Variations on the seventh route to relativity

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Wheeler asked how one might derive the Einstein-Hamilton-Jacobi equation from plausible first principles without any use of the Einstein field equations themselves. In addition to Hojman, Kuchař and Teitelboim's "seventh route to relativity" partial answer to this, there is now a "3-space" partial answer due to Barbour, Foster and Ó Murchadha (BFÓ) which principally differs in that general covariance is no longer presupposed. BFÓ's formulation of the 3-space approach is based on *best-matched* actions such as the lapse-eliminated Baierlein-Sharp-Wheeler (BSW) action of general relativity (GR). These give rise to several branches of gravitational theories including GR on superspace and a theory of gravity on conformal superspace. This paper investigates the 3-space approach further, motivated both by the hierarchies of increasingly well-defined and weakened simplicity postulates present in all routes to relativity, and by the requirement that all the known fundamental matter fields be included. We further the study of configuration spaces of gravity-matter systems upon which BFÓ's formulation leans. We note that in further developments the lapse-eliminated BSW actions used by BFÓ become impractical and require generalization. We circumvent many of these problems by the equivalent use of lapse-uneliminated actions, which furthermore permit us to interpret BFÓ's formulation within Kuchař's generally covariant hypersurface framework. This viewpoint provides alternative reasons to BFÓ's as to why the inclusion of bosonic fields in the 3-space approach gives rise to minimally coupled scalar fields, electromagnetism and Yang-Mills theory. This viewpoint also permits us to quickly exhibit further GR-matter theories admitted by the 3-space formulation. In particular, we show that the spin-$\frac{1}{2}$ fermions of the theories of Dirac, Maxwell-Dirac and Yang-Mills-Dirac, all coupled to GR, are admitted by the generalized 3-space formulation we present. Thus all the known fundamental matter fields can be accommodated. This corresponds to being able to pick actions for all these theories which have less kinematics than suggested by the generally covariant hypersurface framework. For all these theories, Wheeler's thin sandwich conjecture may be *posed*, rendering them timeless in Barbour's sense.

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I. INTRODUCTION

Einstein [1] "derived" his field equations (EFEs)[1]

$$\mathbf{G}_{\alpha\beta}=\mathbf{R}_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}\mathbf{R}=8\pi\mathbf{T}^{\text{Matter}}_{\alpha\beta} \tag{1}$$

by demanding general covariance (GC) and the Newtonian limit; the conservation of energy-momentum requires $\nabla^{\alpha}\mathbf{G}_{\alpha\beta}=0$. Along with these physical considerations, Cartan [2] proved that the derivation requires the following mathematical simplicities: that $\mathbf{G}_{\alpha\beta}$ contains at most second-order derivatives and is linear in these. The EFEs may also be obtained from the Einstein-Hilbert action [3]

$$S_{\text{EH}}=\int d^4x\sqrt{-g}(\mathbf{R}+\mathsf{L}_{\text{Matter}}); \tag{2}$$

an equivalent proof for actions was given by Weyl [3]. Lovelock [4] has shown that the linearity assumption is unnecessary in dimension $D\leqslant 4$.

Arnowitt, Deser and Misner (ADM) [5] split the spacetime metric as follows:

$$g_{\alpha\beta}=\begin{pmatrix}\xi_k\xi^k-N^2 & \xi_j\\ \xi_i & h_{ij}\end{pmatrix},\quad g^{\alpha\beta}=\begin{pmatrix}-\dfrac{1}{N^2} & \dfrac{\xi^j}{N^2}\\ \dfrac{N^i}{N^2} & h^{ij}-\dfrac{\xi^i\xi^j}{N^2}\end{pmatrix} \tag{3}$$

and rearranged the action (2) into the Hamiltonian form

$$S_{\text{ADM}}=\int \mathrm{d}t\int \mathrm{d}^3x(p^{ij}\dot{h}_{ij}-N\mathcal{H}-\xi^i\mathcal{H}_i), \tag{4}$$

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[1]In this paper, spacetime tensors have lower-case Greek indices and space tensors have lower-case Latin indices. Their barred counterparts are local Minkowski and Euclidean indices, respectively. Bold capital Latin letters denote Yang-Mills internal indices. Capital Greek letters denote general indices. The indices N, $\mathbf{N}$, ξ, χ, Φ and Ψ are reserved for other use. Round brackets surrounding more than one index of any type denote symmetrization and square brackets denote antisymmetrization; indices which are not part of this (anti)symmetrization are set between vertical lines. $g_{\alpha\beta}$ is the $(\epsilon+++)$ spacetime metric, with determinant g, where the signature $\epsilon=-s$ is -1 for (Lorentzian) general relativity (GR), 0 for strong gravity and 1 for Euclidean GR. ∇_α is the spacetime covariant derivative, D_a is the spatial covariant derivative and D^2 is the spatial Laplacian. $\mathbf{R}_{\alpha\beta}$ is the spacetime Ricci tensor, $\mathbf{R}$ is the spacetime Ricci scalar, $\mathbf{G}_{\alpha\beta}$ is the spacetime Einstein tensor and $\mathbf{T}_{\alpha\beta}$ is the energy-momentum tensor. h_{ab} is the metric on a spatial hypersurface, with determinant h. p_{ab} is its conjugate momentum, with trace p. R_{ab} is the spatial Ricci tensor and R the spatial Ricci scalar.

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$$\mathcal{H} \equiv G_{ijkl}p^{ij}p^{kl} - \sqrt{h}R = 0, \quad (5)$$

$$\mathcal{H}_i \equiv -2D_j p_i{}^j = 0, \quad (6)$$

up to a divergence term. The lapse N and shift ξ_i have no conjugate momenta. Thus the true gravitational degrees of freedom in general relativity (GR) are contained in Riem, the space of Riemannian 3-metrics on a fixed topology taken here to be closed and without boundary. But the true degrees of freedom are furthermore subjected to the Hamiltonian and momentum constraints $\mathcal{H}$ and $\mathcal{H}_i$, respectively. If one can quotient out the 3-diffeomorphisms (which are generated by ξ_i), one is left with

$$\{\text{Superspace}\} = \frac{\{\text{Riem}\}}{\{\text{3-Diffeomorphisms}\}}, \quad (7)$$

which has naturally defined on it the DeWitt supermetric $G_{ijkl} = 1/\sqrt{h}(h_{i(k|}h_{j|l)} - \frac{1}{2}h_{ij}h_{kl})$ present in the remaining constraint, $\mathcal{H}$.

Wheeler listed six routes to GR in 1973 [6]. The first is Einstein's (plus simplicity postulate upgrades). The second is Hilbert's from Eq. (2). The third and fourth are the two-way working between Eq. (2) and Eqs. (4),(5),(6); these will concern us in this paper as the arena for Wheeler's question [7]: "If one did not know the Einstein-Hamilton-Jacobi equation, how might one hope to derive it straight off from plausible first principles, without ever going through the formulation of the Einstein field equations themselves?" The fifth and sixth routes mentioned are the Fierz-Pauli spin-2 field in an unobservable flat background [8] and Sakharov's idea that gravitation is the elasticity of space that arises from particle physics [9]. One could add some more recent routes to Wheeler's list, such as from the closed string spectrum [10], and the interconnection with Yang-Mills phase space in the Ashtekar variables approach [11]. Among these routes we distinguish three types: to relativity alone, to relativity with all known fundamental matter fields "added on," and to genuinely unified theories (whether partial such as already-unified Rainich-Misner-Wheeler theory [12], Kaluza-Klein theory [13] and the Weyl gravitoelectromagnetic theory [14], or total such as string theory). Finally, some routes will lead to modifications of GR, such as higher derivative theories or Brans-Dicke (BD) theory [15] (it is debatable whether string theory reproduces GR since string theory has a BD or "dilatonic" coupling). Simplicity postulates may be seen as a means of uniquely prescribing GR but there is no reason why nature should turn out to be simple in these ways.

The original "seventh route to relativity" partial answer to Wheeler's question was given by Hojman, Kuchař and Teitelboim (HKT) [16]. As Wheeler suggested, they attached importance to an embeddability condition, which presupposes 4-dimensionally GC spacetime. However, recently Barbour, Foster and Ó Murchadha (BFÓ) [17] have provided a different partial answer *without* this presupposition. In this paper we study whether this is (or can be made) satisfactory, and how it compares to the HKT answer.

HKT required the "representation postulate": that $\mathcal{H}$ and $\mathcal{H}_i$ be such that they close in the same way as the algebra of deformations of a spatial hypersurface embedded in a $(-+++)$ Riemannian spacetime. This algebra is the Dirac Algebra,

$$\{\mathcal{H}(x),\mathcal{H}(y)\} = \mathcal{H}^i(x)\,\delta_{,i}(x,y) + \mathcal{H}^i(y)\,\delta_{,i}(x,y)$$
$$\{\mathcal{H}_i(x),\mathcal{H}(y)\} = \mathcal{H}(x)\,\delta_{,i}(x,y) \quad (8)$$
$$\{\mathcal{H}_i(x),\mathcal{H}_j(y)\} = \mathcal{H}_i(y)\,\delta_{,j}(x,y) + \mathcal{H}_j(x)\,\delta_{,i}(x,y),$$

where $\{,\}$ denote Poisson brackets. Their working is subject to the assumption that the evolution is path-independent, which means that the spacetime containing the hypersurface is foliation-invariant; this is the embeddability assumption. Their further time-reversal assumption is removed in [18]. Weakening their *Ansätze* in stages (cf. the earlier [19]), they obtain as results that $\mathcal{H}$ must be ultralocal[2] and quadratic in its momenta and at most second order in its spatial derivatives (see, however, Sec. II F).

The hope that pure geometrodynamics is by itself a total unified theory has largely been abandoned. So asking about $\mathcal{H}=0$, which corresponds to the Einstein-Hamilton-Jacobi equation [by substituting $p^{ij} = \partial S/\partial h_{ij}$, for Jacobi's principal function S, in Eq. (5)] translates to asking about ${}^{\Psi}\mathcal{H}=0$, including all the known fundamental matter fields, Ψ. We can now assess whether any first principles are truly plausible by seeing if they extend from a route to relativity alone to a route to relativity with all the known fundamental matter fields "added on." The idea of the representation postulate extends additively (at least naively) to matter contributions to $\mathcal{H}$ and $\mathcal{H}_i$. Teitelboim [20] provided a partial extension of HKT's work to include electromagnetism, Yang-Mills theory and supergravity. One must note the absence of spin-$\frac{1}{2}$ fields from this list [21].

In contrast, BFÓ require mere closure in place of closure as the Dirac Algebra. The strength of their method comes from the generalized Hamiltonian dynamics of Dirac [22], which is taken further to provide a highly restrictive scheme based on exhaustion (see [23] for an account). They consider actions constructed according to two principles: best matching and local square roots (see below).

The idea of BFÓ's 3-space approach is to seek for laws of nature that have a relational form. This is taken to mean that relative configurations alone are meaningful and that the time label is to play no role in the formulation. The former is achieved by working indirectly with the relative configuration space via best matching. The latter is emphasized by working with a manifestly reparametrization-invariant Jacobi-type square root action (see Sec. II). Furthermore it is chosen to have a local square root (see below). Then the constraints of GR arise as direct consequences of the implementation of these two principles. The 3-space approach advocates a space rather than spacetime ontology. Rather than being presupposed, 4-dimensional general covariance and the spacetime form of the laws of nature is emergent in the

[2] Ultralocal means no dependence on spatial derivatives.

3-space approach. We now carefully state the two principles for a class of actions from which GR will emerge as essentially singled out.

1: The universal method of *best matching* [17,24,25] is used to implement the *3-dimensional* diffeomorphism invariance by correcting the bare velocities of all bosonic fields B according to the rule $\dot{B}\rightarrow\dot{B}-\pounds_{\xi}B$.[3] For any two 3-metrics on 3-geometries Σ_1, Σ_2, this corresponds to keeping the coordinates of Σ_1 fixed while shuffling around those of Σ_2 until they are as "close" as possible to those of Σ_1.

2: A *local* square root (taken at each space point before integration over 3-space) is used. Thus the pure gravity actions considered are of Baierlein-Sharp-Wheeler (BSW) [26] type,

$$S_{\mathrm{BSW}}=\int d\lambda\int d^3x\sqrt{h}\sqrt{sR+\Lambda}\sqrt{T_{\mathrm{W}}}, \tag{9}$$

where Λ is a cosmological constant.

Writing this form amounts to applying a temporary simplicity postulate

3: The pure gravity action is constructed with at most second-order derivatives[4] in the potential, and with a homogeneously quadratic best-matched kinetic term

$$T_{\mathrm{W}}=\frac{1}{\sqrt{h}}G_{\mathrm{W}}^{abcd}(\dot{h}_{ab}-2D_{(a}\xi_{b)})(\dot{h}_{cd}-2D_{(c}\xi_{d)}), \tag{10}$$

where $G_{\mathrm{W}}^{ijkl}=\sqrt{h}(h^{ik}h^{jl}-Wh^{ij}h^{kl})$, $W\neq\frac{1}{3}$, is the inverse of the most general (invertible) ultralocal supermetric [28], $G^{\mathrm{X}}_{abcd}=(1/\sqrt{h})(h_{ac}h_{bd}-X/2h_{ab}h_{cd})$ for $X=2W/(3W-1)$.

Setting $2N=\sqrt{T_{\mathrm{W}}/(sR+\Lambda)}$, the gravitational momenta are

$$p^{ij}=\frac{\partial\mathsf{L}}{\partial\dot{h}_{ij}}=\frac{\sqrt{h}}{2N}(h^{ic}h^{jd}-Wh^{ij}h^{cd})(\dot{h}_{cd}-2D_{(c}\xi_{d)}). \tag{11}$$

The primary constraint

$$\mathcal{H}\equiv-\sqrt{h}(sR+\Lambda)+\frac{1}{\sqrt{h}}\left(p^{ij}p_{ij}-\frac{X}{2}p^2\right)=0 \tag{12}$$

then follows merely from the form of the Lagrangian. In addition, variation of the action with respect to ξ_i leads to a secondary constraint which is the usual momentum constraint (6).

