Braneworld cosmological models with anisotropy

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For a cosmological Randall-Sundrum braneworld with anisotropy, i.e., of Bianchi type, the modified Einstein equations on the brane include components of the five-dimensional Weyl tensor for which there are no evolution equations on the brane. If the bulk field equations are not solved, this Weyl term remains unknown, and many previous studies have simply prescribed it as *ad hoc*. We construct a family of Bianchi braneworlds with anisotropy by solving the five-dimensional field equations in the bulk. We analyze the cosmological dynamics on the brane, including the Weyl term, and shed light on the relation between anisotropy on the brane and the Weyl curvature in the bulk. In these models, it is not possible to achieve geometric anisotropy for a perfect fluid or scalar field—the junction conditions require anisotropic stress on the brane. But the solutions can isotropize and approach a Friedmann brane in an anti–de Sitter bulk.

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I. INTRODUCTION

High-energy physics theories have recently inspired relatively simple phenomenological models in which one can test some of the consequences of string theories. Randall and Sundrum [1,2] proposed a model that captures some of the essential features of the dimensional reduction of 11-dimensional supergravity introduced by Hořava and Witten [3,4]. The second Randall-Sundrum (RS2) scenario [2] is a five-dimensional anti–de Sitter (AdS_5) "bulk" spacetime with an embedded Minkowski 3-brane where matter fields are confined and Newtonian gravity is effectively reproduced at low energies. The RS2 scenario was generalized to a Friedmann-Robertson-Walker (FRW) brane, showing that the Friedmann equation at high energies gives $H^2 \sim \rho^2$, in contrast with the general-relativistic behavior $H^2 \sim \rho$ [5–8].

As shown in [9], the modified field equations on the brane have two new contributions from extra-dimensional gravity:

$$G_{\alpha\beta} = -\Lambda g_{\alpha\beta} + \kappa^2 T_{\alpha\beta} + 6\frac{\kappa^2}{\lambda}\mathcal{S}_{\alpha\beta} - \mathcal{E}_{\alpha\beta}, \tag{1}$$

where λ is the brane tension (the vacuum energy of the brane when $T_{\alpha\beta}=0$), and Λ and κ are the four-dimensional cosmological and gravitational constants, respectively, given in terms of λ and the fundamental constants of the bulk (Λ_5, κ_5) by

$$\Lambda = \frac{\Lambda_5}{2} + \frac{\lambda^2}{12}\kappa_5^2, \quad \kappa^2 = \frac{\lambda}{6}\kappa_5^4. \tag{2}$$

The term $\mathcal{S}_{\alpha\beta}$ is quadratic in $T_{\alpha\beta}$ and dominates at high energies ($\rho > \lambda$). The five-dimensional Weyl tensor is felt on the brane via its projection $\mathcal{E}_{\alpha\beta}$. This Weyl term is determined by the bulk metric, not by equations on the brane. In FRW braneworlds, the bulk is Schwarzschild-AdS_5 [10–12], and $\mathcal{E}_{\alpha\beta}$ reduces to a simple Coulomb term that gives rise to "dark radiation" on the brane. The simplest generalizations of FRW braneworlds are Bianchi braneworlds.

By making an assumption about the Weyl term on the brane, the dynamics of a Bianchi type I brane was studied in [13], and it was shown that high-energy effects from extra-dimensional gravity remove the anisotropic behavior near the singularity that is found in general relativity. This was extended via a phase space analysis of Bianchi types I and V braneworlds [14,15], showing that the anisotropy is negligible close to the singularity for perfect fluid models with a barotropic linear equation of state $p = w\rho$, with $0 < w < 1$ a constant, opposite to the general relativity case. It was then suggested that this may be generic in cosmological braneworlds, which was supported by subsequent work [16,17] (see also [18–20]). However, a perturbative analysis [21] suggests that this may only be true for homogeneous models.

These studies, and others [22–33], considered only the dynamical equations on the brane, making various assumptions about the Weyl term in the absence of knowledge of the bulk metric. In [34] a bulk metric with a Kasner brane was presented. However, since the Kasner metric is a solution of the four-dimensional Einstein vacuum equations, the bulk metric is a simple warped extension; the general result, with the generic form of the bulk metric, is given in [35].[1] The simplest example of this general result is a Minkowski brane, leading to the RS2 solution. Another example is the Schwarzschild black string solution [36].[2] Up to now, no complete solutions, i.e., for the brane and bulk metrics, have been found for cosmological Bianchi braneworlds.[2] The key difficulty is to find anisotropic generalizations of AdS_5 that can incorporate anisotropy on a cosmological brane, and that are necessarily non-conformally flat.

Previous studies of Bianchi braneworlds have considered the effects of $\mathcal{S}_{\alpha\beta}$ under various assumptions on $\mathcal{E}_{\alpha\beta}$. Here we tackle the question of the construction of complete mod-

[1]Note that this result is sensitive to the form of the bulk field equations, and it breaks down in the presence of a Gauss-Bonnet term in the gravitational action [44].

[2]In [39], solutions with an anisotropic bulk containing a black hole with a non-spherical horizon were found.

els for cosmological braneworlds with anisotropy, that is, we want to construct both the metric in the bulk and on the 3-brane, so that $\mathcal{E}_{\alpha\beta}$ is determined and not assumed *ad hoc*.

