Gauge-invariant Hamiltonian dynamics of cylindrical gravitational waves

Ioannis Kouletsis and Petr Hájíček
Institute of Theoretical Physics, University of Berne, Berne, Switzerland

Jiří Bičák
Institute of Theoretical Physics, Charles University, V Holešovičkách 2, 182 00 Prague 8, Czech Republic

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The model of cylindrical gravitational waves is employed to work out and check a recent proposal of how a diffeomorphism-invariant Hamiltonian dynamics is to be constructed. The starting point is the action by Ashtekar and Pierri because it contains the boundary term that makes it differentiable for nontrivial variations at infinity. With the help of parametrization at infinity, the notion of gauge transformation is clearly separated from that of asymptotic symmetry. The symplectic geometry of asymptotic symmetries and asymptotic time is described and the role of the asymptotic structures in defining a zero-motion frame for the Hamiltonian dynamics of Dirac observables is explained. Complete sets of Dirac observables associated with the asymptotic fields are found and the action of the asymptotic symmetries on them is calculated. The construction of the corresponding quantum theory is sketched: the Fock space, operators of asymptotic fields, the Hamiltonian and the scattering matrix are determined.

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I. INTRODUCTION

A deep problem in quantum gravity is the dependence of the constructed quantum theory on the choice of coordinates. A famous example of a quantization method based on such a choice of gauge is the Arnowitt-Deser-Misner reduction [1]. There have been ideas of how this problem could be avoided, such as Dirac's operator constraint equations [2], Bergmann's work on Dirac observables [3], the Becchi-Rouet-Stora-Tyutin (BRST) method [4], Euclidean quantum gravity [5] etc. None of these methods has as yet been really successful because new problems always emerged.

In this paper, we focus on the method based on the Dirac observables. In a sense, this is the most straightforward one: all variables that are used have to be gauge invariant. However, changes of coordinates include changes of time so that the gauge invariance entails time independence: Dirac observables must be integrals of motion. Two problems that are related to this have been noticed already by Bergmann [3]: "*frozen dynamics*" and *scarcity*: the dynamics of Dirac observables is trivial—they just stay constant, and it is difficult to find *any* such quantity in general relativity. More recently, the *nonlocality* of such variables has been proved [6]: the expression for any Dirac observable in terms of local fields must contain derivatives of all orders.

As for the "frozen dynamics," Rovelli's idea of "evolving constants of motion" [7] has shown one way out of the problem. Another, but not completely unrelated way, is based on the observation that any Hamiltonian formulation of dynamics needs a frame [8]. A possible frame for a nontrivial Hamiltonian dynamics of Dirac observables has been specified in [9] and [10] under the assumption that there is a symmetry. In [11], it is shown that even an asymptotic symmetry suffices, at least for a simple finite-dimensional model.

Concerning the scarcity, there have been different proposals of how Dirac observables could be constructed (e.g., [12]). Also, in the case of asymptotically flat models, there are even complete sets of natural constants of motion, namely the in- and out-fields. These quantities have been described by DeWitt [13] within his covariant perturbation theory and by Ashtekar [14] in general.

Finally, the nonlocality of the kind found need not always lead to a real problem. For example, in any nontrivial quantum field theory in Minkowski spacetime, the expressions for the in- and out-fields in terms of the local fields at a finite time are also badly nonlocal. However, such expressions are not needed. What is needed are the expressions of the in- and out-fields in the in- and out-regions, respectively, which are local.

In the present paper, we are going to strengthen the point made in favor of the Hamiltonian frame based on asymptotic symmetries. We study a field model, extending thus the cases in which the idea works to infinite-dimensional systems (see also [15]). The simplified model that we investigate consists of cylindrical gravitational waves, also called sometimes Einstein-Rosen waves (see, e.g., [16] and [17]). It seems that this model can be used as an example of almost everything. Thus, Kuchař has studied the embedding variables in canonical theory employing the waves [18]. Torre has expressed a complete set of Dirac observables in terms of local fields [19]. Ashtekar and Pierri have found a considerable simplification of Hamiltonian dynamics of the waves using a suitable gauge [20].

Most important, in [20] an asymptotic boundary term analogous to the ADM energy has been added to the action for the first time. This is an achievement because the cylindrical wave spacetimes are not asymptotically flat due to the cylindrical symmetry so that the well-known results in general relativity cannot be used. The model can, nevertheless, be reduced from four to three dimensions by removing the direction of the translational part of the cylindrical symmetry. The resulting system has the form of gravity coupled to a scalar field and the geometry of the three-dimensional spacetime is asymptotically flat in certain sense (locally asymptotically flat). This has been shown in [21] and [22], where the boundary term has been found. Our analysis is, therefore, based on the results of Ashtekar and Pierri.

In Sec. II, the model of free Newtonian particle moving in one-dimensional space is used to explain how a symmetry provides a frame for a nontrivial Hamiltonian dynamics of integrals of motion. The relevant notions and relations are introduced. The method of reduction by a choice of gauge is also described within this framework.

Section III summarizes the well-known results on the cylindrical waves that we shall need, focusing on the asymptotic properties. Section IV reviews and modifies the results of Ashtekar and Pierri so that they become compatible with the theory of Sec. II. The meaning of *asymptotic symmetry* and *asymptotic time* are explained and their relation to "ordinary" symmetry and time is clarified. The way in which the asymptotic symmetry and time determine a frame of a Hamiltonian dynamics of Dirac observables for the cylindrical waves is described. An important new point made in Sec. IV is a clean separation between gauge transformations and symmetries. In [20,23] and [11], the group of diffeomorphisms has been divided into gauges and symmetries according to the action of its elements at infinity. However, it has never been completely clear where the boundary is to be drawn. When we tried to apply this idea to the cylindrical waves, some strange paradoxes have appeared. It has turned out that the action must be parametrized at infinity similarly as in the case of geometrodynamics of Schwarzschild black holes [24]: privileged spacetime coordinates at infinity must be added to the set of canonical coordinates of the phase space. Then, all gauge transformations including reparametrizations at infinity are generated by constraints, while all asymptotic symmetries are generated by expressions in the momenta conjugate to the privileged coordinates at infinity.

The Poisson algebra of the asymptotic Dirac observables listed in Sec. III is calculated in Sec. V. The physical phase space is defined by a complete set of Dirac observables and their Poisson brackets. In Sec. VI, the action of the asymptotic symmetries on the physical phase space is found and the canonical generator of the continuous group of asymptotic time translations is written down. The corresponding Hamiltonian dynamics in the physical phase space is described. It may seem paradoxical that all these gauge-invariant structures (complete set of Dirac observables, their Poisson brackets and the action of asymptotic symmetry on them) can be and, indeed, have been calculated from the gauge-dependent action obtained by Ashtekar and Pierri after a choice of gauge (cf. also [15]). The justification thereof is given in Sec. IV.

Finally, Sec. VII gives a brief sketch of how the results of Secs. V and VI can be employed for one of possible constructions of quantum theory. The Fock space and the basic operators on it are specified. The Hamiltonian operator and the scattering matrix are determined.

II. DYNAMICS OF DIRAC OBSERVABLES IN A ONE-DIMENSIONAL MODEL

A finite-dimensional example can illustrate the main aspects of our method. The underlying geometric framework has been developed for general finite-dimensional systems in Refs. [23,9,25] and [10].

Consider a free Newtonian particle of unit mass in one-dimensional space evolving in the Newtonian time T. Let its position be denoted by Q. The system is described by the action

$$S[Q]=\frac{1}{2}\int_{T_1}^{T_2} dT(Q_{,T})^2. \tag{2.1}$$

This action is invariant under the translation

$$T\rightarrow T+\tau \tag{2.2}$$

for any time T and any real parameter τ. This is a one-dimensional group of symmetry that will play a crucial role in what follows. The symmetry implies via Noether theorem the conservation of energy

$$E=\frac{1}{2}Q_{,T}^2.$$

The corresponding Hamiltonian action is

$$S[Q,P_Q]=\int_{T_1}^{T_2}dT\left(P_Q Q_{,T}-\frac{1}{2}P_Q^2\right). \tag{2.3}$$

The one-form $\Theta=P_Q dQ$ contained in the action is called the Liouville form. The space with the coordinates Q and P_Q is the *physical phase space* Γ_1; it carries the symplectic two-form $\Omega_1=d\Theta=dP_Q\wedge dQ$.

The equations of motion implied by action (2.3) are

$$Q_{,T}=P_Q, \tag{2.4}$$

$$P_{Q,T}=0. \tag{2.5}$$

Their general solution is

$$Q(T)=q+pT, \tag{2.6}$$

$$P_Q(T)=p, \tag{2.7}$$

where $(q,p)\in\mathbb{R}^2$ are constant for a particular solution. We can, therefore, define the *space of solutions* Γ_2 as $\mathbb{R}^2$ with coordinates q and p. An important observation is that Γ_2 also carries a symplectic structure. Indeed, Eqs. (2.6) and (2.7) can be considered as a T-dependent canonical transformation because they imply

$$P_Q dQ=p\,dq+d\frac{(Q-q)^2}{2T},$$

where, of course, the action of the differential d "hits" only the variables Q and q, not T; $(Q-q)^2/2T$ is the generating function. The resulting symplectic form Ω_2 of Γ_2 is $dp\wedge dq$. The dynamical change δ_d within the time increment δT in Γ_2 is trivial:

$$\delta_d p=0,\qquad \delta_d q=0. \tag{2.8}$$

We can also consider the solutions (2.6) and (2.7) as defining two scalar fields on the one-dimensional time manifold $\mathbb{R}$ with coordinate T, the so-called *background manifold*. The push forward by the symmetry transformation (2.2) acts on these fields as follows

$$Q(T) \mapsto Q(T-\tau), \tag{2.9}$$

$$P_Q(T) \mapsto P_Q(T-\tau). \tag{2.10}$$

This symmetry action preserves, of course, the property of being a solution of the equations of motion; if q and p describe the untransformed solution and q' and p' the transformed one, then Eqs. (2.6) and (2.7) yield

$$q' = q - p\tau, \qquad p' = p. \tag{2.11}$$

This action of the symmetry transformation on the space Γ_2 is canonical and is generated by the function $-p^2/2$ (as the Poisson brackets show):

$$\delta_s q = -p\,\delta\tau = \left\{ q, -\frac{\delta\tau}{2} p^2 \right\}, \tag{2.12}$$

$$\delta_s p = 0 = \left\{ p, -\frac{\delta\tau}{2} p^2 \right\}, \tag{2.13}$$

where δ_s is the change caused by the symmetry transformation.

The key observation is now the following. If we compare the time change in q and p due to the dynamics [Eq. (2.8)] with that due to the symmetry [(Eqs. (2.12) and (2.13)], we find that the relative time changes $\delta_d q - \delta_s q$ and $\delta_d p - \delta_s p$ are formally identical to the original dynamics generated by the Hamiltonian $P_Q^2/2$ in Γ_1.