The propagation of $\mathcal{H}$ gives [29]

$$\dot{\mathcal{H}}=\frac{s}{N}D^i(N^2\mathcal{H}_i)+\frac{(3X-2)Np}{2\sqrt{h}}\mathcal{H}+\pounds_{\xi}\mathcal{H}+\frac{2s(1-X)}{N}D_a(N^2D^ap). \tag{13}$$

We require this to vanish in order to have a consistent theory. The first 3 terms of this are said to *vanish weakly* in the sense of Dirac [22], i.e. they vanish by virtue of the constraints $\mathcal{H}$, $\mathcal{H}_i$. The last term has a chance to vanish in three ways, since it has three factors which might be zero. Constraints must be independent of N, so the third factor means that $p/\sqrt{h}=$ *constant*. We require this new constraint to propagate also, but this leads to the lapse being nontrivially fixed by a constant mean curvature (CMC) slicing equation. So, for $s\neq0$, this forces us to have the DeWitt ($W=1$) supermetric of relativity, which is BFÓ's "Relativity Without Relativity" result.

But there is also the $s=0$ possibility regardless of which supermetric is chosen [29], which is a generalization of strong gravity [30]. The HKT program would discard this since it is not a representation of the Dirac Algebra (although Teitelboim did study strong gravity [30]). However, the strong gravity theories meet the 3-space approach's immediate criteria in being dynamically consistent theories of 3-geometries. In this case the theories at most represent nature near singularities (although one can expand about them to obtain GR and Brans-Dicke theory) but it does illustrate that the 3-space approach is a fruitful constructive scheme for alternative theories.

Indeed, Barbour and Ó Murchadha (BO) found alternative conformal theories [31] which are being reformulated by Anderson, Barbour, Foster and Ó Murchadha [32] using a new "free end-point" variational principle [32,33]. *Conformal gravity* has the action

$$S_{\mathrm{C}}=\int d\lambda\frac{\int d^3x\sqrt{h}\phi^4\sqrt{s\left(R-\frac{8D^2\phi}{\phi}\right)+\frac{\Lambda\phi^4}{V(\phi)^{2/3}}}\sqrt{T_{\mathrm{C}}}}{V(\phi)^{2/3}},$$

$$\text{volume}\quad V=\int d^3x\sqrt{h}\phi^6 \tag{14}$$

$$T_{\mathrm{C}}=\frac{1}{\sqrt{h}}G^{abcd}_{(\mathrm{W}=0)}\left(\dot{h}_{ab}-\pounds_{\xi}h_{ab}+\frac{4(\dot{\phi}-\pounds_{\xi}\phi)}{\phi}h_{ab}\right)\times\left(\dot{h}_{cd}-\pounds_{\xi}h_{cd}+\frac{4(\dot{\phi}-\pounds_{\xi}\phi)}{\phi}h_{cd}\right), \tag{15}$$

which is consistent for $s=1$ because it circumvents the above argument about the third factor by independently guaranteeing a new slicing equation for the lapse. Despite its lack of GC, conformal gravity is very similar to GR in the sense that the true configuration space of GR is [35]

[3] λ is the label along curves in superspace; $\partial/\partial\lambda$ is denoted by a dot. $\pounds_{\xi}$ is the Lie derivative with respect to ξ_i.

[4] Furthermore, none of the higher-order derivative potentials considered by BFÓ turn out to be dynamically consistent (but see Sec. II F).

$$\mathrm{CS+V} \equiv \{\text{Conformal Superspace + Volume}\} = \frac{\{\text{Riem}\}}{\{\text{3-Diffeomorphisms}\}\{\text{Volume-preserving Weyl transformations}\}} \quad (16)$$

and conformal gravity arises by considering instead

$$\{\text{Conformal Superspace}\} = \frac{\{\text{Riem}\}}{\{\text{3-Diffeomorphisms}\}\{\text{Weyl transformations}\}}. \quad (17)$$

This has an infinite number of ''shape'' degrees of freedom whereas there is only one volume degree of freedom. Yet removing this single degree of freedom changes one's usual concept of cosmology, and ought to change the problems associated with the quantization of the theory (by permitting the use of a positive-definite inner product and a new interpretation for $\mathcal{H}$) [32]. Setting $s=0$ in Eq. (15) gives strong conformal gravity. One arrives at a further CS+V 3-space theory if one chooses to work on Eq. (16) instead of Eq. (17) [32,34] while retaining a fundamental slicing from the use of free-end-point variation.

To mathematically distinguish GR from these other theories, we use

4: The theory is not conformally invariant, it is obtained by conventional variation and has signature $\epsilon = -s = -1$.

The author's future strategy will involve seeking to overrule these alternative theories by thought experiments and use of current astronomical data, which would tighten the uniqueness of GR as a viable 3-space theory *on physical grounds*. If such attempts persistently fail, these theories will become established as serious alternatives to GR. So far the theories appear consistent with the GR solar system tests, and the CS+V theory will inherit the standard cosmology from GR.

BFÓ furthermore considered ''adding on'' matter to the 3-geometries,[5] subject to the simplicity postulate.

5: The matter potential has at most first-order derivatives and the kinetic term is ultralocal and homogeneous quadratic in the velocities. Apart from the homogeneity, this parallels Teitelboim's matter assumptions [20].

One then discovers in the GR case that the lightcone is universal for bosons, a single 1-form obeys Maxwell's electrodynamics, and sets of interacting 1-forms obey Yang-Mills theory [23]. All these 1-forms have turned out to be massless. Considering a 1-form and scalars simultaneously leads to $U(1)$ gauge theory [36]. The GR matter results carry over to conformal gravity [32].

We sharpen the understanding of what the 3-space approach is because we are interested in why the impressive collection of results in the GR case above arises in BFÓ's approach. We seek for tacit simplicity postulates, survey which assumptions may be weakened and assess the thoroughness and plausibility of BFÓ's principles, results and conjectures. We thus arrive at a number of variations of the 3-space approach. We stress that this is not just about improving the axiomatization. We must be able to find a version that naturally accommodates spin-$\frac{1}{2}$ fermions coupled (1) to GR if the 3-space approach is to provide a set of plausible first principles for GR and (2) to conformal gravity if this is to be a viable alternative. Barbour's work [25,41] has been critically discussed by Butterfield [37] and by Smolin [38] largely from a philosophical point of view. In contrast, this paper discusses (and extends) BFÓ's continuation of this work from a more technical point of view.

In Sec. II, we argue that the BSW principle **2** is problematic. First, Barbour's use of it draws inspiration from the Jacobi formulation of mechanics, but in Sec. II A we point out that the Jacobi formulation itself has limitations and a significant generalization. Furthermore in Secs. II B–D we point out that the differences between the BSW and Jacobi actions are important. Overall, this gives us the ''conformal'' problem in Sec. II C, and the ''notion of distance'' problem in Sec. II D. Second, should the notion of ''BSW-type theories'' not include all the theories that permit the BSW elimination process itself? But when we perform this including fermions in Sec. II E, we find that we obtain not the BSW form but rather its generalization. Thus the inclusion of fermions will severely complicate the use of exhaustive proofs such as those in [17,23]. We furthermore point out that the usual higher derivative theories are not being excluded by BFÓ in Sec. II F. These last two sections include discussion of their HKT counterparts.

In Sec. III, we formalize the second point above by showing that we could just as well use lapse-uneliminated actions for GR and conformal gravity. For GR, these actions may be studied within Kuchař's GC hypersurface framework [27]. This framework brings attention to *tilt* and *derivative coupling* complications in general (Sec. IV A), which are, however, absent for the minimally coupled scalar, and ''accidentally absent'' for the Maxwell and Yang-Mills 1-forms, which are what the 3-space approach picks out. But tilt is present for the massive (Proca) analogues of these 1-forms. We deduce the relation between tilt and the existence of a generalized BSW form. In Sec. IV B we counter BFÓ's hope that *just* the known fundamental matter fields are being picked out by the 3-space approach, by showing that the massless 2-form is also compatible. In Sec. IV C, we find alternative reasons why the Maxwell 1-form is singled out by the 3-space approach, from the point of view of the hypersurface framework. We end by explaining out the complica-

[5]We contest BFÓ's speculation that the matter results might lead to unification in Sec. IV.

tions that would follow were one to permit derivative-coupled 1-forms.

In Sec. V A, we point out that it is consistent to take the bosonic sector of nature to be far simpler than GC might have us believe: best matching suffices for its construction. An alternative scheme to **1** using ''bare'' rather than best-matched velocities to start off with is discussed, in which $\mathcal{H}$ gives rise to all the other constraints as integrability conditions. In Sec. V B, we show how all these results also hold true upon inclusion of spin-$\frac{1}{2}$ fermions. Section V C lists further research topics for fermions in the light of the advances made in this paper.

II. PROBLEMS WITH THE USE OF BSW ACTIONS

A. Insights from mechanics

Suppose the Lagrangian[6]

$$\mathsf{L}(q_{\hat{\Delta}},\dot{q}_{\hat{\Delta}})=\frac{1}{2}M^{\hat{\Delta}\hat{\Gamma}}(q_{\hat{\Pi}})\dot{q}_{\hat{\Delta}}\dot{q}_{\hat{\Gamma}}-V(q_{\hat{\Pi}}) \tag{18}$$

does not depend on q_n. Then q_n is a *cyclic* variable and its Euler-Lagrange equation yields $p^n \equiv \partial\mathsf{L}/\partial\dot{q}_n = c^n$, a constant. Then the Lagrangian may be modified to $\overline{\mathsf{L}}(q_\Delta,\dot{q}_\Delta)\equiv\mathsf{L}-c^n\dot{q}_n$ using the equation for p^n to eliminate $\dot{q}_n$; this is known as *Routhian reduction*.

Next, observe that q_n may be taken to be the time t in a conservative mechanical system; we regard the q_Δ and t as functions of the parameter τ. Then the action takes the parametrized form

$$S=\int_{\tau_1}^{\tau_2}\mathsf{L}\left(q_\Delta,\frac{q'_\Delta}{t'}\right)t'\,\mathrm{d}\tau, \tag{19}$$

and the equation for p^t may be used to eliminate t' from this by Routhian reduction. One thus obtains the Jacobi action

$$S_{\mathrm{J}}=\int_{\tau_1}^{\tau_2}\sqrt{2(E-V)}\,\mathrm{d}\sigma, \tag{20}$$

where $E\equiv c^t$ is the total energy and $\mathrm{d}\sigma^2$ is the line element associated with the Riemannian metric $M_{\Gamma\Delta}$ of the configuration space Q of the configuration variables q_Δ. Minimization of this integral is *Jacobi's principle* [39]. There is then a conformally related line element

$$\mathrm{d}\tilde{\sigma}^2=(E-V)\mathrm{d}\sigma^2 \tag{21}$$

with respect to which the motions of the system are geodesics. The point of this method is the reduction of mechanics problems to the study of well-known geometry.

[6] Newtonian time is denoted by t while τ is a parameter. A dot is used for $\partial/\partial t$ in mechanics workings and a dash for $\partial/\partial\tau$. $\hat{\Delta}$ takes 1 to n and Δ takes 1 to $(n-1)$; n is not to be summed over. $q_{\hat{\Delta}}$ are configuration variables with conjugate momenta $p^{\hat{\Delta}}$.

However, the Jacobi principle in mechanics has a catch: the conformal factor is not allowed to have zeros. If it does then the conformal transformation is only valid in regions where there are no such zeros. These zeros are physical barriers in mechanics. For they correspond to zero kinetic energy by the conservation of energy equation. As the configuration space metric is positive-definite, this means that the velocities must be zero there, so the zeros cannot be traversed.

The Lagrangian (18) is restricted to have a kinetic term homogeneously quadratic in the velocities. Let $\mathsf{L}(q_{\hat{\Delta}},\dot{q}_{\hat{\Delta}})$ be instead a completely general function. Then

$$S=\int_{\tau_1}^{\tau_2}\mathsf{L}\left(q_\Delta,\frac{q'_\Delta}{t'}\right)t'\,\mathrm{d}\tau\equiv\int_{\tau_1}^{\tau_2}\mathcal{L}(q_{\hat{\Delta}},q'_{\hat{\Delta}})d\tau \tag{22}$$

may be modified to

$$S_{\mathrm{J}}=\int_{\tau_1}^{\tau_2}\overline{\mathcal{L}}(q_\Delta,q'_\Delta)\mathrm{d}\tau \tag{23}$$

by Routhian reduction, where $\overline{\mathcal{L}}=F$, some homogeneous linear function of the q'_Δ [39]. For example, F could be a *Finslerian metric function* from which we could obtain a Finslerian metric $f_{\Gamma\Delta}=1/2(\partial^2/\partial q'_\Gamma\partial q'_\Delta)F^2$, provided that F obeys further conditions [40] including the nondegeneracy of $f_{\Gamma\Delta}$. So in general the ''geometrization problem'' of reducing the motion of a mechanical system to a problem of finding geodesics involves more than the study of Riemannian geometry.

To some extent, there is conventional freedom in the choice of configuration space geometry, since we notice that standard maneuvers can alter whether it is Riemannian. This is because one is free in how many redundant configuration variables to include, and in the character of those variables (for example whether they all obey second-order Euler-Lagrange equations).

As a first example, consider the outcome of the Routhian reduction of Eq. (18) more carefully:

$$\overline{\mathsf{L}}(q_\Delta,\dot{q}_\Delta)=\frac{1}{2}\left(M^{\Gamma\Delta}-\frac{M^{\Delta n}M^{\Gamma n}}{M^{nn}}\right)\dot{q}_\Delta\dot{q}_\Gamma+\frac{c^nM^{\Delta n}}{M^{nn}}\dot{q}_\Delta-\overline{V}, \tag{24}$$

where $\overline{V}$ is a modified potential. So Routhian reduction can lead to non-Riemannian geometry, on account of the penultimate ''gyroscopic term'' [39], which is linear in the velocities. We consider the reverse of this procedure as a possible means of arriving at Riemannian geometry to describe systems with linear and quadratic terms. We observe that if the linear coefficients depend on configuration variables, then in general the quadratic structure becomes contaminated with these variables.

