Complete solution of 2D superfield supergravity from graded Poisson-sigma models, and the super point particle

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Recently an alternative description of 2D supergravities in terms of graded Poisson-sigma models (GPSM) has been given. As pointed out previously by the present authors a certain subset of GPSMs can be interpreted as "genuine" supergravity, satisfying the well-known limits of supergravity, albeit deformed by the dilaton field. In our present paper we show that precisely that class of GPSMs corresponds one-to-one to the known dilaton supergravity superfield theories presented a long time ago by Park and Strominger. Therefore, the unique advantages of the GPSM approach can be exploited for the latter: We are able to provide the first complete classical solution for any such theory. On the other hand, the straightforward superfield formulation of the point particle in a supergravity background can be translated back into the GPSM frame, where "supergeodesics" can be discussed in terms of a minimal set of supergravity field degrees of freedom. Further possible applications such as the (almost) trivial quantization are mentioned.

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I. INTRODUCTION

Theories of gravity in 1+1 dimensions naturally emerge from the generalization of (e.g. spherically) reduced Einstein gravity in arbitrary dimensions. Decisive progress in the treatment of their classical and quantum properties were the consequence of the discovery in the early 1990s that a Cartan formulation in a specific lightlike gauge [1], which is equivalent to an Eddington-Finkelstein gauge for the metric, not only simplifies enormously the evaluation of the classical theory, but even allowed an exact (trivial) nonperturbative quantization [2–5]. After the application to a particular model with curvature and torsion [6] it was realized that not only all 2D gravity models but an even larger class of theories may be covered by the concept of Poisson-sigma models (PSMs) [7–11]. There a set of target space coordinates (auxiliary fields on the 2D worldsheet) exists in addition to the gauge degrees of freedom. In this framework the simplicity of 2D classical and quantum gravity becomes manifest. PSM models generalized naturally to the graded case (GPSM) when they are supplemented by anticommuting fields [12,13]. The resulting models exhibit the typical gauge transformation of supergravity theories. However, the fermionic extensions are highly ambiguous. In addition they may introduce new singularities and/or obstructions as compared to the bosonic theory for which they have been derived. This result was obtained for the $N=(1,1)$ superextension, but should hold also for higher N.

Recently the present authors realized [14] that a subset of those GPSMs can be identified, which satisfies a constraint algebra whose structure is very close to the algebra of "genuine" supergravity. In this algebra the modifications by the presence of the dilaton field [and its single fermionic partner for $N=(1,1)$] are, in a sense, minimal. The only GPSMs allowed by that algebra correspond to a *unique* class of (dilaton deformed) $N=(1,1)$ supergravity theories [called "minimal field supergravity" (MFS[1]) in the following] in which—somewhat miraculously—even all singularities and obstructions in the generic fermionic extensions disappear. The bosonic part of physically interesting theories (spherically reduced gravity [15–18], string inspired black hole [19], simplified models [20–24], bosonic potential of supergravity from superspace [25]) are special cases thereof. This is also a nontrivial result, because the "potential" of those bosonic theories must be derivable from a prepotential.

Already in the purely bosonic case, where the PSM is equivalent [26,27] to a general 2D dilaton theory (GDT) with vanishing torsion but dynamical dilaton, the corresponding PSM works with nonvanishing bosonic torsion. If this PSM action shall be extended directly to its supersymmetrized version using the superspace formalism, the generalization of the usual conventional constraints, valid solely for vanishing bosonic torsion, is an imperative step. In consequence, a new solution of Bianchi identities, etc., has to be considered, which turned out to be a highly nontrivial task [28]. Within the GPSM approach this problem is avoided altogether and it suffices to solve a graded Jacobi-type identity (vanishing Nijenhuis tensor) [13].

Therefore the question arises whether, and in what sense, the equivalence of the bosonic PSM and GDT theories can be extended to supergravity. To this end it must be investigated whether a GPSM based MFS has any relation to a genuine dilaton superfield theory, expressed in terms of superspace coordinates for a *dynamical* super-dilaton field. An indication that this may work comes from the known result [13,28–31] that a GPSM model with *vanishing* bosonic torsion [31] is—up to elimination of auxiliary fields—

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[1] Already at this point the authors apologize for the introduction of quite a number of special acronyms. It seems that only in terms of those a reasonably compact formulation of the strategy is possible (cf. also Fig. 1 below).

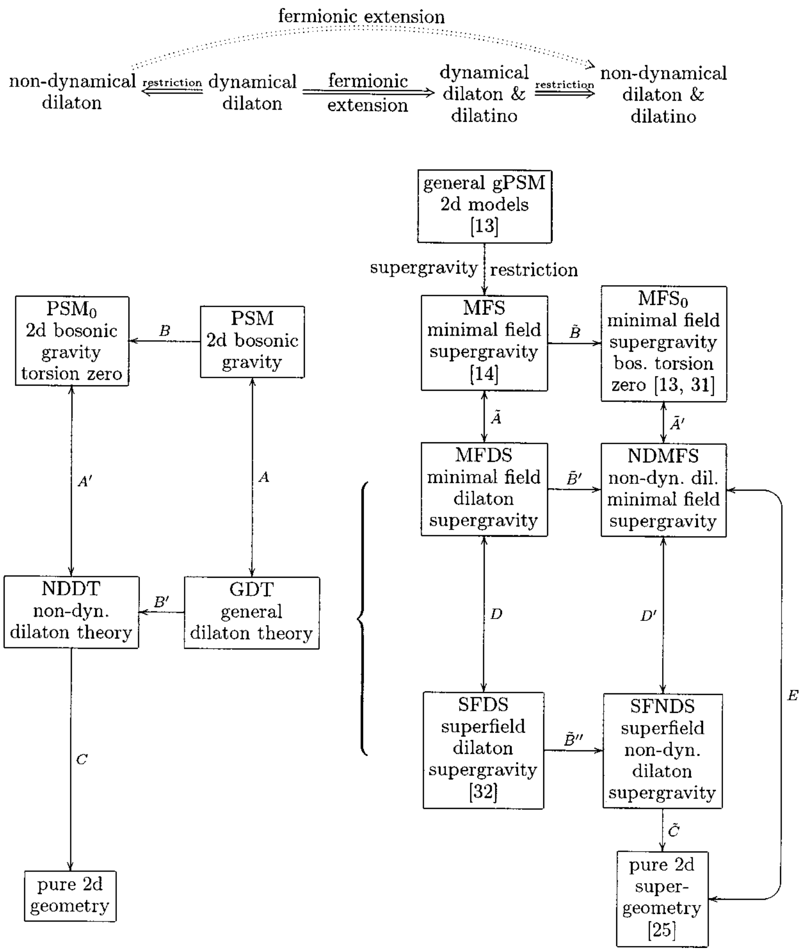

FIG. 1. Relation between different formulations of 2D gravity and 2D supergravity. Explanations are given in the text.

equivalent to a dilaton superfield theory [32] with *nondynamical* dilaton. By further elimination of an auxiliary spinor ("dilatino") this simpler model can be related quite generally to the supergravity model of Howe [25] as well.

One of the main motivations to establish such a relation in the general case is the fact that in the (G)PSM approach the complete exact classical solution can be found for all such models. Also, for bosonic PSMs the quantization is (almost) trivial [1–4][2] as long as no matter interactions are included. But even with matter a meaningful quantum perturbation theory can be developed [34–36], leading to an improved understanding of phenomena like the virtual black hole [37,38]. Most of these results should extend straightforwardly to the GPSM [39], which would allow substantial progress in the understanding of generalized supergravity in two dimensions.

In our present paper we are able to report that, indeed, a detailed equivalence exists between the class of GPSM-supergravities of Ref. [14] (MFS models) and the well-known dilaton superfield supergravities, proposed sometime ago by Park and Strominger [32] [dubbed "superfield dilaton supergravities" (SFDS)]. The equivalence proceeds through different steps which should be transparent in the schematic representation of Fig. 1, an explanation thereof is given first:

The two left hand columns of the figure cover the purely bosonic theories, the ones on the right-hand side (rhs) include their fermionic extensions. The two columns in the middle contain theories with dynamical dilaton, while the two columns at the borders are reserved for the restricted class of models with nondynamical dilaton, respectively.

As indicated by arrows at the top of the figure, different theories displayed in a row are related to each other by means of supersymmetric extension or by restriction to nondynamical dilaton.

Two fermionic extensions (MFDS and SFDS) correspond to GDT, which is indicated by the large bracket.

Relations between different models are described by arrows. Double-headed arrows are used if the corresponding relation indicates complete equivalence, or, at least, holds for the most important class of the connected theories. Simple arrows point from the more general theories towards the restricted ones.

Labels with a tilde indicate that this relation is a straightforward generalization of the corresponding relation among bosonic theories (e.g. $A \leftrightarrow \tilde{A}$). Relations among different arrows within the same part of the figure (bosonic part or supersymmetric part, respectively) are indicated by primes (e.g. $A \leftrightarrow A'$).

The equivalence A between PSM and GDT (respectively A' for nondynamical dilaton) in the bosonic case is well-

[2]For a comprehensive review we suggest [33].

known [26,27,33] so that $\widetilde{A}$ amounts to a trivial generalization of A, when anticommuting fields are included [13]. It connects MFS to minimal field dilaton supergravity (MFDS), the fermionic extension of GDT (the same is true for $\widetilde{A}'$).

The proof of the quite nontrivial equivalence D provides the basis of our present paper. We first establish it between the NDMFS and the SFNDS theory, both *without* dynamical dilaton, following the path D' in Fig. 1. In a second step SFNDS and SFDS are found to be connected by a (super-)conformal transformation ($\widetilde{B}''$ backwards), which in the GPSM frame possesses a counterpart in a special target space diffeomorphism (path $\widetilde{B}$ backwards) between MFS_0 and MFS. That latter transformation turns out to be a generalization of the conformal transformation linking GDT and NDDT in the pure bosonic case (path B). We have found that in this way the more complicated direct relation of the general models (path D) is sufficiently transparent. This strategy is especially important also for keeping track of the proper way the symmetry transformations are mapped upon each other following those successive steps.

Another equivalence is established between theories with nondynamical dilaton (MFS_0 and SFNDS, respectively) and the model of Howe [25]. For the restricted class of actions with invertible (pre-)potential this may be obtained by the elimination of a superfield (path $\widetilde{C}$) or, alternatively, by the path $D' \to E$. This last equivalence also allows one to relate MFS_0 directly (i.e. without using D', but instead $\widetilde{A}' \to E$) to Howe's supergravity [25].

On the basis of those relations the technical advantages of GPSM supergravity can be exploited in full detail for the SFDS theories of Ref. [32]: Proceeding "top down" from the box MFS in Fig. 1 ($\widetilde{A} \to D$) and using the known general solution for the MFS_0 [13] together with (the inverted arrow) $\widetilde{B}$, we are able to give the complete analytic solution for the general superfield dilaton supergravity of Ref. [32], including all fermionic contributions.

Another example where the opposite way, the "bottom up" sequence ($D \to \widetilde{A}$) is to be chosen, is important for the determination of the supergravity generalization of the geodesic within the GPSM formulation because the supersymmetric line element or the super-point particle can be defined straightforwardly in the superfield formulation only.

The paper is organized as follows: In Sec. II at first (Sec. II A) the basic features of GPSMs are reviewed shortly. Then (Sec. II B) the subset MFS of "genuine" supergravities is described as determined in Ref. [14] and the corresponding MFDS (Sec. II C) which is obtained after elimination of certain auxiliary[3] fields. They include the part of the spin-connection which, at the PSM level, depends on the bosonic torsion and the target-space coordinates except the dilaton and the dilatino.

Section III is devoted to the superfield approach of 2D supergravity, where it suffices to consider the standard case with vanishing bosonic torsion. The important role of (super-)conformal transformations is explained in Sec. IV which prepares the ground for the equivalence proof of minimal field supergravity, as deduced from GPSMs, with dilaton superfield supergravity. The proof is presented in Sec. V. All exact classical solutions of 2D superfield supergravity [32] are obtained in Sec.VI. Another application (Sec. VII) is the formulation of a supergeodesic, defined as the motion of a test particle in the background of minimal field supergravity. Here only some very simple special cases are discussed, as e.g. the null-directions and the consequences for the supergravity background generating the Schwarzschild solution. In the Appendixes we collect details of our notation and some lengthy formulas.

II. GRADED POISSON-SIGMA MODEL AND MINIMAL FIELD SUPERGRAVITY

A. Graded Poisson-sigma model

A general GPSM consists of scalar fields $X^I(x)$, which are themselves coordinates of a graded Poisson manifold with Poisson tensor $P^{IJ}(X) = (-1)^{IJ+1} P^{JI}(X)$. The index I, in the generic case, includes commuting as well as anticommuting fields.[4] In addition one introduces the gauge potential $A = \mathrm{d}X^I A_I = \mathrm{d}X^I A_{mI}(x)\mathrm{d}x^m$, a one form with respect to the Poisson structure as well as with respect to the 2D worldsheet coordinates. The GPSM action reads[5]

$$\mathcal{S}_{\mathrm{GPSM}} = \int_M \mathrm{d}X^I \wedge A_I + \frac{1}{2} P^{IJ} A_J \wedge A_I$$

$$= \int e(\partial_0 X^I A_{1I} - \partial_1 X^I A_{0I} + P^{IJ} A_{0J} A_{1I}) \mathrm{d}^2 x. \quad (1)$$

The Poisson tensor P^{IJ} must have a vanishing Nijenhuis tensor (obey a Jacobi-type identity with respect to the Schouten bracket related as $\{X^I, X^J\} = P^{IJ}$ to the Poisson tensor)

$$P^{IL} \partial_L P^{JK} + \text{g-perm}(IJK) = 0, \quad (2)$$

where the sum runs over the graded permutations. Due to Eq. (2) the action (1) is invariant under the symmetry transformations

$$\delta X^I = P^{IJ} \varepsilon_J, \qquad \delta A_I = -\mathrm{d}\varepsilon_I - (\partial_I P^{JK}) \varepsilon_K A_J, \quad (3)$$

where the term $\mathrm{d}\epsilon_I$ in the second of these equations provides the justification for calling A_I "gauge fields."

For a generic (G)PSM the commutator of two transformations (3) is a symmetry modulo the equations of motion (e.o.m.s). Only for P^{IJ} linear in X^I a closed (and linear) Lie algebra is obtained, and Eq. (2) reduces to the Jacobi identity for the structure constants of a Lie group. If the Poisson

[3] These "auxiliary" fields in the (G)PSM approach should not be confused with auxiliary fields in a superfield formulation.

[4] The usage of different indices as well as other features of our notation are explained in Appendix A. For further details one should consult Refs. [13,28].

[5] If the multiplication of forms is evident in what follows, the wedge symbol will be omitted.

tensor is singular—the actual situation in any application to 2D (super-)gravity due to the odd dimension of the bosonic part of the tensor—there exist (one or more) Casimir functions $C(X)$ obeying

$$\{X^I,C\}=P^{IJ}\frac{\partial C}{\partial X^J}=0, \tag{4}$$

which, when determined by the field equations of motion, are constants of motion. The variation of A_I and X^I in Eq. (1) yields the GPSM field equations

$$\mathrm{d}X^I+P^{IJ}A_J=0, \tag{5}$$

$$\mathrm{d}A_I+\frac{1}{2}(\partial_I P^{JK})A_K A_J=0. \tag{6}$$

In the application to two dimensional $N=(1,1)$ supergravity,[6] the gauge potentials comprise the spin connection $\omega_{ab}=\omega\epsilon_{ab}$, the zweibein and the gravitino:

$$A_I=(A_\phi,A_a,A_\alpha)=(\omega,e_a,\psi_\alpha),$$

$$X^I=(X^\phi,X^a,X^\alpha)=(\phi,X^a,\chi^\alpha). \tag{7}$$

The fermionic components ψ_α (‘‘gravitino’’) and χ^α (‘‘dilatino’’) are Majorana spinors. Local Lorentz invariance determines the ϕ-components of the Poisson tensor

$$P^{a\phi}=X^b\epsilon_b{}^a,\quad P^{\alpha\phi}=-\frac{1}{2}\chi^\beta\gamma^3{}_\beta{}^\alpha, \tag{8}$$

and the supersymmetry transformation is encoded in $P^{\alpha\beta}$. In a purely bosonic theory, the only arbitrary component of the Poisson tensor is $P^{ab}=v\epsilon^{ab}$, where the locally Lorentz invariant ‘‘potential’’ $v=v(\phi,Y)$ describes different models $(Y=X^aX_a/2)$. Evaluating Eq. (1) with that P^{ab} and $P^{a\phi}$ from Eq. (8) the action ($\epsilon=\frac{1}{2}\epsilon^{ab}e_b\wedge e_a$ is the volume form, $De_a=\mathrm{d}e_a+\omega\epsilon_a{}^be_b$)

$$\mathcal{S}_{\mathrm{PSM}}=\int_{\mathcal{M}}(\phi\mathrm{d}\omega+X^aDe_a+\epsilon v) \tag{9}$$

is obtained. The physically most interesting models are described by potentials quadratic in X^a

$$v=YZ(\phi)+V(\phi). \tag{10}$$

They include spherically reduced Einstein gravity [15–18], the string inspired black hole [19], the simplified model with $Z=0$ and linear $V(\phi)$ [20–24], the bosonic part of the Howe model [25], etc.