II. THE GEOMETRY OF THE BULK

For a five-dimensional bulk spacetime with a negative cosmological constant, $\Lambda_5<0$, and no additional matter sources, the Einstein equations are

$$^5G_{AB}+\Lambda_5{}^5g_{AB}=0. \tag{3}$$

In order to construct cosmological braneworlds with anisotropy we start from the ansatz used by [5,6] (see also [37,38]):

$$^5ds^2=-n^2(t,y)dt^2+a^2(t,y)d\Sigma_k^2+b^2(t,y)dy^2, \tag{4}$$

where $d\Sigma_k^2$ is the line element of the three-dimensional maximally symmetric surfaces $\{t=t_*,y=y_*\}$, with a curvature index $k=0,\pm1$. Clearly, all the hypersurfaces $\{y=y_*\}$ inherit a FRW metric. Although the Einstein equations (3) can be completely solved for the metric (4), the explicit complete solution (bulk+brane) (see [5,6]) was found for $\dot{b}=0$, which corresponds to Gaussian normal coordinates adapted to the foliation with normal $n_A dx^A=dy$. Since the bulk is Schwarzschild-AdS$_5$, an alternative approach is based on a moving brane in static spherical bulk coordinates [11,12],

$$^5ds^2=-f(r)dT^2+\frac{dr^2}{f(r)}+r^2d\Sigma_k^2, \tag{5}$$

where

$$f(r)=k+\frac{r^2}{\ell^2}-\frac{\mu}{r^2}. \tag{6}$$

Here $\ell^2=-6/\Lambda_5$, so that ℓ is the curvature scale of the bulk. When the parameter μ, the mass of the bulk black hole, vanishes, the solution is simply AdS$_5$, so that the bulk Weyl tensor, and hence the brane Weyl term, vanish. If $\mu\neq0$, then the tidal field of the black hole generates a non-zero Weyl term on the brane. The existence of the black hole horizon requires that $\mu\geqslant0$ for a flat or closed geometry, and $\mu\geqslant-\ell^2/4$ for the open case. The brane trajectory is $r=a(\tau)$, where τ is the cosmological proper time on the brane and $a(\tau)$ is the scale factor, whose evolution is determined by the junction conditions. For a Z_2-symmetric brane, this gives the modified Friedmann equation on the brane,

$$H^2+\frac{k}{a^2}=\frac{\kappa^2}{3}\rho\left(1+\frac{\rho}{2\lambda}\right)+\frac{\Lambda}{3}+\frac{\mu}{a^4}, \tag{7}$$

where the high-energy correction term is ρ^2/λ and the last term on the right is the dark radiation term.

A natural extension of the ansatz in Eq. (4) that will introduce anisotropy is (compare [39] for a similar approach):

$$^5ds^2=-n^2(t,y)dt^2+h_{IJ}(t,y)\omega^I\omega^J+b^2(t,y)dy^2, \tag{8}$$

where $\omega^I=\omega_i^I dx^i$ are one-forms invariant under a Bianchi group (see [40] for details), and $h_{IJ}\omega^I\omega^J$ is the metric induced on the surfaces $\{t=t_*,y=y_*\}$. For simplicity, we consider only Bianchi type I models ($\omega_i^I=\delta_i^I$) (the procedure for non-Abelian Bianchi groups is essentially the same) with a diagonal metric h_{IJ}:

$$^5ds^2=-a_0^2(t,y)dt^2+\sum_{i=1}^{3}a_i^2(t,y)(dx^i)^2+b^2(t,y)dy^2. \tag{9}$$

The field equations (3) for this metric are non-linear partial differential equations (PDEs) in (t,y), like the field equations for the metric (4). In the case of the metric (4) the $\{ty\}$ component of the field equations provides a relation that leads to a set of first integrals. However, this procedure does not work for Eq. (9), and one must deal with non-linear PDEs. We have not been able to find a procedure to solve them analytically.

These difficulties indicate that in order to find analytic solutions we should abandon the generic case and consider special solutions that do not require PDEs. We try a static and Gaussian normal ansatz,

$$^5ds^2=-e^{2A_0(y)}dt^2+\sum_{i=1}^{3}e^{2A_i(y)}(dx^i)^2+dy^2, \tag{10}$$

where we pay the price that the brane is no longer static in the coordinate system. This ansatz can in fact be seen as a five-dimensional generalization of a similar ansatz [41] used in the search for four-dimensional static and cylindrically symmetric spacetimes describing cosmic strings in the presence of a non-vanishing cosmological constant. The field equations for the metric (10) are

$$A''_\alpha+A'_\alpha\sum_{\beta=0}^{3}A'_\beta-\frac{\omega^2}{4}=0, \tag{11}$$

$$\sum_{0\leqslant\alpha<\beta\leqslant3}A'_\alpha A'_\beta-\frac{3}{8}\omega^2=0, \tag{12}$$

where $\omega=4/\ell$.