As a second example, higher-than-quadratic systems may be put into quadratic form by *Ostrogradsky reduction* [42], at the price of introducing extra configuration variables.

We finally note the ordering of the summation and the square root in

$$\mathrm{d}\sigma = \sqrt{\sum_{\Delta,\Gamma=1}^{n-1} \tilde{M}^{\Gamma\Delta}\dot{q}_\Gamma \dot{q}_\Delta}, \tag{25}$$

which we refer to as the "good" or "global square root" ordering.

B. The BSW formulation of GR

GR is an already-parametrized theory. This is because the ADM action (4) (generalized to arbitrary s and Λ at no extra cost) may be rewritten in the Lagrangian form

$$S = \int \mathrm{d}\lambda \int \mathrm{d}^3x \sqrt{h} N \mathsf{L}(h_{ab}, \dot{h}_{ab}; \xi_i; N) = \int \mathrm{d}\lambda \int \mathrm{d}^3x \sqrt{h} N \left(\Lambda + sR + \frac{T_{\mathrm{g}}(\kappa_{ij})}{4N^2} \right), \tag{26}$$

[cf. Eq. (19)] where

$$T_{\mathrm{g}} = \kappa_{ij}\kappa^{ij} - \kappa^2, \quad \kappa_{ij} = \dot{h}_{ij} - 2D_{(i}\xi_{j)}. \tag{27}$$

Then (specifically following BSW [26] or in analogy with Jacobi) extremization with respect to N gives $N = \pm\sqrt{T_{\mathrm{g}}/(\Lambda + sR)}$, which may be used to *algebraically* eliminate N from Eq. (26). Thus one arrives at the BSW action

$$S_{\mathrm{BSW}} = \int \mathrm{d}\lambda \int \mathrm{d}^3x \sqrt{h}\sqrt{(\Lambda + sR)T_{\mathrm{g}}}. \tag{28}$$

Although this looks similar to the Jacobi action in mechanics, there are important differences. First, the GR configuration space is infinite-dimensional; with redundancies, one can consider it to be superspace. The DeWitt supermetric is defined on superspace *pointwise*. By use of a 2-index to 1-index map $G_{abcd} \rightarrow G_{AB}$, DeWitt represented his supermetric as a 6×6 matrix, which is $(-+++++)$ and thus indefinite [28]. As a special case, minisuperspace [43] is the truncation of superspace obtained by considering homogeneous metrics alone. "Minisupermetrics" are $(-++)$, thus they too are indefinite. Second, the BSW action has the "bad" or "local square root" ordering. Below, we first consider minisuperspace, for which this extra complication does not arise, since by homogeneity the "good" Jacobi and "bad" BSW orderings are equivalent.

Finally, BSW's work led to the thin sandwich conjecture [45,46], the solubility of which features as a caveat in BFÓ's original paper. Being able to pose this conjecture for a theory amounts to being able to algebraically eliminate the lapse N from its Lagrangian. This implies that the theory is timeless in Barbour's sense [25,41]. The extension of the conjecture to include fundamental matter fields has only recently begun [46]. This and other investigations are required to assess the robustness of the conjecture to different theoretical settings, to see if in any circumstances it becomes advantageous to base numerical relativity calculations on the algorithm which the conjecture provides.

C. Lack of validity of the BSW form

In perfect analogy with mechanics (21), there is a conformally related line element, $\mathrm{d}\tilde{\sigma}^2 = (\Lambda + sR)\mathrm{d}\sigma^2$ in vacuo, for which the motion associated with Eq. (28) is geodesic [44]. But the observation in mechanics that such conformal transformations are only valid in regions where the conformal factor is nonzero[7] still holds for GR. It is true that the details are different, due to the indefiniteness of the GR supermetric. This causes the zeros to be spurious rather than physical barriers [43]. For while a zero z of the potential corresponds to a zero of the kinetic term by virtue of the Hamiltonian constraint, this now means that the velocity need be null, not necessarily zero, because of the indefiniteness. Thus the motion may continue through z "on the superspace lightcone," which is made up of perfectly reasonable Kasner universes, rather than grind to a halt. Nevertheless, the conformal transformation used to obtain geodesic motion is not valid, so it is questionable whether the BSW form is a "geodesic principle," if in general it describes conformally untransformed *non-geodesic curves* for practical purposes.

To illustrate that the presence of zeros in the potential term is an important occurrence in GR, we note that the Bianchi IX solution has an infinity of such zeros as one approaches the cosmological singularity. This is important because it is conjectured by Belinskii, Khalatnikov and Lifshitz (BKL) [47] that the behavior of Bianchi IX near the cosmological singularity is the generic behavior of a cosmological solution to GR. This sort of conjecture is acquiring numerical support [48]. The above argument was originally put forward by Burd and Tavakol [49] to argue against the validity of the use of the "Jacobi principle" to characterize chaos in GR [50]. Our point is that this argument holds against *any* use, BFO's included, of the BSW form in minisuperspace models of the early universe in GR.

The way out of this argument that we suggest is to abstain from the self-infliction of spurious zeros by not performing the conformal transformation in the first place, thus abandoning the interpretation of the BSW form as a geodesic principle in GR. Conformal gravity, however, is distinct from GR and has no cosmological singularity, so arguments based on the BKL conjecture are not applicable there. Conformal gravity's zeros are real as in mechanics, because T^{C} is positive-definite, and Barbour and Ó Murchadha use this to argue that topologies with $R<0$ at any point are not allowed [31].

D. The BSW form is an unknown notion of distance

BFÓ called the local square root ordering "bad" because it gives one constraint per space point, which would usually render a theory trivial by overconstraining due to the ensuing cascade of secondary constraints. Yet GR contrives to survive this because of its hidden foliation invariance [17]. However Giulini [46] has pointed out another reason why the

[7]In GR, these are regions for which $\Lambda + R < 0$ or for which $\Lambda + R > 0$. We also note that the sign of $\Lambda + R$ plays an important role in the thin sandwich conjecture.

local square root ordering is bad: it does not give rise to known geometry. Below, we extend his finite-dimensional counterexample to the geometry being Finslerian.

The BSW form as a notion of distance provides as the ''full metric'' on superspace

$$\frac{1}{2}\frac{\partial^2}{\partial v^A(u)\partial v^B(w)}S^2_{\rm BSW} = \left[\tilde{G}_{AC}(u)\tilde{G}_{BD}(w)+2\delta^{(3)}(u,w)\times\left(\frac{S_{\rm BSW}}{\sqrt{\tilde{G}_{BD}v^Bv^D}}\tilde{G}_{A[B}\tilde{G}_{C]D}\right)(u)\right]\hat{v}^A\hat{v}^C, \quad (29)$$

where $v^A\equiv\dot{h}^A=\dot{h}^{ab}$ by DeWitt's 2-index to 1-index map and where hats denote unit ''vectors.'' So in general, $\tilde{G}_{A[B}\tilde{G}_{C]D}=0$ is a sufficient condition for the full metric to be degenerate and hence not Finsler (Giulini's example had a 1-dimensional v^A so this always occurred). But if $\tilde{G}_{A[B}\tilde{G}_{C]D}\neq 0$, the full metric is not a function (both in the distributional and functional senses). So using the BSW form as a notion of distance leads to unknown geometry, so there is no scope for the practical application of the BSW form as a geodesic principle.

This is to be contrasted with the global square root, for which the above procedure gives instead (semi-)Riemannian geometry. For minisuperspace, the local square root working presented does indeed collapse to coincide with this global square root working, and the resulting (semi)Riemannian geometry is of considerable use in minisuperspace quantum cosmology [43].

There is also the issue DeWitt raised [44] that in the study of superspace one is in fact considering not single geodesics, but *sheaves* of them. This corresponds to all the different foliations of spacetime in GR, which leads to the problem of time in quantum gravity [51]. Thus there are two difficulties with applying BFO's formulation of GR. The first will still plague conformal gravity whereas the second is absent because there is a preferred lapse rather than foliation invariance.

E. The Fermionic contribution to the action is linear

Since the kinetic terms of the bosons of nature are also quadratic in their velocities, we can use the modifications

$$T_{\rm g}\rightarrow T_{\rm g}+T_{\rm B}, \quad \Lambda+sR\rightarrow\Lambda+sR+U_{\rm B} \quad (30)$$

to accommodate bosonic fields B_Δ in a BSW-type action,

$$S_{\rm B}=\int d\lambda\int d^3x\sqrt{h}\sqrt{\Lambda+sR+U_{\rm B}}\sqrt{T_{\rm g}+T_{\rm B}}. \quad (31)$$

This local square root encodes the correct Hamiltonian constraint for the gravity–boson system. Although the pointwise Riemannian kinetic metric is larger than the DeWitt supermetric, in the case of minimally coupled matter it contains the DeWitt supermetric as an isolated block:

$$\begin{pmatrix} G_{AB}(h_{ab}) & 0 \\ 0 & H^{\rm Matter}_{\Lambda\Sigma}(h_{ab}) \end{pmatrix}. \quad (32)$$

If this is the case, it makes sense to study the pure gravity part by itself, which is a prominent feature of almost all the examples studied in the 3-space approach. We identify this as a tacit simplicity requirement, for without it the matter degrees of freedom interfere with the gravitational ones, so it makes no sense then to study gravity first and then ''add on'' matter. In Brans-Dicke (BD) theory, this is not immediately the case: this is an example in which there are gravity-boson kinetic cross-terms $C_{A\chi}$ in the pointwise Riemannian kinetic metric:

$$\begin{pmatrix} G^{\rm X}_{AB}(h_{ab}) & C_{A\chi}(h_{ab},\chi) \\ C_{\chi B}(h_{ab},\chi) & H^{\rm Matter}_{\chi\chi}(h_{ab},\chi) \end{pmatrix} \quad (33)$$

where χ is the BD field and X is related to the usual BD parameter ω by $X/2=(\omega+1)/(2\omega+3)$. Thus, the metric and dilatonic fields form *together* a theory of gravity with 3 degrees of freedom. However, this is a mild example of nonminimal coupling because redefinition of the metric and scalar degrees of freedom permits blockwise isolation of the form (32). More disturbing examples are considered below and in Sec. IV C.

We now begin to consider whether and how the 3-space formulation can accommodate spin-$\frac{1}{2}$ fermionic fields, F_Δ. Following the strategy employed above for bosons, the BSW working becomes

$$S_{\rm F}=\int d\lambda\int d^3x\sqrt{h}NL(h_{ab},\dot{h}_{ab};\xi_i;N;F_\Delta,\dot{F}_\Delta) =\int d\lambda\int d^3x\sqrt{h}\left[N\left(\Lambda+sR+U_{\rm F}+\frac{T_{\rm g}(\kappa_{ij})}{4N^2}\right)+T_{\rm F}(\dot{F}_\Delta)\right] \quad (34)$$

because $T_{\rm F}$ is linear in $\dot{F}_\Delta$.[8] Then the usual trick for eliminating N does not touch $T_{\rm F}$, which is left outside the square root:

$$S_{\rm F}=\int d\lambda\int d^3x\sqrt{h}(\sqrt{\Lambda+sR+U_{\rm F}}\sqrt{T_{\rm g}}+T_{\rm F}). \quad (35)$$

The local square root constraint encodes the correct gravity-fermion Hamiltonian constraint

$$^{\rm F}\mathcal{H}\equiv-\sqrt{h}(\Lambda+sR+U_{\rm F})+\frac{1}{\sqrt{h}}\left(p_{ij}p^{ij}-\frac{X}{2}p^2\right)=0. \quad (36)$$

[8] We see in Sec. IV that the algebraic dependence on N emergent from such decompositions requires rigorous justification. We provide this for Eq. (34) in Sec. V B.

We postpone the issue of best matching (which is intertwined with gravity-fermion momentum constraint) until Sec. V B. Our concern in this section is the complication of the configuration space geometry due to the inclusion of fermions.

For now the elimination procedure is analogous not to the Jacobi working but rather to its generalization (23). So even the pointwise geometry of the gravity-fermion configuration space is now compromised: $\sqrt{\Lambda + sR + U_{\rm F}}\sqrt{T_{\rm g}} + T_{\rm F}$ could sometimes be a Finslerian metric function. By allowing Eq. (34), we are opening the door to all sorts of complicated possible actions, such as:

(1) $\sqrt[k]{G^{\Sigma_1 \dots \Sigma_k}\dot{q}_{\Sigma_1} \dots \dot{q}_{\Sigma_k}}$.

(2) Arbitrarily complicated compositions of such roots, powers and sums.

(3) More generally, $K_\Delta \dot{q}^\Delta$, where K_Δ is allowed to be an arbitrary function of not only the q_Δ but also of the $\Delta - 1$ independent ratios of the velocities.

(4) The above examples could all be Finslerian or fail to be so by being degenerate. They could also fail to be Finslerian if the K_Δ are permitted to be *functionals* of overall degree 0 in the velocities, which we can take to be a growth of the local-global square root ambiguity.

We would therefore need to modify the BSW principle **2** to a general BSW principle **2G** that includes spin-$\frac{1}{2}$ fermions. This amounts to dropping the requirement of the matter field kinetic term being homogeneously quadratic in its velocities, thus bringing **5** into alignment with Teitelboim's assumptions. We note that with increasing generality the possibility of uniqueness proofs becomes more remote. Although some aims of the 3-space approach such as a full derivation of the universal light-cone would require some level of uniqueness proofs for spin-$\frac{1}{2}$ fermions, the author's strategy is to show in this paper that spin-$\frac{1}{2}$ fermions coupled to GR do possess a 3-space formulation and also to point out that the uniqueness results may have to be generalized in view of the generalization of the BSW form required in this section.