The description of the dynamics of our model can be made generally covariant by *parametrization* (see, e.g., [26]). An arbitrary time $t = t(T)$ is introduced and the action (2.1) is brought into the *constraint-Hamiltonian form* (i.e., the Hamiltonian is a linear combination of constraint functions—a typical property of generally covariant systems, see [4]):

$$S[Q,P_Q,T,P_T;N] = \int_{t_1}^{t_2} dt \left(P_Q Q_{,t} + P_T T_{,t} - N\left(P_T + \frac{1}{2} P_Q^2 \right) \right). \tag{2.14}$$

Let the space $\mathcal{P}$ consist of all initial data Q, P_Q, T and P_T for the field equations implied by action (2.14). The space $\mathcal{P}$ equipped with the symplectic form $\Omega_{\mathcal{P}}$ derived from the Liouville form of action (2.14),

$$\Omega_{\mathcal{P}} = dP_Q \wedge dQ + dP_T \wedge dT,$$

is called the *extended phase space* of the system. N is a Lagrange multiplier and $C=0$ is a constraint with

$$C = P_T + \frac{1}{2} P_Q^2 \tag{2.15}$$

being the constraint function. The surface $\mathcal{C}$ in $\mathcal{P}$ defined by the constraint is called the *constraint surface*. Since equation $C=0$ can be solved for P_T, we can choose the functions Q, P_Q and T as coordinates on $\mathcal{C}$.

The transformations that are canonically generated by the function $\mathcal{H}(N) = NC$ can be considered as reparametrizations—general changes of the time parameter. They are, therefore, analogous to *gauge transformations*. The corresponding gauge group acts along the constraint surface $\mathcal{C}$. At the same time, its *orbits* in $\mathcal{C}$ coincide with the solutions (2.6) and (2.7). The physical phase space Γ_2 of gauge-equivalent solutions can, therefore, be identified with the quotient of the constraint surface by the gauge orbits:

$$\Gamma_2 = \frac{\mathcal{C}}{\mathrm{Orb}}. \tag{2.16}$$

In coordinates (q,p) on Γ_2 and (Q,P_Q,T) on $\mathcal{C}$, the projection $\mathrm{Proj}_{(\mathcal{C}\to\Gamma_2)}$ derived from Eq. (2.16) is

$$(Q,P_Q,T) \in \mathcal{C} \mapsto (q,p) = (Q - P_Q T, P_Q) \in \Gamma_2. \tag{2.17}$$

The symplectic form Ω_2 on Γ_2 can be obtained from $\Omega_{\mathcal{P}}$ as follows. First, $\Omega_{\mathcal{P}}$ is pulled back from $\mathcal{P}$ to $\mathcal{C}$. This yields a two-form $\Omega_{\mathcal{C}}$ degenerated along the gauge orbits in $\mathcal{C}$. The projection $\mathrm{Proj}_{(\mathcal{C}\to\Gamma_2)}$ in Eq. (2.17) determines the symplectic form Ω_2 as the unique solution of the equation $\Omega_{\mathcal{C}} = \mathrm{Proj}^*_{(\mathcal{C}\to\Gamma_2)}\,\Omega_2$ where $*$ denotes the pull-back mapping.

We shall also need the concept of *transversal surface* $\mathcal{T} \subset \mathcal{C}$. This is a section $\sigma_{(\Gamma_2\to\mathcal{C})}: \Gamma_2 \mapsto \mathcal{C}$ in the sense that

$$\mathrm{Proj}_{(\mathcal{C}\to\Gamma_2)} \circ \sigma_{(\Gamma_2\to\mathcal{C})} = \mathrm{Id}_{\Gamma_2} \tag{2.18}$$

with respect to the projection (2.17). Whenever $\mathcal{C}$ admits such a section, the surface $\mathcal{T} = \sigma_{(\Gamma_2\to\mathcal{C})}(\Gamma_2)$ is a copy of the physical phase space Γ_2. A bijection between any surface $\mathcal{T}$ and the physical phase space Γ_2 can be defined by restricting the projection $\mathrm{Proj}_{(\mathcal{C}\to\Gamma_2)}$ to each particular $\mathcal{T}$. The symplectic form Ω_2 induces through each such bijection a unique symplectic form $\Omega_{\mathcal{T}}$. In this way each $\mathcal{T}$ also becomes a phase space with a symplectic structure that is isomorphic to that of Γ_2. Transversal surface $\mathcal{T}$ is called *regular* if if it is not tangential to gauge orbits at any of its points. If $\mathcal{T}$ is defined by the equation $F(Q,P_Q,T) = 0$, then the regularity condition is a non-vanishing Poisson bracket

$$\{F, \mathcal{H}(N)\}|_{\mathcal{C}} \neq 0 \quad \forall N. \tag{2.19}$$

The meaning of the regularity of transversal surfaces simply is that the gauge condition breaks the gauge completely. Systems that are not pathological possess many transversal surfaces.

One way to quantize a generally covariant system is to reduce it to an unconstrained system of the kind (2.3). We shall now show two methods of reduction: by a gauge condition and via Dirac observables and symmetry.

A. Reduction by a choice of gauge

A *gauge condition* is a choice of a particular family of regular transversal surfaces that foliate $\mathcal{C}$. Let the family be given by the set of equations

$$\tilde{F}(Q,P_Q,T)=\tilde{T}. \tag{2.20}$$

For each fixed real $\tilde{T}$, one surface $\mathcal{T}_{\tilde{T}}$ of the family is defined. The reduction using the condition (2.20) can proceed as follows. Suppose that a canonical transformation in $\mathcal{P}$ is known between the original variables (Q,P_Q,T,P_T) and some canonical coordinates $(\tilde{q},\tilde{p},\tilde{T},\tilde{P})$ that have been chosen so as to contain $\tilde{T}$. Then, using the regularity condition of the gauge, one can show that the constraint $C=0$ is solvable with respect to $\tilde{P}$ and so can equivalently be written as

$$\tilde{P}+\tilde{\mathcal{H}}(\tilde{T},\tilde{q},\tilde{p})=0,$$

where $\tilde{\mathcal{H}}$ is some smooth function. The canonical transformation brings action (2.3) to the form

$$S(\tilde{q},\tilde{p},\tilde{T},\tilde{P},\tilde{N})=\int_{t_1}^{t_2}dt[\tilde{p}\tilde{q}_{,t}+\tilde{P}\tilde{T}_{,t}-\tilde{N}(\tilde{P}+\tilde{\mathcal{H}})],$$

where $\tilde{N}$ is a new Lagrange multiplier defined by

$$NC=\tilde{N}(\tilde{P}+\tilde{\mathcal{H}}).$$

Next, $\tilde{T}$ is chosen as the integration variable t and the action is restricted to the constraint surface. The result is

$$\tilde{S}(\tilde{q},\tilde{p})=\int_{\tilde{T}_1}^{\tilde{T}_2}d\tilde{T}(\tilde{p}\tilde{q}_{,\tilde{T}}-\tilde{\mathcal{H}}). \tag{2.21}$$

By this, the reduction is finished.

A problem with this kind of reduction is that the new variables $\tilde{q}$, $\tilde{p}$ as well as the new time $\tilde{T}$ are not the same as the original variables Q, P_Q and T. Classically, the two actions (2.21) and (2.3) are equivalent, because they are related by an extended gauge transformation. The two quantum mechanics, however, that are obtained by the standard quantization method from them, *cannot* be unitarily equivalent [27]: the transformation (2.20) between the respective times involves operators, while each of the times must be a parameter in the respective quantum mechanics.

B. Reduction using Dirac observables and symmetry

A *Dirac observable* $o(\mathcal{P})$ is a function $o:\mathcal{P}\mapsto\mathbb{R}$ whose Poisson bracket with the constraint C vanishes when restricted to $\mathcal{C}$. Dirac observables are gauge invariant.

The correspondence between Dirac observables on $\mathcal{P}$ and functions on Γ_2 is the following: Each function f: $\Gamma_2\mapsto\mathbb{R}$ determines a function $f\circ \mathrm{Proj}_{(\mathcal{C}\to\Gamma_2)}$ on the constraint surface via the projection mapping (2.17). In the chart (Q,T,P_Q) on $\mathcal{C}$ such a function has the form

$$f(Q-P_QT,P_Q). \tag{2.22}$$

The next step is to extend this function from $\mathcal{C}$ to $\mathcal{P}$. Let us denote such an extension by o: $\mathcal{P}\mapsto\mathbb{R}$. The only condition is that o be a smooth function on $\mathcal{P}$ the values of which coincide with $f(Q-P_QT,P_Q)$ at $\mathcal{C}$. In this way, the original function f on Γ_2 defines an equivalence class $\{o\}$ of functions o on $\mathcal{P}$. One can show that any two elements o_1 and o_2 of $\{o\}$ satisfy $o_2=o_1+NC$, where N is a smooth function on $\mathcal{P}$ (see, e.g., [4]). It also follows immediately from the construction that o is constant along orbits so that it is a Dirac observable. Thus, a whole class of Dirac observables corresponds to one function on Γ_2 (one often speaks about Dirac observables meaning these classes).

On the other hand, each Dirac observable o defines via the restriction to $\mathcal{C}$ a function on $\mathcal{C}$ that is constant along orbits. Such a function determines, in turn, a unique function f on Γ_2.

The Poisson brackets between Dirac observables can be calculated using the symplectic structure of the extended phase space $\mathcal{P}$. It is easy to show [9] that the Poisson bracket $\{o_1,o_2\}$ of two Dirac observables is again a Dirac observable and that the Poisson brackets $\{o_1,o_2\}$ and $\{o_1+N_1C,o_2+N_2C\}$ lie in the same class. Thus, the Poisson brackets between the equivalence classes $\{o\}$ are well defined. Moreover, if the class with the representative o_i corresponds to the function f_i, $i=1,2$, on Γ_2, and the class with the representative $\{o_1,o_2\}$ corresponds to f, then

$$\{f_1,f_2\}_{\Gamma_2}=f,$$

as is shown in Ref. [9]. It follows from this that a *complete set of Dirac observables*, together with their Poisson algebra, determine the structure of the physical phase space Γ_2. A complete set of Dirac observables separates gauge orbits, and in simple cases can be used as a coordinate system on the quotient space $\mathcal{C}/\mathrm{Orb}=\Gamma_2$. For our model, a complete set is formed by the functions $o_0=Q-P_QT+N_0C$ and $o_1=P_Q+N_1C$, where N_0 and N_1 are smooth functions on $\mathcal{P}$. A simple calculation gives that the only nontrivial bracket is

$$\{o_0,o_1\}=1+NC,$$

where $N=\{N_0,o_1\}+\{o_0,N_1\}$; the Dirac observables o_0 and o_1 correspond to the functions q and p on Γ_2.

The symmetry group (2.2) acts on the extended phase space as follows

$$(Q,P_Q,T,P_T)\mapsto(Q,P_Q,T+\tau,P_T), \tag{2.23}$$

and is, therefore, generated by the momentum P_T conjugate to T. Observe that P_T itself is a Dirac observable; one can prove [9] that any continuous symmetry group is generated by a Dirac observable. Now, P_T has a nontrivial action on Dirac observables. For example,

$$\{o_0,P_T\}=-o_1+\overline{N}_0C,$$

$$\{o_1,P_T\}=\overline{N}_1 C,$$

where $\overline{N}_0$ and $\overline{N}_1$ are suitable functions on $\mathcal{P}$. The change of Dirac observables *referred to* the symmetry as "zero motion" is, therefore, nontrivial. It is easy to see that this change is generated by the function $-P_T$. The value of P_T at $\mathcal{C}$ is, however,

$$-P_T|_{\mathcal{C}}=\frac{1}{2}P_Q^2,$$

and it lies in the class $o_1^2/2+NC$. The corresponding function on Γ_2 is, therefore, $H=p^2/2$, and it plays the role of the Hamiltonian of the constructed dynamics. In this way, we have recovered the dynamics and the phase space Γ_1 of the original system so that the reduction is accomplished.