In order to solve these equations, we introduce the determinant of the metric,

$$u^2(y)=\exp\left(2\sum_{\alpha=0}^{3}A_\alpha\right). \tag{13}$$

Multiplying Eq. (11) by $u(y)$ and summing over α, we get

$$u''-\omega^2u=0, \tag{14}$$

with the first integral,

$$u'^2-\omega^2u^2+C=0, \tag{15}$$

where C is an arbitrary constant of integration. Once $u(y)$ is known, $A_\alpha(y)$ can be obtained by quadrature:

$$A'_\alpha = \frac{1}{4}\frac{u'}{u} + \frac{C_\alpha}{u}, \tag{16}$$

which comes from the integration of Eq. (11). The constants C_α are constrained by Eqs. (12) and (13):

$$\sum_{\alpha=0}^{3} C_\alpha = 0, \tag{17}$$

$$\sum_{0\leq\alpha<\beta\leq 3} C_\alpha C_\beta = \frac{3}{8}C. \tag{18}$$

These imply

$$(C_1+C_2)^2+(C_1+C_3)^2+(C_2+C_3)^2 = -\frac{3}{4}C. \tag{19}$$

Taking the square of Eq. (17) and using Eq. (18) yields equivalently

$$\sum_{\alpha=0}^{3} C_\alpha^2 = -\frac{3}{4}C. \tag{20}$$

Thus C can never be positive, and the C_α's must be the coordinates of a three-sphere of radius $\sqrt{-3C}/2$, hence $|C_\alpha| \leq \sqrt{-3C}/2$.

When $C=0$ the parameters C_α must all be zero as well. In this particular case,

$$A_\alpha(y) = A_\alpha^o + \frac{\omega}{4}(y-y_o), \tag{21}$$

where A_α^o are integration constants. This model corresponds, as expected, to an exact AdS_5 bulk.

Thus, we are left to consider negative values for C, and we rewrite it as $C=-\omega^2 B^2$. By Eq. (15),

$$u(y) = B\sinh[\omega(y-y_o)], \tag{22}$$

and then Eq. (16) gives

$$A_\alpha(y) = A_\alpha^o + \frac{1}{4}\ln|u(y)| + C_\alpha v(y), \tag{23}$$

where $v' = 1/u$, so that

$$v(y) = \frac{1}{2\omega B}\ln\left\{\frac{\cosh[\omega(y-y_o)]-1}{\cosh[\omega(y-y_o)]+1}\right\}. \tag{24}$$

Finally, we can write the bulk metric solution as

$$e^{2A_\alpha} = N_\alpha^2|\sinh[\omega(y-y_o)]|^{1/2}\left\{\frac{\cosh[\omega(y-y_o)]-1}{\cosh[\omega(y-y_o)]+1}\right\}^{q_\alpha}, \tag{25}$$

where $q_\alpha = C_\alpha/\omega B$ (recall that $\omega B \neq 0$), and N_α are constants whose value can be chosen by rescaling coordinates, but which satisfy the constraint

$$\prod_{\alpha=0}^{3} N_\alpha^2 = B^2, \tag{26}$$

which follows from Eqs. (13) and (22). The exponents q_α are constrained by Eqs. (17) and (20):

$$|q_\alpha| \leq \frac{3}{4}. \tag{27}$$

Note that this is a more restrictive bound than the one found above only from Eq. (20).

We consider first the special case $C_1=C_2=C_3=-C_0/3$, with two possible sets of parameters in Eq. (25), namely $(q_0^\pm, q_i^\pm) = (\mp\frac{3}{4}, \pm\frac{1}{4})$. These two special cases are Schwarzschild-AdS_5, with $k=0$, written in Gaussian normal coordinates. (The $k=-1$ case corresponds to Bianchi V, and the $k=1$ case to Bianchi IX.) We see this via a coordinate transformation in the metric of Eq. (5):

$$r^4 = \frac{8\mu}{\omega^2}\{1\mp\cosh[\omega(y-y_o)]\}, \tag{28}$$

and the remaining coordinates are rescaled by constants that depend on μ, ω, and N_α. It follows that q_α^+ leads to a negative mass μ, which we exclude, so that q_α^- is the physical solution (with a black hole horizon).

This shows that our general five-dimensional bulk solution, Eq. (25), can be seen as an *anisotropic generalization of Schwarzschild-AdS*$_5$. This distinguishes our anisotropic solution from the vacuum Kasner braneworld [34].

We now investigate the character of the singular point $y=y_o$ via curvature scalars. It turns out to be more convenient to use a new set of constants,

$$\frac{d_1}{2} = q_2+q_3,\quad \frac{d_2}{2} = q_1+q_3,\quad \frac{d_3}{2} = q_1+q_2, \tag{29}$$

with

$$d_1^2+d_2^2+d_3^2 = 3. \tag{30}$$

The isotropic cases $q_\alpha^\pm$ correspond to the points $(\pm1,\pm1,\pm1)$ on the 2-sphere (30). The square of the bulk Weyl tensor, $\mathcal{C}^2 = {}^5C_{ABCD}{}^5C^{ABCD}$ is given by

$$\mathcal{C}^2 = \frac{\omega^4}{16\sinh^4[\omega(y-y_o)]}\{21-(d_1^4+d_2^4+d_3^4) + 18\cosh^2[\omega(y-y_o)] + 36 d_1 d_2 d_3 \cosh[\omega(y-y_o)]\}. \tag{31}$$

The behavior near y_o is

$$\mathcal{C}^2 \underset{y\to y_o}{\to} \frac{39+36 d_1 d_2 d_3 - (d_1^4+d_2^4+d_3^4)}{16(y-y_o)^4}. \tag{32}$$

There are four sets of constants d_i that lead to zero numerator: $D_1=(-1,1,1)$, $D_2=(1,-1,1)D_3=(1,1,-1)$, and $D_4=(-1,-1,-1)$. For these cases, $\mathcal{C}^2$ is regular at y_o:

$$\mathcal{C}^2=\frac{9\omega^4}{8\{\cosh[\omega(y-y_o)]+1\}^4}. \tag{33}$$

For all other values of the d_i's, $y=y_o$ is a curvature singularity. The cases D_1, D_2, and D_3 are equivalent in the sense that they represent the same spacetime. The case D_4 is Schwarzschild-AdS$_5$ (with positive mass), and $y=y_o$ corresponds to the horizon of the black hole, since $f(r(y=y_o))=0$, and not to the singularity. Thus the Gaussian coordinates only cover the exterior of the black hole.