Could we not choose to geometrize the gravity-fermion system as a Riemannian geometry instead, by use of the reverse of Routhian reduction? But the coefficients of the linear fermionic velocities in the Einstein-Dirac system contain fermionic variables, so the resulting Riemannian geometry's coefficients would contain the fermionic variables in addition to the metric. We call such an occurrence a *breach of the DeWitt structure*, since it means that contact is lost with DeWitt's study of the configuration space of pure GR [28,44]. So this choice also looks highly undesirable.

For 40 years the natural accommodation of spin-$\frac{1}{2}$ fermions in geometrodynamics [21] has been a source of problems. So this is a big demand on the 3-space approach, and one which must be met if the 3-space approach is truly to describe nature. Our demands here are less than Wheeler's in [21]: we are after a route to relativity with all matter "added on" rather than a complete unified theory. The HKT route appears also to be incomplete at this stage: Teitelboim was unable to find a hypersurface deformation explanation for spin-$\frac{1}{2}$ fermions [20]. Thus when we began this work, all forms of the seventh route to relativity were incomplete with respect to the inclusion of spin-$\frac{1}{2}$ fermions. In Sec. V B, we will point out the natural existence of GR–spin-$\frac{1}{2}$ theory within the 3-space approach.

F. Higher derivative theories

We now argue against the significance of the preclusion of higher derivative theories by BFÓ. The precluded theories are easily seen *not* to be the usual higher derivative theories. There are two simple ways of noticing this. First, the primary constraints encoded by the BFÓ theories with arbitrary $P(h_{ij}, h_{ij,k}, \dots)$ will always be of the form

$$\sqrt{h}\mathcal{H} = -\sqrt{h}P + \frac{1}{\sqrt{h}}\left(p_{ij}p^{ij} - \frac{X}{2}p^2\right) = 0, \qquad (37)$$

which is not what one gets for the usual higher derivative theories. Second, BFÓ's theories have fourth-order terms in their potentials but their kinetic terms remain quadratic in the velocities, while the usual higher derivative theories' kinetic terms are quartic in the velocities. We argue that the mismatch of derivatives between T and P for $P \neq sR + \Lambda$ overrules the theories from within the GC framework, so BFÓ are doing nothing more than GC can do in this case.

It is not clear whether the usual higher derivative theories could be written in some generalized BSW form. The form would either be considerably more complicated than that of pure GR or not exist at all. Which of these is actually true should be checked case by case. We consider this to be a worthy problem in its own right by the final comment in Sec. II B, since this problem may be phrased as "for which higher derivative theories can the thin sandwich formulation be posed?" To illustrate why there is the possibility of nonexistence, consider the simplest example, $\mathbf{R} + \alpha\mathbf{R}^2$ theory. The full doubly contracted Gauss equation is

$$\mathbf{R} = R - s(K_{ab}K^{ab} - K^2) + 2sD_a(n^b D_b n^a - n^a D_b n^b) \qquad (38)$$

and, whereas one may discard the divergence term in the 3+1 split of $\mathbf{R}$, in the 3+1 split of $\mathbf{R}^2$, this divergence is multiplied by $\mathbf{R}$ and so cannot similarly be discarded. So it is unlikely that the elimination of N will be *algebraic* in such theories, which is a requirement for the BSW procedure.[9]

[9] On the other hand, if N occurs only linearly in the action then the variational equation for N contains no N and so cannot be used to eliminate N. If N occurs homogeneously in the action, then the variational equation for N contains N only as an overall factor and so cannot be used to eliminate N either. Also, it *is* permissible for derivatives of N to be present, so long as these terms belong to a total divergence which may then be discarded to leave an action depending only algebraically on N. This might conceivably happen for some cases of higher derivative theories. Finally note that the form in which N appears in the action may change under changes of variables.

Were this algebraic elimination possible, we would get more complicated expressions than the local square root form from it. Indeed, higher derivative theories are known to have considerably more complicated canonical formulations than GR [52]; it is standard to treat them by a variant of Ostrogradsky reduction adapted to constrained systems [52].

It is worth commenting that HKT's derivation of $\mathcal{H}$ being quadratic in its momenta and containing at most second derivatives may also be interpreted as tainted, since it comes about by restricting the gravity to have two degrees of freedom, as opposed to e.g. the three of $\mathbf{R}+\alpha\mathbf{R}^2$ theory or of Brans-Dicke theory. Thus we do not foresee that any variant of the seventh route to relativity will be able to find a way around the second-order derivative assumption of the other routes.

III. LAPSE-UNELIMINATED VARIATIONS ON THE 3-SPACE APPROACH

We have seen that the interpretation of the BSW form as a geodesic principle is subject to considerable complications, and that it may obscure which theories are permitted or forbidden in the 3-space approach. We will now show that the use of the BSW form, and consequently the problems with its interpretation, may be circumvented by the use of lapse-uneliminated actions because the content of GR is not affected by lapse elimination (just as the Jacobi and Euler-Lagrange interpretations of mechanics are equivalent). It is easy to show that the equations of motion that follow from the N-uneliminated 3+1 "ADM" Lagrangian (26) are weakly equivalent to the BSW ones:

$$\begin{aligned}\left(\frac{\partial p^{ij}}{\partial\lambda}\right)_{\mathrm{ADM}} &= \sqrt{h}N\left(h^{ij}\frac{sR+\Lambda}{2}-sR^{ij}\right)-\frac{2N}{\sqrt{h}}\left(p^{im}p_m{}^j-\frac{X}{2}p^{ij}p\right)+\frac{N}{2\sqrt{h}}h^{ij}\left(p_{ab}p^{ab}-\frac{X}{2}p^2\right)+s\sqrt{h}(D^iD^jN-h^{ij}D^2N)+\pounds_\xi p^{ij}\\ &=\sqrt{h}N[h^{ij}(sR+\Lambda)-sR^{ij}]-\frac{2N}{\sqrt{h}}\left(p^{im}p_m{}^j-\frac{X}{2}p^{ij}p\right)+s\sqrt{h}(D^iD^jN-h^{ij}D^2N)+\pounds_\xi p^{ij}\\ &\quad-\frac{N}{2}h^{ij}\left[\sqrt{h}(sR+\Lambda)-\frac{1}{\sqrt{h}}\left(p_{ab}p^{ab}-\frac{X}{2}p^2\right)\right]\\ &=\left(\frac{\partial p^{ij}}{\partial\lambda}\right)_{\mathrm{BSW}}+\frac{N}{2}h^{ij}\mathcal{H},\end{aligned}\tag{39}$$

and similarly when matter terms are included. We use arbitrary s and W above to simultaneously treat the GR and strong gravity cases. The ADM propagation of the Hamiltonian constraint is slightly simpler than the BSW one,

$$\dot{\mathcal{H}}=\frac{s}{N}D^i(N^2\mathcal{H}_i)+\pounds_\xi\mathcal{H}\tag{40}$$

for $W=1$ or $s=0$, where it is understood that the evolution is carried out by the ADM Euler-Lagrange equations or their strong gravity analogues.

We now check that using uneliminated actions does not damage the conformal branch of the 3-space approach. The conformal gravity action (15) is equivalent to

$$S=\int \mathrm{d}\lambda\,\frac{\int \mathrm{d}^3x\sqrt{h}N\phi^4\left[s\left(R-\frac{8D^2\phi}{\phi}\right)+\frac{\Lambda\phi^4}{V^{2/3}(\phi)}+\frac{T_{\mathrm{C}}}{4N^2}\right]}{V(\phi)^{2/3}}\tag{41}$$

where the lapse is $N=\frac{1}{2}\sqrt{T_{\mathrm{C}}/[s(R-8D^2\phi/\phi)+\Lambda\phi^4/V^{2/3}]}$. The following equivalent of Eq. (39) holds:

$$\left(\frac{\partial p^{ij}}{\partial\lambda}\right)_{N\text{-uneliminated}}=\left(\frac{\partial p^{ij}}{\partial\lambda}\right)_{N\text{-eliminated}}+h^{ij}\left(\frac{N\mathcal{H}^{\mathrm{C}}}{2}-\frac{\sqrt{h}\phi^6}{3V}\int \mathrm{d}^3xN\mathcal{H}^{\mathrm{C}}\right)\tag{42}$$

for

$$\mathcal{H}^{\mathrm{C}}\equiv-\frac{\sqrt{h}\phi^4}{V^{2/3}}\left[s\left(R-\frac{8D^2\phi}{\phi}\right)+\frac{\Lambda\phi^4}{V^{2/3}}\right]+\frac{V^{2/3}}{\sqrt{h}\phi^4}p^{ab}p_{ab}\tag{43}$$

the conformal gravity equivalent of the Hamiltonian constraint.

We now develop a strategy involving the study of lapse-uneliminated actions. This represents a first step in disentangling Barbour's no time [25,41] and no scale [32,33] ideas. It also permits us to investigate which standard theories exist according to the other 3-space approach rules, by inspection of formalisms of these theories. We could then choose to algebraically eliminate the lapse where possible to show which of these theories can be formulated in the original BFO 3-space approach. We emphasize that existence is by no means guaranteed: some perfectly good GC formulations of theories are not best-matched, or do not permit a BSW reformulation because they cannot be made to depend algebraically on the lapse. Thus the uneliminated form can be used to help test whether the 3-space approach is or can be made to be a satisfactory scheme for all of nature.

We can furthermore use this lapse-uneliminated formulation to interpret the GR branch of the 3-space approach within Kuchař's hypersurface framework, which has striking interpretational consequences, to which we now turn.

IV. THE 3-SPACE APPROACH AND THE HYPERSURFACE FRAMEWORK

A. Nonderivatively coupled 1-forms

In his series of four papers, Kuchař [27] considers (I) the deformation of a hypersurface, (II) the kinematics of tensor fields on the hypersurface, (III) the dynamics of the fields on the hypersurface, and (IV) geometrodynamics of the fields.[10] The fields are decomposed into perpendicular and tangential parts. We are mainly concerned with 1-forms in this section, for which the decomposition is[11] $A_\alpha = n_\alpha A_\perp + e^a_\alpha A_a$; we also require the decomposition of the metric, $g_{\alpha\beta} = g_{ab} e^a_\alpha e^b_\beta - n_\alpha n_\beta$. A deformation at a point x of a hypersurface Σ may be decomposed into two parts: the *tilt*, for which $N(x)=0$, $[\partial_a N](x) \neq 0$ and the *translation*, for which $N(x)\neq 0$, $[\partial_a N](x)=0$. We follow Kuchař's use of first-order actions. For the 1-form, this amounts to rewriting the second-order action $S_{\mathrm{A}} = \int d^4x \sqrt{-g}\,\mathsf{L}(A_\alpha, \nabla_\beta A_\alpha, g_{\alpha\beta})$ by setting $\lambda^{\alpha\beta} = \partial L/\partial(\nabla_\beta A_\alpha)$ and using the Legendre transformation $(A_\alpha, \nabla_\beta A_\alpha, \mathsf{L}) \rightarrow (A_\alpha, \lambda_{\alpha\beta}, L)$, where the "Lagrangian potential" is $L = [\lambda^{\alpha\beta}\nabla_\beta A_\alpha - \mathsf{L}](A_\alpha, \lambda_{\alpha\beta}, g_{\alpha\beta})$. Then the "hypersurface Lagrangian" is

$$\delta_N S_{\mathrm{A}} = \int_\Sigma d^3x (\pi^\perp \delta_{\mathbf{N}} A_\perp + \pi^a \delta_{\mathbf{N}} A_a - N\,{}_{\mathrm{A}}\mathcal{H}^{\mathrm{o}} - N^a\,{}_{\mathrm{A}}\mathcal{H}^{\mathrm{o}}_a) \tag{44}$$

where δ_N is the normal change in the projection, the A-contribution to the momentum constraint ${}_{\mathrm{A}}\mathcal{H}^{\mathrm{o}}_a$ is obtained from $\delta_N = \delta_{\mathbf{N}} - \pounds_\xi$ (see Fig. 1) integrating by parts where necessary, and the A-contribution to the Hamiltonian constraint on a fixed background ${}_{\mathrm{A}}\mathcal{H}^{\mathrm{o}}$ may be further decomposed into its translation and tilt parts,

$${}_{\mathrm{A}}\mathcal{H}^{\mathrm{o}} = {}_{\mathrm{A}}\mathcal{H}^{\mathrm{o}}_{\mathrm{t}} + {}_{\mathrm{A}}\mathcal{H}^{\mathrm{o}}_{\not t}. \tag{45}$$

[10]References to these complicated papers are pinned down by these Roman numerals followed by the relevant section numbers. We restrict attention to $s=1$ in this section.

[11]We use e^α_a for the projector onto the hypersurface and n_α for the perpendicular vector to the hypersurface, and K_{ab} for the extrinsic curvature. The index perpendicular to the hypersurface is denoted by the subscript $\perp$, the subscript $\not t$ denotes the tilt part and the subscript t denotes the translational part.

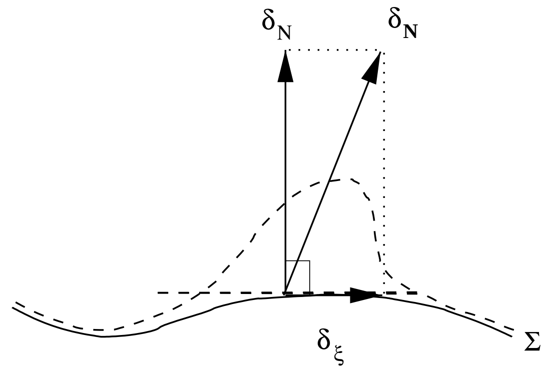

FIG. 1. The change along an arbitrary deformation of the hypersurface Σ is split according to $\delta_{\mathbf{N}} = \delta_N + \delta_\xi$. Kuchař showed that $\pounds_\xi = \delta_\xi$ when acting on spatial tensors (see Kuchař I.7 and III.5).