Mathematically, any symmetry of $\mathcal{P}$ that is not pure gauge transformation can generate a nontrivial evolution of Dirac observables on Γ_2 because it defines a nontrivial mapping between gauge orbits in $\mathcal{C}$. By projection to Γ_2 a symmetry is obtained which can be interpreted as the generator of a dynamical evolution on Γ_2. Physically, it must be additionally required that the symmetry be privileged by the situation at hand. Only then its role as a true Hamiltonian on Γ_2 can be justified. Here, the constant translation (2.23) is physically privileged by the arguments leading from Eqs. (2.1) to (2.14) and in particular by the fact that it yields through Noether's theorem the energy of the Newtonian particle in the privileged reference system T. The transformation (2.23) is indeed a symmetry of $\mathcal{P}$ because the Poisson bracket of its generator, $-P_T$, with the constraint function (2.15) vanishes on $\mathcal{C}$.

The following observation is very important. If P_T generates a symmetry that leads to the Hamiltonian H in Γ_2, then so does $P_T+N'C$ for any smooth N': the dynamics of Dirac observables is uniquely determined by the whole class of symmetry generators. Why is this important: In our simple model, we have a unique symmetry and it is generated by P_T. The reason is that our model is a so-called "already parametrized system" with a privileged time T. Indeed, there also is a privileged choice of gauge due to this fact: $F(Q,P_Q,T)\equiv T$, which leads to the "right" action (2.3) by the reduction procedure of Sec. II A. However, many models of real interest, such as general relativity, are not already parametrized systems [28]. For such models, there is no privileged time and no symmetry in general (cf. [29]). But in asymptotically flat cases, there is a privileged asymptotic time and an asymptotic symmetry. As it is shown in [23], such symmetries do not determine their generators in the extended phase space $\mathcal{P}$ uniquely but only up to addition of a linear combination of constraints. Despite that, they still define a unique dynamics of Dirac observables.

III. POLARIZED CYLINDRICAL WAVES: SOLUTIONS AND ASYMPTOTIC BEHAVIOR

A vacuum spacetime describing cylindrical gravitational waves with a fixed state of polarization (one degree of freedom per point) has two commuting, hypersurface-orthogonal spacelike Killing vectors $\partial/\partial\varphi$ and $\partial/\partial z$; the Killing field $\partial/\partial\varphi$ is rotational and it keeps a timelike axis fixed; $\partial/\partial z$ is translational; coordinates φ and z are invariantly defined up to a translation $z\mapsto z+a$. The metric can be written in the form

$$ds^2=e^{\gamma-\psi}(-dT^2+dR^2)+e^{\psi}dz^2+R^2e^{-\psi}d\varphi^2, \quad (3.1)$$

where T and R are invariantly defined, T up to a translation $T\mapsto T+a$. In the above equation, $\psi=\psi(T,R)$ and $\gamma=\gamma(T,R)$. [To obtain Eq. (3.1) one uses a consequence of vacuum field equations—see, e.g., [18,30].]

It is well known that because of the translational symmetry $\partial/\partial z$, the four-dimensional Einstein equations are equivalent to the three-dimensional Einstein equations with certain matter sources (see, e.g., [22,20] and [17]). In our case of cylindrical symmetry ($\partial/\partial\varphi$ is a further Killing field) the four-dimensional Einstein vacuum equations the solutions of which give Einstein–Rosen waves are equivalent to Einstein equations in three dimensions with a zero-rest-mass scalar field ψ as a source. It is, however, more advantageous for the canonical formulation to work with the physical Klein-Gordon field $\phi=\psi/\sqrt{8G}$, G being the Newton constant. Hence, we formulate everything with the help of the field ϕ.

In three dimensions, the metric is given by (cf. [22] and [17])

$$ds^2=g_{ab}dx^a dx^b=e^{\gamma}(-dT^2+dR^2)+R^2d\varphi^2, \quad (3.2)$$

and the Einstein field equations become

$$\frac{\partial^2\gamma}{\partial R^2}-\frac{\partial^2\gamma}{\partial T^2}+\frac{1}{R}\frac{\partial\gamma}{\partial R}=8G\left(\frac{\partial\phi}{\partial T}\right)^2, \quad (3.3)$$

$$-\frac{\partial^2\gamma}{\partial R^2}+\frac{\partial^2\gamma}{\partial T^2}+\frac{1}{R}\frac{\partial\gamma}{\partial R}=8G\left(\frac{\partial\phi}{\partial R}\right)^2, \quad (3.4)$$

$$\frac{1}{R}\frac{\partial\gamma}{\partial T}=8G\frac{\partial\phi}{\partial R}\frac{\partial\phi}{\partial T}. \quad (3.5)$$

The field equation for ϕ,

$$-\frac{\partial^2\phi}{\partial T^2}+\frac{\partial^2\phi}{\partial R^2}+\frac{1}{R}\frac{\partial\phi}{\partial R}=0, \quad (3.6)$$

is the wave equations for the nonflat metric (3.2) *as well as* for the flat (Minkowski) metric obtained by putting $\gamma=0$ in Eq. (3.2). This crucial simplification implies that the scalar field ϕ is decoupled from the equations satisfied by the metric. Equations (3.3)–(3.5) reduce to two simple equations

$$\frac{\partial\gamma}{\partial R}=4GR\left[\left(\frac{\partial\phi}{\partial T}\right)^2+\left(\frac{\partial\phi}{\partial R}\right)^2\right], \quad (3.7)$$

$$\frac{\partial\gamma}{\partial T}=8GR\frac{\partial\phi}{\partial T}\frac{\partial\phi}{\partial R}, \quad (3.8)$$

the wave equation (3.6) is their integrability condition. We can thus solve the axisymmetric—in three dimensions "spherically" symmetric—wave equation (3.6) on Minkowski space and then solve Eqs. (3.6) and (3.8) for the metric function $\gamma(T,R)$ by quadratures. These well-known facts are of key importance in the canonical and quantum theory since all physical degrees of freedom are contained in the scalar field.

We shall now briefly review some of the results on the asymptotics obtained in [22] and [31]. We shall extend the discussion by including both future and past null infinities, and later also by employing a Fourier-type decomposition.

The Cauchy data for the scalar field ϕ, given on the Cauchy surface topologically $\mathbb{R}^2$, suffice to determine the whole spacetime metric. For data which fall off appropriately, the three-dimensional Lorentzian geometry is asymptotically flat both at spatial [21] and null infinity [22] although in four dimensions the Einstein-Rosen spacetimes are not asymptotically flat (see [31] for a detailed investigation of cylindrical waves at null infinity in four dimensions).

By employing the "method of descent" from the Kirchhoff formula in four dimensions one can find the representation of the solution $\phi(T,R)$ of the wave equation (3.6) in three dimensions in terms of Cauchy data $\phi_0=\phi(0,R)$ and $\phi_1=\phi_{,T}(0,R)$. This has been used in [22] to find the asymptotic behavior of the field ϕ and the whole metric (3.2) at the future null infinity for the data of compact support (see Sec. II in [22]). By applying the same procedure one can analyze the solutions at the past null infinity. Introducing retarded and advanced time coordinates

$$U=T-R, \qquad V=T+R \tag{3.9}$$

[notice that these are null coordinates for both flat Minkowski metric and the curved metric (3.2)], one obtains expansions in the powers of $R^{-1/2}$ along null hypersurfaces $U=$const and $V=$const of the form

$$\phi(V,R)=\frac{1}{\sqrt{R}}g(V)+\sum_{k=1}^{\infty}\frac{g_k(V)}{R^{k+1/2}}, \tag{3.10}$$

$$\phi(U,R)=\frac{1}{\sqrt{R}}f(U)+\sum_{k=1}^{\infty}\frac{f_k(U)}{R^{k+1/2}}. \tag{3.11}$$

The coefficients in the expansions are determined by the Cauchy data. By rewriting the Einstein field equations (3.7) and (3.8) in terms of U and R (respectively V and R), we obtain the asymptotic behavior of the metric function γ at $\mathcal{I}^+$ in the form

$$\gamma(U,R)=\gamma_\infty-8G\int_{-\infty}^{U}dU\left(\frac{df}{dU}\right)^2+O(R^{-2}), \tag{3.12}$$

and similarly at $\mathcal{I}^-$. Here the constant γ_∞, which will play a key role in the following, is determined uniquely by the Cauchy data for ϕ [cf. Eq. (3.7)]

$$\gamma_\infty=\gamma(0,\infty)=4G\int_0^\infty dRR\left[\left(\frac{\partial\phi}{\partial T}\right)^2+\left(\frac{\partial\phi}{\partial R}\right)^2\right]. \tag{3.13}$$

The value of γ_∞ represents the total energy of the scalar field ϕ computed by using the Minkowski metric. For any nontrivial data, γ_∞ is positive. Hence, the metric at spatial infinity, given by

$$ds^2=e^{\gamma_\infty}(-dT^2+dR^2)+R^2d\varphi^2, \tag{3.14}$$

has a conical singularity because the distance of the circles with radii R and $R+dR$ is different by a factor e^{γ_∞} from the difference of their circumferences divided by 2π. It can be shown [22] that as one approaches $\mathcal{I}^+$ ($R\to\infty$, $U=$const), one finds [cf. Eq. (3.12)]

$$\gamma(U,\infty)=\gamma_\infty-8G\int_{-\infty}^{U}du\left(\frac{df}{du}\right)^2, \tag{3.15}$$

and γ to vanish at the timelike infinity i^+. Hence,

$$\gamma_\infty=8G\int_{-\infty}^{\infty}dU\left(\frac{df}{dU}\right)^2. \tag{3.16}$$

The conical singularity, present at spacelike infinity, is thus "radiated out," and the future timelike infinity i^+ becomes smooth. Equation (3.15) plays the role of the well-known Bondi mass-loss formula, the function df/dU being analogous to the Bondi news function [see also [32], Eq. (3.6), for an analysis in four dimensions]. Clearly, analogous formulas to Eqs. (3.15) and (3.16) are valid for incoming waves, with df/dU replaced by dg/dV:

$$\gamma(V,\infty)=8G\int_{-\infty}^{V}dv\left(\frac{dg}{dv}\right)^2$$

and

$$\gamma_\infty=8G\int_{-\infty}^{\infty}dV\left(\frac{dg}{dV}\right)^2. \tag{3.17}$$

Here we assume smooth past timelike infinity i^- and incoming waves from the past null infinity $\mathcal{I}^-$ with a null data $g(V)$ bring in mass-energy which reveals itself as a conical singularity characterized by $\gamma(V,\infty)$. At spatial infinity i_0 ($V=\infty$, $R=\infty$) this becomes just the constant γ_∞ given in Eq. (3.13). The fluxes of radiation, the analogues of the news function, as well as conical singularities are observable quantities at the past and future null infinities. Both are given by the asymptotic null data $g(V)$ and $f(U)$. The asymptotic null data will be important in the following.

Starting from the representation of the solutions of the three-dimensional wave equation (3.6) in terms of the Kirchhoff-type formula obtained by the "method of descent" from four dimensions one can, for the Cauchy data of compact support, obtain not only expansions (3.10) and (3.11), but also the explicit expression for the null data $f(U)$ [respectively $g(V)$] as the integral over the Cauchy data ϕ_0 and

ϕ_1. However, these integrals become simple only for retarded times U so large that the support of the data is completely in the interior of the past cone (similarly for the advanced times at $\mathcal{I}^-$); see [22]. Here we need the null data for all U's at $\mathcal{I}^+$ and V's at $\mathcal{I}^-$.