Potentials of type (10) allow the integration of the (single) Casimir function C in Eq. (4)

$$C=e^{Q(\phi)}Y+W(\phi),\quad Q(\phi)=\int_{\phi_1}^{\phi}\mathrm{d}\varphi\, Z(\varphi),$$

$$W(\phi)=\int_{\phi_0}^{\phi}\mathrm{d}\varphi e^{Q(\varphi)}V(\varphi), \tag{11}$$

where e.g., in spherically reduced gravity C on-shell is proportional to the ADM-mass in the Schwarzschild solution.

The auxiliary variables X^a and the torsion-dependent part of the spin connection ω can be eliminated by *algebraic* equations of motion (path A in Fig. 1). Then the action reduces to the familiar generalized dilaton theory in terms of the dilaton field ϕ and the metric:

$$\mathcal{S}_{\mathrm{GDT}}=\int \mathrm{d}^2x e\left(\frac{1}{2}R\phi-\frac{1}{2}Z\partial^m\phi\partial_m\phi+V(\phi)\right). \tag{12}$$

Both formulations are equivalent at the classical [26,27] as well as at the quantum level [34–36].

For theories with nondynamical dilaton [$Z=0$ in Eq. (10)] a further elimination of ϕ is possible if the potential $V(\phi)$ is invertible. In this way one arrives at a theory solely formulated in terms of the zweibein e_m^a (path C in Fig. 1).

B. Minimal field supergravity

For $N=(1,1)$ supergravity [cf. Eq. (7)] a generic fermionic extension of the action (9) is obtained by making general Lorentz invariant *Ansätze* for $P^{a\alpha},P^{\alpha\beta}$ together with the fermionic extension of $P^{ab}=\epsilon^{ab}(v+\chi^2v_2)$ of the bosonic case $(\chi^2=\chi^\alpha\chi_\alpha)$. Then the Jacobi identity (2) is solved. Here Eq. (8) and the bosonic potential v are a given input. This leads to an algebraic, albeit highly ambiguous solution with several arbitrary functions [13]. In addition, the fermionic extensions generically exhibit new singular terms. Also not all bosonic models permit such an extension for the whole range of their bosonic fields, sometimes even no extension is allowed.

As shown by the present authors [14], it is, nevertheless, possible to select ‘‘genuine’’ supergravity from this huge set of theories. This is possible by a generalization of the standard requirements for a ‘‘true’’ supergravity [41–45] to the situation, where deformations from the dilaton field ϕ are present. To this end the nonlinear symmetry (3), which is closed on-shell only, is—in a first step—related to a more convenient (off-shell closed) algebra of Hamiltonian constraints $G^I=\partial_1X^I+P^{IJ}(X)A_{1J}$. The Hamiltonian obtained from Eq. (1) in terms of these constraints takes the form [2,14,46]

$$H=\int \mathrm{d}x^1\, G^IA_{0I}. \tag{13}$$

In a second step a certain linear combination of the G^I, suggested by the ADM parametrization [14,47,48], maps the G^I algebra upon a deformed version of the superconformal algebra (deformed Neuveu-Schwarz, respectively, Ramond algebra). This algebra is appropriate to impose restrictions, which represent a natural generalization of the requirements

[6]More complicated identifications of the 2D Cartan variables with A_I are conceivable [40].

from supergravity to theories deformed by the dilaton field. It turned out that the subset of models allowed by these restrictions uniquely leads to the GPSM *supergravity* class of theories [called "minimal field supergravity" (MFS) in our present paper] with the Poisson tensor[7]

$$P^{ab}=\left(V+YZ-\frac{1}{2}\chi^2\left(\frac{VZ+V'}{2u}+\frac{2V^2}{u^3}\right)\right)\epsilon^{ab}, \tag{14}$$

$$P^{\alpha b}=\frac{Z}{4}X^a(\chi\gamma_a\gamma^b\gamma^3)^\alpha+\frac{iV}{u}(\chi\gamma^b)^\alpha, \tag{15}$$

$$P^{\alpha\beta}=-2iX^c\gamma_c^{\alpha\beta}+\left(u+\frac{Z}{8}\chi^2\right)\gamma^{3\alpha\beta}, \tag{16}$$

where the three functions V, Z and the "prepotential" u depend on the dilaton field ϕ only. Besides the fixed components of P^{IJ} according to Eq. (8) supergravity requires the existence of supersymmetry transformations, which are generated by the first term in Eq. (16). It has been a central result of Ref. [14] that $P^{\alpha\beta}$ must be of the form (16), i.e. the generator of supersymmetry transformations is not allowed to receive any deformations with respect to its form from rigid supersymmetry. Furthermore in order to satisfy the condition (2) V, Z and u must be related by ($u'=du/d\phi$)

$$V(\phi)=-\frac{1}{8}((u^2)'+u^2Z(\phi)). \tag{17}$$

Thus, starting from a certain bosonic model with potential (10) in Eq. (14), the only restriction remains that it must be expressible in terms of a prepotential u by Eq. (17). This happens to be the case for most physically interesting theories [15–25]. Inserting the Poisson tensor (8), (14)–(16) into Eq. (1) the ensuing action becomes [the covariant derivatives are defined in Eq. (A7)]

$$\begin{aligned}\mathcal{S}_{\mathrm{MFS}}=\int_{\mathcal{M}}\Bigg(&\phi\,\mathrm{d}\omega+X^aDe_a+\chi^\alpha D\psi_\alpha\\&+\epsilon\left(V+YZ-\frac{1}{2}\chi^2\left(\frac{VZ+V'}{2u}+\frac{2V^2}{u^3}\right)\right)\\&+\frac{Z}{4}X^a(\chi\gamma_a\gamma^be_b\gamma^3\psi)+\frac{iV}{u}(\chi\gamma^ae_a\psi)+iX^a(\psi\gamma_a\psi)\\&-\frac{1}{2}\left(u+\frac{Z}{8}\chi^2\right)(\psi\gamma_3\psi)\Bigg).\end{aligned} \tag{18}$$

For later reference we also define the simpler model with $\bar{Z}=0$ (MFS_0), where the fields are denoted by ($\bar{\phi},\bar{X}^a,\bar{\chi}^\alpha$) and ($\bar{\omega},\bar{e}_a,\bar{\psi}_\alpha$)

$$\mathcal{S}_{\mathrm{MFS}_0}(\bar{\phi},\bar{X}^a,\bar{\chi}^\alpha;\bar{\omega},\bar{e}_a,\bar{\psi}_\alpha)=\mathcal{S}_{\mathrm{MFS}}|_{Z=0;\ \text{fields}\to\overline{\text{fields}}}. \tag{19}$$

In terms of Eq. (8) and Eqs. (14)–(16) the supersymmetry transformations of the MFS model, according to Eq. (3), read:

$$\delta\phi=\frac{1}{2}(\chi\gamma^3\varepsilon), \tag{20}$$

$$\delta X^a=-\frac{Z}{4}X^b(\chi\gamma_b\gamma^a\gamma^3\varepsilon)-\frac{iV}{u}(\chi\gamma^a\varepsilon), \tag{21}$$

$$\delta\chi^\alpha=2iX^c(\varepsilon\gamma_c)^\alpha-\left(u+\frac{Z}{8}\chi^2\right)(\varepsilon\gamma^3)^\alpha, \tag{22}$$

$$\begin{aligned}\delta\omega=&\frac{Z'}{4}X^b(\chi\gamma_b\gamma^a\gamma^3\varepsilon)e_a+i\left(\frac{V}{u}\right)'(\chi\gamma^a\varepsilon)e_a\\&+\left(u'+\frac{Z'}{8}\chi^2\right)(\varepsilon\gamma^3\psi),\end{aligned} \tag{23}$$

$$\delta e_a=\frac{Z}{4}(\chi\gamma_a\gamma^b\gamma^3\varepsilon)e_b-2i(\varepsilon\gamma_a\psi), \tag{24}$$

$$\begin{aligned}\delta\psi_\alpha=&-(D\varepsilon)_\alpha+\frac{Z}{4}X^a(\gamma_a\gamma^b\gamma^3\varepsilon)_\alpha e_b+\frac{iV}{u}(\gamma^b\varepsilon)_\alpha e_b\\&+\frac{Z}{4}\chi_\alpha(\varepsilon\gamma^3\psi).\end{aligned} \tag{25}$$

We list neither here nor below transformations with the three bosonic parameters ε_i. The symmetry transformation generated by Eq. (8) corresponds to the local Lorentz transformations, the other two, by the field-dependent choice of the symmetry parameter $\varepsilon_a=\xi^mA_{ma}$, describe 2D diffeomorphisms ξ^m [49]. Clearly, the invariance with respect to the latter three transformations is also evident from the explicit form of the action (18).

C. Minimal field dilaton supergravity

The PSM form of the action (18) represents a theory with nonvanishing bosonic torsion. This can be seen easily from the e.o.m. obtained by variation of X^a. Nevertheless, it is (locally and globally) equivalent to a theory with dynamical dilaton field and vanishing bosonic torsion. We recall the basic steps of this relation (path $\tilde{A}$ in Fig. 1) as applied already to the GPSM in [13]. For this purpose the action (18) is most conveniently abbreviated as

$$\mathcal{L}_{\mathrm{MFS}}=\int_{\mathcal{M}}\left(\phi\,\mathrm{d}\omega+X^aDe_a+\chi^\alpha D\psi_\alpha+\frac{1}{2}P^{AB}e_Be_A\right), \tag{26}$$

where now $A=(a,\alpha)$ only includes the zweibein e_a and the gravitino $e_\alpha=\psi_\alpha$ components (cf. Appendix A). Varying Eq. (26) with respect to X^a leads to the torsion equation

[7]The constant $\tilde{u}_0$ in Ref. [14] has been fixed as $\tilde{u}_0=-2$. This is in agreement with standard supersymmetry conventions.

$$De_a + \frac{1}{2}(\partial_a P^{AB}) e_B e_A = 0, \tag{27}$$

which can be used to substitute the independent spin connection ω by the dependent[8] supersymmetry covariant connection $\tilde{\omega}$ and by the torsion $\tilde{\tau}$:

$$\omega_a = e_a^m \omega_m = \tilde{\omega}_a - \tilde{\tau}_a, \tag{28}$$

$$\tilde{\omega}_a = \epsilon^{mn} \partial_n e_{ma} - i\epsilon^{mn}(\psi_n \gamma_a \psi_m), \tag{29}$$

$$\tilde{\tau}_a = -\frac{1}{2}(\partial_a \hat{P}^{AB}) \epsilon^{mn} e_{Bn} e_{Am}, \tag{30}$$

$$\hat{P}^{AB} = P^{AB} + 2i\delta_\alpha^A \delta_\beta^B X^c \gamma_c^{\alpha\beta}. \tag{31}$$

By partially integrating the torsion dependent part of Eq. (26) some derivatives are moved onto the dilaton field ϕ and the action reads (up to total derivatives):

$$\mathcal{S}_{\text{MFS}} = \int d^2x\, e\left(\frac{1}{2}\tilde{R}\phi + (\chi\tilde{\sigma}) - \frac{1}{2}\hat{P}^{AB}\epsilon^{mn} e_{Bn} e_{Am} + \left(X^a + e_m^a \epsilon^{mn}(\partial_n \phi) + \frac{1}{2} e_m^a \epsilon^{mn}(\chi\gamma^3\psi_n)\right)\tilde{\tau}_a\right). \tag{32}$$

The curvature scalar

$$\tilde{R} = 2 * d\tilde{\omega} = 2\epsilon^{mn}\partial_n \tilde{\omega}_m \tag{33}$$

through Eq. (29) depends on the torsion free spin connection $\tilde{\omega}$ which, in turn, may be expressed as well by the metric $g_{mn} = e_m^a e_{na}$. In addition, the fermionic partner of the curvature scalar has been introduced, which is defined as

$$\tilde{\sigma}_\alpha = *(\tilde{D}\psi)_\alpha = \epsilon^{mn}\left(\partial_n \psi_{m\alpha} + \frac{1}{2}\tilde{\omega}_n (\gamma^3 \psi_m)_\alpha\right). \tag{34}$$

Varying again with respect to X^a finally allows one to eliminate this field as well:

$$X^a = -e^a{}_m \epsilon^{mn}\left((\partial_n \phi) + \frac{1}{2}(\chi\gamma^3\psi_n)\right). \tag{35}$$

Inspecting the original action (26) one realizes that this is the e.o.m. of the independent spin connection ω. It is important to notice that the structure of Eq. (35) does not depend on the details of the Poisson tensor, but is determined solely by the condition of local Lorentz invariance. Equations (28) and (35) are algebraic and even linear in the variables to be eliminated. Therefore, they may be reinserted into the action (32):

$$\mathcal{S}_{\text{MFDS}} = \int d^2x\, e\left(\frac{1}{2}\tilde{R}\phi + (\chi\tilde{\sigma}) - \frac{1}{2}\hat{P}^{AB}\Big|_{X^a}\epsilon^{mn} e_{Bn} e_{Am}\right). \tag{36}$$

Here X^a indicates that this field should be replaced by Eq. (35).

Because in the MFS the Poisson tensor P^{ab} depends quadratically on X^a [cf. Eq. (14)], according to Eq. (35) the usual quadratic dynamical term for the dilaton field ϕ is produced. Thus, reinserting the Poisson tensor (14)–(16) with Eq. (35) into Eq. (36) yields the minimal field *dilaton* supergravity (MFDS):

$$\begin{aligned}\mathcal{S}_{\text{MFDS}} = \int d^2x\, e\Bigg(&\frac{1}{2}\tilde{R}\phi + (\chi\tilde{\sigma}) \\ &+ V - \frac{1}{4u}\chi^2\left(VZ + V' + 4\frac{V^2}{u^2}\right) \\ &- \frac{1}{2}Z\left(\partial^m\phi\partial_m\phi + \frac{1}{2}(\chi\gamma^3\psi^m)\partial_m\phi + \frac{1}{2}\epsilon^{mn}\partial_n\phi(\chi\psi_m)\right) - \frac{iV}{u}\epsilon^{mn}(\chi\gamma_n\psi_m) \\ &+ \frac{u}{2}\epsilon^{mn}(\psi_n\gamma^3\psi_m)\Bigg).\end{aligned} \tag{37}$$

This action describes dilaton supergravity theories with minimal field content and vanishing bosonic torsion: The bosonic variable e_m^a appears explicitly, but it also is contained in the dependent spin connection $\tilde{\omega}$ according to Eq. (29). Beside the dilaton field ϕ the fermionic dilatino χ^α remains. Clearly for $Z=0$ (NDMFS in Fig. 1) the dynamical terms for the dilaton field disappear [$\bar{V} = \bar{V}(\bar{\phi})$, etc.]:

$$\mathcal{S}_{\text{NDMFS}} = \int d^2x\, \bar{e}\left(\frac{1}{2}\overline{\tilde{R}\phi} + (\overline{\chi\tilde{\sigma}}) + \bar{V} - \frac{1}{4\bar{u}}\bar{\chi}^2\left(\bar{V}' + 4\frac{\bar{V}^2}{\bar{u}^2}\right) - \frac{i\bar{V}}{\bar{u}}\epsilon^{mn}(\bar{\chi}\gamma_n\bar{\psi}_m) + \frac{\bar{u}}{2}\epsilon^{mn}(\bar{\psi}_n\gamma^3\bar{\psi}_m)\right). \tag{38}$$

While a further elimination of the dilatino is possible for quite general NDMFS models (discussed in Sec. V A), one can get rid of ϕ in certain very special (simple) cases of Eq. (38) only, namely for invertible potential terms V, respectively, u [cf. Eq. (17)].

The supersymmetry transformations of the MFDS model follow by eliminating X^a and ω in Eqs. (20), (22), (24) and (25). Except for Eq. (25) the new transformation rules are immediate by substituting X^a by Eq. (35). In Eq. (25) we use the explicit formula of the covariant torsion [cf. Eq. (30) with Eqs. (14)–(16)]

$$\tilde{\tau}_a = -Z\left(X_a + \frac{1}{4}(\chi\gamma_a\gamma^b\psi_n)e_b^n\right). \tag{39}$$

[8] Here and also in the superfield approach below supersymmetry covariant quantities acquire a tilde when they denote *dependent* variables.