The translational part ${}_{\mathrm{A}}\mathcal{H}^{\mathrm{o}}_{\mathrm{t}}$ may contain a term $2\,{}_{\mathrm{A}}P^{ab}K_{ab}$ due to the possibility of *derivative coupling* of the metric to the 1-form, while the remainder of ${}_{\mathrm{A}}\mathcal{H}^{\mathrm{o}}_{\mathrm{t}}$ is denoted by ${}_{\mathrm{A}}\mathcal{H}_{\mathrm{t}}$:

$${}_{\mathrm{A}}\mathcal{H}^{\mathrm{o}}_{\mathrm{t}} = {}_{\mathrm{A}}\mathcal{H}_{\mathrm{t}} + 2\,{}_{\mathrm{A}}P^{ab}K_{ab}. \tag{46}$$

For the 1-form field, using the decomposition $\lambda^{\alpha\beta} = \lambda^{\perp\perp} n^\alpha n^\beta + \lambda^{a\perp} e^\alpha_a n^\beta + \lambda^{\perp b} n^\alpha e^\beta_b + \lambda^{ab} e^\alpha_a e^\beta_b$ and $\lambda^{a\perp} = \pi^a$, $\lambda^{\perp\perp} = \pi^\perp$ (by the definition of canonical momentum), one obtains

$${}_{\mathrm{A}}\mathcal{H}_{\mathrm{t}} = L + \sqrt{h}(\lambda^{\perp a} D_a A_\perp - \lambda^{ab} D_a A_b). \tag{47}$$

We also require

$${}_{\mathrm{A}}P^{ab} = \frac{\sqrt{h}}{2}(-A^{(a|}\lambda^{\perp|b)} + A_\perp \lambda^{(ab)} - A^{(a}\pi^{b)}). \tag{48}$$

For the 1-form, $\lambda^{\perp a}$ and λ^{ab} play the role of Lagrange multipliers; one would then use the corresponding multiplier equations to attempt to eliminate the multipliers from Eq. (44). In our examples below, $A_\perp$ will also occur as a multiplier, but this is generally not the case.

The above sort of decomposition holds for any rank of tensor field. $\mathcal{H}^{\mathrm{o}}_{\not t}$, P^{ab} and $\pounds_\xi$ are universal for each rank, whereas $\mathcal{H}_{\mathrm{t}}$ contains L, which has further details of the particular field in question. These three universal features represent the kinematics due to the presupposition of spacetime. The $\pounds_\xi$ contribution is "shift kinematics," while the tilt contribution is "lapse kinematics."

The point of Kuchař's papers is to construct very general consistent matter theories by presupposing spacetime and correctly implementing the resulting kinematics. We are able to show below that in not presupposing spacetime, BFO are attempting to construct consistent theories by using shift kinematics (which is the best matching principle) alone, and thus attempting to deny the presence of any "lapse kinematics" in nature. This turns out to be remarkably successful for the bosonic theories of nature.

We begin by noting that nonderivative-coupled fields are a lot simpler to deal with than derivative-coupled ones. We then ask which fields are included in this simpler case, in which the matter fields do not affect the gravitational part of the Hamiltonian constraint so that the gravitational momenta remain independent of the matter fields. Now, we realize that this is a *tacit assumption* in almost all[12] of BFÓ's work.

0: The implementation of "adding on" matter is for matter contributions that do not interfere with the structure of the gravitational theory.

This amounts to the absence of Christoffel symbols in the matter Lagrangians, which is true of minimally coupled scalar fields ($D_a\chi=\partial_a\chi$) and of Maxwell and Yang-Mills theories and their massive counterparts (since $D_aA_b-D_bA_a=\partial_aA_b-\partial_bA_a$). Thus it suffices to start off by considering the nonderivative-coupled case on the grounds that it includes all the fields hitherto thought to fit in with the BFÓ scheme, and also the massive 1-form fields which do not.

Consider then the Proca 1-form. Its Lagrangian is

$$\mathsf{L}_{\text{Proca}}=-\nabla_{[\alpha}A_{\beta]}\nabla^{[\alpha}A^{\beta]}-\frac{m^2}{2}A_\alpha A^\alpha, \tag{49}$$

with corresponding Lagrangian potential

$$L=-\frac{1}{4}\lambda^{[\alpha\beta]}\lambda_{[\alpha\beta]}+\frac{m^2}{2}A_\alpha A^\alpha. \tag{50}$$

Whereas ${}_{\mathrm{A}}\mathcal{H}^{\mathrm{o}}_{\not\perp}$ has in fact been completed to a divergence, ${}_{(\mathrm{A})}\mathcal{H}^{\mathrm{o}}_{\not\perp}=A^aD_a\pi^\perp+A_\perp D_a\pi^a$ suffices to generate the tilt change of $A_\perp$ and A_a for the universal 1-form (see Kuchař III.6). The first term of this vanishes since $\pi^\perp=0$ by antisymmetry for the 1-forms described by Eq. (49). Also ${}_{\mathrm{A}}P^{ab}=0$ by antisymmetry so

$$ {}_{\mathrm{A}}\mathcal{H}^{\mathrm{o}}=\sqrt{h}\left[-\frac{1}{4}\lambda_{ab}\lambda^{ab}+\frac{1}{2h}\pi_a\pi^a+\frac{m^2}{2}(A_aA^a-A_\perp^2)-\lambda^{ab}A_{[a,b]}\right]+A_\perp D_a\pi^a \tag{51}$$

by Eqs. (45), (46). The multiplier equation for λ_{ab} gives

$$\lambda_{ab}=-2D_{[b}A_{a]}\equiv B_{ab}. \tag{52}$$

For $m\neq0$, the multiplier equation for $A_\perp$ gives

$$A_\perp=-\frac{1}{m^2\sqrt{h}}D_a\pi^a, \tag{53}$$

and elimination of the multipliers in Eq. (51) using Eqs. (52), (53) gives

$$ {}_{\mathrm{A}}\mathcal{H}^{\mathrm{o}}=\frac{1}{2\sqrt{h}}\pi_a\pi^a+\frac{\sqrt{h}}{4}B_{ab}B^{ab}+\frac{m^2\sqrt{h}}{2}A_aA^a+\frac{1}{2m^2\sqrt{h}}(D_a\pi^a)^2, \tag{54}$$

which is nonultralocal in the momenta. We note that this does nothing to eliminate the remaining term in the tilt: the Proca field has nonzero tilt.

But, for $m=0$, the $A_\perp$ multiplier equation gives instead the Gauss constraint of electromagnetism

$$\mathcal{G}\equiv D_a\pi^a\approx0. \tag{55}$$

This would not usually permit $A_\perp$ to be eliminated from Eq. (54) but the final form of ${}_{\mathrm{A}}\mathcal{H}^{\mathrm{o}}$ for $m=0$ is

$$ {}_{\mathrm{A}}\mathcal{H}^{\mathrm{o}}=\frac{\sqrt{h}}{4}B^{ab}B_{ab}+\frac{1}{2\sqrt{h}}\pi_a\pi^a+A_\perp(D_a\pi^a\approx0), \tag{56}$$

so the cofactor of $A_\perp$ in Eq. (44) weakly vanishes by 55, so $A_\perp$ may be taken to "accidentally" drop out. This means that the tilt of the Maxwell field may be taken to be zero. The tilt is also zero for the metric and for the scalar field. So far all these fields are allowed by BFÓ and have no tilt, whereas the disallowed Proca field has tilt.

We can begin to relate this occurrence to the BSW principle **2** or **2G**. Suppose an action has a piece depending on ∂_aN in it. Then the immediate elimination of N from it is *not* algebraic, so the procedure of BSW is not possible. By definition, the tilt part of the Hamiltonian constraint is built from the ∂_aN contribution using integration by parts. But, for the $A_\perp$-eliminated Proca Lagrangian, this integration by parts gives a term that is nonultralocal in the momenta, $(D_a\pi^a)^2$, which again contain ∂_aN within. Thus, for this formulation of Proca theory, one cannot build a BSW–Proca action to start off with. Of importance, this problem with spatial derivatives was not foreseen in the simple analogy with the Jacobi principle in mechanics, where there is only one independent variable.

The above argument requires refinement from the treatment of further important physical examples. This is a fast method of finding matter theories compatible with the 3-space approach by the following argument. If there is no derivative coupling and if one can arrange for the tilt to play no part in a formulation of a matter theory, then all that is left of the hypersurface kinematics is the shift kinematics, which is the best-matching principle. But complying with hypersurface kinematics is a guarantee for consistency so in these cases best matching suffices for consistency.

First, we consider K interacting 1-forms $A_a^{\mathbf{K}}$ with Lagrangian.[13]

$$\mathsf{L}^{\mathcal{A}}=-\left(\nabla_{[\alpha}A_{\beta]}^{\mathbf{A}}+\frac{g}{2}C^{\mathbf{A}}{}_{\mathbf{BC}}A_\beta^{\mathbf{B}}A_\alpha^{\mathbf{C}}\right)\left(\nabla^{[\alpha}A_{\mathbf{A}}^{\beta]}+\frac{g}{2}C_{\mathbf{ADE}}A^{\mathbf{D}\beta}A^{\mathbf{E}\alpha}\right)-\frac{m^2}{2}A_{\alpha\mathbf{M}}A^{\alpha\mathbf{M}}. \tag{57}$$

[12]We have argued in Sec. II E that the exception, Brans-Dicke theory, is a mild one.

[13]$\mathbf{D}_a$ is the Yang-Mills covariant derivative and $C_{\mathbf{ABC}}$ are the Yang-Mills structure constants. By $gC^{\mathbf{A}}{}_{\mathbf{BC}}$ we strictly mean $g^{\mathcal{A}}C^{\mathbf{A}}_{\mathcal{A}\mathbf{BC}}$ where $\mathcal{A}$ indexes each gauge subgroup in a direct product. Then each such gauge subgroup can be associated with a distinct coupling constant $g^{\mathcal{A}}$.

We define $\lambda_{\mathbf{M}}^{\alpha\beta}=\partial\mathrm{L}/\partial(\nabla_\beta A_\alpha^{\mathbf{M}})$ and the corresponding Lagrangian potential is

$$L=-\frac{1}{4}\lambda_{\mathbf{M}}^{[\alpha\beta]}\lambda^{\mathbf{M}}_{[\alpha\beta]}-\frac{g}{2}C_{\mathbf{BDE}}A_\beta^{\mathbf{D}}A_\alpha^{\mathbf{E}}\lambda^{\alpha\beta\mathbf{B}}+\frac{m^2}{2}A_{\alpha\mathbf{M}}A^{\alpha\mathbf{M}}. \tag{58}$$

The overall tilt contribution is now the sum of the tilt contributions of the individual fields, so ${}_{(A_{\mathbf{M}})}\mathcal{H}^{\mathrm{o}}_{\not t}=A_\perp^{\mathbf{M}}D_a\pi^a_{\mathbf{M}}$ suffices to generate the tilt change. Again, ${}_{\mathrm{A}_{\mathbf{M}}}P^{ab}=0$ by antisymmetry so

$$\begin{aligned}{}_{\mathrm{A}_{\mathbf{M}}}\mathcal{H}^{\mathrm{o}}=\sqrt{h}\Big[&-\frac{1}{4}\lambda^{\mathbf{M}}_{ab}\lambda^{ab}_{\mathbf{M}}+\frac{1}{2h}\pi_a^{\mathbf{M}}\pi^a_{\mathbf{M}}+\frac{m^2}{2}(A_a^{\mathbf{M}}A^a_{\mathbf{M}}\\ &-A_\perp^{\mathbf{M}}A_{\perp\mathbf{M}})-\lambda^{ab}_{\mathbf{M}}A^{\mathbf{M}}_{[a,b]}\Big]+A_\perp^{\mathbf{M}}\mathbf{D}_a\pi^a_{\mathbf{M}}\\ &-\frac{g}{2}C_{\mathbf{MPQ}}(\sqrt{h}\lambda^{ab\mathbf{M}}A_b^{\mathbf{P}}A_a^{\mathbf{Q}}+2\pi^{\mathbf{M}}A_\perp^{\mathbf{P}}A_a^{\mathbf{Q}})\end{aligned} \tag{59}$$

by Eqs. (45),(46). The multipliers are $\lambda^{ab}_{\mathbf{M}}$ and $A_\perp^{\mathbf{M}}$, with corresponding multiplier equations

$$\lambda^{\mathbf{M}}_{ab}=-2D_{[b}A^{\mathbf{M}}_{a]}\equiv B^{\mathbf{M}}_{ab}, \tag{60}$$

$$\begin{aligned}A_{\mathbf{M}\perp}&=-\frac{1}{m^2\sqrt{h}}\mathbf{D}_a\pi^a_{\mathbf{M}}\\ &\equiv-\frac{1}{m^2\sqrt{h}}(D_a\pi^a_{\mathbf{M}}+gC_{\mathbf{LMP}}\pi^{\mathbf{L}a}A_a^{\mathbf{P}})\end{aligned} \tag{61}$$

for $m\neq0$. We thus obtain the eliminated form

$$\begin{aligned}{}_{\mathrm{A}_{\mathbf{M}}}\mathcal{H}^{\mathrm{o}}=&\frac{1}{2\sqrt{h}}\pi_{\mathbf{M}a}\pi^{\mathbf{M}a}+\frac{\sqrt{h}}{4}B_{\mathbf{M}ab}B^{\mathbf{M}ab}+\frac{m^2\sqrt{h}}{2}A_{\mathbf{M}a}A^{\mathbf{M}a}\\ &+\frac{1}{2m^2\sqrt{h}}(\mathbf{D}_a\pi^{\mathbf{M}a})(\mathbf{D}_b\pi_{\mathbf{M}b})\end{aligned} \tag{62}$$

and the massive Yang-Mills field is left with nonzero tilt. For $m=0$, the second multiplier equation gives instead the Yang-Mills Gauss constraint

$$\mathcal{G}^{\mathbf{M}}\equiv\mathbf{D}_a\pi^{\mathbf{M}a}\approx0. \tag{63}$$