To achieve this, we start from a Fourier-type decomposition. This, in three dimensions, means to write the solutions in terms of the Bessel functions of zero order provided that we require the solutions to be regular everywhere, in particular at $R=0$ (see, e.g., [33]).

Thus, we start from the solutions of the form

$$\phi(T,R)=\frac{1}{\sqrt{2}}\int_0^\infty d\omega[A(\omega)J_0(\omega R)e^{-i\omega T}+\text{c.c.}]. \tag{3.18}$$

As usual, we write just "c.c." instead of the second term, meaning the complex conjugate of the first one. Using the asymptotic expansion of the Bessel function at $R\to\infty$ (see, e.g., [33]), we obtain

$$\phi(T,R)=\frac{1}{2\sqrt{R\pi}}\int_0^\infty \frac{d\omega}{\sqrt{\omega}}\{[A(\omega)e^{-i(\pi/4)-i\omega U}+\text{c.c.}] +[A(\omega)e^{i(\pi/4)-i\omega V}+\text{c.c.}]\}+O(R^{-3/2}), \tag{3.19}$$

where U and V are retarded and advanced time coordinates given by Eq. (3.9). Hence, the null data at the future and past null infinities read as follows:

$$f(U)=\frac{1}{2\sqrt{\pi}}\int_0^\infty \frac{d\omega}{\sqrt{\omega}}[A(\omega)e^{-i(\pi/4)-i\omega U}+\text{c.c.}], \tag{3.20}$$

$$g(V)=\frac{1}{2\sqrt{\pi}}\int_0^\infty \frac{d\omega}{\sqrt{\omega}}[A(\omega)e^{i(\pi/4)-i\omega V}+\text{c.c.}]. \tag{3.21}$$

It is easy to invert the last equations by writing, for example,

$$g(V)+g(-V)=\sqrt{\frac{2}{\pi}}\int_0^\infty d\omega[\tilde{A}(\omega)+\tilde{A}^*(\omega)]\cos\omega V, \tag{3.22}$$

$$g(V)-g(-V)=-i\sqrt{\frac{2}{\pi}}\int_0^\infty d\omega[\tilde{A}(\omega)-\tilde{A}^*(\omega)]\sin\omega V, \tag{3.23}$$

where $\tilde{A}(\omega)=(2\omega)^{-1/2}A(\omega)e^{i\pi/4}$. Using Fourier cosine and sine (inverse) transforms to express $\tilde{A}\pm\tilde{A}^*$, we find

$$A(\omega)=\sqrt{\frac{\omega}{\pi}}e^{-i\pi/4}\int_0^\infty dV[g(V)e^{i\omega V}+g(-V)e^{-i\omega V}], \tag{3.24}$$

$A^*(\omega)$ being given by complex conjugation. Alternatively, we can write

$$A(\omega)=\sqrt{\frac{\omega}{\pi}}e^{-i\pi/4}\int_{-\infty}^\infty dVg(V)e^{i\omega V}. \tag{3.25}$$

Similarly, from Eq. (3.20) we get

$$A(\omega)=\sqrt{\frac{\omega}{\pi}}e^{i\pi/4}\int_0^\infty dU[f(U)e^{i\omega U}+f(-U)e^{-i\omega U}], \tag{3.26}$$

or

$$A(\omega)=\sqrt{\frac{\omega}{\pi}}e^{i\pi/4}\int_{-\infty}^\infty dUf(U)e^{i\omega U}. \tag{3.27}$$

Since according to Eq. (3.18) the functions $A(\omega)$ determine the solutions $\phi(T,R)$ everywhere, Eqs. (3.24)–(3.27) imply that either the null data $g(V)$ at $\mathcal{I}^-$ or $f(U)$ at $\mathcal{I}^+$ determine $\phi(T,R)$ uniquely in the spacetime.

The amplitudes $A(\omega)$ can be expressed also in terms of the Cauchy data $\phi_0=\phi(0,R)$ and $\phi_1=\phi_{,T}(0,R)$ directly from Eq. (3.18). Using the Hankel transform (see, e.g., [33]): for two functions $X(x)$ and $Y(y)$,

$$X(x)=\int_0^\infty dyY(y)\sqrt{xy}J_0(xy) \tag{3.28}$$

is equivalent to

$$Y(y)=\int_0^\infty dxX(x)\sqrt{xy}J_0(xy). \tag{3.29}$$

Expressing ϕ_1 from Eq. (3.18) we obtain

$$A(\omega)=\frac{1}{\sqrt{2}}\int_0^\infty dR(\omega\phi_0-i\phi_1)RJ_0(\omega R). \tag{3.30}$$

Hence, as expected, we need both ϕ_0 and ϕ_1 to determine the solution of the wave equation (3.6) everywhere. For time-symmetric initial data, $\phi_1=0$, the amplitudes $A(\omega)$ become real.

Although for the Cauchy data of compact support and even for more general data falling off sufficiently rapidly at spatial infinity we get $\phi\sim 1/\sqrt{R}$ at null infinities as in Eqs. (3.10) and (3.11) (see [22]), at spatial infinity, i.e. in the limit $R\to\infty$, T fixed, the solutions fall off more rapidly:

$$\phi\sim O(1/R), \qquad \phi_{,R}\sim O(1/R^2). \tag{3.31}$$

This will be needed in the canonical theory. To demonstrate the fall-off, employ again the asymptotic form of $J_0(\omega R)$ at $R\to\infty$ in Eq. (3.18),

$$\phi=\frac{\text{const}}{\sqrt{R}}\int_0^\infty \frac{d\omega}{\sqrt{\omega}}\cos\left(\omega R-\frac{\pi}{4}\right)[A(\omega)e^{-i\omega T}+\text{c.c.}],$$

and put $\omega R=\omega'$ in the integral. Then we get

$$\phi = \frac{\text{const}}{\sqrt{R}} \int_0^\infty \frac{d\omega'}{\sqrt{R\omega'}} \cos\left(\omega' - \frac{\pi}{4}\right)\left[A\left(\frac{\omega'}{R}\right)e^{-i\omega' T/R} + \text{c.c.}\right],$$

which at large R and fixed T leads to $\phi \sim 1/R$.

Finally, let us illustrate previous results by a simple example. The Weber-Wheeler-Bonnor pulse [34,35] represents an exact, time-symmetric vacuum solution of the Einstein equations with cylindrical symmetry which satisfies all regularity conditions required above. The pulse comes in from the past null infinity, concentrates around the axis of symmetry (in three dimensions around the center $R=0$) at $T=0$, and then reexpands to future null infinity. The real amplitude

$$A(\omega) = Ce^{-a\omega}, \tag{3.32}$$

where C and a are constants, implies, by using Eq. (3.18), solution

$$\phi = C\left\{\frac{[(a^2+R^2-T^2)^2+4a^2T^2]^{1/2}+a^2+R^2-T^2}{(a^2+R^2-T^2)^2+4a^2T^2}\right\}^{1/2}, \tag{3.33}$$

regular everywhere. [Due to the factor $1/\sqrt{2}$ in Eq. (3.18) ϕ here must be multiplied by $\sqrt{2}$ to get Eq. (3.15) in [31].] At spatial infinity, $R\to\infty$, T fixed, we see that

$$\phi = C\sqrt{2}\,\frac{1}{R} + O(1/R^2), \tag{3.34}$$

in accordance with Eq. (3.31). At the past null infinity ($R \to \infty$, $V=T+R$ fixed), we find

$$\phi = C\frac{\sqrt{2}}{2}\left[\frac{V+(V^2+a^2)^{1/2}}{V^2+a^2}\right]^{1/2}\frac{1}{\sqrt{R}} + O(1/R^{3/2}). \tag{3.35}$$

(At future null infinity, $R\to\infty$, $U=T-R$ fixed, the same expression, with V replaced by U follows.) A simple calculation, starting from the formula (3.21) for the profile $g(V)$ and using $A(\omega)$ from Eq. (3.32), yields exactly the factor at $1/\sqrt{R}$ in Eq. (3.35) [for integrals $\int_0^\infty dx e^{-ax}(1/\sqrt{x})\cos bx$ and $\int_0^\infty dx e^{-ax}(1/\sqrt{x})\sin bx$ needed in the calculation, see e.g., [36], formulas 3.944, 13 and 14].

With ϕ given by Eq. (3.35) one can find the explicit expression for function γ by solving Eqs. (3.7) and (3.8). It reads [34,31] as follows:

$$\gamma = 4GC^2\left\{\frac{1}{a^2} - \frac{2R^2[(a^2+R^2-T^2)^2-4a^2T^2]}{[(a^2+R^2-T^2)^2+4a^2T^2]^2} + \frac{1}{a^2}\,\frac{R^2-a^2-T^2}{[(a^2+R^2-T^2)^2+4a^2T^2]^{1/2}}\right\}. \tag{3.36}$$

The conicity at spatial infinity is thus given by

$$\gamma_\infty = 8G(C/a)^2. \tag{3.37}$$

With this explicit solution one can verify directly relations (3.15)–(3.17) for the conicity as it is radiated "in" and "out," similarly to the Bondi mass in four dimensions.

IV. HAMILTONIAN FORMULATION

Let us essentially repeat the canonical treatment of Ashtekar and Pierri given in [20] but with a slight modification in order to establish the analogy with the model of Sec. II. Let us start by considering the volume part of the canonical action derived from the Einstein-Hilbert action by assuming cylindrical symmetry in [20]:

$$S = \frac{1}{8G}\int dt \int_0^\infty dr(p_\gamma\gamma_{,t} + p_R R_{,t} + p_\psi\psi_{,t} - NC - N^r C_r), \tag{4.1}$$

where

$$C = e^{-\gamma/2}(2R_{,rr} - \gamma_{,r}R_{,r} - p_\gamma p_R + R^{-1}p_\psi^2/2 + R\psi_{,r}^2/2)$$

and

$$C_r = e^{-\gamma}(-2p_{\gamma,r} + p_\gamma\gamma_{,r} + p_R R_{,r} + p_\psi\psi_{,r})$$

are the constraint functions, γ, R, and ψ are defined by Eq. (3.1), p_γ, p_R and p_ψ are the conjugate momenta, while N and N^r are Lagrange multipliers—the so-called lapse and shift functions. One should add to this action the boundary energy term

$$-\frac{1}{4G}\int dt(1-e^{-\gamma_\infty/2}), \tag{4.2}$$

and specify the fall-off of N according to $\lim_{r\to\infty}N=1$ so that the action be differentiable [20]. However, this term is not invariant under reparametrizations of the label time t as there is no temporal density present in the integrand in Eq. (4.2). Addition of the bare term (4.2) to the action (4.1) would imply a privileged choice of asymptotic time. The total action would then not be in the constraint-Hamiltonian form but rather in an already reduced form at spatial infinity.