After some algebra the result

$$\delta\phi=\frac{1}{2}(\chi\gamma^3\varepsilon), \tag{40}$$

$$\delta\chi^\alpha=-2i\epsilon^{mn}\left(\partial_n\phi+\frac{1}{2}(\chi\gamma^3\psi_n)\right)(\varepsilon\gamma_m)^\alpha - \left(u+\frac{Z}{8}\chi^2\right)(\varepsilon\gamma^3)^\alpha, \tag{41}$$

$$\delta e_m^a=\frac{Z}{4}(\chi\gamma^a\gamma^b\gamma^3\varepsilon)e_{mb}-2i(\varepsilon\gamma^a\psi_m), \tag{42}$$

$$\delta\psi_{m\alpha}=-(\tilde{D}\varepsilon)_\alpha+\frac{iV}{u}(\gamma_m\varepsilon)_\alpha+\frac{Z}{4}\left(\partial^n\phi(\gamma_m\gamma_n\varepsilon)_\alpha + \frac{1}{2}(\psi_m\gamma^n\chi)(\gamma_n\gamma^3\varepsilon)_\alpha\right) \tag{43}$$

is obtained.

The action (37) with its symmetry transformations (40)–(43) is most convenient for a comparison with a superfield formulation of 2D supergravity, because in Eq. (37) the bosonic torsion vanishes and it is precisely this case for which the standard supergravity has been developed.

III. SUPERFIELD DILATON SUPERGRAVITY

Any formulation of dilaton supergravity in superspace is embedded in the background of pure 2D super-geometry. The simplest nontrivial superfield extension of the topological bosonic 2D action $\int d^2x\, eR$ is obtained by promoting the determinant $e=\sqrt{-g}$ to the superdeterminant E, and the curvature R to a component of a real superfield S, which appears in a function $\mathcal{F}(S)$. At the same time the integration is extended to an integral over $N=(1,1)$ superspace $[z^M=(x^m,\theta^\mu)]$:

$$\mathcal{S}_{\rm Howe}=\int d^2x\, d^2\theta\, E\mathcal{F}(S). \tag{44}$$

In the following Eq. (44) will be referred to as the "Howe-action" because the analysis of 2D supergravity in terms of superfields goes back to the seminal paper [25] of this author. In the notation and conventions of the Appendixes (cf. also Ref. [13] and the superspace conventions of [28]) the respective θ-expansions read

$$E=e\left(1-2i(\theta\underline{\zeta})+\frac{1}{2}\theta^2(\underline{A}+2\underline{\zeta}^2+\underline{\lambda}^2)\right), \tag{45}$$

$$S=\underline{A}+2(\theta\gamma^3\underline{\tilde{\sigma}})+2i\underline{A}(\theta\underline{\zeta})+\frac{1}{2}\theta^2(\epsilon^{mn}\partial_n\underline{\tilde{\omega}}_m - \underline{A}(\underline{A}+2\underline{\zeta}^2+\underline{\lambda}^2)-4i(\underline{\zeta}\gamma^3\underline{\tilde{\sigma}})). \tag{46}$$

For reasons that will become clear in Sec. V B superfield components are consistently expressed by underlined letters, except e_m^a. $\underline{\zeta}_\alpha$ and $\underline{\lambda}_\alpha^a$ are the components of the Lorentz covariant decomposition of the gravitino $\underline{\psi}_a^\alpha=e_a^m\underline{\psi}_m^\alpha$ according to Eq. (B7). The dependent variables $\underline{\tilde{\omega}}$, $\underline{\tilde{R}}$ and $\underline{\tilde{\sigma}}$ are defined by the Eqs. (29), (33) and (34), when substituting all variables therein by underlined ones.

The independent variables in the Howe-action (44) are the components of the zweibein e^a, of its fermionic partner $\underline{\psi}^a$ and an auxiliary field $\underline{A}$. Inserting the decomposition (B7) of $\underline{\psi}$ and integrating out superspace Eq. (44) reduces to [cf. Eq. (B1), derivatives with respect to $\underline{A}$ are indicated by a dot]:

$$\mathcal{S}_{\rm Howe}=\int d^2x\, e\left(\frac{1}{2}\dot{\mathcal{F}}\underline{\tilde{R}}-\underline{A}(\underline{A}\dot{\mathcal{F}}-\mathcal{F})+2\ddot{\mathcal{F}}\underline{\tilde{\sigma}}^2 - 2i\underline{A}\ddot{\mathcal{F}}(\underline{\psi}^a\gamma_a\gamma^3\underline{\tilde{\sigma}})-\frac{1}{2}\underline{A}^2\ddot{\mathcal{F}}(\underline{\psi}^m\underline{\psi}_m) + \left(\frac{1}{2}\underline{A}^2\ddot{\mathcal{F}}-(\underline{A}\dot{\mathcal{F}}-\mathcal{F})\right)\epsilon^{mn}(\underline{\psi}_n\gamma^3\underline{\psi}_m)\right). \tag{47}$$

Here $\mathcal{F}(\underline{A})$ is $\mathcal{F}(S)|_{\theta=0}$, the body of the function $\mathcal{F}(S)$ in Eq. (44). The action (47) remains invariant under the supergravity transformations[9] [as in the notation for the fields, $\underline{\varepsilon}$ is used to distinguish that transformation parameter from ε in Eqs. (40)–(43)]:

$$\delta e_m{}^a=-2i(\underline{\varepsilon}\gamma^a\underline{\psi}_m), \quad \delta e^m{}_a=2i(\underline{\varepsilon}\gamma^m\underline{\psi}_a), \tag{48}$$

$$\delta\underline{\psi}_m{}^\alpha=-\left((\underline{\tilde{D}}\underline{\varepsilon})^\alpha+\frac{i}{2}\underline{A}(\underline{\varepsilon}\gamma_m)^\alpha\right), \tag{49}$$

$$\delta\underline{A}=-2\left((\underline{\varepsilon}\gamma^3\underline{\tilde{\sigma}})-\frac{i}{2}\underline{A}e^m{}_a(\underline{\varepsilon}\gamma^a\underline{\psi}_m)\right). \tag{50}$$

As it stands, Eq. (47) cannot be equivalent to a more general supergravity like $\mathcal{S}_{\rm MFDS}$ in Eq. (37). Only for Eq. (19), the special case of a nondynamical dilaton, a relation will be worked out in Sec. V A, but Eq. (47) is clearly insufficient to represent the general theory with dynamical dilaton field.

In order to describe the superfield generalization of all bosonic GDT with *dynamical* dilaton [as exemplified by Eq. (12)], ϕ is promoted to a superfield as well and one arrives at the general superfield dilaton supergravities (SFDS, cf. Fig. 1) of Park and Strominger [32]

$$\mathcal{S}_{\rm SFDS}=\int d^2x\, d^2\theta\, E(J(\Phi)S+K(\Phi)D^\alpha\Phi D_\alpha\Phi+L(\Phi)). \tag{51}$$

The general dilaton supergravity model of this type is described by three functions $J(\Phi)$, $K(\Phi)$ and $L(\Phi)$ of the dilaton superfield

[9] In agreement with our systematic notation e.g. the covariant derivative $\underline{\tilde{D}}$ refers to the dependent spin connection (29) for the underlined components of the superfield.

$$\Phi=\underline{\phi}+\frac{1}{2}\theta\gamma^3\underline{\chi}+\frac{1}{2}\theta^2\underline{F}. \tag{52}$$

In Eq. (52) a scalar dilaton field $\underline{\phi}$ appears as the lowest component. From superspace geometry the standard transformation rules [25,28]

$$\delta\underline{\phi}=-\frac{1}{2}\underline{\varepsilon}\gamma^3\underline{\chi}, \tag{53}$$

$$\delta\underline{\chi}_\alpha=-2(\gamma^3\underline{\varepsilon})_\alpha\underline{F}+i(\gamma^3\gamma^b\underline{\varepsilon})_\alpha(\underline{\psi}_b\gamma^3\underline{\chi}) -2i(\gamma^3\gamma^m\underline{\varepsilon})_\alpha\partial_m\underline{\phi}, \tag{54}$$

$$\delta\underline{F}=-2i(\underline{\varepsilon}\,\underline{\zeta})\underline{F}-\frac{i}{2}(\underline{\varepsilon}\gamma^m\gamma^3(\underline{\tilde{D}}_m\underline{\chi})) +(\underline{\varepsilon}\,\underline{\lambda}^m)((\underline{\psi}_m\gamma^3\underline{\chi})-2\partial_m\underline{\phi}) \tag{55}$$

are an immediate consequence. Integrating out superspace and elimination of the auxiliary fields $\underline{F}$ and $\underline{A}$ by their (algebraic) e.o.m.s is straightforward but leads to rather lengthy expressions. We, therefore, relegate some relevant formulas to Appendix B. Furthermore, we assume in the following that the reparametrization $J(\Phi)\rightarrow\Phi$ is possible, so that only K and L remain as two free functions. This agrees with the appearance of only Z and V and with the simple factor ϕ in front of $\tilde{R}$ in the bosonic part of $\mathcal{S}_{\mathrm{MFDS}}$ in Eq. (37).[10]

Then Eq. (51) becomes [$L(\underline{\phi})$ and $K(\underline{\phi})$ are the body of $L(\Phi)$ and $K(\Phi)$, derivatives thereof are taken with respect to $\underline{\phi}$]

$$\begin{aligned}\mathcal{S}_{\mathrm{SFDS}}=\int d^2x\,e\Bigg(&\frac{1}{2}\underline{\tilde{R}}\,\underline{\phi}+(\underline{\chi}\,\underline{\tilde{\sigma}})+2K\Bigg(\partial^m\underline{\phi}\partial_m\underline{\phi}-\frac{i}{4}\underline{\chi}\gamma^m\partial_m\underline{\chi}\\&-(\underline{\psi}_n\gamma^m\gamma^n\gamma^3\underline{\chi})\partial_m\underline{\phi}\Bigg)+2KL^2-LL'\\&+L\epsilon^{mn}(\underline{\psi}_n\gamma^3\underline{\psi}_m)+iL'(\underline{\zeta}\gamma^3\underline{\chi})+\frac{1}{4}\Bigg(\frac{1}{2}L''-K'L\\&+K(\underline{\psi}_n\gamma^m\gamma^n\underline{\psi}_m)\Bigg)\underline{\chi}^2\Bigg),\end{aligned} \tag{56}$$

with the corresponding symmetry transformations

$$\delta e_m{}^a=-2i(\underline{\varepsilon}\gamma^a\underline{\psi}_m),\qquad \delta e^m{}_a=2i(\underline{\varepsilon}\gamma^m\underline{\psi}_a), \tag{57}$$

$$\delta\underline{\psi}_m{}^\alpha=-(\underline{\tilde{D}}\,\underline{\varepsilon})^\alpha-\frac{i}{2}\left(4KL-L'-\frac{1}{4}K'\underline{\chi}^2\right)(\underline{\varepsilon}\gamma_m)^\alpha, \tag{58}$$

$$\delta\underline{\phi}=-\frac{1}{2}\underline{\varepsilon}\gamma^3\underline{\chi}, \tag{59}$$

$$\delta\underline{\chi}_\alpha=2L(\gamma^3\underline{\varepsilon})_\alpha+i(\gamma^3\gamma^b\underline{\varepsilon})_\alpha(\underline{\psi}_b\gamma^3\underline{\chi})-2i(\gamma^3\gamma^m\underline{\varepsilon})_\alpha\partial_m\underline{\phi}. \tag{60}$$

We shall need below also the special case $K=0$ of the action (56), called SFNDS in Fig. 1:

$$\mathcal{S}_{\mathrm{SFNDS}}=\int d^2x\,\bar{e}\left(\frac{1}{2}\overline{\underline{\tilde{R}}}\,\overline{\underline{\phi}}+(\overline{\underline{\chi}}\,\overline{\underline{\tilde{\sigma}}})-\overline{LL'}+\bar{L}\epsilon^{mn}(\overline{\underline{\psi}}_n\gamma^3\overline{\underline{\psi}}_m) +i\overline{L'}(\overline{\underline{\psi}}\gamma^3\overline{\underline{\chi}})+\frac{1}{8}\overline{L''}\,\overline{\underline{\chi}^2}\right). \tag{61}$$

It is written in terms of barred variables $\overline{\underline{\phi}}$, $\overline{\underline{\chi}}$, $\overline{e}_m^a$ and $\overline{\underline{\psi}}$ in analogy to the notation of NDMFS, Eq. (38).

The basic task (path D in Fig. 1) of Sec. V is to show the equivalence of $\mathcal{S}_{\mathrm{MFDS}}$ in Eq. (37) with $\mathcal{S}_{\mathrm{SFDS}}$ in Eq. (56), *together* with a correct translation of the transformation laws (40)–(43) into Eqs. (57)–(60). In view of the quite different structures this clearly has no obvious answer, although the number of fields and their type [(e,ϕ,ψ,χ) for MFDS, respectively $(e,\underline{\phi},\underline{\psi},\underline{\chi})$ for SFDS] coincide. Therefore, first the transformations connecting theories "horizontally" in Fig. 1 must be discussed.

IV. TARGET SPACE DIFFEOMORPHISMS AND CONFORMAL TRANSFORMATIONS

Transformations of fields in a certain action generically lead to new theories when those transformations contain singularities. A famous case is the string inspired black hole model [19] which, even in interaction with minimally coupled matter, by a dilaton field dependent (singular) conformal transformation can be brought to flat space. In fact, this is the basic reason for being able to find the classical solution in that model. The black hole singularity disappears in flat space, and thus the global geometric properties of the theory experience a profound change. Nevertheless, as long as such a transformation is performed only locally in function space and if, at the end of the day, for the physical interpretation one returns to the variables of the original theory, this detour can be a very valuable mathematical tool.

A. Target space diffeomorphism in GPSMs

Different GPSMs can be mapped upon each other by the target space diffeomorphism

$$X^I\Rightarrow\overline{X^I}=\overline{X^I}(X). \tag{62}$$

It is straightforward to check that the action (1) is form-invariant under this diffeomorphism when the gauge potentials and the Poisson tensor are transformed according to

$$\overline{A_I}=\frac{\partial X^J}{\partial\overline{X^I}}A_J, \tag{63}$$

$$\overline{P^{IJ}}=(\overline{X^I}\overleftarrow{\partial}_K)P^{KL}(\overrightarrow{\partial}_L\overline{X^J}). \tag{64}$$

[10] For models of the form (51) that do not allow a global reparametrization of this type, the equivalence to a GPSM discussed below holds patch-wise, only.

Here $\vec{\partial}_I$ [cf. Eq. (A10)] is the usual derivative acting to the right and $\overleftarrow{\partial}_I$ acts as

$$f\overleftarrow{\partial}_I=(-1)^{I(f+1)}\vec{\partial}_I f. \tag{65}$$

We emphasize again at this point that Eq. (62) need not hold globally and thus physics may be different in two models connected by such a transformation, when, e.g. in the case of gravity theories, the $\bar{A}_I$ are identified with the Cartan variables associated to the new gauge field coordinates.

The two models from P^{IJ} and $\bar{P}^{IJ}$ clearly obey two *different* sets of symmetry transformations, cf. Eq. (3). The relation among them can be written as

$$\overline{\delta X^I}=\delta\bar{X}^I(X), \tag{66}$$

$$\overline{\delta A_I}=\delta\bar{A}_I(A,X)+\text{e.o.m.s}, \tag{67}$$

$$\bar{\varepsilon}_I=\frac{\partial X^J}{\partial \bar{X}^I}\varepsilon_J. \tag{68}$$

The necessity for the appearance of the e.o.m.s in Eq. (67) is easily seen when inserting the transformed ε in the characteristic derivative term of Eq. (3):

$$\delta\bar{A}_I(A,X)=-\frac{\partial X^J}{\partial\bar{X}^I}\mathrm{d}\left(\frac{\partial\bar{X}^K}{\partial X^J}\bar{\varepsilon}_K\right)+\cdots$$
$$=-\mathrm{d}\bar{\varepsilon}_I-(-1)^K\frac{\partial X^J}{\partial\bar{X}^I}\mathrm{d}\left(\frac{\partial\bar{X}^K}{\partial X^J}\right)\bar{\varepsilon}_K+\cdots. \tag{69}$$

Obviously this produces terms of the form $\mathrm{d}\bar{X}$, which are absent in the rest of the transformation. This indicates that each $\mathrm{d}\bar{X}^I$ has to be removed by the e.o.m.s (5) to arrive at the transformation law as given in Eq. (3) for $\bar{\varepsilon}(\varepsilon,X)$. Finally we note that the e.o.m.s (5) transform into the same ones for $\bar{X}$ and $\bar{A}$, while the e.o.m.s (6) transform into e.o.m.s of both types (5) and (6) in terms of $\bar{X}$ and $\bar{A}$.