Far from $y=y_o$, $\mathcal{C}^2$ decays exponentially,

$$\mathcal{C}^2 \underset{y\to\infty}{\longrightarrow} \frac{9}{2}\omega^4 e^{-2\omega y}. \tag{34}$$

This behavior, which is completely independent of the parameters d_i (or C_i), means that our general anisotropic solution is asymptotically AdS$_5$. The square of the bulk Riemann tensor, the Kretschmann scalar $\mathcal{R}^2={}^5R_{ABCD}{}^5R^{ABCD}$, is

$$\mathcal{R}^2=\mathcal{C}^2+\frac{5}{32}\omega^4. \tag{35}$$

III. EMBEDDING OF THE BRANE

In order to obtain Bianchi I cosmological models the embedding of the brane must respect the Bianchi I symmetries, so the most general embedding is

$$t=S(\tau), \quad x^i=X^i, \quad y=Y(\tau), \tag{36}$$

where $\{\tau,X^i\}$ are local coordinates on the brane. The normal to the brane is

$$n_A dx^A=\frac{\epsilon}{\sqrt{1-V^2}}(-Ve^{A_0}dt+dy). \tag{37}$$

Here $\epsilon=\pm 1$ determines the orientation of the normal, and V is a function defined by the coordinate velocity of the brane,

$$V^2=\frac{\dot{Y}^2}{1+\dot{Y}^2}, \tag{38}$$

so that $|V|\leqslant 1$. The functions S and Y are not independent; since $n_A dx^A$ must vanish identically on the brane,

$$\dot{S}^2=(1+\dot{Y}^2)e^{-2A_0(Y)}. \tag{39}$$

We use a local foliation of the bulk such that the brane is itself a hypersurface of the foliation. This foliation is described by the normal (37), with V being now a function of y. The brane is then determined by the choices (36) and (38). An alternative way of determining the location of the brane, which will be also useful, is to prescribe the function $V(y)$ and then the embedding is given by Eqs. (38) and (39) up to an integration constant.

We introduce the vectors

$$u_A dx^A=\frac{1}{\sqrt{1-V^2}}(-e^{A_0}dt+Vdy), \tag{40}$$

$$\mathrm{e}^i_A dx^A=e^{A_i}dx^i, \tag{41}$$

which together with n_A form an orthonormal basis for the bulk. The vector u^A is a four-velocity tangent to the foliation, and hence to the brane. The condition (39) ensures that τ is the proper time on the brane of the observers with four-velocity u^A.

The metric inherited by the brane and other hypersurfaces of the foliation is the first fundamental form,

$$g_{AB}={}^5g_{AB}-n_An_B, \tag{42}$$

so that

$$g_{tt}=-\frac{e^{2A_0}}{1-V^2}, \quad g_{ij}=e^{2A_i}\ \delta_{ij}, \tag{43}$$

$$g_{ty}=-Ve^{-A_0}g_{tt}, \quad g_{yy}=V^2e^{-2A_0}g_{tt}. \tag{44}$$

The extrinsic curvature (second fundamental form) is

$$K_{AB}=\frac{1}{2}\pounds_n g_{AB}=g_A^C g_B^D\, {}^5\nabla_C n_D, \tag{45}$$

where $\pounds$ is the Lie derivative and $K_{AB}=K_{(AB)}$, $K_{AB}n^B=0$. Then

$$K_{tt}=-\epsilon\frac{e^{2A_0}}{(1-V^2)^{3/2}}\left\{A_0'+\frac{VV'}{(1-V^2)}\right\}, \tag{46}$$

$$K_{ij}=\epsilon\frac{e^{2A_i}}{\sqrt{1-V^2}}A_i'\,\delta_{ij}, \tag{47}$$

$$K_{ty}=-Ve^{-A_0}K_{tt}, \tag{48}$$

$$K_{yy}=V^2e^{-2A_0}K_{tt}, \tag{49}$$

with trace

$$K=\frac{\epsilon}{\sqrt{1-V^2}}\left\{\frac{u'}{u}+\frac{VV'}{(1-V^2)}\right\}. \tag{50}$$

Using Eq. (38), one can give a geometrical interpretation to terms in the extrinsic curvature. The factor $\sqrt{1-V^2}$ is $\sqrt{1+\dot{Y}^2}$, i.e., the inverse of the arclength of the embedding function $Y(\tau)$, whereas the VV' term can be written as $\ddot{Y}/(1+\dot{Y}^2)$, i.e., the curvature times the arclength.