In order to recover the full constraint-Hamiltonian framework of our model in Sec. II, we need to justify the inclusion of a temporal density in Eq. (4.2). This can be done following the general approach by Beig and O' Murchadha [23]. First, one considers fall-off conditions for the configuration space data at $r\to\infty$. These have been specified in [20]. The configuration space fields approach infinity according to

$$\begin{aligned}\gamma(t,r) &\to \gamma_\infty(t) + O(1/r),\\ R(t,r) &\to r(1+O(1/r)),\\ \psi(t,r) &\to O(1/r),\end{aligned} \tag{4.3}$$

where $rO(1/r)$, $r^2\,O(1/r^2)$, etc. admit limits at $r\to\infty$. (These limits generally depend on the time t.) One then requires that the action of the Liouville form

$$\int_0^\infty dr(p_\gamma\, d\gamma + p_R dR + p_\psi d\psi)$$

on vector fields of the form

$$\int_0^\infty dr \left(\delta\gamma\frac{\partial}{\partial\gamma} + \delta R\frac{\partial}{\partial R} + \delta\psi\frac{\partial}{\partial\psi}\right)$$

should be finite. The resulting integral

$$\int_0^\infty dr(p_\gamma\delta\gamma + p_R\delta R + p_\psi\delta\psi)$$

is finite if the momenta satisfy the following fall-off conditions at $r\to\infty$:

$$p_\gamma(t,r)\to O(1/r^2),$$
$$p_R(t,r)\to O(1/r^2), \tag{4.4}$$
$$p_\psi(t,r)\to O(1/r).$$

Next, concerning the behavior of the lapse and shift, there are several aspects that ought to be kept in mind. First, it is the finiteness and differentiability of the Hamiltonian part

$$\mathcal{H}_1[N,N^r] = \int_0^\infty dr(NC + N^r C_r) \tag{4.5}$$

of action (4.1) (cf. [23]). Second, $\mathcal{H}_1[N,N^r]$ has to generate a transformation within the phase space defined by the boundary conditions (4.3) and (4.4) [23]. Finally, if we are going to have a full analogy to action (2.14) of Sec. II, we have to parametrize the model also at infinity, as it is done in [24] for a spherically symmetric model.

The constraints functional (4.5) remains finite even if $N(r)$ and $N^r(r)$ approach arbitrary temporal densities at $r\to\infty$; namely,

$$N(t,r)\to N_\infty(t) + O(1/r),$$
$$N^r(t,r)\to N^r_\infty(t) + O(1/r), \tag{4.6}$$

where $N_\infty(t)$ has to be non-negative. The condition that the lapse and shift should approach temporal densities at $r\to\infty$ is the minimum requirement that is compatible with the invariance of the action (4.1) under reparametrizations of t.

Conditions (4.3)–(4.6) now imply that the action is not differentiable. The problem comes from the variation of $\gamma(t,r)$ leading to the boundary term

$$\delta S \to \frac{1}{8G}\int dt \int_0^\infty dr(Ne^{-\gamma/2}\ R_{,r}\delta\gamma)_{,r}$$
$$= \frac{1}{8G}\int dt(N_\infty e^{-\gamma_\infty/2}\delta\gamma_\infty) \tag{4.7}$$

at spatial infinity. In order to have a consistent canonical theory, one needs to add to the action the boundary term

$$\frac{1}{4G}\int dt(N_\infty e^{-\gamma_\infty/2}),$$

whose variation with respect to $\gamma(t,r)$ cancels the boundary term in Eq. (4.7). The differentiable action is therefore

$$S = \frac{1}{8G}\int dt\int_0^\infty dr(p_\gamma\gamma_{,t} + p_R R_{,t} + p_\psi\psi_{,t} - NC - N^r C_r)$$
$$-\frac{1}{4G}\int dt N_\infty(1 - e^{-\gamma_\infty/2}). \tag{4.8}$$

The boundary term in Eq. (4.8) has now been modified by the addition of a constant in order that it coincides with the asymptotic energy derived from first principles in [20]. There is no boundary term involving the shift in spite of the fact that its asymptotic value may be nonzero. The corresponding constraint functional generates an "even supertranslation" (in the language of [23]) and is differentiable without any correction, similarly to the case of four-dimensional general relativity, cf. [23].

When varying the action (4.8) with respect to the lapse $N(t,r)$, it is important to keep the ends of its variation fixed. Indeed, if $N_\infty(t)$ is varied in Eq. (4.8) then one gets an unwanted field equation implying that the asymptotic energy vanishes. It follows that the action (4.8) is not yet in true constraint-Hamiltonian form. Following Kuchař [24], this can be improved by the "parametrization at infinity": $N_\infty(t)$ should be replaced by a differentiated asymptotic time $dT_\infty/dt = T_{\infty,t}(t)$. The asymptotic time is determined by the asymptotic metric: it must hold $N_\infty = 1$ if the parameter t is chosen to be T_∞. The time $T_\infty(t)$ can be varied in the ensuing action. Its variation leads to a redundant equation amounting to the conservation of the asymptotic energy. One should next introduce the momentum P_∞ and add the associated constraint (which is linear in P_∞) to the action by a new Lagrange multiplier $N_\infty(t)$.

In this way the action is brought into the true constraint-Hamiltonian form

$$S = \frac{1}{8G}\int dt(P_\infty T_{\infty,t}) + \frac{1}{8G}\int dt\int_0^\infty dr(p_\gamma\gamma_{,t} + p_R R_{,t}$$
$$+ p_\psi\psi_{,t}) - \int dt N_\infty\left(P_\infty + \frac{1}{4G}(1 - e^{-\gamma_\infty/2})\right)$$
$$-\frac{1}{8G}\int dt\int_0^\infty dr(NC + N^r C_r). \tag{4.9}$$

The multipliers N and N^r obey the asymptotic condition (4.6). The action (4.9) is the analogue of the action (2.14) for the Newtonian particle. One can verify that the field equations derived from the variations of Eq. (4.9) coincide with those of Sec. III, preserve the fall-off conditions (4.3)–(4.6) and imply the conservation of the asymptotic energy.

Action (4.9) is our starting point for the canonical theory. Although it corresponds to action (2.14) of Sec. II, observe that there is no a priori analogue of action (2.3) of Sec. II. We have to begin with the extended phase space $\mathcal{P}$ with coordinates $\gamma(r)$, $p_\gamma(r)$, $R(r)$, $p_R(r)$, $\psi(r)$, $p_\psi(r)$, T_∞, P_∞ and the symplectic form $\Omega_{\mathcal{P}}$,

$$\Omega_{\mathcal{P}}=\frac{1}{8G}\, dP_\infty\wedge dT_\infty+\frac{1}{8G}\int_0^\infty dr[dp_\gamma(r)\wedge d\gamma(r)+dp_R(r)\wedge dR(r)+dp_\psi(r)\wedge d\psi(r)].$$

The constraint surface $\mathcal{C}$ is defined by $\mathcal{H}[N,N^r]=0$ for all $N(r)$ and $N^r(r)$ satisfying the fall-off conditions, where

$$\mathcal{H}[N,N^r]=N_\infty\left(P_\infty+\frac{1}{4G}\,(1-e^{-\gamma_\infty/2})\right)+\frac{1}{8G}\times\int_0^\infty dr(NC+N^rC_r). \tag{4.10}$$

The orbits are defined by the canonical action of the constraint functional $\mathcal{H}[N,N^r]$. The canonical transformations generated by Eq. (4.10) are considered as a gauge transformation. Within the gauge group, there is no distinction between the ''symmetry'' and the ''proper gauge'' as, e.g., in [20,23] and [11]. The functional (4.10) generates only reparametrizations both ''inside'' the spacetime *and* at infinity. Symmetries are now generated by different functions. Indeed, the functional (4.10) has vanishing Poisson brackets with P_∞ for any $N(r)$ and $N^r(r)$ satisfying the conditions (4.6) because the variable γ_∞ is asymptotic value of the canonical coordinate γ so that $\{P_\infty,\gamma_\infty\}=0$. Hence, it is the function P_∞ that generates the symmetry. In general, we conjecture that one can introduce privileged coordinates at infinity and that asymptotic symmetries are generated by their conjugate momenta or suitable combinations of the momenta and the coordinates (like, e.g., boosts).

The variable T_∞ to which P_∞ is conjugate is a kind of a ''privileged time'' but the surface $T_\infty=$const is neither a transversal surface in the phase space, nor a Cauchy surface in each solution spacetime. Indeed, the function $T_\infty-c$, where c is a constant, has vanishing Poisson brackets with $\mathcal{H}[N,N^r]$ for all $N(r)$ and $N^r(r)$ whose asymptotic values vanish; hence, the duly generalized regularity condition (2.19) is not satisfied. It follows that an infinite-dimensional submanifold of each orbit lies in the surface $T_\infty=$const (Fig. 1). This is connected to the fact that the condition $T_\infty=$const defines only a particular section of infinity in each cylindrical wave spacetime but not a Cauchy surface of the whole spacetime; there is a relation between Cauchy and transversal surfaces, cf. [28].

The reduction by gauge condition, analogous to that described in Sec. II A, starts by a choice of a one-dimensional family of transversal surfaces. Let us denote the manifold formed by all chosen transversal surfaces in $\mathcal{C}$ by $\mathcal{G}$. In Sec. II, a privileged choice of gauge has been possible: $\mathcal{G}$ has been the family of surfaces $T=t$, $t\in\mathbb{R}$, where T is the privileged time. The nearest to this we can come is to choose the transversal surfaces in $\mathcal{G}$ to be the intersections of $\mathcal{G}$ and $T_\infty=$const (Fig. 1). There are, of course, many choices of $\mathcal{G}$. One example of such a choice is carried out in [20]. Let us describe an analogous choice for our action (4.9).

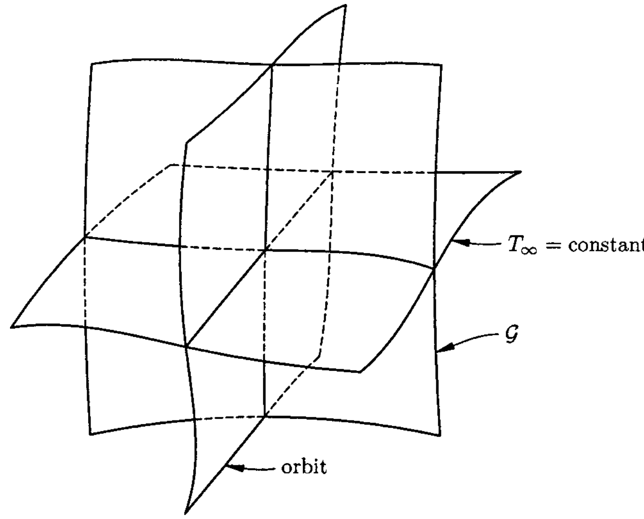

FIG. 1. Important surfaces in the constraint manifold $\mathcal{C}$. The intersection of $T_\infty=$const with any orbit is infinite-dimensional. The gauge condition surface $\mathcal{G}$ intersects each orbit in a (one-dimensional) dynamical trajectory of the reduced theory. The points common to $\mathcal{G}$ and each surface $T_\infty=$const is a (infinite-dimensional) transversal surface $\mathcal{T}_{T_\infty}$.

Following Ashtekar and Pierri, one may fix the part of the gauge associated with the constraints $C(r)=0$ and $C_r(r)=0$ by imposing the gauge-fixing conditions

$$R(r)=r,\qquad p_\gamma(r)=0. \tag{4.11}$$

These are the defining equations for $\mathcal{G}$. Viewed as constraints, these conditions form together with the constraints $C(r)=0$ and $C_r(r)=0$ a second-class system. The remaining constraint $P_\infty+(1/4G)(1-e^{-\gamma_\infty/2})=0$ can be taken care for by the gauge-fixing condition

$$T_\infty=\text{const}. \tag{4.12}$$

The surface $\mathcal{T}_{T_\infty}$ in $\mathcal{C}$ defined by Eqs. (4.11), (4.12) selects an initial datum from each gauge orbit in $\mathcal{C}$. The gauge-condition surface $\mathcal{G}$ is swept by all $\mathcal{T}_{T_\infty}$.