It is worth mentioning a specialty of the GPSM structure at this point. In an action based on linear symmetry transformations new related actions are usually obtained by a rearrangement of invariant functions—e.g. the rearrangement of superfields to obtain the general Park-Strominger model from the special case with $K=0$ as discussed below. On the other hand, the GPSM action is not constructed by the composition of invariant functions and supergravity invariant derivatives, but the invariant is always the whole action. Thus, modifying a GPSM action *necessarily* implies the modification of the symmetry transformations [cf. Eq. (68)].

B. Conformal transformation for MFS and MFDS

In GPSM-theories conformal transformations are a special type of target space diffeomorphisms. They, in particular, may be used to connect (path $\tilde{B}$ in Fig. 1) the MFS models (18) to MFS_0 models (19) with vanishing bosonic torsion ($\bar{Z}=0$) or, equivalently, the MFDS models (37) to related models without dynamical dilaton (NDMFS, path $\tilde{B}'$ in Fig. 1). The MFS_0 action (19) is mapped upon Eq. (18) of MFS by [cf. Ref. [13], Eqs. (5.42), (5.48)]

$$\phi=\bar{\phi},\quad X^a=e^{-Q(\phi)/2}\bar{X}^a,\quad \chi^\alpha=e^{-Q(\phi)/4}\bar{\chi}^\alpha, \tag{70}$$

$$\omega=\bar{\omega}+\frac{Z}{2}\left(\bar{X}^b\bar{e}_b+\frac{1}{2}\bar{\chi}^\beta\bar{\psi}_\beta\right),\quad e_a=e^{Q(\phi)/2}\bar{e}_a,$$
$$\psi_\alpha=e^{Q(\phi)/4}\bar{\psi}_\alpha, \tag{71}$$

with Q defined in Eq. (11). After the fields X^a and the part of ω dependent on bosonic torsion have been eliminated the ensuing NDMFS action (38) is connected with the general MFDS action (37) by the same transformation rules for ϕ, χ, e_a, and ψ as given in Eqs. (70) and (71). The prepotential u transforms according to

$$u=e^{-Q(\phi)/2}\bar{u}, \tag{72}$$

which leads to a canonical transformation of $\bar{V}(\phi)=-\frac{1}{4}\overline{uu'}$ into Eq. (17), such that the combination $\bar{e}\bar{V}=eV$ remains invariant.

The symmetry transformations of the MFS models (20)–(25) with respect to the variables with and without a bar, respectively, are equivalent up to equations of motion of ω (or just as well $\bar{\omega}$). In contrast, applying Eqs. (70),(71) to the symmetry transformations of the MFDS model [Eqs. (40)–(43)] with $\bar{Z}=0$ reproduces the ones for $Z\neq0$ without recourse to the e.o.m.s of ω. Indeed, the latter have been used explicitly therein to eliminate the independent part of the spin connection.

In the NDMFS action (38) also the dilatino no longer represents a dynamical field. Variation with respect to $\bar{\chi}^\alpha$ leads to

$$\bar{\chi}_\alpha=-\frac{8}{\bar{u}''}+2i\frac{\bar{u}'}{\bar{u}''}\epsilon_m{}^n(\gamma^m\bar{\psi}_n)_\alpha, \tag{73}$$

and (provided $\bar{u}''\neq0$) the dilatino may be eliminated altogether. The resulting action in terms of e_a, ψ and ϕ [$\tilde{\bar{R}}$ and $\tilde{\bar{\sigma}}$ are dependent variables as in Eqs. (33),(34), but with $\bar{Z}=0$]

$$\mathcal{S}^{(2)}_{\text{NDMFS}}=\int \mathrm{d}^2x\Bigg(\frac{1}{2}\tilde{\bar{R}}\phi-\frac{4}{\bar{u}''}\tilde{\bar{\sigma}}^2-\frac{1}{8}(\bar{u}^2)'-2i\frac{\bar{u}'}{\bar{u}''}$$
$$\times\bar{e}^m_a(\bar{\psi}_m\gamma^a\gamma^3\tilde{\bar{\sigma}})+\left(\frac{\bar{u}}{2}-\frac{1}{4}\frac{(\bar{u}')^2}{\bar{u}''}\right)$$
$$\times\epsilon^{mn}(\bar{\psi}_n\gamma^3\bar{\psi}_m)+\frac{1}{4}\frac{(\bar{u}')^2}{\bar{u}''}(\bar{\psi}^m\bar{\psi}_m)\Bigg) \tag{74}$$

is invariant under the symmetry transformations

$$\delta\bar{e}_m^a = -2i(\varepsilon\gamma^a\bar{\psi}_m), \tag{75}$$

$$\delta\bar{\psi}_m^\alpha = -(\widetilde{\bar{D}}\varepsilon)^\alpha + \frac{i\bar{u}'}{4}(\varepsilon\gamma_m)^\alpha, \tag{76}$$

$$\delta\phi = -\frac{4}{\bar{u}''}(\widetilde{\bar{\sigma}}\gamma^3\varepsilon) - \frac{i\bar{u}'}{\bar{u}''}\bar{e}_a^m(\bar{\psi}_m\gamma^a\gamma^3\varepsilon). \tag{77}$$

C. Conformal transformation in superspace

A similar conformal transformation connecting the general dilaton superfield action SFDS (51) to a model with nondynamical dilaton field [NDMFS, $K=0$ in Eq. (51)] is also known in superspace [32] (path $\tilde{B}''$ in Fig. 1).[11] It must contain a multiplication of the superzweibein E_M^A with a factor $\Lambda(\Phi)$ depending on the full superspace multiplet Φ. The resulting action is again an integral over superspace. One consequence thereof is the fact that a kinetic term for $\underline{\phi}$ necessarily implies a kinetic term for $\underline{\chi}$. It is known that a super-Weyl transformation preserving the constraints on the supertorsion (with vanishing bosonic torsion) has the form [25]

$$\bar{E}_M{}^a = \Lambda E_M{}^a, \quad \bar{E}_M{}^\alpha = \Lambda E_M{}^\alpha + iE_M{}^a\gamma_a^{\alpha\beta}\underline{D}_\beta\Lambda^{1/2}. \tag{78}$$

In our case we are interested in the consequence of that transformation on the action (61) of SFNDS. Choosing in Eq. (78)

$$\Lambda = \exp[\sigma(\Phi)], \quad \sigma' = -2K, \tag{79}$$

in the action (61) produces the general SFDS action (56). The different components are related by

$$\underline{\phi} = \underline{\bar{\phi}}, \quad \underline{\chi} = e^{-\sigma/2}\underline{\bar{\chi}}, \quad e_m^a = e^\sigma \bar{e}_m^a, \tag{80}$$

$$\underline{\psi}_m^\alpha = e^{\sigma/2}\underline{\bar{\psi}}_m^\alpha + \frac{i}{2}Ke^{\sigma/2}\bar{e}_m^a(\underline{\bar{\chi}}\gamma_a\gamma^3)^\alpha, \tag{81}$$

where for σ, respectively, K the body $\sigma(\underline{\phi})$, respectively, $K(\underline{\phi})$, is understood.

V. EQUIVALENCE OF GPSM AND SUPERFIELD DILATON SUPERGRAVITY

A. Equivalence for nondynamical dilaton

Inspection of the SFNDS superfield action (61) without dynamical dilaton and of the action $\mathcal{S}_{\mathrm{NDMFS}}$ in Eq. (38), which originated from the GPSM formulation of supergravity, shows that in this special case the two theories are the same, identifying

$$\underline{\bar{\psi}} = \bar{\psi}, \quad \underline{\bar{\chi}} = \bar{\chi}, \quad \underline{\phi} = \phi, \quad \underline{\bar{L}}(\phi) = \frac{\bar{u}(\phi)}{2}. \tag{82}$$

It can be checked straightforwardly that the equivalence holds as well at the level of the symmetry transformations when ε and $\underline{\varepsilon}$ are identified [cf. Eqs. (40)–(43) with $Z=0$ and Eqs. (57)–(60) with $K=0$]. Indeed, this relation of the Park-Strominger model with $K=0$ to a GPSM (interpreted as a model with nonlinear super-Poincaré algebra) had been observed already before [30,50]. In the GPSM-based formalism the identification corresponds to the sequence of paths $\tilde{A}' \to D'$ in Fig. 1 [13,31].

For a nondynamical dilaton ϕ we observe yet another identification between $\mathcal{S}_{\mathrm{NDMFS}}$ of Eq. (38) and $\mathcal{S}_{\mathrm{Howe}}$ of Eq. (47), when the dilatino in the former case has been eliminated as in Eq. (74). Indeed Eqs. (74) and (47), as well as the corresponding symmetry transformations (75)–(77) and (48)–(50), are identical for

$$\underline{\psi} = \bar{\psi}, \quad \underline{A} = -\frac{\bar{u}'}{2},$$

$$\mathcal{F}(\underline{A}) = \frac{1}{2}(\bar{u}(\phi(\underline{A})) - \phi(\underline{A})\bar{u}'(\phi(\underline{A}))). \tag{83}$$

In Eq. (73) we had to assume that $u'' \neq 0$ and thus the invertibility of the first equation of Eq. (83) is guaranteed.

The equivalence (83) corresponds to the steps $\tilde{A}' \to E$ in Fig. 1. Alternatively the path E establishes a relation $D' \to E$ between two superspace actions, namely superfield dilaton supergravity with nondynamical dilaton [SFNDS, Eq. (61)] and the model of Howe (47). $D' \to E$ and $\tilde{C}$ are not identical: Relation $\tilde{C}$ can entirely be formulated in superspace and holds [as its bosonic counterpart C, cf. comment below Eq. (12)] in very special cases only (invertible potentials or prepotentials, respectively). On the other hand, the path $D' \to E$ does not correspond to the elimination of a superfield. Instead the two superfields in the version of Eq. (51) with[12] $K=0$, $\bar{E}^a = (\bar{e}^a, \underline{\bar{\psi}}^a, \underline{\bar{A}})$ and $\bar{\Phi} = (\underline{\bar{\phi}}, \underline{\bar{\chi}}, \underline{\bar{F}})$, are related to the superfield in the model of Howe $E^a = (e^a, \psi^a, A)$ by the steps

$$\text{SFNDS}\begin{pmatrix}\bar{e}^a, \underline{\bar{\psi}}^a, \underline{\bar{A}}\\ \underline{\bar{\phi}}, \underline{\bar{\chi}}, \underline{\bar{F}}\end{pmatrix} \xrightarrow[D']{\text{elimination of } \underline{\bar{A}} \text{ and } \underline{\bar{F}}} \text{NDMFS}\begin{pmatrix}\bar{e}^a, \underline{\bar{\psi}}^a\\ \underline{\bar{\phi}}, \underline{\bar{\chi}}\end{pmatrix},$$

$$\text{NDMFS}\begin{pmatrix}\bar{e}^a, \underline{\bar{\psi}}^a\\ \underline{\bar{\phi}}, \underline{\bar{\chi}}\end{pmatrix} \xrightarrow[E]{\substack{\text{elimination of } \underline{\bar{\chi}}\\ \text{reinterpretation } \underline{\phi}\to A}} \text{Howe}(e^a, \psi^a, A).$$

Obviously this equivalence combines components of different superfields in Eq. (51) into the components of the superfield S of Eq. (44). Whenever the last equation in Eq. (83) can be solved explicitly for A, $D' \to E$ is equivalent to $\tilde{C}$.

[11] We reemphasize that at the level of dilaton theories with vanishing bosonic torsion all known results [25,32] from 2D supergravity can be taken over.

[12] This action corresponds to the sum of Eqs. (B8) and (B10).

However, $\tilde{C}$ is meaningful if and only if such a solution exists, while Eq. (74)—the result of $D' \to E$—does not depend on the latter.

B. Dynamical dilaton

One may think that by means of (super-)conformal transformations, proceeding along the paths $\tilde{B}$ respectively, $\tilde{B}''$, also in the general case the identification (path D) can be established in a straightforward manner. However, with a *dynamical* dilaton the problem remains how to relate the fields (ψ,χ), respectively $(\underline{\psi},\underline{\chi})$, because no obvious identification thereof is apparent. Also the relation between the symmetry transformations is far from trivial. Indeed, comparison of Eq. (37) and Eqs. (40)–(43) with Eq. (56) and Eqs. (57)–(60) immediately leads to two important observations.

While the SFDS action (56) includes standard kinetic terms for both the dilaton field $\underline{\phi}$ and its supersymmetric partner $\underline{\chi}$, in the MFDS formulation for ϕ such a term is generated too, but not for the dilatino.

The transformations of the zweibeine, Eq. (57) in the SFDS action and Eq. (42) in the MFDS action, are different. But in any comparison of the two models we had to assume that the zweibeine should be the same. Thus the gravitini ψ and $\underline{\psi}$ appearing on the rhs of these transformations must be different.

In contrast to the super-Weyl transformation in superspace (path $\tilde{B}''$ in Fig. 1), the conformal transformation leading from MFS_0 (19) with $Z=0$ to the MFS (37) ($Z\neq 0$, path $\tilde{B}$ in Fig. 1) depends on the dilaton field ϕ alone [cf. Eqs. (70) and (71)]. One could, of course, try to introduce more complicated transformations including also a dependence on the dilatino χ. It turns out that this is neither necessary nor possible: as pointed out above, the relation (35), leading to the kinetic term for ϕ, does not depend on the details of the Poisson tensor and hence not on a peculiarity of some special class of models. In fact *any* GPSM with local Lorentz invariance after elimination of X^a and ω exhibits at best a kinetic term in ϕ, but never in χ. A similar conclusion holds for the symmetry transformations (cf. the comment at the end of Sec. IV A).

On the other hand, the missing kinetic term for χ in MFDS could be generated by an appropriate mixing of ψ and χ in $\underline{\psi}$. It is not difficult to find the correct relation. When the SFNDS model (61) is transformed according to Eq. (80) one could try to replace Eq. (81) by the simpler rule

$$\psi = e^{\sigma/2}\overline{\underline{\psi}}. \tag{84}$$

This implies a new definition of the gravitino ψ. On the other hand, in this way a connection with the MFDS action (37), the one following from the GPSM approach, can be established. Namely, identifying the gravitino ψ in Eq. (84) with the gravitino of that action produces all terms there, provided

$$\underline{\phi}=\phi, \quad \underline{\chi}=\chi, \quad K(\phi)=-\frac{1}{4}Z(\phi), \quad L(\phi)=\frac{u}{2}. \tag{85}$$

Now all terms on the left-hand side of Eq. (85) refer to superfield supergravity, whereas on the rhs we find quantities defined in the GPSM-based MFDS approach. This is not surprising, as the transformation rules (80) together with Eq. (84) of SFDS by taking into account $K=-\frac{1}{4}Z$ are equivalent[13] to the transformations of ϕ, e, χ and ψ in Eqs. (70) and (71).

So far we followed the paths $D' \to \tilde{B}'$ in Fig. 1. In order to establish the relation D between SFDS and MFDS, the main goal of this section, the *Ansatz*

$$\underline{\psi}_m^{\alpha} = \psi_m^{\alpha} + \frac{i}{8}Z(\phi)e_m^a\epsilon_{ab}(\chi\gamma^b)^{\alpha} \tag{86}$$

together with Eq. (85) suggests itself by comparison of Eq. (84) with Eq. (81). It follows when the conformal factors in the two terms on the rhs of Eq. (81) are absorbed first in a redefinition of $\overline{\underline{\psi}}$; then the conformal transformations (70) and (71) for χ^{α}, e_a and ψ_{α} are taken into account. Not surprisingly, all contributions to Eq. (37) linear in Z are reproduced. But also terms proportional to Z^2 are found to cancel.

There seems to remain a difference in the symmetry transformations. Assuming $\underline{\varepsilon}=\varepsilon$ one obtains ($\Delta=\delta_{\mathrm{MFDS}} - \delta_{\mathrm{SFDS}}$)

$$\Delta\phi=0, \quad \Delta\chi_{\alpha}=\frac{Z}{8}\chi^2(\gamma^3\varepsilon)_{\alpha},$$
$$\Delta e_m^a=\frac{Z}{2}\epsilon^{ab}\chi\varepsilon e_{mb},$$
$$\Delta\psi_{m\alpha}=-\frac{Z}{4}\chi\varepsilon(\gamma^3\psi_m)_{\alpha}. \tag{87}$$

However, this is nothing else but a local Lorentz transformation [cf. Eq. (A7)] with field dependent parameter $\varepsilon_{\phi}=\frac{Z}{2}\chi\varepsilon$. An analogous transformation emerges as well in the GPSM based formalism: the application of Eq. (68) leads to ($\bar{\varepsilon}$ denotes the symmetry parameter of the MFS_0 model with $\bar{Z}=0$)

$$\varepsilon_{\phi}=\bar{\varepsilon}_{\phi}+\frac{Z}{2}\left(X^b\varepsilon_b+\frac{1}{2}\chi^{\beta}\varepsilon_{\beta}\right), \quad \varepsilon_a=e^{Q(\phi)/2}\bar{\varepsilon}_a,$$
$$\varepsilon_{\alpha}=e^{Q(\phi)/4}\bar{\varepsilon}_{\alpha}. \tag{88}$$

The superspace parameters $\underline{\varepsilon}$ obey a similar relation, but with the opposite sign in front of $Z/2$ in the first equation. Thus, a supersymmetry transformation $\bar{\varepsilon}_{\alpha}=\bar{\underline{\varepsilon}}_{\alpha}$ in MFS_0, respectively SFNDS, under the conformal transformations (70), (71), and (80) becomes $[\varepsilon=(\varepsilon_{\phi},\varepsilon_a,\varepsilon_{\alpha})]$

$$\bar{\varepsilon}=(0,0,\bar{\varepsilon}_{\alpha})\to\varepsilon=\left(\frac{Z}{4}\chi\varepsilon,0,\varepsilon_{\alpha}\right), \tag{89}$$

$$\bar{\underline{\varepsilon}}=(0,0,\bar{\underline{\varepsilon}}_{\alpha})\to\varepsilon=\left(-\frac{Z}{4}\chi\underline{\varepsilon},0,\underline{\varepsilon}_{\alpha}\right). \tag{90}$$

[13] Notice the similarity between Eqs. (11) and (79).