A. Projection of the bulk Weyl tensor onto the brane

The modified Einstein equations (1) on the brane contain the projection of the bulk Weyl tensor [9],

$$\mathcal{E}_{AC}={}^{5}C_{ABCD}n^{B}n^{D}, \tag{51}$$

which is symmetric, tracefree and orthogonal to n^A. Relative to any observer, and in particular the observer with the preferred four-velocity u^A, this can be decomposed as [42,43]

$$\mathcal{E}_{AB}=-\kappa^2\left\{\mathcal{U}\left(u_Au_B+\frac{1}{3}h_{AB}\right)+2\mathcal{Q}_{(A}u_{B)}+\mathcal{P}_{AB}\right\}, \tag{52}$$

where $h_{AB}=g_{AB}+u_Au_B$ projects into the comoving rest space of u^A, $\mathcal{U}$ is the Weyl energy density on the brane, $\mathcal{Q}_A$ is the Weyl momentum flux on the brane, and $\mathcal{P}_{AB}$ is the Weyl anisotropic stress on the brane.

Bianchi-I symmetry enforces $\mathcal{Q}_A=0$, while

$$\mathcal{U}=\frac{1}{2\kappa^2u^2}\left\{C_0(u'-2C_0)+\frac{3}{8}\omega^2B^2\right\}, \tag{53}$$

$$\mathcal{P}_{AB}=\frac{1}{\kappa^2}\sum_{i=1}^{3}\mathcal{P}_i\mathrm{e}_{iA}\mathrm{e}_{iB},\quad \sum_{i=1}^{3}\mathcal{P}_i=0, \tag{54}$$

where

$$\mathcal{P}_i=\frac{C_0+3C_i}{3u}\left(\frac{u'}{4u}+\frac{C_0}{u}\right)-\frac{1}{u(1-V^2)}\left[(C_0+3C_i)\left(\frac{u'}{4u}+\frac{C_i}{u}\right)+\frac{\omega^2B^2-16C_i^2}{4u}\right]. \tag{55}$$

Clearly, $\mathcal{P}_{AB}=0$ for the isotropic case.

IV. BRANEWORLD MATTER FIELDS

While the induced metric is continuous, there are discontinuities in its first derivatives across the brane, so that there is a jump in the extrinsic curvature. In the case of Z_2 symmetry with the brane as a fixed point, the junction conditions determine the brane energy-momentum tensor in terms of the extrinsic curvature:

$$T_{AB}-\lambda g_{AB}=-\frac{2}{\kappa_5^2}(K_{AB}-Kg_{AB}). \tag{56}$$

The energy-momentum tensor can be decomposed, relative to observers with four-velocity u^A, as

$$T_{AB}=\rho u_Au_B+ph_{AB}+2q_{(A}u_{B)}+\pi_{AB}, \tag{57}$$

where ρ, p, q_A, and π_{AB} are, respectively, the energy density, isotropic pressure, momentum density, and anisotropic stresses measured by u^A.

For a Bianchi I braneworld, the symmetries enforce $q_A=0$. From Eqs. (46)–(49) and (56), we find that for our Bianchi I models,

$$\rho+\lambda=\frac{2\epsilon}{\kappa_5^2\sqrt{1-V^2}}\left(\frac{C_0}{u}-\frac{3}{4}\frac{u'}{u}\right), \tag{58}$$

$$p-\lambda=\frac{2\epsilon}{\kappa_5^2\sqrt{1-V^2}}\left\{\frac{3}{4}\frac{u'}{u}+\frac{1}{3}\frac{C_0}{u}+\frac{VV'}{(1-V^2)}\right\}. \tag{59}$$

$$\pi_{AB}=\sum_{i=1}^{3}\pi_i\mathrm{e}_{iA}\mathrm{e}_{iB},\quad \sum_{i=1}^{3}\pi_i=0, \tag{60}$$

where

$$\pi_i=-\frac{2\epsilon}{\kappa_5^2\sqrt{1-V^2}}\left(\frac{C_0+3C_i}{3u}\right). \tag{61}$$

The anisotropic directional pressures $p_i=p+\pi_i$ are

$$p_i-\lambda=\frac{2\epsilon}{\kappa_5^2\sqrt{1-V^2}}\left\{\frac{3}{4}\frac{u'}{u}-\frac{C_i}{u}+\frac{VV'}{(1-V^2)}\right\}. \tag{62}$$

In our case, since we do not have momentum density, the vanishing of the anisotropic stresses implies a perfect-fluid energy-momentum tensor. This happens if and only if $C_0=-3C_i$, for all i; i.e., *the brane can support a perfect fluid if and only if the metric is isotropic*. Furthermore, Eqs. (55) and (61) show that *the Weyl anisotropic stresses $\mathcal{P}_i$ vanish if and only if the matter anisotropic stresses π_i vanish*. Therefore, geometric anisotropy enforces, via the extrinsic curvature and the junction conditions, anisotropy in the matter fields. This may be a peculiar feature of our solution, based on the ansatz Eq. (10). However, it may be a generic feature of anisotropic cosmological braneworlds.

