIT EXACT INHOMOGENEOUS COSMOLOGICAL SOLUTIONS

As an illustration of the solution generating method we consider some classes of explicit exact inhomogeneous scalar-tensor cosmologies with $\Lambda=0$.

A. Class 1

Let us consider Senovilla's solution [29] (see also [28]):

$$ds_E^2=T^4(t)\cosh^2(3ax)[-dt^2+dx^2] +B_E(t,x)[T^3(t)\sinh(3ax)dy^2 +T^{-3}(t)\sinh^{-1}(3ax)dz^2], \tag{32}$$

$$8\pi G_*\rho_E=15a^2T^{-4}(t)\cosh^{-4}(3ax), \tag{33}$$

$$u_E=T^{-2}(t)\cosh^{-1}(3ax)\frac{\partial}{\partial t}, \tag{34}$$

where

$$T(t)=\lambda_1\cosh(at)+\lambda_2\sinh(at), \tag{35}$$

$$B_E(t,x)=T(t)\sinh(3ax)\cosh^{-2/3}(3ax), \tag{36}$$

and $a>0$, λ_1, and λ_2 are arbitrary constants.

The solution generating method gives the following scalar-tensor solution:

$$ds^2=T^4(t)\cosh^2(3ax)[-dt^2+dx^2] +B_E(t,x)[T^{3\cos(\theta)}(t)\sinh^{\cos(\theta)}(3ax)dy^2 +T^{-3\cos(\theta)}(t)\sinh^{-\cos(\theta)}(3ax)dz^2], \tag{37}$$

$$\varphi=-\frac{1}{2}\sin(\theta)\ln[T^3(t)\sinh(3ax)]. \tag{38}$$

B. Class 2

Wainwright and Goode's solution [27] is given by

$$ds_E^2=\sinh^4(2qt)\cosh^2(3qx)[-dt^2+dx^2] +B_E(t,x)[\tanh^3(qt)dy^2+\tanh^{-3}(qt)dz^2], \tag{39}$$

$$8\pi G_*\rho_E=15q^2\sinh^{-4}(2qt)\cosh^{-4}(3qt), \tag{40}$$

$$u_E=\sinh^{-2}(2qt)\cosh^{-1}(3qx)\frac{\partial}{\partial t}, \tag{41}$$

where

$$B_E(t,x)=\sinh(2qt)\cosh^{-2/3}(3qx) \tag{42}$$

and $q>0$ is an arbitrary constant.

The corresponding scalar-tensor image of that solution is the following:

$$ds^2=\sinh^4(2qt)\cosh^2(3qx)[-dt^2+dx^2]+B_E(t,x)\times[\tanh^{3\cos(\theta)}(qt)dy^2+\tanh^{-3\cos(\theta)}(qt)dz^2], \quad (43)$$

$$\varphi(t,x)=-\frac{3}{2}\sin(\theta)\ln[\tanh(qt)]. \quad (44)$$

C. Class 3

The solution found by Davidson is the following [31]:

$$ds_E^2=-(1+x^2)^{6/5}dt^2+t^{4/3}(1+x^2)^{2/5}dx^2+B_E(t,x)[(tx)dy^2+(tx)^{-1}dz^2], \quad (45)$$

$$8\pi G_*\rho_E=\frac{12}{5}t^{-4/3}(1+x^2)^{-12/5}, \quad (46)$$

$$u_E=(1+x^2)^{-3/5}\frac{\partial}{\partial t}, \quad (47)$$

where

$$B_E(t,x)=t^{1/3}(1+x^2)^{-2/5}x. \quad (48)$$

Its scalar-tensor image is given by

$$ds^2=-(1+x^2)^{6/5}dt^2+t^{4/3}(1+x^2)^{2/5}dx^2+B_E(t,x)[(tx)^{\cos(\theta)}dy^2+(tx)^{-\cos(\theta)}dz^2], \quad (49)$$

$$\varphi(t,x)=-\frac{1}{2}\sin(\theta)\ln(tx). \quad (50)$$

D. Class 4

Here as a seed solution we take Collins's solution of Bianchi type VI_h [35]:

$$ds_E^2=-d\tau^2+\tau^2dx^2+B_E(\tau,x)[\tau^{\sqrt{3}b/2}e^{\sqrt{3}x/2}dy^2+\tau^{-\sqrt{3}b/2}e^{-\sqrt{3}x/2}dz^2], \quad (51)$$

$$8\pi G_*\rho_E=\frac{3}{8}\frac{1-b^2}{\tau^2}, \quad (52)$$

$$u_E=\frac{\partial}{\partial\tau} \quad (53)$$

where $B_E(\tau,x)$ is given by

$$B_E(\tau,x)=\tau^{1/2}e^{bx/2} \quad (54)$$

and $0<b<1$.

The corresponding Einstein frame scalar-tensor solution is the following:

$$ds^2=-d\tau^2+\tau^2dx^2+B_E(\tau,x)[\tau^{\sqrt{3}b\cos(\theta)/2}\times e^{\sqrt{3}\cos(\theta)x/2}dy^2+\tau^{-\sqrt{3}b\cos(\theta)/2}e^{-\sqrt{3}\cos(\theta)x/2}dz^2], \quad (55)$$

$$\varphi(\tau,x)=-\frac{\sqrt{3}}{4}\sin(\theta)(b\ln\tau+x). \quad (56)$$

The Einstein frame metric is homogeneous while the dilaton field is not constant over the surface of homogeneity. So we have a "tilted" cosmological solution in the Einstein frame. The Jordan frame solution, however, is inhomogeneous.

We could generate many more examples of exact solutions which are images of the known G_2 Einstein cosmologies (see, for example, the solutions given in [30,36], and [37]). However, the explicit solutions given here are representative and qualitatively cover the general case.

It should be noted that the properties of the solutions found in the physical Jordan frame depend strongly on the particular scalar-tensor theory and need a separate investigation.

IV. CONCLUSION

In this paper we have presented a simple and effective method for generating exact G_2 cosmologies in scalar-tensor theories with a constant dilaton potential, $V(\varphi)=\text{const}$, and coupled to a perfect fluid with an equation of state $\rho=3p$. Several classes of explicit exact solutions have been given. These solutions are the only known explicit perfect fluid scalar-tensor G_2 cosmologies.

It is worth noting that the solutions can be found by assuming separation of variables of the metric components [38]. However, the way we derived the solutions here is much more elegant and is applicable to more general cases when the metric components are not separable.

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