Adding the two contributions to ε_ϕ and $\underline{\varepsilon}_\phi$, respectively, yields the result found in Eq. (87): $\varepsilon_\phi = Z/2\,\chi\varepsilon$. This terminates the proof that the minimal field supergravity in the sense of Ref. [14] is—up to elimination of auxiliary fields—equivalent to SFDS, the superfield dilaton gravity of Park and Strominger. The symmetry transformations are mapped correctly upon each other, modulo a local field-dependent Lorentz transformation.

It may be useful to conclude this section with a compilation of the relevant formulas which, in agreement with the corresponding sequence of steps in Fig. 1, relate minimal field supergravity with the superfield dilaton theory of Ref. [32].

Actions: Seven different actions that describe in some sense 2D supergravity have been presented. These are

(1) the GPSM based MFS of Eq. (18) and the special version MFS_0 thereof with vanishing bosonic torsion (19),
(2) general dilaton supergravity MFDS in Eq. (37) with its special version with nondynamical dilaton (38) (NDMFS),
(3) SFDS of Eq. (56) and SFNDS of Eq. (61), which both originate from the general dilaton superfield theory by Park and Strominger (51),
(4) the model of Howe in Eqs. (44) and (47) which, when derived from NDMFS (38) by elimination of the dilatino, takes the form (74).

Transformations: The dilaton field ϕ and the zweibeine e_m^a coincide for all models.

path $\tilde{\mathbf{A}}$: The MFS fields (ϕ, X^a, χ^α) and $(\omega, e_a, \psi_\alpha)$ are reduced to the set $(\phi, \chi^\alpha, e_a, \psi_\alpha)$ of MFDS by Eqs. (28)–(30) and (33)–(35).

paths $\tilde{\mathbf{B}}, \tilde{\mathbf{B}}', \tilde{\mathbf{B}}''$: At each level (MFS, MFDS and SFDS) a special target space transformation connects the models with nondynamical dilaton (barred variables: MFS_0, MFNDS, SFNDS) to the general ones. For MFS this relation turns out to be the conformal transformation of Eqs. (70) and (71) which, when restricted to the fields $(\phi, \chi^\alpha, e_a, \psi_\alpha)$, also holds for MFDS vs NDMFS. For SFDS the super-Weyl transformations (78) and (80),(81) are applied.

path D: After elimination of the auxiliary fields in SFDS [Eqs. (B11) and (B12)], this theory is equivalent to MFDS: the identification of the remaining fields and (pre-)potentials is contained in Eqs. (85) and (86), the supersymmetry transformations are equivalent up to a local Lorentz transformation (87).

path E: The NDMFS action allows the elimination of the dilatino [Eq. (73)], leading to a theory that may be identified with the model of Howe [Eq. (83)]. Only in certain cases path E is equivalent to the superfield relation $\tilde{C}$. Therefore, a combination of the paths $E \to \tilde{C}$ cannot be used as an alternative to D'.

VI. SOLUTION OF THE GENERAL DILATON SUPERGRAVITY MODEL

The close relation between the general GPSM describing MFS supergravity and the general superfield supergravity (51) can be used to combine the advantages of both approaches. We first use the fact that in MFS, as a GPSM, it is manifestly simpler to arrive at the complete (classical and quantum) solution of a 2D gravity system. Using the list of formulas described in the last paragraph of the preceding section it can be mapped directly into the complete exact solution of the Park-Strominger supergravity (51), where we assume that a redefinition of Φ by the replacement $J(\Phi) \to \Phi$ is possible everywhere.

As supergravity in two dimensions without matter has no propagating degrees of freedom the physical content of the system is encoded in the Casimir functions (4). Every GPSM gravity possesses at least one Casimir function, as the bosonic part of the tensor has odd dimension [cf. Eq. (11)]. For the MFS_0 model [Eq. (18) with $Z=0$] this function can be chosen as [13]

$$\bar{C} = \bar{Y} - \frac{1}{8}\bar{u}^2 + \frac{1}{16}\chi^2 \bar{u}'. \tag{91}$$

Because the on-shell Casimir function is a constant it must be conformally invariant. Thus a simple change of variables according to Eqs. (70) and (71) leads to

$$C = e^Q\left(Y - \frac{1}{8}u^2 + \frac{1}{16}\chi^2 C_\chi \right), \tag{92}$$

$$C_\chi = u' + \frac{1}{2}uZ. \tag{93}$$

Eliminating the auxiliary field X^a by Eq. (35), the Casimir function of the MFDS model of Eq. (37) becomes [the special case of NDMFS (38) is found by setting $Q=Z=0$]

$$C_{\mathrm{MFDS}} = e^Q\Bigg(-\frac{1}{2}\partial^n\phi\,\partial_n\phi - \frac{1}{8}u^2 - \frac{1}{2}\partial^n\phi(\chi\gamma^3\psi_n) + \frac{1}{16}\chi^2\left(u' + \frac{1}{2}uZ - \psi^n\psi_n \right)\Bigg). \tag{94}$$

For explicit calculations the expressions are written more conveniently in terms of light-cone coordinates (cf. Appendix A). Assuming $X^{++} \neq 0$ one can introduce the Lorentz-scalars

$$\rho^{(+)} = \frac{\chi^+}{\sqrt{|X^{++}|}}, \quad \rho^{(-)} = \sqrt{|X^{++}|}\,\chi^-. \tag{95}$$

The solution of the MFS_0 model (19) has been derived already in Ref. [13] Sec. 8, the conformal transformation (70),(71) of which yields the general solution of MFS (18). The strategy to obtain in a straightforward way the general solution for a (G)PSM model consists in following the steps set out in Ref. [51] (cf. [33]). The final result is best parametrized in terms of (almost-)Casimir-Darboux coordinates, which can be identified *a posteriori*. Indeed, introducing new gauge-potentials $\mathcal{A}_I = (A_C, A_\phi, A_{++}, A_{(+)}, A_{(-)})$ that correspond to the target space variables $\mathcal{X}^I = (C, \phi, X^{++}, \rho^{(+)}, \rho^{(-)})$ [cf. Eqs. (62) and (63)], all $\mathcal{A}_I$ can be expressed in terms of the $\mathcal{X}^I$ by the solution of their e.o.m.s, except for A_C. The e.o.m. of A_C simply reads

$$\mathrm{d}C=0 \tag{96}$$

and therefore we introduce a new integration function F:

$$\mathrm{d}C=0\Rightarrow \mathrm{d}A_C=0\Rightarrow A_C=-\mathrm{d}F. \tag{97}$$

Thus the solution is parametrized in terms of the target space variables $\mathcal{X}^I$ and the free function F. Denoting by $\mathcal{V}$ the component of $P^{ab}=\mathcal{V}\epsilon^{ab}$ [cf. Eq. (14)]

$$\mathcal{V}=V+YZ-\frac{1}{2}\chi^2\left(\frac{VZ+V'}{2u}+\frac{2V^2}{u^3}\right), \tag{98}$$

the general analytic solution on a patch with $X^{++}\neq 0$ and $C=\text{const}\neq 0$ can be written as

$$\omega=\frac{\mathrm{d}X^{++}}{X^{++}}-e^Q\mathcal{V}A_\Upsilon-\frac{e^Q}{8C}C_\chi\rho^{(+)}\mathrm{d}\Upsilon -\frac{Z}{4}e^Q\left(\frac{uZ}{8C}\rho^{(-)}\rho^{(+)}\mathrm{d}\phi+\frac{1}{4}C_\chi\rho^{(-)}\rho^{(+)}\mathrm{d}F -\frac{\sigma}{\sqrt{2}C}\rho^{(-)}\mathrm{d}\Upsilon\right), \tag{99}$$

$$X^{++}e_{++}=-\mathrm{d}\phi-e^QX^{++}X^{--}A_\Upsilon-e^Q\frac{u}{16C}\rho^{(-)}\rho^{(+)}\mathrm{d}\phi +\frac{\sigma}{4\sqrt{2}}\left(\frac{e^Q}{C}\left(\rho^{(-)}+\frac{\sigma u}{2\sqrt{2}}\rho^{(+)}\right)\mathrm{d}\Upsilon -\rho^{(+)}\mathrm{d}\rho^{(+)}\right), \tag{100}$$

$$\frac{e_{--}}{X^{++}}=-e^QA_\Upsilon, \tag{101}$$

$$\sqrt{|X^{++}|}\psi_+=\frac{e^Q}{8}C_\chi\rho^{(-)}A_\Upsilon +\frac{\sigma}{2\sqrt{2}}\left(\mathrm{d}\rho^{(+)}-e^Q\frac{u\sigma}{2\sqrt{2}C}\left(\mathrm{d}\Upsilon+\frac{1}{2}Z\Upsilon\,\mathrm{d}\phi\right)\right), \tag{102}$$

$$\frac{\psi_-}{\sqrt{|X^{++}|}}=-\frac{e^Q}{8}C_\chi\rho^{(+)}A_\Upsilon+e^Q\frac{\sigma}{2\sqrt{2}C}\left(\mathrm{d}\Upsilon+\frac{Z}{2}\Upsilon\,\mathrm{d}\phi\right). \tag{103}$$

Beside the abbreviation C_χ in Eq. (93) a new variable and its gauge potential, namely ($\sigma=\operatorname{sign}X^{++}$)

$$\Upsilon=\rho^{(-)}-\frac{\sigma u}{2\sqrt{2}}\rho^{(+)},\quad A_\Upsilon=\mathrm{d}F+e^Q\frac{\sigma}{4\sqrt{2}C^2}\Upsilon\,\mathrm{d}\Upsilon, \tag{104}$$

have been introduced. It should be noticed that in Eqs. (102) and (103) half of the terms produced by Υ in A_Υ vanish due to the Grassmann property $(\rho^{(+)})^2=(\rho^{(-)})^2=0$.

In Eqs. (99)–(103) X^{--} and Υ are dependent variables according to Eq. (92) with $Y=X^{++}X^{--}$ at $C=\text{const}\neq 0$ and Eq. (105). Further arbitrary functions are $\phi,\mathrm{d}F$ and the fermionic $\rho^{(+)},\rho^{(-)}$.

It is straightforward to check that the spinor $\tilde{c}$ defined as ($\sigma=\operatorname{sign}X^{++}$)

$$\tilde{c}=e^{Q/2}\Upsilon \tag{105}$$

commutes[14] with everything but itself. From the Schouten bracket

$$\{\tilde{c},\tilde{c}\}=-2\sqrt{2}\sigma e^QC \tag{106}$$

it follows that for $C\equiv 0$ an additional fermionic Casimir function $\tilde{c}$ arises. On a patch with $X^{++}=0$, but $X^{--}\neq 0$, we can define an analogous quantity $\hat{c}$ with $\rho^{(-)}$ and $\rho^{(+)}$ interchanged. $\tilde{c}=\text{const}$ relates $\rho^{(+)}$ and $\rho^{(-)}$ [cf. Eq. (105)]. Its associated gauge potential is [cf. Eq. (97)] $\tilde{A}=-\mathrm{d}f$. The general solution reads

$$\omega=\frac{\mathrm{d}X^{++}}{X^{++}}-e^Q\mathcal{V}\,\mathrm{d}F+e^{Q/2}\frac{\sigma}{2\sqrt{2}}C_\chi\rho^{(+)}\mathrm{d}f -\frac{Z}{2}e^{Q/2}\left(\rho^{(-)}\mathrm{d}f+e^{Q/2}\frac{1}{8}C_\chi\rho^{(-)}\rho^{(+)}\mathrm{d}F\right), \tag{107}$$

$$X^{++}e_{++}=-\mathrm{d}\phi-e^QX^{++}X^{--}\,\mathrm{d}F-\frac{1}{2}\left(\frac{\sigma}{2\sqrt{2}}\rho^{(+)}\mathrm{d}\rho^{(+)} +e^{Q/2}\left(\rho^{(-)}+\frac{\sigma u}{2\sqrt{2}}\rho^{(+)}\right)\mathrm{d}f\right), \tag{108}$$

$$\frac{e_{--}}{X^{++}}=-e^Q\,\mathrm{d}F, \tag{109}$$

$$\sqrt{|X^{++}|}\psi_+=\frac{e^Q}{8}C_\chi\rho^{(-)}\,\mathrm{d}F+\frac{\sigma}{2\sqrt{2}}(\mathrm{d}\rho^{(+)}+e^{Q/2}u\,\mathrm{d}f), \tag{110}$$

$$\frac{\psi_-}{\sqrt{|X^{++}|}}=-\frac{e^Q}{8}C_\chi\rho^{(+)}\,\mathrm{d}F-e^{Q/2}\,\mathrm{d}f. \tag{111}$$

[14] All commutators refer to the definition below Eq. (1).

Beside the anticommuting constant $\tilde{c}$ the free functions of this solution are $\mathrm{d}F$, $\mathrm{d}f$, ϕ, X^{++} and $\rho^{(+)}$. X^{--} and $\rho^{(-)}$ are dependent variables according to Eq. (92) with $C=0$ and Eq. (105) with $\tilde{c}=\mathrm{const}$.

For certain potentials (10) also solutions with $X^{++}=X^{--}=0$ may appear, which can describe a ''supersymmetric ground-state'' [32]. Then $\chi=0$ and the discussion reduces to the pure bosonic case (cf. e.g. Ref. [52] for a situation where such a solution appears).

VII. COUPLING OF SUPERSYMMETRIC TEST-PARTICLE IN MINIMAL FIELD SUPERGRAVITY

To find the proper invariant coupling of a test-particle to supergravity seems to be a hopeless task when one stays within the GPSM related MFS model. On the other hand, in order to study global properties for any of the solutions obtained above a ''super-geodesic'' is needed. For a problem of this type, where simple access to an invariant expression is needed, the superfield approach is the method of choice. The path of a super-particle is described by the map $\tau\rightarrow z^M(\tau)$ with coordinates[15]

$$z^M=(x^m,\theta^\mu). \tag{112}$$

Holonomic indices are transformed into anholonomic ones according to

$$x^a=e_m^a x^m, \quad \theta^\alpha=\delta_\mu^\alpha\theta^\mu. \tag{113}$$

Due to the second equation (113) no separate notation [cf. Eq. (A8)] for the components of θ^μ is needed and $\theta^\mu=(\theta^+,\theta^-)$.

The action of a super-particle with mass m moving along the curve $z^M(\tau)$ may be written as [53–57]

$$\mathcal{S}_{PP}=\int \mathrm{d}\tau\left(g^{-1}(\dot{z}^M E_M{}^{++}\dot{z}^N E_N{}^{--}) + \frac{m^2}{2}g - m\dot{z}^M E_M{}^A\Gamma_A\right). \tag{114}$$

It exhibits the well-known additional fermionic κ symmetry [55]:

$$\delta_\kappa z^M E_M{}^a=0,$$

$$\delta_\kappa z^M E_M{}^+=-\left(m\kappa^+ + \sqrt{2}\frac{\kappa^-}{g}\dot{z}^M E_M{}^{++}\right),$$

$$\delta_\kappa z^M E_M{}^-=-\left(m\kappa^- + \sqrt{2}\frac{\kappa^+}{g}\dot{z}^M E_M{}^{--}\right), \tag{115}$$

$$\delta_\kappa g=-4\dot{z}^M(E_M{}^+\kappa^- + E_M{}^-\kappa^+).$$

[15] $x^m=x^m(\tau)$ and $\theta^\mu=\theta^\mu(\tau)$ are taken in this section without explicitly indicating the difference to the free variables x and θ in the preceding sections.

The action (114) is a condensed expression which includes both the massless and the massive case. The limit $m\rightarrow 0$ of Eq. (114) leads to the standard action of the massless point particle, while the formula for the massive particle to be found in most of the literature is recovered by rescaling $\tilde{g}=-mg$ and $\tilde{\kappa}^\pm=-m\kappa^\pm$.