The fluid kinematics of the matter are described by the expansion, $\Theta=\nabla_Au^A$, the shear, $\sigma_{AB}=[h^C_{(A}h^D_{B)}-\frac{1}{3}h^{CD}h_{AB}]\nabla_Cu_D$, the vorticity, $\omega_{AB}=h^C_{[A}h^D_{B]}\nabla_Du_C$, and the acceleration, $\dot{u}^A=u^B\nabla_Bu^A$. For Bianchi symmetry, the matter flow is geodesic and irrotational, $\omega_{AB}=0=\dot{u}_A$. The expansion and shear for our Bianchi I braneworlds are given by

$$\Theta=\frac{V}{\sqrt{1-V^2}}\left(\frac{3}{4}\frac{u'}{u}-\frac{C_0}{u}\right), \tag{63}$$

$$\sigma_{AB}=\sum_{i=1}^{3}\sigma_i\mathrm{e}_{iA}\mathrm{e}_{iB},\quad \sum_{i=1}^{3}\sigma_i=0, \tag{64}$$

where

$$\sigma_i=\frac{V}{\sqrt{1-V^2}}\left(\frac{C_0+3C_i}{3u}\right). \tag{65}$$

Equations (58) and (63) imply

$$\rho+\lambda=-\frac{2\epsilon}{\kappa_5^2V}\Theta. \tag{66}$$

To ensure that the brane is expanding for positive energy density, we require $\epsilon/V<0$. Equations (61) and (65) also imply

$$\pi_{AB}=-\frac{2\epsilon}{\kappa_5^2 V}\sigma_{AB}. \tag{67}$$

We have checked that our expressions satisfy the generalized Friedmann equation for a Bianchi brane [13,42]:

$$\frac{1}{9}\Theta^2=\frac{\kappa^2}{3}\rho\left(1+\frac{\rho}{2\lambda}\right)-\frac{\kappa^2}{4\lambda}\pi^{AB}\pi_{AB}+\frac{\Lambda}{3}+\frac{\kappa^2}{3}\mathcal{U}+\frac{1}{6}\sigma^{AB}\sigma_{AB}. \tag{68}$$

V. SOME EXPLICIT MODELS

We can construct explicit cosmological models using the freedom still available in embedding the 3-brane. Choosing the parameters q_α, which are subject to the constraints (17) and (18), defines the bulk spacetime. The embedding $Y(\tau)$ is a function of one variable (proper time), and involves the freedom to choose the direction of the normal n^A, and the sign ϵ which defines its orientation.

One way to construct a particular cosmology on the 3-brane is to prescribe the density ρ. Using Eq. (58), V can then be obtained as a function of y,

$$V^2=1-\left[\frac{2}{\kappa_5^2(\rho+\lambda)}\left(\frac{C_0}{u}-\frac{3}{4}\frac{u'}{u}\right)\right]^2. \tag{69}$$

Then the embedding is completely determined by integrating,

$$\tau-\tau_o=\pm\int^y\frac{\sqrt{1-V^2}}{V}dy, \tag{70}$$

which gives an implicit form of the function $Y(\tau)$. However, one cannot use any physical argument or intuition in order to start with a cosmologically relevant density ρ as a function of the coordinate of the extra dimension.

A more appealing procedure is to start by prescribing the embedding function $Y(\tau)$, or equivalently the redefined function

$$x(\tau)=\omega[y(\tau)-y_o]. \tag{71}$$

Then $V(\tau)$ and $S(\tau)$ are given by Eqs. (38) and (39), respectively, and $\rho(\tau)$ by Eq. (58). Using this approach, we investigate under what circumstances a Minkowski brane can be embedded in our anisotropic bulk, how we can recover the standard embedding of a FRW brane in the isotropic case, and, finally, several examples of the embedding of anisotropic branes in a general anisotropic bulk.

A. Embedding of a Minkowski brane

The simplest embedding is $x(\tau)=x_*=\text{const}\ (>0)$, which implies $V=0$, so that the $g_{AB}(\tau,X^i,x_*)$ are constant, by Eqs. (43) and (44), and the 3-brane is Minkowskian. However, the matter variables are constants that do not vanish in general,

$$\rho_*=\lambda_c\left(\frac{3\cosh x_*-4q_0}{3\sinh x_*}\right)-\lambda, \tag{72}$$

$$p_*=\lambda-\lambda_c\left(\frac{9\cosh x_*+4q_0}{9\sinh x_*}\right), \tag{73}$$

so the models obtained in this way are not empty. Here

$$\lambda_c=-\epsilon\frac{6}{\ell\kappa_5^2}, \tag{74}$$

so that $|\lambda_c|$ is the critical tension corresponding to the RS fine tuning [1,2,9], i.e., for which $\Lambda=0$:

$$\Lambda=\frac{3}{\ell^2}\left[\left(\frac{\lambda}{\lambda_c}\right)^2-1\right]. \tag{75}$$

In general the matter fluid will not be perfect because the anisotropic stresses (61) only vanish when the bulk spacetime is isotropic ($q_0+3q_i=0$, for all i). Note that taking $\epsilon=-1$ [see Eq. (74)], there always exists a finite positive λ such that $\rho>0$. For instance, in the isotropic case where $q_0=\frac{3}{4}$, this condition is satisfied by any λ such that

$$\lambda<\lambda_c\sqrt{\frac{\cosh x_*-1}{\cosh x_*+1}}. \tag{76}$$

In general, the brane cannot be empty: if $\rho_*=0=p_*$, $\pi_{AB}=0$, then $q_\alpha=0$, which is incompatible with the constraint equation (18).