In this case, the tilt is nonzero, but the Yang-Mills Gauss constraint "accidentally" enables the derivative part of the tilt to be converted into an algebraic expression, which then happens to cancel with part of the Lagrangian potential:

$$\begin{aligned}{}_{\mathrm{A}_{\mathbf{M}}}\mathcal{H}^{\mathrm{o}}=&\frac{\sqrt{h}}{4}B^{ab}_{\mathbf{M}}B^{\mathbf{M}}_{ab}+\frac{1}{2\sqrt{h}}\pi_a^{\mathbf{M}}\pi^a_{\mathbf{M}}\\ &+A_\perp^{\mathbf{M}}(D_a\pi^a_{\mathbf{M}}+gC_{\mathbf{LMP}}\pi^{\mathbf{L}a}A_a^{\mathbf{P}}\approx0).\end{aligned} \tag{64}$$

Second, we consider $U(1)$ 1-form–scalar gauge theory, with interactions of the form $\chi^*A^\mu\partial_\mu\chi$ and $\chi^*\chi A^\mu A_\mu$. This could be viewed either as the interaction of a (strongly favored but still hypothetical) Higgs field with the electromagnetic field, or as a warm-up exercise toward the inclusion of the interaction term of Maxwell-Dirac theory [the classical theory behind quantum electrodynamics (QED)] and its standard model generalization (see Sec. V B). The Maxwell-scalar Lagrangian is[14]

$$\begin{aligned}\mathrm{L}^{\mathrm{U}(1)}_{\mathrm{MS}}=&-\nabla_{[\alpha}A_{\beta]}\nabla^{[\alpha}A^{\beta]}+(\partial_\mu\chi-ieA_\mu\chi)(\partial^\mu\chi^*+ieA^\mu\chi^*)\\ &-\frac{m_\chi^2}{2}\chi^*\chi.\end{aligned} \tag{65}$$

Now, in addition to $\lambda^{\alpha\beta}$, define $\mu^\alpha=\partial\mathrm{L}/\partial(\nabla_\alpha\chi)$ and $\nu^\alpha=\partial\mathrm{L}/\partial(\nabla_\alpha\chi^*)$, so the Lagrangian potential is

$$\begin{aligned}L=&-\frac{1}{4}\lambda^{[\alpha\beta]}\lambda_{[\alpha\beta]}+\frac{m^2}{2}A_\alpha A^\alpha+\mu^\alpha\nu_\alpha-ieA_\alpha(\chi^*\nu^\alpha-\chi\mu^\alpha)\\ &+\frac{m_\chi^2}{2}\chi^*\chi.\end{aligned} \tag{66}$$

${}_{(\mathrm{A})}\mathcal{H}^{\mathrm{o}}_{\not t}=A_\perp D_a\pi^a$ still suffices to generate the tilt (as scalars contribute no tilt), we have ${}_{A,\chi,\chi^*}P^{ab}=0$, and

$$\begin{aligned}{}_{A,\chi,\chi^*}\mathcal{H}^{\mathrm{o}}_{\mathrm{t}}=\sqrt{h}\Big[&-\frac{1}{4}\lambda_{ab}\lambda^{ab}+\mu_a\nu^a+\frac{1}{h}\left(\frac{1}{2}\pi_a\pi^a-\pi_\chi\pi_{\chi^*}\right)\\ &+\frac{m_\chi^2}{2}\chi^*\chi-ie\Big(A_a[\chi^*\nu^a-\chi\mu^a]\\ &-\frac{A_\perp}{\sqrt{h}}[\chi^*\pi_{\chi^*}-\chi\pi_\chi]\Big)\Big].\end{aligned} \tag{67}$$

The λ_{ab} multiplier equation is Eq. (52) again, while the $A_\perp$ multiplier equation is now

$$\mathcal{G}_{\mathrm{U}(1)}\equiv D_a\pi^a+ie(\chi^*\pi_{\chi^*}-\chi\pi_\chi)=0, \tag{68}$$

which can be explained in terms of electromagnetism now having a fundamental source. In constructing ${}_{A,\chi,\chi^*}\mathcal{H}^{\mathrm{o}}$ from Eqs. (45),(46),(67), we can convert the tilt to an algebraic expression by the sourced Gauss law (68) which again happens to cancel with a contribution from the Lagrangian potential:

$$\begin{aligned}{}_{A,\chi,\chi^*}\mathcal{H}^{\mathrm{o}}=&-\lambda^{ab}A_{[a,b]}-(\mu^a+\nu^a)\phi_{,a}+{}_{(A)}\mathcal{H}^{\mathrm{o}}_{\not t}+{}_{A,\chi,\chi^*}\mathcal{H}^{\mathrm{o}}_{\mathrm{t}}\\ =&\Big[\frac{1}{4}B_{ab}B^{ab}-\mu_a\nu^a+\frac{1}{h}\left(\frac{1}{2}\pi_a\pi^a-\pi_\chi\pi_{\chi^*}\right)\\ &+\frac{m_\chi^2}{2}\chi^*\chi\Big]\\ &+A_\perp[D_a\pi^a+ie(\chi^*\pi_{\chi^*}-\chi\pi_\chi)\approx0].\end{aligned} \tag{69}$$

[14]This working is unaffected by inclusion of a scalar field potential function.

It is not too hard to show that the last two accidents also accidentally conspire together to wipe out the tilt contribution in Yang-Mills 1-form–scalar gauge theory. This theory is also obviously nonderivative-coupled.

We now present a more general treatment about the occurrence of these accidents. They arise from eliminating $A_\perp$ from its multiplier equation. For this to make sense, $A_\perp$ must *be* a multiplier, thus $\pi^\perp = 0$. Then for general L, the multiplier equation is

$$\frac{\partial L}{\partial A_\perp} + D_a \pi^a = 0. \tag{70}$$

Then the requirement that $A_\perp D_a \pi^a + L$ be independent of $A_\perp$ on using Eq. (70) means that $-A_\perp(\partial L/\partial A_\perp) + L$ is independent of $A_\perp$. Thus the accidents occur whenever the Lagrangian potential is linear in $A_\perp$.

From the broadening of our understanding due to the above two examples, we can precisely reformulate the BSW principle **2** within the GC hypersurface framework as

2U: We use lapse-uneliminated actions homogeneously quadratic in their velocities and permit only those for which the matter contributes a weakly vanishing tilt.

We can combine this with dropping the requirement for homogeneously quadratic actions (Principle **2G**) to obtain a Principle **2UG**, in anticipation of the inclusion of spin-$\frac{1}{2}$ fermions.

So for Einstein-Maxwell theory, Einstein-Yang-Mills theory, and their corresponding scalar gauge theories, (1) the absence of derivative coupling guarantees that they can be coupled to GR without disrupting its canonical structure as tacitly assumed by BFÓ. (2) The absence of tilt guarantees that the resulting coupled theories can be put into BSW form. Because the theories have homogeneously quadratic kinetic terms, this is indeed the BSW form **2** (as opposed to its generalization **2G**), (3) now, the GC hypersurface framework guarantees consistency if all the required kinematics are included. But the only sort of kinematics left is best matching. Thus, all these theories are guaranteed to exist as theories in BFÓ's original formulation of the 3-space approach.

These workings begin to show (if one presupposes spacetime), what sort of obstacles in Kuchař's spacetime ontology might be regarded as responsible for the uniqueness results for bosonic matter when one starts from BFÓ's 3-dimensional ontology (see also Sec. IV C).

There is a slight procedural complication in (3), which we illustrate for the BFÓ formulation of Einstein-Maxwell theory. One starts off with

$$S_{\mathrm{BSW_A}} = \int \mathrm{d}\lambda \int \mathrm{d}^3x \sqrt{h}\sqrt{R - D_{[a}A_{b]}D^{[a}A^{b]}} \times \sqrt{T_{\mathrm{g}} + h^{ab}(\dot{A}_a - \pounds_\xi A_a)(\dot{A}_b - \pounds_\xi A_b)}, \tag{71}$$

and then one discovers the Gauss constraint of electromagnetism $\mathcal{G}$ is enforced, which one then encodes by the corresponding ''electromagnetic'' best matching. This amounts to the introduction of an auxiliary velocity Θ (variation of the action with respect to this Θ yields $\mathcal{G}$), according to

$$\dot{A}_a \rightarrow \dot{A}_a - \partial_a \Theta. \tag{72}$$

B. The 3-space approach allows more than the fields of nature

We have described how the fields hitherto known to be permitted by the 3-space approach may be identified within the GC approach. These fields all have the universal kinematic feature called best matching by BFÓ, and no other significant universal feature (tilt or derivative coupling). Are these fields then the known fundamental matter fields, which somehow have less universal kinematic features than GC would lead one to expect? This question may be subdivided as follows. Does the 3-space approach single out *only* the known fundamental matter fields? Does the 3-space approach single out *all* the known fundamental matter fields? Kuchař makes no big deal about the simplified form weakly equivalent to his decomposition of the electromagnetic field, because it does not close to reproduce the Dirac Algebra (see Kuchař III.11,12); it only does so *modulo* the Gauss constraint of electromagnetism, $\mathcal{G}$. He takes this to be an inconvenience, one which can be got around by adhering to the form directly obtained from the decomposition, whereas BFÓ take it as a virtue that the simplified form ''points out'' the new constraint, $\mathcal{G}$, as an integrability condition.

The first question can be answered by counterexample. One should interpret the question as coarsely as possible; for example one could argue that the 3-space approach is not capable of restricting the possibility of Yang-Mills theory to the gauge groups conventionally used to describe nature, or that by no means is massless 1-form–scalar gauge theory guaranteed to occur in nature. Rather than such subcases or effects due to interaction terms, we find it more satisfactory to construct a distinct matter theory which is not known to be present in nature. The last section has put us into a good position to do this.

Consider the 2-form $\Phi_{\alpha\beta}$ Lagrangian

$$\mathrm{L} = -\nabla_{[\gamma}\Phi_{\alpha\beta]}\nabla^{[\gamma}\Phi^{\alpha\beta]} - \frac{m^2}{2}\Phi_{\alpha\beta}\Phi^{\alpha\beta}, \tag{73}$$

define $\lambda^{\alpha\beta\gamma} = \partial L/\partial(\nabla_\gamma \Phi_{\alpha\beta})$ and use the Legendre transformation to obtain the Lagrangian potential

$$L = -\frac{1}{4}\lambda^{[\alpha\beta\gamma]}\lambda_{[\alpha\beta\gamma]} + \frac{m^2}{2}\Phi_{\alpha\beta}\Phi^{\alpha\beta}. \tag{74}$$

Then ${}_{(\Phi)}\mathcal{H}^{\mathrm{o}}_{\not\perp} = 2\Phi_{\perp b}D_a\pi^{ab}$ suffices to generate the 2-form tilt and ${}_{\Phi}P^{ab} = 0$ by antisymmetry. The multipliers are λ^{abc} and $A_{\perp a}$ with corresponding multiplier equations $\lambda_{abc} = -2D_{[b}\Phi_{ab]} \equiv B_{abc}$ and, for $m \neq 0$,

$$\Phi^b_\perp = -\frac{1}{m^2\sqrt{h}}D_a\pi^{ab}, \tag{75}$$

which may be used to eliminate the multipliers, giving rise to the nonultralocal form

$$ {}_{\Phi}\mathcal{H}^{\rm o}=\frac{\sqrt{h}}{4}B^{abc}B_{abc}+\frac{3}{4\sqrt{h}}\pi^{ab}\pi_{ab}+\frac{7}{8m^2\sqrt{h}}h_{bd}(D_a\pi^{ab})\times(D_c\pi^{cd})+\frac{m^2}{2}\Phi_{ab}\Phi^{ab}. \tag{76}$$

But for $m=0$, the $\Phi_{\perp b}$ multiplier constraint is

$$\mathcal{G}^b\equiv D_a\pi^{ab}\approx 0 \tag{77}$$

and

$$ {}_{\Phi}\mathcal{H}^{\rm o}=\frac{h}{4}B^{abc}B_{abc}+\frac{3}{4\sqrt{h}}\pi^{ab}\pi_{ab}+2\Phi_{\perp b}(D_a\pi^{ab}\approx 0). \tag{78}$$

So our massless 2-form's tilt is zero and this leads to the elimination of $\Phi_{\perp b}$ by the same sort of "accident" that permits $A_\perp$ to be eliminated in electromagnetism. So, for this massless 2-form, best matching is equivalent to all the GC hypersurface kinematics, and as this guarantees closure, we deduce that there exists a resulting 3-space approach theory starting with

$$S_\Phi=\int d\lambda\int d^3x\sqrt{h}\sqrt{R+D_{[c}\Phi_{ab]}D^{[c}\Phi^{ab]}}\times\sqrt{T_{\rm g}+h^{ab}h^{cd}(\dot{\Phi}_{[ab]}-\pounds_\xi\Phi_{ab})(\dot{\Phi}_{[cd]}-\pounds_\xi\Phi_{cd})}, \tag{79}$$

which leads to the enforcement of Eq. (77), which is subsequently encoded by the introduction of an auxiliary variable Θ_b. This working should also hold for any p-form for $p\leqslant d$, the number of spatial dimensions. Yet only the $p=1$ case, electromagnetism, is known to occur. This is evidence against BFÓ's speculation that the 3-space approach hints at "partial unification" of gravity and electromagnetism, since these extra unknown fields would also be included as naturally as the electromagnetic field. Note also that the ingredients of low energy string theory are getting included rather than excluded: p-forms, the dilatonic coupling . . . . These are signs that the 3-space approach is not as restrictive as BFÓ originally hoped.

The second question must be answered exhaustively. It is the minimal requirement for the 3-space approach to be taken seriously as a description of nature. The 3-space approach gives gravity, electromagnetism and Yang-Mills theories such as the $SU(2)\times U(1)$ theory of the electroweak bosons and the $SU(3)$ theory of the gluons of the strong force. One may argue that disallowing fundamental Proca fields is unimportant, because the photon and gluons are believed to be massless and the observed masses of the W^+, W^- and Z^0 weak bosons are thought to be not fundamental but rather acquired by spontaneous symmetry breaking [53]. The next problem is the inclusion of spin-$\frac{1}{2}$ fermions (see Sec. V B), in order to complete the 3-space approach for the theories of the simplest free fundamental fields that can account for nature. One could then investigate all the interactions involved in the standard model [53]. We note that one cannot be sure whether it is these simplest field theories that are present in nature, since our particle accelerators are located in a rather flat region. Thus our results are subject to our ignorance of nature's unexplored high-curvature regime. The notion of "simplest" includes relying on replacing partial derivatives with covariant derivatives to find the curved analogues of the flat laws. Yet this procedure could in principle be ambiguous [6] or not realized in nature due to putative further symmetry reasons [54].