In contrast to bosonic gravity the action (114) does not contain the full super-line element

$$(ds)^2=\mathrm{d}z^M\otimes\mathrm{d}x^N G_{NM}=2\,\mathrm{d}z^M E_M{}^{++}\otimes\mathrm{d}z^N E_N{}^{--} + 2\,\mathrm{d}z^M E_M{}^{+}\otimes\mathrm{d}z^N E_N{}^{-}. \tag{116}$$

The standard super-particle (114) with $m=0$ only considers the first part of Eq. (116), including bosonic anholonomic indices to be summed over [53–55], which by itself is invariant under supergravity transformations. We do not provide a detailed comparison of the consequences of the two approaches (114) and (116) within this work. But it is important to notice that even the case $m=0$ in Eq. (114) leads to different equations of motion in the supersymmetry sector than the ones following from Eq. (116).[16]

In the standard gauge (B3)–(B6) the connection Γ^A reduces to the result in flat superspace with the only nonvanishing components

$$\Gamma_+=\theta^-, \quad \Gamma_-=\theta^+. \tag{117}$$

To explore the global structure of two-dimensional supergravity with this super-particle the solutions (99)–(103) and (107)–(111) must be inserted in Eq. (114). To this end the θ-expansion of Eq. (114) must be calculated explicitly. After some super-algebra the first term of Eq. (114) (relevant for the massless super-particle) takes the form

$$\begin{aligned} g^{-1}(\dot{z}^M E_M{}^{++}\dot{z}^N E_N{}^{--}) = g^{-1}(&\dot{x}^m e_m^{++} + \sqrt{2}\dot{\theta}^+\theta^+ \\ &+ 2\sqrt{2}\dot{x}^m \underline{\psi}_m^+\theta^+ + \theta^-\theta^+\underline{A}\dot{x}^m e_m^{++}) \\ &\times(\dot{x}^n e_n^{--} + \sqrt{2}\dot{\theta}^-\theta^- \\ &+ 2\sqrt{2}\dot{x}^n\underline{\psi}_n^-\theta^- + \theta^-\theta^+\underline{A}\dot{x}^n e_n^{--}), \end{aligned} \tag{118}$$

while the Wess-Zumino contribution becomes

$$\dot{z}^M E_M{}^A\Gamma_A=\dot{\theta}^+\theta^- + \dot{\theta}^-\theta^+ + \dot{x}^m\underline{\psi}_m^+\theta^- + \dot{x}^m\underline{\psi}_m^-\theta^+ + \dot{x}^m\underline{\tilde{\omega}}_m\theta^-\theta^+. \tag{119}$$

When inserting the classical solution for $\underline{A}$ [Eq. (B12)] and the explicit expression for the dependent spin-connection $\underline{\tilde{\omega}}$

[16] What is meant by ''global'' properties of a solution is well-known to depend on the ''device'' by which (super-)geodesics are defined. Already in the purely bosonic case with nonvanishing torsion the use of ''geodesics'' (depending on Christoffel symbols only) or ''autoparallels'' (depending also on the contorsion) may lead to different global properties.

[cf. Eq. (29)] in Eqs. (118) and (119), the action of the supersymmetric test-particle (114) is parametrized in terms of the zweibein e_m, the gravitino $\underline{\psi}_m$, the dilaton field $\underline{\phi}$ and the dilatino $\underline{\chi}$. By means of the identification (85) and (86) the action (114) turns into a function of e_m, ψ_m, ϕ and χ:

$$\mathcal{S}_{PP}=\int \mathrm{d}\tau\left\{g^{-1}A^{++}A^{--}+\frac{m^2}{2}g-m(B^{+-}+B^{-+})\right\}, \tag{120}$$

$$A^{++}=\dot{x}^m e_m^{++}\left(1+\frac{1}{2}Z\chi^-\theta^+-\frac{1}{2}\theta^2\left(\frac{1}{2}uZ+\frac{1}{2}u'-\frac{1}{16}Z'\chi^2\right)\right)+\sqrt{2}\dot{\theta}^+\theta^++2\sqrt{2}\dot{x}^m\psi_m^+\theta^+, \tag{121}$$

$$B^{+-}=\dot{\theta}^+\theta^-+\frac{1}{4}\theta^2\dot{x}^m\tilde{\omega}_m+\dot{x}^m\psi_m^+\theta^-+\frac{Z}{4\sqrt{2}}\dot{x}^m e_m^{++}\chi^-\theta^-. \tag{122}$$

Here A^{--} and B^{-+} are defined through Eqs. (121) and (122) by the interchange of all *explicit* anholonomic indices $+\to-$, $-\to+$.

A. Gauge choice

When the supersymmetric test-particle moves on the supergravity background, the zweibein and the gravitino in Eq. (120) are replaced by their classical solutions (100)–(103) or (108)–(111) respectively. In principle, "super-geodesics" could then be obtained from variation of Eq. (114) or (120), respectively. This task simplifies considerably when an appropriate gauge-fixing is used:

(1) The solutions from MFS depend, among others, on the variable X^{++}, which is not present in superspace. The supersymmetric test-particle being manifestly invariant under local Lorentz transformations, we can eliminate this dependence by a (finite) local Lorentz transformation. Thus on any patch with $X^{++}\neq 0$ we can fix its value to $X^{++}=1$ or $X^{++}=-1$, depending on the sign of the original configuration. For $X^{++}=0$ we have to parametrize the solution analogously in terms of X^{--} (cf. Sec. VI).

(2) κ-symmetry can be used to gauge one of the fermionic variables to a constant [58]. It turns out that $\dot{\theta}^-\equiv 0$ is the preferable choice for $X^{++}\neq 0$.

(3) It has been argued in Ref. [12] that the classical solution (99)–(103) is equivalent to the corresponding solution of the purely bosonic model up to local supersymmetry transformations. Thus locally all fermionic target-space degrees of freedom could be gauged away: $\rho^{(+)}\equiv 0$, $\rho^{(-)}\equiv 0$ and consequently $\psi_\pm\equiv 0$. One might ask whether this zero fermion (ZF) gauge is accessible and allowed.

Concerning the question whether the ZF gauge is allowed, one should consult the situation in the purely bosonic case. There the line element, after elimination of $Y=X^{++}X^{--}$ by means of the Casimir constant, is determined by two arbitrary functions F and ϕ. The "gauge" $\mathrm{d}F=0$ is forbidden by the requirement of nonsingular gravity, namely that the determinant of the metric should be different from zero. It would be natural, although not strictly necessary, to transfer these arguments directly to supergravity, i.e. demanding a nonsingular super-determinant. However, as can be seen from Eq. (45), the vanishing of that determinant is controlled by its bosonic contributions. Thus within this line of arguments the ZF gauge is allowed.

Typically the accessibility of a gauge is more difficult to answer than the question whether it is allowed. Beside the mere mathematical challenge to describe finite gauge transformations, accessibility involves subtle physical questions as well. For MFS the mathematical aspect found a definite answer [12]: By means of a finite GPSM gauge transformation any solution can be brought to ZF gauge.[17]

The physical aspect is more involved. Indeed, under a finite transformation not only the specific solution of the field equations itself, but also the "device" defining the (super-)geodesics (e.g., the super-point particle action) must be transformed. This last step can be omitted if and only if the *new* solution together with the *old* device turns out to have the same physics as the *new* solution with some "invariant" device.[18] This does not necessarily imply that the solution does not transform at all, but solely that the two systems are physically (albeit not mathematically) equivalent. This is often the case for broken *bosonic* symmetries, where states (field configurations) exhibit such a degeneracy. On the other hand, some well-known symmetries do not allow the simplification of using the old device: The conformal transformation discussed in Sec. IV is a bosonic example for that. In the present context it is important that *broken* supersymmetry does not allow the above shortcut as well. Indeed, breaking of supersymmetry never leads to an equivalent class of states with the same physical properties.[19] But, as may be checked easily by inserting any of the solutions of Sec. VI, including the one considered in Ref. [12], into the supersymmetry transformations (22) and (25), many of them break at least half of the supersymmetries (cf. Ref. [32] and

[17]The explicit proof had been performed in Ref. [12] for MFS_0 only, but it generalizes straightforwardly to MFS by the use of the conformal transformations (70) and (71).

[18]A simple example is a wave packet solution in classical field theory. Clearly the solution breaks (global) rotation symmetry as the wave packet moves in a certain direction. In an "invariant" system of detectors, the latter are (or can be) arranged in a rotationally symmetric way. Then physics (measurement of the wave packet) remains the same.

[19]In contrast to the example in footnote 18 broken supersymmetry acting on a bosonic wave packet produces fermions. Then it is obviously relevant whether the detector has been transformed too. If this is the case it would still register "bosons" although it would receive fermionic contributions as well.

the systematic study of supersymmetric solutions in Ref. [59]). Therefore, with respect to the transformation proposed in Ref. [12] the device must be transformed as well. Thus we expect that in the generic case the global properties of the solutions of Sec. VI do depend on fermionic background fields, if these fields cannot be transformed away by means of *unbroken* supersymmetries.

Despite the problems of physical accessibility of the ZF gauge we will, for simplicity, restrict our analysis below to this specific class of solutions. This choice also correlates with the observation that classical field equations usually possess solutions with vanishing fermion fields.[20] Inserting it into the super-determinant (45) yields a nontrivial result, namely

$$E=e\left(1-\frac{1}{2}\theta^-\theta^+(uZ+u')\right). \tag{123}$$

For nonsingular gauges—in the usual sense—the superdeterminant is nonvanishing and does *not* reduce to the purely bosonic result, although the dilatino and the gravitino of the background have been gauged away, because the fermionic partner $\theta^+(\tau)$ of the bosonic geodesic $x^m(\tau)$ survives.

Besides the drastic simplification of the pointparticle action, this gauge is very convenient also from the technical point of view, as it permits an easy application for both solutions (99)–(103) and (107)–(111) derived in the previous section. It should be kept in mind, though, that the ZF gauge is problematic. Nevertheless, for a first cursory exploration of super-geodesics derived from Eqs. (120)–(122) it certainly is a convenient starting point.

(4) In the bosonic sector an Eddington-Finkelstein like gauge is the most convenient one [33]. To this end the worldsheet coordinates x^m are chosen such that the remaining target space variables $F(x)$ and $\phi(x)$ describe the trivial embedding

$$F(x)\equiv x^0(\tau)=F(\tau),\quad \phi(x)\equiv x^1(\tau)=\phi(\tau). \tag{124}$$

These will be our bosonic coordinates in the following.

The pointparticle action (120) with the solution (99)–(103) in the gauge choice as proposed in (1)–(4) above, together with Eq. (124) simplifies to

$$\mathcal{S}_{PP}=\int d\tau\left(\mathcal{S}_{PP}^{\text{bosonic}}+\mathcal{S}_{PP}^{\text{SUSY}}+\frac{m^2}{2}g\right), \tag{125}$$

$$\mathcal{S}_{PP}^{\text{bosonic}}=g^{-1}e^Q\dot{F}(\dot{\phi}+\xi(\phi)\dot{F}), \tag{126}$$

$$\begin{aligned}\mathcal{S}_{PP}^{\text{SUSY}}=&g^{-1}[\theta^-\theta^+\mu(\phi)\cdot e^Q\dot{F}(\dot{\phi}+\xi(\phi)\dot{F})\\&-\sqrt{2}\dot{\theta}^+\theta^+(\dot{\phi}+\xi\dot{F})]\\&-m[\theta^-\dot{\theta}^++\theta^-\theta^+[\dot{F}(Z\cdot\xi(\phi)+\xi'(\phi))+Z\cdot\dot{\phi}]].\end{aligned} \tag{127}$$

The dependence on the dilaton ϕ occurs in the coefficient of bosonic torsion Z and in two functions $\xi(\phi)$ and $\mu(\phi)$ defined as

$$\xi(\phi)=C+\frac{1}{8}e^Qu^2,\quad \mu(\phi)=-u\cdot Z-u'. \tag{128}$$

B. Orbits of the massless point particle

To explore the global structure of a certain gravitational background the equations of motion from Eqs. (125)–(127) have to be derived. To this end the variation with respect to the super-coordinates $\{z^M\}=\{F,\phi,\theta^+\}$ has to be calculated. It is convenient to reparametrize the curve $z^M(\tau)$ in such a way that $\dot{g}\equiv 0$. Then the variations can be written as

$$\begin{aligned}g\cdot\delta_F\mathcal{S}_{PP}=&\frac{\partial}{\partial\tau}\{-e^Q(\dot{\phi}+2\xi\dot{F})(1+\theta^-\theta^+\mu)+\sqrt{2}\dot{\theta}^+\theta^+\xi\\&+mg\theta^-\theta^+(Z\xi+\xi')\}\end{aligned} \tag{129}$$

$$\begin{aligned}g\cdot\delta_\phi\mathcal{S}_{PP}=&e^Q(1+\theta^-\theta^+\mu)((\xi'+Z\xi)\dot{F}^2-\ddot{F})\\&+e^Q\theta^-\theta^+\mu'\xi\dot{F}^2-e^Q\theta^-\dot{\theta}^+\mu\dot{F}\\&-\sqrt{2}\dot{\theta}^+\theta^+\xi'\dot{F}+\sqrt{2}\ddot{\theta}^+\theta^+\\&-mg(\theta^-\theta^+\dot{F}(Z'\xi+Z\xi'+\xi'')-\theta^-\dot{\theta}^+Z)\end{aligned} \tag{130}$$

$$\begin{aligned}g\cdot\delta_{\theta^+}\mathcal{S}_{PP}=&-e^Q\theta^-\mu\dot{F}(\dot{\phi}+\xi\dot{F})+\sqrt{2}\theta^+(\ddot{\phi}+\xi'\dot{\phi}\dot{F}+\xi\ddot{F})\\&+2\sqrt{2}\dot{\theta}^+(\dot{\phi}+\xi\dot{F})+mg\theta^-(\dot{F}(Z\xi+\xi')+\dot{\phi}Z).\end{aligned} \tag{131}$$

Equation (129) is a total derivative as $\partial/\partial F$ is a Killing field. The expression in the curly brackets corresponds to the related constant of motion. To solve Eqs. (130) and (131) with the constant of motion (129) in full generality is a daunting task which we do not attempt in the present work. For illustrative purposes we find it sufficient to consider special cases with some physical relevance. The massless particle ($m=0$) together with its supersymmetric orbit $\theta^+(\tau)$ already allows the discussion of different physically interesting analytic solutions. In the cases to be treated below further restrictions will be made:

(1) "Minkowski ground state models": Within bosonic theories of gravity a special subset is determined by the condition that at $C=0$ in Eq. (11) the metric of these theories reduces to the one of Minkowski space [26,33]. This implies a relation between the functions $Z(\phi)$ and $V(\phi)$ in Eq. (10), which for supergravity becomes just the condition $\mu=0$, i.e. the relation

$$\frac{u'}{u}=-Z \tag{132}$$

[20] There are, of course, counterexamples, e.g., the solution (107)–(111) for $\tilde{c}\neq 0$ in Eq. (105).

between the prepotential and the function which determines nonvanishing bosonic torsion. Spherically reduced gravity from $d=4$ belongs to this class with (l is an arbitrary constant)

$$Z_{\mathrm{SRG}}=-(2\phi)^{-1}, \quad u_{\mathrm{SRG}}=-l\sqrt{\phi}, \tag{133}$$

but also more general models which are asymptotically flat, if $u(\infty)\to\infty$.

(2) The (bosonic) light-like directions $\mathcal{S}_{PP}^{\mathrm{bosonic}}=0$ correspond to especially simple solutions. They are characterized by

(2a) $\dot{F}=0$,

(2b) $\dot{\phi}+\xi\dot{F}=0$.

1. Minkowski ground-state models

The solution of the purely bosonic model with $\mu=0$ is regarded as a given input in this section. Actually, the interesting cases will be covered by solutions of the type (2a) and (2b) above.

For $\mu=0$ and $m=0$ the variation (131) vanishes if

$$\partial_\tau(\dot{\phi}+\xi\dot{F})\theta^+ + 2(\dot{\phi}+\xi\dot{F})\dot{\theta}^+=0 \tag{134}$$

holds. The solutions can be classified as follows:

(A) $\dot{\phi}+\xi\dot{F}=0$. This is the special case $\mu=0$ of (2b) above and will be discussed together with the generic solutions of this type below.

(B) $\dot{\theta}^+=\frac{1}{2}\partial_\tau \log(\dot{\phi}+\xi\dot{F})\theta^+$.