If we embed the brane at $x_*\ll 1$, then

$$\rho_*\sim\lambda_c\left[\frac{3-4q_0}{3x_*}+\frac{x_*}{2}\right]-\lambda. \tag{77}$$

Then, for $-\frac{3}{4}\leqslant q_0<\frac{3}{4}$, the matter density grows very large unless the brane tension λ is also unrealistically large. In the isotropic case, it decreases as x_* approaches the black hole horizon at $x=0$, where it becomes negative. On the other hand, if we place the brane at a large distance, $x_*\gg 1$,

$$\rho_*\sim\lambda_c-\lambda,\quad p_*\sim-\rho_*. \tag{78}$$

This result is independent of the parameters q_α so it means we can have a nearly vacuum brane embedded in our anisotropic bulk solution for large enough x_* if we choose λ as the critical brane tension λ_c. The existence of this embedding is something one should have expected *a priori*, because our five-dimensional solution asymptotically approaches an AdS_5 spacetime for large x.

To sum up, we have shown that we can *embed a non-vacuum Minkowskian brane in a general anisotropic bulk*. In order to make the 3-brane empty we have to locate it asymptotically far from $y=y_0$. These results generalize the findings of [10] that a Minkowski brane can be embedded in a (isotropic) Schwarzschild-AdS_5 bulk.

B. Embedding of a FRW brane

In the isotropic case, for a bulk spacetime with q_α^+ which correspond to the point D_4 on the sphere (30), we follow [11,12] and choose

$$x(\tau)=\operatorname{arccosh}\left[\frac{\omega^2}{8\mu}a^4(\tau)-1\right]. \tag{79}$$

Substituting in Eq. (58), we get the effective Friedmann equation (7) with $k=0$, thus recovering the results of [6,9,11,12] for an isotropic bulk. Note that in this case the anisotropic stress tensor (60) is identically zero and the matter on the brane is a perfect fluid. A combination of Eqs. (58) and (59) also leads to the effective Raychaudhuri equation [42],

$$\dot{H}=-H^2-\frac{\kappa^2}{6}(\rho+3p)-\frac{\kappa^2}{6\lambda}\rho(2\rho+3p)-\frac{\mu}{a^4}+\frac{\Lambda}{3}. \tag{80}$$

The cosmological dynamics of this case have been extensively investigated for a barotropic linear equation of state [14,15].

C. Embedding of an anisotropic brane

The metric tensor on the brane has Bianchi I form,

$$ds^2=-d\tau^2+\sum_{i=1}^{3}a_i^2(\tau)(dx^i)^2, \tag{81}$$

and the mean scale factor of the universe is $a(\tau)=(a_1a_2a_3)^{1/3}$. There are infinite ways of constructing these models as there are infinite ways of prescribing the embedding. Here we just present a few examples.

Example I. We have

$$x(\tau)=\Omega\tau, \tag{82}$$

with $0\leqslant\tau<\infty$ and $\Omega>0$. In this case ρ grows very large at early times and asymptotically reaches a constant value at late times. Positivity of the energy density requires that

$$0<\lambda\leqslant\lambda_c\sqrt{1+\left(\frac{\Omega}{\omega}\right)^2}, \tag{83}$$

where the equality corresponds to an asymptotically vacuum universe. The anisotropic stress vanishes at late times and the fluid becomes perfect. The universe expands exponentially in the far future,

$$a(\tau)\underset{\tau\to\infty}{\longrightarrow}e^{\Omega\tau/4}, \tag{84}$$

independent of the constants q_α defining the bulk spacetime. In the early universe,

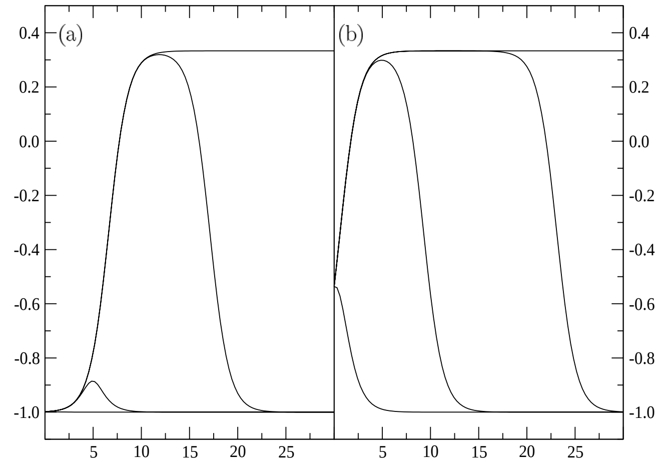

FIG. 1. Evolution of the equation of state w as a function of proper time τ/Ω, Eq. (86), for (a) $q_0=-0.001$ and (b) $q_0=-0.4$. The curves in both graphs correspond to values $\alpha=1$, 0.999 999 999, 0.9999, 0.1 from top to bottom.

$$a(\tau)\underset{\tau\to 0}{\longrightarrow}\tau^{(3-4q_0)/12}, \quad a_i(\tau)\underset{\tau\to 0}{\longrightarrow}\tau^{(1-4q_i)/4}. \tag{85}$$

[The exponent in $a(\tau)$ is always positive.] These cosmological models do not isotropize as we approach the initial singularity, in contrast with the results of [13–15], where assumptions were imposed on the Weyl anisotropic stresses. This example shows that *the Weyl anisotropic stresses can affect significantly the dynamical behavior near the singularity*.