C. Derivative coupling and the 3-space 1-form *Ansatz*

In their study of 1-forms, BFÓ used a BSW-type action with the potential term

$$U_{\rm A}=C^{abcd}D_bA_aD_dA_c+\frac{M^2}{2}A_aA^a, \tag{80}$$

(where $C^{abcd}=C_1h^{ac}h^{bd}+C_2h^{ad}h^{bc}+C_3h^{ab}h^{cd}$ for constant C_1, C_2, C_3, M), which is natural within their 3-space ontology. They then obtain ${}^{\rm A}\mathcal{H}$ and ${}^{\rm A}\mathcal{H}_i$ in the usual 3-space way (from the local square root and from ξ_i-variation). Then the propagation of ${}^{\rm A}\mathcal{H}$ enforces $C_1=-C_2$, $C_3=0$ and also the Gauss constraint of electromagnetism $\mathcal{G}$, whose propagation then enforces $M=0$. Having thus discovered that a new (Abelian) gauge symmetry is present, $\mathcal{G}$ is then encoded by the corresponding "electromagnetic" best matching, by introduction of an auxiliary velocity Θ [see Eq. (72)]. Identifying $\Theta=A_0$, this is a derivation of Einstein-Maxwell theory for $A_\alpha=[A_0,A_i]$.

We find it profitable to also explain this occurrence starting from the 4-dimensional ontology of the GC hypersurface framework. The natural choice of 1-form potential and kinetic terms would then arise from the decomposition of

$$\mathrm{L}=-C^{\alpha\beta\gamma\delta}\nabla_\beta A_\alpha\nabla_\delta A_\gamma-\frac{M^2}{2}A_\alpha A^\alpha. \tag{81}$$

Using the following set of four results from (Kuchař II.2),

$$\nabla_bA_\perp=D_bA_\perp-K_{bc}A^c, \tag{82}$$

$$N\nabla_\perp A_a=-\delta_NA_a-NK_{ab}A^b-A_\perp\partial_aN \tag{83}$$

$$\nabla_bA_a=D_bA_a-A_\perp K_{ab} \tag{84}$$

$$N\nabla_\perp A_\perp=-\delta_NA_\perp-A^a\partial_aN, \tag{85}$$

we obtain that

$$\mathrm{L}=-(C_1+C_2+C_3)\left(\frac{\delta_N A_\perp + A^a\partial_a N}{N}\right)^2 + C_1\left[\left(\frac{\delta_N A_a + A_\perp\partial_a N}{N}+K_{ac}A^c\right)^2+(D_aA_\perp - K_{ac}A^c)^2\right] + 2C_2\left(\frac{\delta_N A_a + A_\perp\partial_a N}{N}+K_{ac}A^c\right)(D^aA_\perp - K^a{}_cA^c) - 2C_3\left(\frac{\delta_N A_\perp + A^c\partial_c N}{N}\right)(D^aA_a - A_\perp K) - C^{abcd}(D_bA_a - A_\perp K_{ab})(D_dA_c - A_\perp K_{cd}) - \frac{M^2}{2}(A_aA^a - A_\perp A^\perp). \quad (86)$$

Then, if one chooses to prefer the 4-dimensional ontology and then to import BFÓ's 3-space assumptions into it, one finds the following explanations for BFÓ's uniqueness results from a 4-dimensional perspective.

First, BFÓ's tacit assumption that addition of a 1-form A_a does not affect the 3-geometry part of the action can be phrased as there being no derivative coupling, ${}_{\mathrm{A}}P^{ab}=0$, which using Eq. (48) implies that $\lambda^{(ab)}=0$, $\pi^b=-\lambda^{\perp b}$. Since $\lambda^{\alpha\beta}=-2C^{\alpha\beta\gamma\delta}\nabla_\delta A_\gamma$, this *by itself* implies $C_1=-C_2$, $C_3=0$.

If $A_\perp$ were a velocity as Barbour would argue [33] (following from its auxiliary status, just as N and ξ_i are velocities), it makes sense for the 3-space ansatz to contain no $\delta_N A^\perp$. But we now see from Eq. (86) that this by itself is also equivalent to $C_1=-C_2$, $C_3=0$ from the 4-dimensional perspective. Also, inspecting Eq. (86) for Maxwell theory reveals that

$$\mathrm{L}=\frac{C_1}{N^2}[\delta_N A_a - D_a(-NA_\perp)]^2 - C_1D^bA^a(D_bA_a - D_aA_b). \quad (87)$$

So in fact $\Theta=-NA_\perp$, so $A_\perp$ itself is not a velocity. Notice in contrast that the issue of precisely what Θ is does not arise in the 3-space approach because it is merely an auxiliary velocity that appears in the last step of the working.

One argument for the 3-space 1-form field *Ansatz* is simplicity: consideration of a 3-geometry and a single 3D 1-form leads to Maxwell's equations. However, we argue that in the lapse uneliminated form, provided that one is willing to accept the additional kinematics, we can extend these degrees of freedom to include a dynamical $A_\perp$. The 3-space approach is about *not* accepting kinematics other than best matching, but the GC hypersurface framework enables us to explore what happens when tilt and derivative-coupling kinematics are "switched on." Working within the GC hypersurface framework, if $A_\perp$ is allowed to be dynamical, there is derivative coupling, and consistency would require the presence of 2 further bunches of terms, with coefficients proportional to C_1+C_2 and to C_3. The first bunch consists of the following sorts of terms:

$$D^bA^aA_\perp\delta_N h_{ab}, \quad A^b\left(D^aA_\perp - A_\perp\frac{\partial^a N}{N}\right)\delta_N h_{ab}, \quad \frac{1}{N}h^{ab}A^c\delta_N A_a\delta_N h_{bc}, \quad A^bA^dh^{ac}\delta_N h_{ab}\delta_N h_{cd}, \quad A_\perp A_\perp h^{ac}h^{bd}\delta_N h_{ab}\delta_N h_{cd}. \quad (88)$$

The second bunch consists of the following sorts of terms:

$$h^{ab}\left(A_\perp D^cA_c + A^c\frac{\partial_c N}{N}\right)\delta_N h_{ab}, \quad \frac{1}{N}h^{ab}A_\perp\delta_N A_\perp\delta_N h_{ab}, \quad A_\perp A_\perp h^{ab}h^{cd}\delta_N h_{ab}\delta_N h_{cd}. \quad (89)$$

The naive blockwise Riemannian structure of the configuration space of GR and nonderivative-coupled bosonic fields (32) can get badly broken by derivative coupling (cf. Kuchař IV.5). Either of the above bunches by itself exhibits all the unpleasant configuration space features we mentioned in Sec. II E: the first two terms of Eq. (88) are linear and hence the geometry is not Riemannian, the third is a metric-matter cross-term, and the last two terms breach the DeWitt structure; likewise the first term of Eq. (89) is linear, the second is a cross-term and the third is a breach of the DeWitt structure. If the DeWitt structure is breached in nature, then the study of pure canonical gravity and of the isolated configuration space of pure gravity are undermined. Whereas there is no evidence for this occurrence, we have argued at the end of the last section that some forms of derivative coupling are only manifest in experimentally unexplored high-curvature regimes.

In the hypersurface framework, if $A_\perp$ were dynamical, then it would not be a Lagrange multiplier, and so it would not have a corresponding multiplier equation with which the tilt could be "accidentally" removed, in which case there would not exist a corresponding BSW form containing $A_\perp$. This argument, however, is not watertight, because it does not prevent some other BSW form from existing since variables other than $A_\perp$ could be used in attempts to write down actions that obey the 3-space principles. As an example of such an attempt, we could use the N-dependent variable A_0 to put Proca theory into BSW form. In this case the attempt fails as far as the 3-space approach is concerned, because A_0 features as a non-best-matched velocity in contradiction with principle **1**. This shows, however, that criteria for whether a matter theory can be coupled to GR in the 3-space approach are unfortunately rather dependent on the formalism used for the matter field. The 3-space approach would then amount to attaching particular significance to formalisms meeting its description. This is similar in spirit to how those formalisms which close precisely as the Dirac Algebra are favored in the hypersurface framework and the HKT and Teitelboim [20] papers. In both cases one is required to find at least one compatible formalism for all the known fundamental matter fields.

V. DISCUSSION AND THE INCLUSION OF SPIN-$\frac{1}{2}$ FERMIONS

A. Variations on the seventh route to relativity

The split [Eqs. (45),(46)] of ${}_{\rm A}\mathcal{H}^{\rm o}$ or perhaps more simply the equations (82), (83), (84), (85) [and their analogues for higher-rank tensors (see e.g. Kuchař III.9)], sum up the position of best matching within the GC hypersurface framework. The required presupposition of embeddability in the GC hypersurface framework leads to three sorts of kinematics for tensor fields: best matching, tilt and derivative coupling. All three of these are required in general in order to guarantee consistency and Kuchař's papers are a recipe for the computation of all the terms required for this consistency. Thus in GR where it is available, the GC hypersurface framework is powerful and advantageous as a means of writing down consistent matter theories. If conformal gravity is regarded as a competing theory to GR, it makes sense therefore to question what the 4-geometry of conformal gravity is, and whether its use could lead to a more illuminating understanding of matter coupling than offered by the 3-space approach. We are thus free to ask how special GR is in admitting a constructive kinematic scheme for coupled consistent tensorial matter theories.

As BFÓ formulate it, the 3-space approach denies the primary existence of the lapse. But we have demonstrated that whether or not the lapse is eliminated does not affect the mathematics, so we would prefer to think of the 3-space approach as denying ''lapse kinematics.'' BFÓ's use of BSW forms does lead to a more restrictive scheme than GC, but we have demonstrated in Sec. IV that this restriction can be understood in terms of when the GC hypersurface framework has no tilt. Furthermore, we have unearthed the tacit simplicity postulate **0** and have rephrased this and the generalized BSW postulate **2G** as nonderivative coupling and the no tilt condition **2UG**, respectively, within the GC hypersurface framework.

Working in the GC hypersurface framework (with lapse-uneliminated actions with only shift kinematics) has the additional advantage that we are immediately able to turn on and hence investigate the mathematical and physical implications of the tilt and derivative-coupling kinematics. Nevertheless, it is striking that best matching kinematics suffice to describe all of the known fundamental bosonic fields coupled to GR. The absence of other kinematics includes the absence of the derivative-coupled theories whose presence in nature would undermine the study of pure canonical gravity of DeWitt and others. We see our work as support for this study. The less structure is assumed in theoretical physics, the more room is left for predictability. Could it really be that nature has less kinematics than the GC hypersurface framework of GR might have us believe?

We next question whether the best-matching kinematics itself should be presupposed, since it is also striking that the additional constraints of the GR-boson system $(\mathcal{H}_i,\mathcal{G},\mathcal{G}^J,\dots)$ are interpretable as integrability conditions for $\mathcal{H}$. This allows the following alternative to starting with the best-matching principle **1**, which could in principle allow more complicated shift kinematics than the current formulation.[15]

1I: Start with a 3-dimensional action with bare velocities. $\mathcal{H}$ can be deduced immediately from the action, and demanding $\dot{\mathcal{H}}\approx 0$ leads to a number of other constraints. These are all then to be encoded by use of auxiliary variables.

This has the immediate advantage of treating the gravitational best matching on the same footing as the encoding of Gauss constraints. The 3-space approach has recently been reformulated this way by Ó Murchadha [55].

We present caveats to this approach both here and in Sec. V B. Here, we note that for strong gravity, **1** and **1I** lead to inequivalent theories because $\mathcal{H}$ and $\mathcal{H}_i$ propagate independently. So starting from some constraint and the demand of integrability might miss out independent but compatible constraints. **1** and **1I** are, however, equivalent (by inspection of the constraint algebras) for GR coupled to the known fundamental bosonic fields.

So far, at least the bosonic sector of nature appears to be much simpler than the GC hypersurface framework of GR might suggest, and the 3-space approach may be formulated in two equivalent ways **1** and **1I** as regards best matching. We now consider both **1** and **1I** for spin-$\frac{1}{2}$ fermions.

B. Fermions and the 3-space approach

Whereas it is true that the spinorial laws of physics may be rewritten in terms of tensors [56], the resulting equations are complicated and it is not clear if and how they may be obtained from action principles. Thus we are almost certainly compelled to investigate coupled spinorial and gravitational fields by attaching local flat frames to our manifolds.

There are two features we require for the analysis of the spin-$\frac{1}{2}$ laws of nature coupled to gravity. First, we want the analysis to be clear in terms of shift and lapse kinematics, given our success in this paper with this approach. However, one should expect the spinors to have further sorts of kinematics not present for tensor fields. Second, we want to explicitly build $SO(3,1)$ (spacetime) spinors out of $SO(3)$ (spatial) ones.[16] We hope to perform this first-principles analysis in the future. In this paper, we consider the first feature in the following 4-component spinor formalism.

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[15] We consider the difference between shift kinematics and lapse kinematics to be particularly significant because of their association with linear and quadratic constraints, respectively. We have no doubt in the correctness of handling linear constraints in physics so it would not be a problem if the concept of best matching requires refinement.

[16] This is standard use of representation theory, based on the accidental Lie algebra relation $SO(4)\cong SO(3)\oplus SO(3)$, which depends on the dimension of space being 3. This relation is a common source of tricks in the particle physics and quantum gravity literatures. By $SO(3,1)$ and $SO(3)$ spinors, we strictly mean spinors corresponding to their universal covering groups, $SL(2,\mathrm{C})$ and $SU(2)$, respectively. We are not yet concerned in this paper with the differences between $SO(4)$ and $SO(3,1)$ from a quantization perspective, which render Euclidean quantum programs easier in some respects.