The general solution of (B) is

$$\theta^+=\frac{1}{\sqrt{|\dot{\phi}+\xi\dot{F}|}}\lambda, \tag{135}$$

where λ is an arbitrary constant spinor. The space of anticommuting variables of this class of solutions can be parametrized by the two constant spinors λ and θ^-. In a bosonic superfunction A with body A_B and soul A_S therefore the decomposition

$$A(\tau)=A_B(\tau)+A_S(\tau)=A_B(\tau)+\theta^-\lambda a(\tau) \tag{136}$$

can be introduced. Here $A_B(\tau)$ and $a(\tau)$ are ordinary bosonic functions.

As Eq. (135) does not depend on θ^- any τ-derivative of θ^+ is again proportional to θ^+ (or zero), which especially means that $\theta^+(\tau)$ has no simple zeros. The singularity in Eq. (135) corresponds to a light-like direction (A) of the bosonic line element.

Evaluating the variations (129) and (130) with the solution (135) they simplify to

$$\partial_\tau(e^Q(\dot{\phi}+2\xi\dot{F}))=0, \tag{137}$$

$$(\xi'+Z\xi)\dot{F}^2-\ddot{F}=0, \tag{138}$$

which are the relations from the purely bosonic model. The motion of θ^+ is determined up to the initial value according to Eq. (134), i.e. up to the numerical value of λ in Eq. (135). Due to the absence of the θ's in Eqs. (137) and (138), the evolution of ϕ and F will not depend on the fermionic variables. But as all bosonic quantities must be regarded as (commuting) superfunctions, respectively, supernumbers, a soul can still be introduced in them by an appropriate choice of the initial values for these fields. An example of this type is evaluated below.

2. Light-like solution (2a) with $\mu\neq 0$

The vanishing of Eq. (131) relates the motion in the direction θ^+ to $\dot{\phi}$ and $\ddot{\phi}$:

$$\dot{\theta}^+=-\frac{1}{2}\frac{\ddot{\phi}}{\dot{\phi}}\theta^+. \tag{139}$$

Again the evolution of $\theta^+(\tau)$ does not depend on θ^- and the general solution

$$\theta^+(\tau)=(\dot{\phi})^{-1/2}\lambda \tag{140}$$

with an arbitrary constant spinor λ is an immediate consequence. The space of anticommuting coordinates is again two-dimensional and may be parametrized by (θ^-,λ).

As a consequence of Eq. (140) the term $\propto\dot{\theta}^+\theta^+$ in the action (127) vanishes. Thus the complete action (125) is identically zero for $\dot{F}=0$ and these orbits are not only null directions of the bosonic part of the action, but of the whole super-particle action (125).

In terms of the solution (140) Eq. (130) vanishes identically, whereas the constant of motion (129) can be brought into the form

$$e^Q(1+\theta^-\theta^+\mu)\dot{\phi}=k. \tag{141}$$

Here k is some constant super-number. After some straightforward super-algebra the τ derivative of Eq. (141) is found as

$$\ddot{\phi}=-\dot{\phi}^2\left(Z+\frac{1}{2}\theta^-\theta^+\mu Z+\theta^-\theta^+\mu'\right). \tag{142}$$

The body of this equation is seen to yield the correct light-like geodesic $\ddot{\phi}=-Z\dot{\phi}^2$. Inserting this relation into Eq. (139) [the terms $\propto\mu$ in Eq. (142) vanish due to $(\theta^+)^2=0$] leads to

$$\dot{\theta}^+=\frac{1}{2}Z\theta^+\dot{\phi}. \tag{143}$$

As $Z=dQ/d\phi$ [cf. Eq. (11)] Eq. (143) can be transformed into a total derivative and the general solution for θ^+ [Eq. (140)] can be expressed alternatively as

$$\theta^+(\tau)=e^{Q/2}\lambda. \tag{144}$$

The constant spinor λ appearing in this solution is the same as the one in Eq. (140).

To solve Eq. (141) a decomposition according to Eq. (136) is necessary. Using the relations

$$k=k_B+k_S, \quad \partial_\tau k_B=\partial_\tau k_S=0, \tag{145}$$

$$e^{Q(\phi)}=e^{Q(\phi_B)}[1+Z(\phi_B)\phi_S], \tag{146}$$

the body and soul of Eq. (141) become

$$e^{Q(\phi_B)}\dot{\phi}_B=k_B, \tag{147}$$

$$e^{Q(\phi_B)}\dot{\phi}_S+k_B Z(\phi_B)\phi_S+k_B\theta^-\theta^+\mu(\phi_B)=k_S, \tag{148}$$

where in Eq. (148) $\dot{\phi}_B$ has been eliminated by means of Eq. (147). The latter equation is equivalent to the one of the bosonic model, but the value of the complete $\phi(\tau)$ receives contributions from ϕ_S as well. With the souls $\phi_S=\theta^-\lambda\varphi(\tau)$ and $k_S=\theta^-\lambda\tilde{k}$ for ϕ and k in Eq. (148) become

$$e^{Q(\phi_B)}\dot{\varphi}+k_B Z(\phi_B)\varphi+k_B e^{Q(\phi_B)/2}\mu(\phi_B)=\tilde{k}. \tag{149}$$

For a complete set of initial values for super-coordinates $z^M(\tau=0)$ and for the constant of motion k, Eqs. (144), (147) and (149) uniquely determine the evolution of the two dynamical variables ϕ and θ^+. Certainly, Eqs. (147) and (149) cannot be solved in general. In certain cases, however, a simple solution can be obtained:

Due to Eq. (143) the purely bosonic solution corresponds to a special choice of initial values, namely $\theta^+=0$. Indeed, if $\theta^+=0$ for any $\tau=\tau_0$, all τ-derivatives on $\theta^+(\tau_0)$ vanish as well and consequently θ^+ is zero everywhere. In this case the constant of motion has a vanishing soul as well.

Models with vanishing bosonic torsion $(Z=0)$ possess a simple solution with a nontrivial fermionic sector. As both θ^+ and θ^- are constant in this case Eq. (141) becomes a total derivative [cf. Eq. (128)]:

$$\frac{\partial}{\partial\tau}(\phi-\theta^-\theta^+u)=k\Rightarrow\phi-\theta^-\theta^+u-c=k\tau. \tag{150}$$

Both k and c are supernumbers, where c is determined by the initial value of ϕ as

$$c=\phi_0-\theta^-\theta^+u(\phi_0). \tag{151}$$

The explicit solution is

$$\phi_B=k_B\tau+\phi_0, \tag{152}$$

$$\phi_S=\theta^-\theta^+(u(\phi_B)-u(\phi_0))+k_S\tau, \tag{153}$$

where it has been assumed that the initial value of ϕ is bosonic: $\phi_0=\phi_{B0}$. This class of solutions is especially interesting as it determines the null-directions of a massless super point particle on a background described by the model of Howe [25].

As an example with nonvanishing bosonic torsion we consider spherically reduced gravity from four dimensions, i.e., a combination of (2a) with $\mu=0$ and Eq. (133). As $\dot{\phi}+\xi\dot{F}\neq 0$ here, the situation is given by (B) above. Inserting Eq. (133) into Eqs. (147) and (144) and after integration of the former relation, one finds after a simple redefinition of $(1/2)(k_B\tau+c_B)\to\tau$

$$\phi_B=\tau^2, \quad \theta^+(\tau)=\frac{\lambda}{\tau}. \tag{154}$$

The determination of the soul of ϕ simplifies in this special case, as $\mu(\phi)\equiv 0$. The integrating factor of the differential equation (149) is given by

$$\rho(\tau)=\frac{1}{2\tau}, \tag{155}$$

which yields the last dynamical quantity as

$$\phi_S=\theta^-\lambda\tau\left(c_S+\tilde{k}\,\frac{2\tau-c_B}{k_B}\right). \tag{156}$$

Obviously the value of $\phi_S(\tau)$ does not depend on the evolution of $\theta^+(\tau)$ in Eq. (154). This is a consequence of $\mu=0$ and, as already discussed above, holds in any Minkowski ground-state theory. Therefore, with the special choice $c_S=\tilde{k}=0$ the soul of ϕ vanishes for all τ, but of course, one is not forced to choose the integration constants like that. Nevertheless, a nontrivial coupling of the bosonic variable ϕ to the θ variables is somehow artificial. On the other hand, $\theta^+(\tau)$ does depend on the behavior of ϕ_B and thus even in this simple example the bosonic and the fermionic part of the pointparticle do not decouple from each other. The evolution of $\theta^+(\tau)$ shows a singularity at $\tau=0$, a point where with our choice of τ the singularity of the Schwarzschild black hole at $\phi=0$ is encountered.

3. Light-like Solution (2b) with $\mu\neq 0$

Equation (131) vanishes trivially in this case, while Eqs. (129) and (130) (except for $\xi=0$) are related by

$$\text{Eq. (129)}=\xi\cdot\text{Eq. (130)}. \tag{157}$$

The remaining only differential equation reads

$$e^{Q}(1+\theta^-\theta^+\mu)\dot{\phi}+\sqrt{2}\xi\dot{\theta}^+\theta^+=k. \tag{158}$$

If we had still $\dot{\theta}^+\propto\theta^+$, Eq. (158) does reduce to Eq. (141), which, of course, is expected to hold for the body of the two equations. Thus the solution (139) is one of the solutions for Eq. (158), with the same $\phi(\tau)$ as derived in the previous section (notice, however, the different behavior of F).

Nevertheless, Eq. (158) allows solutions with $\dot{\theta}^+$ not proportional to θ^+. This does not lead to a nonvanishing action (125), as the latter is proportional to $\dot{\phi}+\xi\dot{F}$. Again one observes that any bosonic null-direction is a null-direction of the whole super-particle.

To discuss the general solution the split into soul and body is made again:

$$e^{Q(\phi_B)}\dot{\phi}_B = k_B, \tag{159}$$

$$e^{Q(\phi_B)}\dot{\phi}_S + k_B Z(\phi_B)\phi_S + k_B\theta^-\theta^+\mu(\phi_B) + \sqrt{2}\xi(\phi_B)\dot{\theta}^+\theta^+ = k_S. \tag{160}$$

By introducing the general decomposition for θ^+

$$\theta^+(\tau) = f(\tau)e^{Q/2}\lambda + g(\tau)\theta^-, \quad f(0)=1, \quad g(0)=0, \tag{161}$$

and using again $\phi_S(\tau) = \theta^-\lambda\varphi(\tau)$, Eq. (160) becomes

$$e^{Q(\phi_B)}\dot{\varphi} + k_B Z(\phi_B)\varphi + k_B e^{Q(\phi_B)/2}\mu(\phi_B)f(\tau) + \sqrt{2}\xi(\phi_B)\left(e^{Q(\phi_B)/2}[\dot{g}(\tau)f(\tau) - \dot{f}(\tau)g(\tau)] - e^{-Q(\phi_B)/2}\frac{1}{2}k_B Z f(\tau)g(\tau)\right) = \tilde{k}. \tag{162}$$

Obviously the system is underdetermined, because three free functions φ, f, and g obey one single first order differential equation. The special situation of $\mu=0$ follows straightforwardly by setting this function to zero in all equations of this section.

VIII. CONCLUSION AND OUTLOOK

The formulation of supergravity as a superfield (dilaton) theory is well-known since the seminal works of Howe [25] and Park and Strominger [32]. This approach uses the second order formalism with vanishing bosonic torsion. Unless the dilaton superfield is nondynamical, this field appears in a second order action, which applies to most models with a bosonic potential of direct physical relevance like Einstein gravity, spherically reduced from D dimensions. Also the string inspired CGHS model [19] (the formal limit D→∞) belongs to this class.

The complicated structure of the equations of motion in a second order formalism, together with the large number of (auxiliary) field variables in superfield supergravity, probably has been the main reason why to this day no full solution of generic 2D supergravity with dynamical dilaton [32] has been published.[21]

Our present work closes this lacuna. Also for the first time—we believe—a manageable treatment is provided for ‘‘supergeodesics,’’ the motion of a super-point particle in a very general supergravity background solution which, nonetheless, is described by a minimal set of fields.

In order to achieve these results—which by far do not exhaust the list of further applications—one relies heavily upon the knowledge gained in first order 2D bosonic gravity [1,2] and the more general concept of Poisson-sigma models [7,8].

Within this approach not only in bosonic gravity many (without interaction with matter: essentially all) classical and quantum problems have found a complete solution [33], but the graded extension of Poisson-sigma models also opens the door towards a similar treatment of supergravity-like theories [13]. A subset of those graded Poisson-sigma models (MFS in Fig. 1), already identified by the present authors as ‘‘genuine’’ supergravities from their algebra of Hamiltonian constraints [14], is equivalent to a second order superdilaton theory (MFDS). The latter, in a more roundabout way (path $\tilde{B}'\to D'\to\tilde{B}''$ in Fig. 1) is shown here to be *identical* to the superfield dilaton theory of Ref. [32] (SFDS in Fig. 1), when different auxiliary fields in the latter are eliminated, when certain conformal transformations are made and when the gravitino field is redefined appropriately. We also were able to prove that during these procedures the corresponding supersymmetric transformations are mapped exactly upon each other—with a local Lorentz transformation mixed in only. The chain of formulas which provides the identification of superfield dilaton supergravity [32] with the corresponding class of supergravity theories derived from the graded Poisson-sigma approach is compiled at the end of Sec. V. These formulas should be consulted together with a flow diagram of Fig. 1 and the corresponding list of actions.

Of course, once this identification is established, the general solution of MFS, obtained after a conformal transformation (target space diffeomorphism in GPSM parlance) from the solution published already in Ref. [13], represents the general solution of superfield supergravity of Ref. [32] (cf. Sec. VI). Our only technical restriction has been that a further arbitrary function of the dilaton superfield Φ [$J(\Phi)$ in the action of Ref. [32], respectively our Eq. (51)] has been assumed to be invertible so that the replacement $J(\Phi)\to\Phi$ is possible. This restriction is not serious as on the one hand all physically interesting theories (spherically reduced gravity, etc.) are of this form anyhow. On the other hand, this inversion is permitted locally in function space, so that the only further complication can be that the general solution is valid only in a certain patch in this case.

This general solution of superfield supergravity (after a suitable choice of gauge-fixings among the bosonic symmetries) depends on a bosonic constant Casimir function C and on two bosonic functions dF and ϕ, which may be interpreted as the coordinates of the world sheet. For $C\neq 0$ the anticommuting space is parametrized by two free fermionic functions, at $C\equiv 0$ one of them is replaced by an anticommuting constant Casimir function and an anticommuting df.

As argued in Ref. [12] the two fermionic gauge-degrees of freedom of $N=(1,1)$ supergravity could be used locally to put those functions to zero [zero fermion gauge (ZF gauge)] so that the bosonic solution survives after all as the only ‘‘nontrivial’’ one, as long as no interactions with matter are considered. Indeed, by only demanding a nonsingular determinant of the supermetric—generalizing the nonsingularity restriction of bosonic gravity—this ZF gauge would be permitted, and, in this philosophy, pure supergravity would become ‘‘trivial,’’ because the gravitino can be ‘‘gauged away.’’ However, whether such a gauge is ‘‘physically’’ accessible depends on the way a ‘‘measuring device’’ (e.g. supergeodesic) is transformed in this process. Only for a very restricted class of solutions an ‘‘invariant’’ device can be

[21]Cf. the comment below Eq. (50) in [32].

constructed, namely when such a solution does not break supersymmetry. In the generic case our solution, as well as the bosonic one of Ref. [32], show explicit supersymmetry breaking. The general solution provided in the present work covers all cases, but in the applications presented in our present paper—for simplicity—backgrounds in the ZF gauge are considered, only.

The formulation of the massive super-point particle, moving along a ‘‘super-geodesic’’ is straightforward in the superfield formalism, but so far—for lack of a general solution of the supergravity background—has been very difficult to work with in practice. Mapping the action of that particle into the GPSM-based formalism by means of the identification found in our present work, leads to a system of supergeodesic equations (Sec. VII) which describes the motion in a background solution. Interestingly enough, even in the ZF gauge (with gravitino set to zero identically) the bosonic geodesics (the ‘‘body’’) are accompanied by an orbit in the superpartner (‘‘soul’’) of the bosonic coordinates. This opens an interesting field of detailed investigations of such systems where body and soul may mutually influence each other in the quest for a globally complete solution (described by a ‘‘super-Penrose diagram’’). We only present some simple examples here with massless (light-like) movement and especially concentrate also on theories with Minkowski ground state models in their bosonic sector. Such backgrounds become flat for $C=0$. They include e.g. the Schwarzschild solution, but also other models which are not asymptotically flat.

Clearly the range of further studies, made possible by our present work, is very broad because the (G)PSM technology is extremely powerful, not only at the classical, but in particular also at the quantum level [1–4,33–36,60]. On the other hand, superspace techniques allow one to couple matter fields in a straightforward way to MDFS and NDMFS (cf. Fig. 1). Therefore a path integral quantization of MFS coupled to matter fields becomes manageable [39]. Also questions like the one concerning a fermionic counterpart of the (bosonic) virtual Black Hole [37,38] can be expected to find an answer—as well as a plethora of further research directions, e.g., the inclusion of Yang-Mills fields, just to name one of them [51,52,61].