In order to confirm that this reduction is admissible, let us add Eqs. (4.11) and (4.12) to the action (4.9) by Lagrange multipliers M, M^r, and find out if the ensuing action determines unique values for N, N^r. One obtains the action

$$S=\frac{1}{8G}\int dt(P_\infty\, T_{\infty,t})+\frac{1}{8G}\int dt\int_0^\infty dr(p_\gamma\gamma_{,t}+p_RR_{,t}+p_\psi\,\psi_{,t})-\int dtN_\infty\left(P_\infty+\frac{1}{4G}(1-e^{-\gamma_\infty/2})\right)-\int dtM_\infty(T_\infty-t)-\frac{1}{8G}\int dt\int_0^\infty dr(NC+N^rC_r)-\int dt\int_0^\infty dr(M(R-r)+M^rp_\gamma), \tag{4.13}$$

where the set of conditions (4.12) is implemented by the expression $\int dt M_\infty (T_\infty - t)$. Indeed, it is not difficult to check that all redundant variables in Eq. (4.13) can be expressed uniquely in terms of the canonical pair (ψ, p_ψ) by solving the set of equations derived from the variation of these variables in Eq. (4.13). This confirms that the gauge-fixing conditions (4.11) are regular. In particular, the unique reduced expressions for the multipliers N, N^r and N_∞ are

$$N(T_\infty, R) = \exp\left[-\frac{1}{4}\int_R^\infty dr (r^{-1} p_\psi^2(T_\infty, r) + r\psi_{,r}^2(T_\infty, r))\right],$$

$$N^r(T_\infty, R) = 0, \tag{4.14}$$

those for the canonical pairs (γ, p_γ), (R, p_R) and $(T_\infty, p_{T\infty})$ read

$$\gamma(T_\infty, R) = \frac{1}{2}\int_0^R dr(r^{-1}p_\psi^2(T_\infty, r) + r\psi_{,r}^2(T_\infty, r)),$$

$$p_\gamma(T_\infty, R) = 0,$$

$$R(T_\infty, r) = r,$$

$$p_R(T_\infty, R) = -p_\psi(T_\infty, R)\psi_{,R}(T_\infty, R),$$

$$T_\infty(t) = t,$$

$$P_\infty(T_\infty) = -\frac{1}{4G}\left[1 - \exp\left(-\frac{1}{2}\gamma_\infty\right)\right],$$

where

$$\gamma_\infty = \frac{1}{2}\int_0^\infty dR(R^{-1}p_\psi^2(T_\infty, R) + R\psi_{,R}^2(T_\infty, R)), \tag{4.15}$$

and unique expressions also follow for the multipliers M, M^r, M_∞. The uniqueness of these expressions partially relies on the conditions imposed on the canonical fields at $r=0$ (see, e.g., [20]) which force $\gamma(T_\infty, 0)$ to vanish for all T_∞.

The reduced action for the remaining canonical pair $(\psi(T_\infty, R), p_\psi(T_\infty, R))$ on $\mathcal{G} \subset \mathcal{C}$, parametrized by the values of the asymptotic time, is therefore

$$S = \frac{1}{8G}\int dT_\infty \int_0^\infty dR\ (p_\psi(T_\infty, R)\psi_{,T_\infty}(T_\infty, R)) - \frac{1}{4G}\int dT_\infty (1 - e^{-\gamma_\infty(T_\infty)/2}), \tag{4.16}$$

where $\gamma_\infty(T_\infty)$ is expressed as a functional of $\psi(T_\infty, R)$ and $p_\psi(T_\infty, R)$ in Eq. (4.15). The action (4.16) is analogous to the reduced action (2.3). The phase space Γ_1 is described by coordinates $\psi(R)$ and $p_\psi(R)$, while the symplectic form is

$$\Omega_1 = \int_0^\infty dR dp_\psi(R) \wedge d\psi(R).$$

The action (4.16) is precisely the reduced action of Ashtekar and Pierri. In particular, the Ashtekar-Pierri time t in Eq. (19) of [20] corresponds to the time T_∞ here.

Geometrically, $\psi(r)$, $p_\psi(r)$ and T_∞ are coordinates on the gauge-condition surface $\mathcal{G}$. The surfaces defined in $\mathcal{G}$ by the equation $T_\infty =$ constant are transversal surfaces in the *phase space* $\mathcal{P}$. However, they also determine a family of Cauchy surfaces of constant Ashtekar–Pierri time in each solution *spacetime* (cf. [20]). In this sense, the part (4.11) of the gauge condition determines a particular extension of the points at infinity defined by $T_\infty = \text{const}$ to whole Cauchy surfaces in the spacetimes. However, different choices of $\mathcal{G}$ lead to different Cauchy surface extensions of these points at infinity. Hence, two different choices of $\mathcal{G}$ entail two different choices of time so that the transformation between the times has again the character of Eq. (2.20) even if the part (4.12) of gauge conditions remains always the same—only the asymptotic values of these times have then to coincide. As noted at the end of Sec. II (A), it is the transformation (2.20) between respective times which causes difficulties in constructing a unique plausible quantum theory.

Considering the privileged symmetry generated by P_∞, we can see that it remains a symmetry of the reduced theory. It acts in $\mathcal{G}$ as follows:

$$(\psi(r), p_\psi(r), T_\infty) \mapsto (\psi(r), p_\psi(r), T_\infty + \tau), \tag{4.17}$$

while the original action of P_∞ in $\mathcal{P}$ is

$$(\gamma(r), p_\gamma(r), R(r), p_R(r), \psi(r), p_\psi(r), T_\infty) \mapsto (\gamma(r), p_\gamma(r), R(r), p_R(r), \psi(r), p_\psi(r), T_\infty + \tau). \tag{4.18}$$

The map (4.18) is *tangential* to $\mathcal{G}$ and the map (4.17) is just the restriction of Eq. (4.18) to $\mathcal{G}$. This follows from the fact that the constraints as well as relations (4.11) that define $\mathcal{G}$ are independent of T_∞.

The dynamics defined by action (4.16) determines a foliation of $\mathcal{G}$ by one-dimensional dynamical trajectories represented by two functions of two variables $\psi(R, T_\infty)$ and $p_\psi(R, T_\infty)$. These are identical with the intersections of $\mathcal{G}$ with the orbits. In this way, we obtain a bijection between integrals of motion of the reduced theory and Dirac observables. On one hand, any Dirac observable is constant along each orbit. Hence, it must also be constant along each dynamical trajectory of action (4.16). On the other, any function on $\mathcal{G}$ that is constant along each dynamical trajectory defines a unique extension to $\mathcal{C}$ that is constant along each orbit.

This relation between the Dirac observables of the extended system and the integrals of motion of the reduced theory, together with the compatibility of the symmetry groups generated by P_∞ in the extended and reduced theories, justify the approach of Secs. V and VI, where we shall construct the gauge-invariant dynamics starting from the gauge-dependent action (4.16).

V. PHYSICAL PHASE SPACE Γ_2

In this section, we choose a complete set of Dirac observables, find their Poisson algebra and calculate their Poisson brackets with the symmetry generator P_∞. This task is simplified if we start from Ashtekar–Pierri reduced action (4.16) instead of the original parametrized action (4.9). According to what has been shown in the previous sections, the result is independent of the gauge chosen to reduce the action (4.9).

The reduced action (4.16) can be rewritten in terms of the rescaled field $\phi = \psi/\sqrt{8G}$ introduced in Sec. III as follows:

$$S = \int dt dR (\pi_\phi \dot{\phi} - H),$$

where

$$\gamma_\infty = 4G \int_0^\infty dR \left(\frac{1}{R} \pi_\phi^2 + R \phi'^2 \right) \tag{5.1}$$

enters the Hamiltonian

$$H = \frac{1}{4G}(1 - e^{-(1/2)\gamma_\infty}). \tag{5.2}$$

For simplicity, the notation for our time T_∞ has been changed to the Ashtekar-Pierri notation t. The Hamiltonian depends on t only through π_ϕ and ϕ so that H and γ_∞ are constants of motion,

$$\dot{\gamma}_\infty = 0. \tag{5.3}$$

The canonical equations that follow from the action are

$$\dot{\pi}_\phi = e^{-(1/2)\gamma_\infty}(R\phi')', \tag{5.4}$$

$$\dot{\phi} = e^{-(1/2)\gamma_\infty}\frac{1}{R}\pi_\phi. \tag{5.5}$$

Equations (5.5) and (5.3) imply

$$\ddot{\phi} = e^{-(1/2)\gamma_\infty}\frac{1}{R}\dot{\pi}_\phi$$

so that

$$e^{\gamma_\infty}\ddot{\phi} = \frac{\partial^2 \phi}{\partial R^2} + \frac{1}{R}\frac{\partial \phi}{\partial R}. \tag{5.6}$$

If we use the relation between the Einstein-Rosen time T and the Ashtekar-Pierri time t (see [20]),

$$T(t) = e^{-(1/2)\gamma_\infty} t, \tag{5.7}$$

then Eq. (5.6) becomes the wave equation (3.6). The general solution to Eq. (3.6) is given by Eq. (3.18), which can be written in terms of time t as

$$\phi(t,R) = \frac{1}{\sqrt{2}}\int_0^\infty d\omega [A(\omega) J_0(\omega R) e^{-i\omega T(t)} + \text{c.c.}], \tag{5.8}$$

and Eq. (5.5) yields

$$\pi_\phi(t,R) = \frac{R}{\sqrt{2}}\int_0^\infty d\omega [-i\omega A(\omega) J_0(\omega R) e^{-i\omega T(t)} + \text{c.c.}]. \tag{5.9}$$

Equations (5.8) and (5.9) describe the general solution to the canonical equations (5.4) and (5.5) in terms of the set of constants $A(\omega)$. Hence, the parameters $A(\omega)$ can serve as coordinates on the physical phase space Γ_2.

The physical phase space is a symplectic manifold. Its full structure can be obtained if we find a transversal surface. As has been explained in Sec. II, any transversal surface, together with the symplectic form that results from pulling back the symplectic form from the extended phase space to the transversal surface, form the structure that is isomorphic to the physical phase space. In our case, the initial data ϕ_0 and $\pi_{\phi 0}$ of the canonical coordinates ϕ and π_ϕ at the Cauchy surface $t=0$ determine a unique solution (5.8) and (5.9) so that they can also be considered as coordinates on the physical phase space Γ_1. Moreover, the surface $\mathcal{T}_0$ defined by the Ashtekar and Pierri gauge (4.11) together with the condition $T_\infty = 0$ *is* a transversal surface. Hence, the symplectic form Ω on the physical phase space with respect to the coordinates ϕ_0 and $\pi_{\phi 0}$ is

$$\Omega_2 = \int_0^\infty dR d\pi_{\phi 0}(R) \wedge d\phi_0(R) \tag{5.10}$$

because this is the pull back of $\Omega_{\mathcal{P}}$ to $\mathcal{T}_0$ by the injection map of $\mathcal{T}_0$ into $\mathcal{P}$; the manifold $\mathcal{T}_0$ with this symplectic form is isomorphic to the physical phase space Γ_2.

The relations between the parameters $A(\omega)$ and ϕ_0, $\pi_{\phi 0}$ can be obtained from Eqs. (5.8) and (5.9):

$$\phi_0(R) = \frac{1}{\sqrt{2}}\int_0^\infty d\omega J_0(\omega R)[A(\omega) + A^*(\omega)], \tag{5.11}$$

and

$$\pi_{\phi 0}(R) = -\frac{iR}{\sqrt{2}}\int_0^\infty d\omega \omega J_0(\omega R)[A(\omega) - A^*(\omega)], \tag{5.12}$$

while the inverse transformation is analogous to Eq. (3.30):

$$A(\omega) = \frac{1}{\sqrt{2}}\int_0^\infty dR J_0(\omega R)[\omega R \phi_0(R) - i\pi_{\phi 0}(R)]. \tag{5.13}$$

A further set of parameters to determine points of the physical phase space are the $\mathcal{I}^-$ null data $g(V)$ or $\mathcal{I}^+$ null data $f(U)$. The transformations between $A(\omega)$ and $g(V)$ is given by Eqs. (3.21) and (3.25), those between $A(\omega)$ and $f(V)$ by Eqs. (3.20) and (3.27).