In Géhéniau and Henneaux's (GH) [57] 4-component spinor study of the Einstein-Dirac (ED) system, the term $\bar{\psi}\gamma^{\bar{\lambda}}\nabla^{s}_{\bar{\lambda}}\psi$ is decomposed as follows[17]

$$\sqrt{|g|}\bar{\psi}\gamma^{\bar{\lambda}}\nabla^{s}_{\bar{\lambda}}\psi = i\sqrt{h}\psi^{\dagger}\Big[N\gamma^{\bar{0}}\gamma^{\bar{l}}D^{s}_{\bar{l}}\psi + \frac{NK}{2}\psi + N_{,\bar{l}}\gamma^{\bar{0}}\gamma^{\bar{l}}\psi - (\dot{\psi} - \pounds^{s}_{\xi}\psi - \partial_{\mathrm{R}}\psi)\Big], \tag{90}$$

where

$$\pounds^{s}_{\xi}\psi = \xi^{i}\psi_{,i} - \frac{1}{4}E^{i}_{[\bar{r}}\pounds_{\xi}E_{\bar{s}]i}\gamma^{\bar{s}}\gamma^{\bar{r}}\psi, \tag{91}$$

$$\partial_{\mathrm{R}}\psi = \frac{1}{4}E^{i}_{[\bar{r}}\dot{E}_{\bar{s}]i}\gamma^{\bar{r}}\gamma^{\bar{s}}\psi. \tag{92}$$

First, observe that the tensorial Lie derivative $\pounds_{\xi}\psi = \xi^{i}\psi_{,i}$ is but a piece of the spinorial Lie derivative (91) [57,58]. There is also an additional triad rotation correction (92) to the velocities in addition to the 3-diffeomorphism-dragging Lie derivative correction. The notion **1** of best matching must be generalized to accommodate this additional, very natural geometric correction: given two spinor-bundle 3-geometries Σ_1, Σ_2, the (full spinorial) drag shufflings of Σ_2 (keeping Σ_1 fixed) are accompanied by the rotation shufflings of the triads glued to it. The triad rotation correction is associated with a further "locally Lorentz" constraint $\mathcal{J}_{\bar{\mu}\bar{\nu}}$ [59].

In thinking from first principles about best matching in sufficiently general terms to include the treatment of spinors, it is not clear whether the triad rotations need be included from the start. One might "discover and encode" these as occurs with the Gauss laws for 1-forms. Also, use of the "bare" principle **1I** may not require a conceptual advance on best matching: the Dirac procedure beginning with $\mathcal{H}$ would provide us with the correct $\mathcal{H}_i$, whose encoding would yield the full ξ^i correction for spinors. Pursuing this last line of approach, Nelson and Teitelboim's work [60] may be taken to imply that $\mathcal{H}_i$ and $\mathcal{J}_{\bar{\mu}\bar{\nu}}$ are indeed integrability conditions for $\mathcal{H}$. For in terms of Dirac brackets $\{,\}^*$, starting from $\mathcal{H}$, $\{\mathcal{H},\mathcal{H}\}^*$ gives $\mathcal{H}_i$ and then we can form $\{\mathcal{H},\mathcal{H}_i\}^*$ which gives $\mathcal{J}_{\bar{\mu}\bar{\nu}}$ (and $\mathcal{H}$) so we have recovered all the constraints as integrability conditions for $\mathcal{H}$. One does not recover $\mathcal{H}$ if one starts with $\mathcal{H}_i$ or $\mathcal{J}_{\bar{\mu}\bar{\nu}}$, so in some sense $\mathcal{H}$ is privileged. However, this does highlight our other caveat for the integrability idea: one might choose to represent the constraint algebra differently by mixing up the usual generators. For example, a linearly-related set of constraints is considered in [60], for which the integrability of any of the constraints forces the presence of all the others. Our defense against this is to invoke again that we only require one formulation of the 3-space approach to work, so we would begin with the quadratic constraint $\mathcal{H}$ nicely isolated.

Second, although derivative coupling (second term) and tilt (third term) appear to be present in Eq. (90), GH observed that these cancel in the Dirac field contribution to the Lagrangian density,

$$\sqrt{|g|}\mathrm{L}_{\mathrm{D}} = \sqrt{|g|}\Big[\frac{1}{2}(\bar{\psi}\gamma^{\bar{\lambda}}\nabla^{s}_{\bar{\lambda}}\psi - \nabla^{s}_{\bar{\lambda}}\bar{\psi}\gamma^{\bar{\lambda}}\psi) - m_{\psi}\bar{\psi}\psi\Big]. \tag{93}$$

While Nelson and Teitelboim [60] do not regard their formulation's choice of absence of derivative coupling as a deep simplification (they adhere to the HKT school of thought and the simplification is not in line with the hypersurface deformation algebra), the GH result is clearly encouraging for the 3-space approach. For, once Eq. (90) has been used in Eq. (93), we obtain an action of the form **2UG**, so we can cast ED theory into the **2G** generalized BSW form (35).

Finally, we comment on the inclusion of 1-form-fermion interaction terms of the Einstein-standard model theory

$$i f_{\mathcal{A}}\tau^{\mathcal{A}}_{\mathbf{I}}\bar{\psi}\gamma^{\bar{\beta}}E^{\mu}_{\bar{\beta}}A^{\mathbf{I}}_{\mu}\psi \tag{94}$$

where $\mathcal{A}$ takes the values $U(1)$, $SU(2)$ and $SU(3)$ and $\tau^{\mathcal{A}}_{\mathbf{I}}$ are the generators of these groups. The decomposition of these into spatial quantities is trivial. No additional complications are expected from the inclusion of such terms, since (1) they contain no velocities so the definitions of the momenta are unaffected (this includes there being no scope for derivative coupling) and (2) they are part of gauge-invariant combinations, unlike the Proca term which breaks gauge invariance and significantly alters the Maxwell canonical theory. In particular, the new terms clearly contribute linearly in $A_{\perp}$ to the Lagrangian potential, so by the argument at the end of Sec. IV A, an accident occurs ensuring that tilt kinematics is not necessary. Also, clearly the use of the form (93) is compatible with the inclusion of the interactions (94) since, acting on $\bar{\psi}$ the gauge correction is the opposite sign. So our proposed formulation's combined standard model matter Lagrangian is

[17] We use barred Greeks for Minkowski indices and barred Latins for Euclidean indices. The Minkowski metric is denoted by $\eta_{\bar{\mu}\bar{\nu}}$. The $\gamma^{\bar{\lambda}}$ are Dirac matrices, obeying the Dirac Algebra $\gamma^{\bar{\mu}}\gamma^{\bar{\nu}} + \gamma^{\bar{\nu}}\gamma^{\bar{\mu}} = 2\eta^{\bar{\mu}\bar{\nu}}$, which is not to be confused with the Dirac algebra (8). Dirac's suited triads are denoted by $E^{\mu}_{\bar{\lambda}}$; these obey $E_{\bar{0}a}=0$, $E_{\bar{0}\bar{0}} = -N$, $E^{\sigma}_{\bar{\mu}}E_{\bar{\nu}\sigma} = \eta_{\bar{\mu}\bar{\nu}}$ and $E_{\bar{\lambda}\alpha}E^{\bar{\lambda}}_{\beta} = g_{\alpha\beta}$. ψ is a 4-component spinor, with conjugate $\bar{\psi} = i\gamma^{\bar{0}}\psi^{\dagger}$. The spacetime spinorial covariant derivative is $\nabla^{s}_{\bar{\nu}}\psi = \psi_{,\bar{\nu}} - \frac{1}{4}\Omega_{\bar{\rho}\bar{\sigma}\bar{\nu}}\gamma^{\bar{\rho}}\gamma^{\bar{\sigma}}\psi$, where $\Omega_{\bar{\rho}\bar{\sigma}\bar{\nu}} = (\nabla_{\beta}E_{\bar{\rho}\alpha})E^{\alpha}_{\bar{\sigma}}E^{\beta}_{\bar{\nu}}$ is the spacetime spin connection. The spatial spinorial covariant derivative is $D^{s}_{\bar{p}}\psi = \psi_{,\bar{p}} - \frac{1}{4}\omega_{\bar{r}\bar{s}\bar{p}}\gamma^{\bar{r}}\gamma^{\bar{s}}\psi$, where $\omega_{\bar{r}\bar{s}\bar{p}} = (D_b E_{\bar{r}a})E^{a}_{\bar{s}}E^{b}_{\bar{p}}$ is the spatial spin connection.

$$\mathrm{L}^{\mathcal{A}}_{\mathrm{SM}}=\Bigg[\frac{1}{2}[\overline{\psi}\overline{\gamma^{\lambda}}(\nabla^{\mathrm{s}}_{\lambda}-i\mathrm{f}_{\mathcal{A}}\tau^{\mathcal{A}}_{\mathbf{I}}E^{\mu}_{\lambda}A^{\mathbf{I}}_{\mu})\psi - (\nabla^{\mathrm{s}}_{\lambda}+i\mathrm{f}_{\mathcal{A}}\tau^{\mathcal{A}}_{\mathbf{I}}E^{\mu}_{\lambda}A^{\mathbf{I}}_{\mu})\overline{\psi}\overline{\gamma^{\lambda}}\psi]-m_{\psi}\overline{\psi}\psi\Bigg]+\mathrm{L}^{\mathcal{A}}_{\mathrm{YM}}. \tag{95}$$

Here $\mathrm{L}^{\mathcal{A}}_{\mathrm{YM}}$ is given by the $m=0$ version of Eq. (57) and we would need to sum the square bracket over all the known fundamental fermionic species, which thus simultaneously incorporates all the required accidents. There is also no trouble with the incorporation of the Yukawa interaction term $\overline{\psi}\chi\psi$ which could be required for some fermions to pick up mass from a Higgs scalar.

Thus the Lagrangian for all the known fundamental matter fields can be built by assuming best-matching kinematics and that the DeWitt structure is respected. The thin sandwich conjecture can be posed for all these fields coupled to GR. The classical physics of all these fields is timeless in Barbour's sense.

C. Future developments

We end by suggesting further work toward answering Wheeler's question in the Introduction stimulated by the advances in this paper. It remains to explicitly build a best-matched generalized BSW ED action starting from a pair of spatial $SO(3)$ spinors. Use of Eq. (90) in Eq. (93) still has remnants of 4-dimensionality in its appearance: it is in terms of 4-component spinors and Dirac matrices. However, recall that the Dirac matrices are built out of the Pauli matrices associated with $SO(3)$, and choosing to work in the chiral representation, the 4-component spinors may be treated as $\psi=[\psi_{\mathrm{D}},\psi_{\mathrm{L}}]$, i.e in terms of right-handed and left-handed $SO(3)$ 2-component spinors. Thus a natural formulation of ED theory in terms of 3-dimensional objects exists. To accommodate neutrino (Weyl) fields, one would consider a single $SO(3)$ spinor, that is set $\psi=[0,\psi_{\mathrm{L}}]$, $m=0$ before the variation is carried out. While we are free to accommodate all the known fundamental fermionic fields in the 3-space approach, one cannot predict the number of Dirac and Weyl fields present in nature nor their masses nor the nongravitational forces felt by each field. So, consider actions with integrands such as $\sqrt{R+U_{\mathrm{F}}}\sqrt{T_{\mathrm{g}}+T_{\mathrm{F}}}$ or $N(R+U_{\mathrm{F}})+(1/4N)T_{\mathrm{g}}+T_{\mathrm{F}}$ for U_{F} and T_{F} *built from spatial first principles using SO(3) spinors*. Obtain $\mathcal{H}$ and treat its propagation exhaustively to obtain constraint algebras. Is a universal light-cone recovered? Is Einstein-Dirac theory singled out? One could attempt this work for a bare T_{F} or (more closely to BFÓ's original work) for a best-matched T_{F}. In connection with the latter, how is the thin sandwich conjecture for Einstein-Dirac theory well-behaved? On coupling a 1-form field, do these results hold for Einstein-Maxwell-Dirac theory? On coupling K 1-form fields, do they hold for Einstein-Yang-Mills-Dirac theories such as the Einstein-standard model? There is also the issue of whether conformal gravity can accommodate spin-$\frac{1}{2}$ fermions.

It is worth considering whether any of our ideas for generalizing the 3-space approach extend to canonical supergravity [61]. This could be seen as a robustness test for our ideas and possibly lead to a new formulation of supergravity. Also, supersymmetry is proposed to resolve the hierarchy problem and help with many other problems of theoretical physics [62]. Furthermore, if the hierarchy problem is to be resolved in this way, the forthcoming generation of particle accelerators are predicted to see superparticles. Hence there is another reason for asking if the 3-space approach extends to supergravity with supersymmetric matter: this may well be soon required to describe nature. The supergravity constraint algebra is not known well enough [64] to comment whether the new supersymmetric constraint $\mathcal{S}^{-}_{\mu}$ arises as an integrability condition for $\mathcal{H}$. Note, however, that Teitelboim was able to treat $\mathcal{S}^{-}_{\mu}$ as arising from the square root of $\mathcal{H}$ [63]; however, this means that the bracket of $\mathcal{S}^{-}_{\mu}$ and its conjugate gives $\mathcal{H}$, so it is questionable whether the supergravity $\mathcal{H}$ retains all of the primary importance of the GR $\mathcal{H}$.

Finally, given the competition from [17] and this paper, it would be interesting to see whether any variant[18] of HKT can be made to accommodate spin-$\frac{1}{2}$ fermions, and also to refine Teitelboim's GR-matter postulates to the level of HKT's pure GR postulates.

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[18]Kouletsis' recent work [65], comparison with which we consider beyond the scope of this paper, is a variation on HKT's work using the generally covariant history formalism [66]. This work does not explicitly mention spin-$\frac{1}{2}$ fermions either.

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