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APPENDIX A: NOTATIONS AND CONVENTIONS

These conventions are identical to [13,28], where additional explanations can be found.

Generic indices (used in the context of GPSM's) are chosen from the middle of the alphabet:

$I,J,K,\ldots$ include both commuting and anticommuting objects. Generalized commutation relations are written in the standard way

$$v^I w^J = (-1)^{IJ} w^J v^I, \tag{A1}$$

where the indices in the exponent take values 0 (commuting object) or 1 (anticommuting object).

$i,j,k,\ldots$ are generic commuting indices.

To label *holonomic* coordinates, letters from the middle of the alphabet are used:

$M,N,L,\ldots$ can be both commuting and anticommuting.

$m,n,l,\ldots$ are commuting.

$\mu,\ \nu,\ \rho,\ldots$ are anticommuting.

Anholonomic coordinates are labeled by letters from the beginning of the alphabet:

$A,B,C,\ldots$ can be both commuting and anticommuting.

$a,b,c,\ldots$ are commuting.

$\alpha,\beta,\gamma,\ldots$ are anticommuting.

The index ϕ is used to indicate the dilaton component of the GPSM fields:

$$X^\phi = \phi, \quad A_\phi = \omega. \tag{A2}$$

The summation convention is always $NW \rightarrow SE$, especially for a fermion χ: $\chi^2 = \chi^\alpha \chi_\alpha$. Our conventions are arranged in such a way that almost every bosonic expression is transformed trivially to the graded case when using this summation convention and replacing commuting indices by general ones. This is possible together with exterior derivatives acting *from the right*, only. Thus the graded Leibniz rule is given by

$$\mathrm{d}(AB) = A\mathrm{d}B + (-1)^B (\mathrm{d}A)B. \tag{A3}$$

In terms of anholonomic indices the metric and the symplectic 2×2 tensor are defined as

$$\eta_{ab} = \eta^{ab} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \epsilon_{ab} = -\epsilon^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$\epsilon_{\alpha\beta} = \epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{A4}$$

The metric in terms of holonomic indices is obtained by $g_{mn} = e_n^b e_m^a \eta_{ab}$ and for the determinant the standard expression $e = \det e_m^a = \sqrt{-\det g_{mn}}$ is used. The volume form reads $\epsilon = \frac{1}{2}\epsilon^{ab} e_b \wedge e_a$.

The γ matrices are used in a chiral representation:

$$\gamma^0{}_\alpha{}^\beta=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \gamma^1{}_\alpha{}^\beta=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},$$

$$\gamma^3{}_\alpha{}^\beta=(\gamma^1\gamma^0)_\alpha{}^\beta=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{A5}$$

The matrices $(\gamma^a)^{\alpha\beta}=\epsilon^{\alpha\delta}\gamma^a{}_\delta{}^\beta$ and $(\gamma^3)^{\alpha\beta}$ are symmetric in $\{\alpha,\beta\}$. The most important relations among the γ matrices are

$$\gamma^a\gamma^b=\eta^{ab}\mathbf{1}+\epsilon^{ab}\gamma^3,\quad \gamma^a\gamma^3=\gamma^b\epsilon_b{}^a. \tag{A6}$$

Covariant derivatives of anholonomic indices with respect to the geometric variables $e_a=\mathrm{d}x^m e_{am}$ and $\psi_\alpha=\mathrm{d}x^m\psi_{\alpha m}$ include the two-dimensional spin-connection one form $\omega^{ab}=\omega\epsilon^{ab}$. When acting on lower indices the explicit expressions read ($\frac{1}{2}\gamma^3$ is the generator of Lorentz transformations in spinor space):

$$(De)_a=\mathrm{d}e_a+\omega\epsilon_a{}^b e_b,\quad (D\psi)_\alpha=\mathrm{d}\psi_\alpha-\frac{1}{2}\omega\gamma^3{}_\alpha{}^\beta\psi_\beta. \tag{A7}$$

Finally light-cone components are introduced. As we work with spinors in a chiral representation we can use

$$\chi^\alpha=(\chi^+,\chi^-),\quad \chi_\alpha=\begin{pmatrix}\chi_+\\ \chi_-\end{pmatrix}. \tag{A8}$$

For Majorana spinors upper and lower chiral components are related by $\chi^+=\chi_-$, $\chi^-=-\chi_+$, $\chi^2=\chi^\alpha\chi_\alpha=2\chi_-\chi_+$. Vectors in light-cone coordinates are given by

$$v^{++}=\frac{i}{\sqrt{2}}(v^0+v^1),\quad v^{--}=\frac{-i}{\sqrt{2}}(v^0-v^1). \tag{A9}$$

Derivatives with respect to these components are written compactly as

$$\partial_K=\frac{\partial}{\partial X^K}=(\partial_\phi,\partial_{++},\partial_{--},\partial_+,\partial_-). \tag{A10}$$

The additional factor i in Eq. (A9) permits a direct identification of the light-cone components with the components of the spin-tensor $v^{\alpha\beta}=iv^c\gamma_c^{\alpha\beta}/\sqrt{2}$. This implies that $\eta_{++|--}=1$ and $\epsilon_{--|++}=-\epsilon_{++|--}=1$. The γ matrices in light-cone coordinates become

$$(\gamma^{++})_\alpha{}^\beta=\sqrt{2}i\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},\quad (\gamma^{--})_\alpha{}^\beta=-\sqrt{2}i\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}. \tag{A11}$$

APPENDIX B: SUPERSPACE INTEGRATION

As for fermionic fields the square of the superspace variables is abbreviated by $\theta^2=\theta^\alpha\theta_\alpha$. Superspace is integrated out by

$$\frac{1}{2}\int \mathrm{d}\theta^2\,\theta^2=1. \tag{B1}$$

Our covariant derivative with respect to anholonomic indices is

$$D_\alpha=\partial_\alpha+i(\gamma^a\theta)_\alpha\partial_a. \tag{B2}$$

With these conventions and the particular choice of gauge used in this work the components[22] of the super-zweibein read [13]

$$E_m{}^a=e_m{}^a+2i(\theta\gamma^a\underline{\psi}_m)+\frac{1}{2}\theta^2\underline{A}e_m{}^a, \tag{B3}$$

$$E_m{}^\alpha=\underline{\psi}_m{}^\alpha-\frac{1}{2}\underline{\tilde{\omega}}_m(\theta\gamma^3)^\alpha+\frac{i}{2}\underline{A}(\theta\gamma_m)^\alpha-\frac{1}{2}\theta^2\left(\frac{3}{2}\underline{A}\,\underline{\psi}_m{}^\alpha+i(\underline{\tilde{\sigma}}\gamma_m\gamma^3)^\alpha-\underline{A}(\underline{\zeta}\gamma_m)^\alpha\right), \tag{B4}$$

$$E_\mu{}^a=i(\theta\gamma^a)_\mu, \tag{B5}$$

$$E_\mu{}^\alpha=\delta_\mu{}^\alpha\left(1-\frac{1}{4}\theta^2\underline{A}\right), \tag{B6}$$

where $\underline{\tilde{\omega}}$ and $\underline{\tilde{\sigma}}$ are defined as in Eqs. (29) and (34), respectively, however, expressed in terms of underlined fields. In superspace it is often useful to introduce the Lorentz covariant decomposition of the gravitino field

$$\underline{\psi}_\alpha{}^a=(\underline{\zeta}\gamma^a)_\alpha+\underline{\lambda}_\alpha{}^a,\quad \underline{\zeta}_\alpha=\frac{1}{2}(\underline{\psi}^a\gamma_a)_\alpha,$$

$$\underline{\lambda}_\alpha{}^a=\frac{1}{2}(\underline{\psi}^b\gamma^a\gamma_b)_\alpha. \tag{B7}$$

Finally we provide the relevant superspace integrations in the general dilaton model of Ref. [32]. According to Eq. (51) the following integrations have to be performed:

$$\int \mathrm{d}^2\theta\, EJ(\Phi)S=e\left(\frac{1}{2}\underline{\tilde{R}}J+J'(\underline{\chi}\,\underline{\tilde{\sigma}})+J'\underline{F}\,\underline{A}+\frac{1}{8}J''\underline{A}\,\underline{\chi}^2\right), \tag{B8}$$

[22] Field components in a superfield are denoted by underlined symbols throughout.

$$\int d^2\theta\, E K(\Phi) D^\alpha \Phi D_\alpha \Phi$$
$$= e\Bigg(2K\bigg(\partial^m \underline{\phi}\partial_m \underline{\phi} - \frac{i}{4}\underline{\chi}\gamma^m \partial_m \underline{\chi} + \underline{F}^2 - (\underline{\psi}_n \gamma^m \gamma^n \gamma^3 \underline{\chi})\partial_m \underline{\phi}\bigg) + \frac{1}{4}K\underline{\chi}^2(\underline{\psi}_n\gamma^m\gamma^n\underline{\psi}_m) + \frac{1}{4}K'\,\underline{\chi}^2\underline{F}\Bigg), \tag{B9}$$

$$\int d^2\theta\, E L(\Phi)$$
$$= e\bigg((\underline{A} + 2\underline{\zeta}^2 + \underline{\lambda}^2)L + L'(\underline{F} + i(\underline{\zeta}\gamma^3\underline{\chi})) + \frac{1}{8}L''(\underline{\chi}\,\underline{\chi})\bigg). \tag{B10}$$

Evaluating Eq. (51) with Eqs. (B8)–(B10), the variation of the action with respect to the auxiliary fields $\underline{A}$ and $\underline{F}$ yields the elimination conditions for those fields:

$$\underline{F} = -\frac{1}{J'}\left(L + \frac{1}{8}J''\underline{\chi}^2\right), \tag{B11}$$

$$\underline{A} = 4\frac{KL}{(J')^2} - \frac{L'}{J'} + \left(\frac{KJ''}{2(J')^2} - \frac{1}{4}\frac{K'}{J'}\right)\underline{\chi}^2. \tag{B12}$$

[1] W. Kummer and D.J. Schwarz, Nucl. Phys. **B382**, 171 (1992).
[2] F. Haider and W. Kummer, Int. J. Mod. Phys. A **9**, 207 (1994).
[3] P. Schaller and T. Strobl, ''Integrable models and strings,'' gr-qc/9406027.
[4] P. Schaller and T. Strobl, Class. Quantum Grav. **11**, 331 (1994).
[5] D. Louis-Martinez, J. Gegenberg, and G. Kunstatter, Phys. Lett. B **321**, 193 (1994).
[6] M.O. Katanaev and I.V. Volovich, Ann. Phys. (N.Y.) **197**, 1 (1990).
[7] P. Schaller and T. Strobl, ''Finite dimensional integrable systems,'' hep-th/9411163.
[8] P. Schaller and T. Strobl, Mod. Phys. Lett. A **9**, 3129 (1994).
[9] T. Klösch and T. Strobl, Class. Quantum Grav. **13**, 965 (1996).
[10] T. Klösch and T. Strobl, Class. Quantum Grav. **13**, 2395 (1996).
[11] T. Klösch and T. Strobl, Class. Quantum Grav. **13**, 1191 (1996).
[12] T. Strobl, Phys. Lett. B **460**, 87 (1999).
[13] M. Ertl, W. Kummer, and T. Strobl, J. High Energy Phys. **01**, 042 (2001).
[14] L. Bergamin and W. Kummer, J. High Energy Phys. **05**, 074 (2003).
[15] P. Thomi, B. Isaak, and P. Hájcek, Phys. Rev. D **30**, 1168 (1984).
[16] P. Hájcek, Phys. Rev. D **30**, 1178 (1984).
[17] H.-J. Schmidt, Gravitation Cosmol. **3**, 185 (1997).
[18] H.-J. Schmidt, Int. J. Mod. Phys. D **7**, 215 (1998).
[19] C.G. Callan, Jr., S.B. Giddings, J.A. Harvey, and A. Strominger, Phys. Rev. D **45**, 1005 (1992).
[20] B.M. Barbashov, V.V. Nesterenko, and A.M. Chervyakov, Theor. Math. Phys. **40**, 572 (1979); Teor. Mat. Fiz. **40**, 15 (1979); J. Phys. A **13**, 301 (1980).
[21] C. Teitelboim, Phys. Lett. **126B**, 41 (1983).
[22] C. Teitelboim, in *Quantum Theory of Gravity: Essays in Honor of the 60th Birthday of Bryce S. DeWitt*, pp. 327–344.
[23] R. Jackiw, in *Quantum Theory of Gravity: Essays in Honor of the 60th Birthday of Bryce S. DeWitt* [22], pp. 327–344.
[24] R. Jackiw, Nucl. Phys. **B252**, 343 (1985).
[25] P.S. Howe, J. Phys. A **12**, 393 (1979).
[26] M.O. Katanaev, W. Kummer, and H. Liebl, Phys. Rev. D **53**, 5609 (1996).
[27] M.O. Katanaev, W. Kummer, and H. Liebl, Nucl. Phys. **B486**, 353 (1997).
[28] M. Ertl, Ph.D. thesis, Technische Universität Wien, hep-th/0102140.
[29] N. Ikeda and K.I. Izawa, Prog. Theor. Phys. **90**, 237 (1993).
[30] N. Ikeda, Ann. Phys. (N.Y.) **235**, 435 (1994).
[31] J.M. Izquierdo, Phys. Rev. D **59**, 084017 (1999).
[32] Y.-C. Park and A. Strominger, Phys. Rev. D **47**, 1569 (1993).
[33] D. Grumiller, W. Kummer, and D.V. Vassilevich, Phys. Rep. **369**, 327 (2002).
[34] W. Kummer, H. Liebl, and D.V. Vassilevich, Nucl. Phys. **B493**, 491 (1997).
[35] W. Kummer, H. Liebl, and D.V. Vassilevich, Nucl. Phys. **B513**, 723 (1998).
[36] W. Kummer, H. Liebl, and D.V. Vassilevich, Nucl. Phys. **B544**, 403 (1999).
[37] D. Grumiller, W. Kummer, and D.V. Vassilevich, Nucl. Phys. **B580**, 438 (2000).
[38] D. Grumiller, W. Kummer, and D.V. Vassilevich, ''Virtual black holes in generalized dilaton theories and their special role in string gravity,'' hep-th/0208052.
[39] L. Bergamin, D. Grumiller, and W. Kummer (in preparation).
[40] T. Strobl, ''Gravity in Lie algebra morphisms,'' Commun. Math. Phys. (to be published).
[41] D.Z. Freedman, P. van Nieuwenhuizen, and S. Ferrara, Phys. Rev. D **13**, 3214 (1976).
[42] D.Z. Freedman and P. van Nieuwenhuizen, Phys. Rev. D **14**, 912 (1976).
[43] S. Deser and B. Zumino, Phys. Lett. **62B**, 335 (1976).
[44] S. Deser and B. Zumino, Phys. Lett. **65B**, 369 (1976).
[45] R. Grimm, J. Wess, and B. Zumino, Phys. Lett. **73B**, 415 (1978).
[46] H. Grosse, W. Kummer, P. Presnajder, and D.J. Schwarz, J. Math. Phys. **33**, 3892 (1992).
[47] M.O. Katanaev, Nucl. Phys. **B416**, 563 (1994).
[48] M.O. Katanaev, Ann. Phys. (N.Y.) **B296**, 1 (2002).
[49] T. Strobl, ''Gravity in two spacetime dimensions,'' hep-th/0011240.
[50] N. Ikeda, Int. J. Mod. Phys. A **9**, 1137 (1994).
[51] W. Kummer and P. Widerin, Phys. Rev. D **52**, 6965 (1995).
[52] D. Grumiller and W. Kummer, ''The classical solutions of the

dimensionally reduced gravitational chern-simons theory,'' hep-th/0306036.
[53] R. Casalbuoni, Phys. Lett. **62B**, 49 (1976).
[54] L. Brink and J.H. Schwarz, Phys. Lett. **100B**, 310 (1981).
[55] W. Siegel, Phys. Lett. **128B**, 397 (1983).
[56] A.V. Galajinsky and A.A. Deriglazov, Theor. Math. Phys. **106**, 84 (1996).
[57] M.E. Knutt-Wehlau and R.B. Mann, Nucl. Phys. **B514**, 355 (1998).
[58] U. Lindstrom, M. Rocek, W. Siegel, P. van Nieuwenhuizen, and A.E. van de Ven, Phys. Lett. B **224**, 285 (1989).
[59] L. Bergamin, D. Grumiller, and W. Kummer, ''Supersymmetric black holes are extremal and bald in 2D dilaton supergravity,'' hep-th/0310006.
[60] T. Strobl, Phys. Rev. D **50**, 7346 (1994).
[61] G. Guralnik, A. Iorio, R. Jackiw, and S.Y. Pi, ''Dimensionally reduced gravitational chern-simons term and its kink,'' hep-th/0305117.