The quantity γ_∞ is a function on the physical phase space given, in terms of the four different coordinate systems, by Eqs. (5.1), (3.16) and (3.17). Equation (5.1), into which Eqs. (5.11) and (5.12) are substituted, yields after some simple transformations the expression for γ_∞ in terms of $A(\omega)$:

$$\gamma_\infty = 8G \int_0^\infty d\omega \omega A^*(\omega) A(\omega). \tag{5.14}$$

We can also express the symplectic form (5.10) in terms of $A(\omega)$. If Eqs. (5.11) and (5.12) are substituted into Eq. (5.10), we obtain

$$\Omega_2 = -\frac{i}{2}\int_0^\infty d\omega \int_0^\infty d\omega' \int_0^\infty dR\, \omega' R J_0(\omega R) J_0(\omega' R)$$
$$\times[-dA(\omega)\wedge dA(\omega') + dA(\omega)\wedge dA^*(\omega')$$
$$-dA^*(\omega)\wedge dA(\omega') + dA^*(\omega)\wedge dA^*(\omega')].$$

The formulas (3.28) and (3.29) imply, however, that

$$\int_0^\infty dRRJ_0(\omega R)J_0(\omega' R) = \frac{1}{\sqrt{\omega\omega'}}\delta(\omega-\omega'). \tag{5.15}$$

Hence, using the antisymmetry of the wedge product, we obtain finally

$$\Omega_2 = i\int_0^\infty d\omega dA^*(\omega)\wedge dA(\omega). \tag{5.16}$$

Let us also express the symplectic form of the physical phase space in terms of the asymptotic null data $g(V)$ and $f(U)$. Equations. (5.16) and (3.25) give

$$i\int_0^\infty d\omega dA^*(\omega)\wedge dA(\omega) = -\frac{i}{\pi}\int_{-\infty}^\infty dV \int_{-\infty}^\infty d\bar{V} dg(V)$$
$$\wedge dg(\bar{V})\int_0^\infty d\omega\omega e^{i\omega(V-\bar{V})}.$$

Since the wedge product is antisymmetric in V and V', only the antisymmetric part of the integral over ω contributes to the result. However,

$$\frac{i}{2\pi}\int_0^\infty d\omega\omega[e^{i\omega(V-\bar{V})} - e^{-i\omega(V-\bar{V})}]$$
$$= \frac{1}{2\pi}\int_{-\infty}^\infty d\omega i\omega e^{i\omega(V-\bar{V})}$$
$$= \frac{1}{2\pi}\frac{d}{dV}\int_{-\infty}^\infty d\omega e^{i\omega(V-\bar{V})}$$
$$= \frac{d}{dV}\delta(V-\bar{V}).$$

Hence,

$$\Omega_2 = \int_{-\infty}^\infty dV dg'(V)\wedge dg(V). \tag{5.17}$$

By analogous calculation, Eqs. (5.16) and (3.27) yield

$$\Omega_2 = \int_{-\infty}^\infty dU df'(U)\wedge df(U). \tag{5.18}$$

Finally, let us calculate the transformation between $f(U)$ and $g(V)$ if $f(U)$ is defined by the solution determined by $g(V)$. Such a transformation is, therefore, entailed by Eqs. (3.20) and (3.25):

$$f(U) = \frac{1}{2\pi}\int_{-\infty}^\infty dVg(V)\int_0^\infty d\omega[ie^{i\omega(U-V)} - ie^{-i\omega(U-V)}].$$

The distribution $D(U-V)$ defined by the integral over ω can be approximated by a convergent series of distributions $D_\epsilon(U-V)$ (see [37]),

$$\lim_{\epsilon\to 0} D_\epsilon(U-V) = D(U-V),$$

where $\epsilon > 0$ and

$$D_\epsilon(U-V) = \int_0^\infty d\omega[ie^{i\omega(U-V)-\omega\epsilon} - ie^{-i\omega(U-V)-\omega\epsilon}]$$
$$= -2\frac{U-V}{(U-V)^2+\epsilon^2}.$$

However,

$$\lim_{\epsilon\to 0}\int_{-\infty}^\infty dV\left[-2\frac{U-V}{(U-V)^2+\epsilon^2}\right]g(V)$$
$$= -2P\int_{-\infty}^\infty dV\,\frac{g(V)}{U-V},$$

where P denotes the principal value. Hence,

$$f(U) = -\frac{1}{\pi}P\int_{-\infty}^\infty dV\,\frac{g(V)}{U-V}. \tag{5.19}$$

VI. REPRESENTATION OF SYMMETRIES IN THE PHYSICAL PHASE SPACE

There are two interesting symmetries to be represented in the physical phase space. The first is the infinitesimal time translation, and the second is the map

$$\sigma: \mathcal{I}^- \mapsto \mathcal{I}^+,$$

defined by $U=V$ in terms of coordinates U at $\mathcal{I}^+$ and V at $\mathcal{I}^-$ (an analogous symmetry transformation has been studied in [25]).

The push-forward action of the infinitesimal translation $t \mapsto t+\delta t$ on the solution fields $\phi(t,R)$ and $\pi_\phi(t,R)$ is given by

$$\phi(t,R)\mapsto\phi(t-\delta t,R), \tag{6.1}$$

$$\pi_{\phi}(t,R)\mapsto\pi_{\phi}(t-\delta t,R). \tag{6.2}$$

Substituting $t-\delta t$ for t into Eqs. (5.8) and (5.9) and comparing the results with these equations leads to $A(\omega)\mapsto A(\omega)+\delta A(\omega)$, where

$$\delta A(\omega)=i\omega e^{-(1/2)\gamma_\infty}A(\omega)\,\delta t. \tag{6.3}$$

The same result can be obtained, if we put $\phi_0(R)+\delta\phi_0(R)$ and $\pi_{\phi 0}(R)+\delta\pi_{\phi 0}(R)$ into Eq. (5.13) and calculate the corresponding $\delta A(\omega)$. We must utilize the fact that

$$\delta\phi_0(R)=-\dot{\phi}_0(R)\,\delta t,\qquad \delta\pi_{\phi 0}(R)=-\dot{\pi}_{\phi 0}(R)\,\delta t,$$

express the time derivatives with the help of the equations of motion (5.4) and (5.5), transfer the r-derivatives from $\phi_0(R)$ to $J_0(\omega R)$, and use the Bessel equation that is satisfied by $J_0(\omega R)$,

$$-\frac{1}{R}(RJ_0'(\omega R))'=\omega^2 J_0(\omega R).$$

It follows that the action of the infinitesimal time translation is canonically generated by the function $-H$ defined by Eq. (5.2), with γ_∞ given by Eq. (5.1), where ϕ and π_ϕ are replaced by ϕ_0 and $\pi_{\phi 0}$. (Indeed, the momentum conjugate to t is $P_\infty=-H$.) We thus have

$$\delta\phi_0(R)=\{\phi_0(R),-H\},$$

$$\delta\pi_{\phi 0}(R)=\{\pi_{\phi 0}(R),-H\},$$

$$\delta A(\omega)=\{A(\omega),-H\},$$

and obtain analogously

$$\delta f(U)=\{f(U),-H\}, \tag{6.4}$$

$$\delta g(V)=\{g(V),-H\}. \tag{6.5}$$

Let us express $\delta f(U)$ and $\delta g(V)$ explicitly from the action of translations. Since the whole solution is shifted along the background manifold defined by the coordinates t and R by $t\mapsto t+\delta t$, $R\mapsto R$, we have, regarding Eq. (5.7),

$$U=T-R\mapsto e^{-(1/2)\gamma_\infty}(t+\delta t)-R=U+e^{-(1/2)\gamma_\infty}\delta t,$$

and similarly for V:

$$V\mapsto V+e^{-(1/2)\gamma_\infty}\delta t.$$

Hence,

$$\delta f(U)=-f'(U)e^{-(1/2)\gamma_\infty}\delta t, \tag{6.6}$$

$$\delta g(V)=-g'(V)e^{-(1/2)\gamma_\infty}\delta t. \tag{6.7}$$

The same relations result from the Poisson brackets (6.4) and (6.5), if Eqs. (3.20), (3.21), (5.2), (5.14) and (5.16) are used; notice that Eq. (5.16) implies

$$\{A(\omega),A(\omega')\}=0,\qquad \{A^*(\omega),A^*(\omega')\}=0, \tag{6.8}$$

and

$$\{A(\omega),A^*(\omega')\}=-i\,\delta(\omega-\omega'). \tag{6.9}$$

As has been explained in Sec. II (see also [11]), the dynamics of the Dirac observables is defined by the comparison of the equations of motion with the action of symmetry. Since the equations of motion for the Dirac observables are trivial (the observables remain constant), the symmetry action alone gives the total dynamical change.

The second symmetry σ: $\mathcal{I}^-\mapsto\mathcal{I}^+$ is a purely asymptotic one, similarly to $T_\infty\mapsto T_\infty+\tau$. Its action on solutions can be found in an analogous way via the Cauchy data for solutions at $\mathcal{I}$'s. We consider Cauchy null datum $g_1(V)$ at $\mathcal{I}^-$ as defining solution $\phi_1(t,R)$. Then we push forward the field $g_1(V)$ at $\mathcal{I}^-$ to $\mathcal{I}^+$ by σ_*, which results in Cauchy datum $f_2(U)$ at $\mathcal{I}^+$. The Cauchy datum $f_2(U)$ determines another solution $\phi_2(t,R)$, and we define it as the image of $\phi_1(t,R)$ by σ. The corresponding map in Γ_2 can be calculated by using coordinates $f(U)$ in Γ_2. Solution $\phi_2(t,R)$ has coordinate $f_2(U)$; let $\phi_1(t,R)$ have coordinate $f_1(U)$. Then the point $f_2(U)$ in Γ_2 is the image of the point $f_1(U)$ of Γ_2 by map σ. The dynamics defined by σ is the inverse map because it compares the evolution by the wave equation (which is trivial because Dirac observables remain constant) with the map by σ (cf. [25]).

The push forward map σ_* of fields at $\mathcal{I}^-$ to those at $\mathcal{I}^+$ acts as follows:

$$\sigma_* g(V)=f(U),$$

where $f(U)=g(U)$. It follows immediately that the dynamical evolution defined by the "zero motion" σ is represented by transformation (5.19).

We can also introduce Fourier amplitudes $a(\omega)$ and $b(\omega)$ of the asymptotic data by

$$a(\omega)=A(\omega)e^{i\pi/4},\qquad b(\omega)=A(\omega)e^{-i\pi/4},$$

so that Eqs. (3.20) and (3.21) become

$$f(U)=\frac{1}{2\sqrt{\pi}}\int_0^\infty\frac{d\omega}{\sqrt{\omega}}[b(\omega)e^{-i\omega U}+\text{c.c.}],$$

$$g(V)=\frac{1}{2\sqrt{\pi}}\int_0^\infty\frac{d\omega}{\sqrt{\omega}}[a(\omega)e^{-i\omega V}+\text{c.c.}].$$

The push forward of the amplitudes is clearly given by

$$\sigma_* a(\omega)=b(\omega),$$

where $b(\omega)=a(\omega)$. Canonical representation of σ is, therefore:

$$b(\omega)=a(\omega)e^{-i\pi/2}=-ia(\omega). \tag{6.10}$$

This map [or Eq. (5.19)] becomes the S-matrix of the one-particle sector in the quantum theory of the model.