Note also that despite the fact that the universe is collapsing in the past, there exist models within the family of solutions for which at least one spatial dimension could be expanding (e.g., $q_3=-3/4, q_0=q_1=q_2=1/4$). In the future the $a_i(\tau)$ approach the mean radius $a(\tau)$ and all the models become isotropic. For the embedding (82) the equation of state has a simple analytical expression,

$$w(\tau)=-\left[\frac{(1-\alpha)e^{\Omega\tau}+(1+\alpha)e^{-\Omega\tau}+8q_0/9}{(1-\alpha)e^{\Omega\tau}+(1+\alpha)e^{-\Omega\tau}-4q_0/3}\right], \tag{86}$$

where $\alpha=\lambda/\lambda_c\sqrt{1+(\Omega/\omega)^2}$ is a normalized brane tension, with $0<\alpha\leqslant 1$ by Eq. (83). For $q_0>0$ the equation of state becomes singular as τ increases. For $q_0=0$ and any value of α, we have $w=-1$. As α approaches its maximum value, the equation of state has a transient period with $w>0$ before reaching the constant value -1. However, when $\alpha=1$, w tends to 1/3, i.e. the matter behaves as a radiation fluid, even though the expansion is increasing exponentially due to geometrical effects. Some examples of the fluid behavior admitted by the embedding are shown in Fig. 1.

Example II. We have

$$x(\tau)=4\beta\ln(\Omega\tau), \tag{87}$$

with $\Omega^{-1}\leqslant\tau<\infty$ and $\Omega,\beta>0$. The qualitative behavior is very similar to that of example I. Here the brane tension has to satisfy the new condition $0<\lambda\leqslant\lambda_c$ instead of Eq. (83) in order to have $\rho>0$. The universe isotropizes in the future, with mean radius

$$a(\tau)\underset{\tau\to\infty}{\to}(\Omega\tau)^{\beta}, \tag{88}$$

which includes radiation-domination ($\beta=\frac{1}{2}$), matter-domination ($\beta=\frac{2}{3}$), and power-law inflation ($\beta>1$).

Example III. We have

$$x(\tau)=\operatorname{arccosh}\left[\frac{(\Omega\tau)^{2\beta}+1}{(\Omega\tau)^{2\beta}-1}\right], \tag{89}$$

with $\Omega^{-1}\leqslant\tau<\infty$ and $\Omega,\beta>0$. The scale factors are

$$a_i(\tau)=\left[\frac{2(\Omega\tau)^{\beta(1-4q_i)}}{(\Omega\tau)^{2\beta}-1}\right]^{1/4}, \tag{90}$$

so that each spatial direction can have different rates of expansion/contraction, and the universe does not isotropize in the future, unlike examples I and II. However, the models do isotropize in the past. The mean scale factor shows that all these models are expanding in the past and collapsing in the future. (In [15] a similar qualitative behavior was found in a Bianchi I brane when the mass of the bulk black hole is negative.) In this case the matter content never behaves as a perfect fluid.

VI. DISCUSSION

We have constructed complete (brane+bulk geometry) cosmological braneworlds with anisotropy. These solutions are the first such models with matter content. Our ansatz starts from a static form for the bulk metric, Eq. (10), with the brane moving relative to the static frame. The anisotropy arises from imposing Bianchi symmetries on a family of homogeneous 3-surfaces. For the sake of simplicity, we only considered the Abelian Bianchi I case, but other groups can be treated following the same approach.

There are two important aspects of the construction of cosmological braneworlds.

The bulk geometry. In our case, this is given by Eq. (25), where the parameters q_α control the anisotropy. The anisotropic bulk curvature produces a nonzero Weyl anisotropic tensor $\mathcal{P}_{AB}$ which, as shown in the examples of the previous section, can have a fundamental impact on the dynamics. Previous studies of Bianchi braneworld dynamics which impose *ad hoc* assumptions on $\mathcal{P}_{AB}$ are unable to treat consistently the relation between the bulk and brane geometries.

The embedding. This is where most of the freedom arises (a function of one variable). As shown by the examples in the previous section, the dynamics are very sensitive to the embedding. From the physical point of view, this leads to the question of what is the most natural state of movement for a brane. However, this question cannot be answered in the phenomenological context of the RS2 scenarios.

In choosing the embedding of the brane it is very important to consider the following general feature of our models: when the brane is close to $y=y_o$, the effects of the anisotropy are important for the cosmological dynamics, whereas when it is located far from $y=y_o$, we have an effectively FRW cosmological model (in an anisotropic bulk). This fact can be seen from the relative shear eigenvalues,

$$\frac{\sigma_i}{\Theta}\underset{x\gg1}{\to}\frac{\ell}{9}\frac{C_0+3C_i}{u}\underset{x\to\infty}{\to}0. \tag{91}$$

This is illustrated by examples I and II, where in the future the brane isotropizes, and also by example III, where by contrast, the brane approaches FRW in the past.

A striking feature of our models is that geometric anisotropy on the brane, from the Bianchi symmetry, imposes via the bulk curvature and the junction conditions, anisotropy on the matter content of the brane. In other words, it is not possible within our family of models to have a perfect fluid matter content (including the case of a minimally coupled scalar field)—anisotropic pressure in the matter is unavoidable unless the brane geometry reduces to Friedmann isotropy. This feature may be a consequence of the simplicity of the bulk metric ansatz that we used, but it raises an interesting challenge, i.e., to find complete Bianchi braneworld solutions with perfect fluid matter and nonzero anisotropy.

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