VII. QUANTUM THEORY

It is easy to construct the Hilbert space, the operators representing the Dirac observables, the Hamiltonian, and to define the scattering matrix in the standard way of quantization of linear field theories (see, e.g., [38] or [39]). A sketch thereof will be described in this section.

Let us start from the Poisson brackets (6.8) and (6.9) for the observables $A(\omega)$. Roughly, in the canonical quantization, Poisson brackets are replaced by commutators multiplied by i (the units are chosen so that the Planck constant is 1). Then, we have

$$[\hat{A}(\omega'),\hat{A}(\omega)]=0,\quad [\hat{A}(\omega'),\hat{A}^\dagger(\omega)]=\delta(\omega'-\omega). \tag{7.1}$$

These are commutators of the *annihilation* and *creation operators* of a quantum field theory for a continuous spectrum. They form our starting point.

For many constructions it is favorable to use a smeared version of the operators. We choose any complete orthonormal basis of (complex) functions $X_n(\omega)$, where $\omega\in(0,\infty)$. This means that any complex function f can be decomposed,

$$f(\omega)=\sum_n f_n X_n(\omega),$$

where f_n are complex coefficients, and that

$$\int_0^\infty d\omega X_n^*(\omega)X_m(\omega)=\delta_{nm}. \tag{7.2}$$

Defining

$$\hat{A}_n=\int_0^\infty d\omega X_n^*(\omega)\hat{A}(\omega), \tag{7.3}$$

we obtain

$$\hat{A}(\omega)=\sum_n X_n(\omega)\hat{A}_n, \tag{7.4}$$

and

$$[\hat{A}_n,\hat{A}_m]=0,\qquad [\hat{A}_n,\hat{A}_m^\dagger]=\delta_{nm}. \tag{7.5}$$

Then we can define the vacuum state $|0\rangle$ by

$$\hat{A}_n|0\rangle=0\quad \forall n,\qquad \langle 0|0\rangle=1, \tag{7.6}$$

which also implies that

$$\hat{A}(\omega)|0\rangle=0\quad \forall\omega. \tag{7.7}$$

The elements of a complete basis in the Hilbert space are obtained by application of any number of creation operators $\hat{A}_m^\dagger$ to $|0\rangle$; if the total number of the creation operators is N, then the state is an N-graviton state. The scalar product is defined by scalar products of the basis elements, which, in turn, are determined by the commutation rules (7.5) and the conditions (7.6). For example,

$$(\hat{A}_m^\dagger|0\rangle,\hat{A}_n^\dagger|0\rangle)=\langle 0|\hat{A}_m\hat{A}_n^\dagger|0\rangle$$
$$=\langle 0|\hat{A}_m^\dagger\hat{A}_n+\delta_{nm}|0\rangle=\delta_{nm}.$$

The Hilbert space defined in this way is often called the *Fock space* and we denote it by $\mathcal{F}$.

Those Dirac observables defined in Sec. V that are linear in the variables $A(\omega)$ and $A^\dagger(\omega)$ can be associated with operators on $\mathcal{F}$ that are linear combinations of the operators $\hat{A}(\omega)$ and $\hat{A}^\dagger(\omega)$ with the same coefficients. This definition preserves the relation between Poisson brackets and commutators. For example, we define

$$\hat{f}(U)=\frac{1}{2\sqrt{\pi}}\int_0^\infty\frac{d\omega}{\sqrt{\omega}}[\hat{A}(\omega)e^{-i(\pi/4)-i\omega U}$$
$$+\hat{A}^\dagger(\omega)e^{i(\pi/4)+i\omega U}].$$

The matrix elements of $\hat{f}(U)$ with respect to the Fock basis are easily calculated by using the decomposition (7.4). In such a way, we have a Hilbert space and the operators that correspond to the basic quantities.

In order to construct the Hamiltonian, we start from Eqs. (5.2) and (5.14). We define the quadratic operator $\hat{\gamma}_\infty$ by the normal factor ordering:

$$\hat{\gamma}_\infty=8G\int_0^\infty d\omega\omega\hat{A}^\dagger(\omega)\hat{A}(\omega)=\sum_{nm}\omega_{nm}\hat{A}_n^\dagger\hat{A}_m,$$

where

$$\omega_{nm}=8G\int_0^\infty d\omega\omega X_n^*(\omega)X_m(\omega).$$

Then $\hat{\gamma}_\infty|0\rangle=0$. The operator $\hat{\gamma}_\infty$ is self-adjoint on $\mathcal{F}$; it has a continuous spectrum. Its (generalized) eigenvectors form a δ-normalized basis of $\mathcal{F}$, elements of which are obtained from the vacuum by application of any number of the creation operators $\hat{A}^\dagger(\omega)$ (and a normalization factor). For example,

$$\hat{\gamma}_\infty(\hat{A}^\dagger(\omega)|0\rangle)=8G\omega(\hat{A}^\dagger(\omega)|0\rangle).$$

Then, any function of $\hat{\gamma}_\infty$ can be defined by the spectral theorem (see, e.g., [40]): it has the same eigenvectors, and its eigenvalues are the values that the function has on the corresponding eigenvalues of $\hat{\gamma}_\infty$. In this way, the Hamilton operator

$$\hat{H}=\frac{1}{4G}\left[1-\exp\left(-\frac{1}{2}\hat{\gamma}_\infty\right)\right]$$

is well-defined. For example,

$$\hat{H}|0\rangle=\frac{1}{4G}\left[1-\exp\left(-\frac{1}{2}\times 0\right)\right]=0,$$

because $\hat{\gamma}_\infty$ has the eigenvalue zero on $|0\rangle$.

Finally, we can define the scattering matrix $\hat{S}$. In order to do that, we have to determine what are the in- and out-states. It seems natural to take the states that result from applying any number of the operators $\hat{a}^{\dagger}(\omega)$ to $|0\rangle$ corresponding to the observables $a(\omega)$ of Sec. VI as the in-states. Similarly, the out-states can be defined by $b(\omega)$. From Eq. (6.10), we have a simple Bogolyubov transformation between $\hat{a}(\omega)$ and $\hat{b}(\omega)$:

$$\hat{a}(\omega)=i\hat{b}(\omega).$$

The construction of the scattering matrix that implements a given Bogolyubov transformation is described in [39] or [38]. We shall skip it because it lies outside the scope of this paper.

Another interesting question is what is the relation between the Hamiltonian $\hat{H}$ and the scattering operator $\hat{S}$? There are methods of calculating $\hat{S}$ from $\hat{H}$: one has to take some limits within the Euclidean regime (see, e.g., [13]). However, an application, or even an applicability, of these methods to our case also lies outside the scope of this work.

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[1] R. Arnowitt, S. Deser, and C. W. Misner, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (Wiley, New York, 1962).
[2] P. A. M. Dirac, *Lectures on Quantum Mechanics* (Yeshiva University Press, New York, 1964).
[3] P. G. Bergmann, Rev. Mod. Phys. **33**, 510 (1961).
[4] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems* (Princeton University Press, Princeton, NJ, 1992).
[5] S. W. Hawking, in *General Relativity: An Einstein Centenary Survey*, edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1979).
[6] C. G. Torre, Phys. Rev. D **48**, R2373 (1993).
[7] C. Rovelli, Phys. Rev. D **42**, 2638 (1990); **43**, 442 (1991); **44**, 1339 (1991).
[8] J. Kijowski and W. M. Tulczyjev, *A Symplectic Framework for Field Theories*, Lecture Notes in Physics Vol. 107 (Springer, Berlin, 1979).
[9] P. Hájíček, J. Math. Phys. **36**, 4612 (1995).
[10] P. Hájíček, Class. Quantum Grav. **13**, 1353 (1996).
[11] P. Hájíček, in Proceedings of the *271. WE-Heraeus-Seminar Aspects of Quantum Gravity*, Bad Honnef, Germany, 2002 (Springer, Berlin, 2003), gr-qc/0204049.
[12] R. Gambini and J. Pullin, Class. Quantum Grav. **17**, 4515 (2000).
[13] B. C. DeWitt, *Dynamical Theory of Groups and Fields* (Gordon and Breach, Inc., New York, 1965).
[14] A. Ashtekar, *Asymptotic Quantization* (Bibliopolis, Napoli, 1987).
[15] J. Bičák and P. Hájíček, Phys. Rev. D (to be published), gr-qc/0308013.
[16] In fact, cylindrical gravitational waves were found more than ten years before Einstein and Rosen by Guido Beck, Z. Phys. **33**, 713 (1925). Beck was mainly interested in time-independent axisymmetric fields but he realized that through changes $z\mapsto it$, $t\mapsto iz$ one obtains cylindrical waves.
[17] J. Bičák, in *Einstein's Field Equations and Their Physical Implications*, edited by B. G. Schmidt, Lecture Notes in Physics Vol. 540 (Springer, Berlin, 2000).
[18] K. V. Kuchař, Phys. Rev. D **4**, 955 (1971).
[19] C. G. Torre, Class. Quantum Grav. **8**, 1895 (1991).
[20] A. Ashtekar and M. Pierri, J. Math. Phys. **37**, 6250 (1996).
[21] A. Ashtekar and M. Varadarajan, Phys. Rev. D **50**, 4944 (1994).
[22] A. Ashtekar, J. Bičák, and B. G. Schmidt, Phys. Rev. D **55**, 669 (1997).
[23] R. Beig and N. O'Murchadha, Ann. Phys. (N.Y.) **174**, 463 (1987).
[24] K. V. Kuchař, Phys. Rev. D **50**, 3961 (1994).
[25] P. Hájíček and C. J. Isham, J. Math. Phys. **37**, 3522 (1996).
[26] K. V. Kuchař, in *Relativity, Astrophysics and Cosmology*, edited by W. Israel (Reidel, Dordrecht, 1973).
[27] P. Hájíček, Nucl. Phys. B (Proc. Suppl.) **80** (2000), CD-ROM supplement, gr-qc/9903089.
[28] P. Hájíček and J. Kijowski, Phys. Rev. D **61**, 024037 (2000).
[29] K. V. Kuchař, J. Math. Phys. **22**, 2640 (1981).
[30] J. Bičák and B. G. Schmidt, Phys. Rev. D **40**, 1827 (1989).
[31] A. Ashtekar, J. Bičák, and B. G. Schmidt, Phys. Rev. D **55**, 687 (1997).
[32] J. J. Stachel, J. Math. Phys. **7**, 1321 (1966).
[33] J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975).
[34] J. Weber and J. A. Wheeler, Rev. Mod. Phys. **29**, 509 (1957).
[35] W. B. Bonnor, J. Math. Mech. **6**, 203 (1957).
[36] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic, New York, 1969).
[37] I. M. Gel'fand and G. E. Shilov, *Generalized Functions* (Academic, New York, 1964), Vol. 1.
[38] B. C. DeWitt, Phys. Rep., Phys. Lett. **19C**, 295 (1975).
[39] R. M. Wald, *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics* (The University of Chicago Press, Chicago, 1994).
[40] J. Dieudonné, *Foundation of Modern Analysis* (Academic, New York, 1969).