Phantom cosmologies

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We discuss a class of phantom ($p < -\varrho$) cosmological models. Except for the phantom we admit various forms of standard types of matter and discuss the problem of singularities for these cosmologies. The singularities are different from those of standard matter cosmology since they appear for infinite values of the scale factor. We also find an interesting relation between the phantom models and standard matter models which is like the duality symmetry of string cosmology.

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I. INTRODUCTION

The observation of distant supernovae [1,2] has given evidence for an accelerating universe filled with matter which violates the strong energy condition $\varrho + 3p \geqslant 0$, where p is the pressure and ϱ is the energy density. However, a more detailed check of the data suggests that also the matter which violates the weak energy condition $\varrho + p \geqslant 0, \varrho \geqslant 0$ is admissible at a high confidence level [3–10].

A possible violation of the energy conditions puts in doubt some fundamental achievements of general relativity since most of its basic theorems rely on various energy conditions (null, strong, weak, dominant, averaged null, averaged strong, and averaged dominant). As we know the Hawking-Penrose singularity theorems, the positive mass or Bondi mass theorems, the laws of black hole thermodynamics, and the cosmic censorship conjecture all rely on the energy conditions [11]. Also, for spacetimes which violate the weak energy condition wormholes can exist and so causality violation emerges. However, a necessity to explain observational data makes us revise these fundamentals and address some more general theories.

The physical background for strongly negative pressure matter (phantom) may be looked for in string theory [12]. The point is that the group velocity of a wave packet is negative which leads to behavior of the packet which is somewhat unusual—namely, it moves in the direction opposite to the momentum. In such circumstances a wave packet which leaves the comoving volume transfers momentum into the volume which makes negative contribution to the pressure [13,14].

Formally, one can get the phantom by switching the sign of kinetic energy of the standard scalar field Lagrangian, i.e., by taking $\mathcal{L} = -(1/2)\partial_\mu \phi \partial^\mu \phi - V(\phi)$ which gives the energy density $\varrho = -(1/2)\dot{\phi}^2 + V(\phi)$ and the pressure $p = -(1/2)\dot{\phi}^2 - V(\phi)$ and leads to $\varrho + p = -\dot{\phi}^2 < 0$ [3,15–21]. The phantom type of matter may also arise from a bulk viscous stress due to particle production [22] or in higher-order theories of gravity [23], Brans-Dicke and non-minimally coupled scalar field theories [24,25].

The cosmological models which allow for phantom matter appear naturally in the mirage cosmology of the brane-world scenario [26] and in kinematically driven quintessence (k-essence) models [27].

It has been shown that the matter which violates the strong energy condition allows the cyclic (oscillating, non-singular) universes [28]. It seems that similar solutions should also appear for the phantom matter which violates the weak energy condition.

A different idea of a cyclic universe in which the contracting big crunch phase can be connected with an expanding big bang phase [29] has been revived recently in the context of M-theory cosmology motivated ekpyrotic scenario [30]. In this reference it has been shown in Ref. [31] that the only way to make a transition from a contracting phase to an expanding phase in a flat universe is to violate the weak energy condition and this gives another motivation for studying phantom cosmologies.

Some other proposals which may have something in common with the phantom are Cardassian models in which one adds an extra power in the energy density term to the Friedmann equation [32–34], or modified Einstein-Hilbert action models in which an extra term of arbitrary (both positive and negative) power of scalar curvature appears in the gravitational action [35–37]. Both these possibilities may be directly motivated by string/M theory [38,39].

II. BASIC SYSTEM OF EQUATIONS WITH THE PHANTOM

The phantom is a new type of cosmological fluid which has a very strong negative pressure which violates the weak energy condition [4,5,12], i.e., it obeys the equation of state

$$p = (\gamma - 1)\varrho = w\varrho, \tag{1}$$

with negative barotropic index

$$\gamma = w + 1 < 0, \tag{2}$$

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and this implies that $p < -\varrho$.

In order to study phantom cosmologies, we start our discussion with the basic system of equations for an isotropic and homogeneous Friedmann universe which reads as

$$\kappa^2 \varrho = -\Lambda + 3\frac{k}{a^2} + 3\frac{\dot{a}^2}{a^2}, \tag{3}$$

$$\kappa^2 p = \Lambda - 2\frac{\ddot{a}}{a} - \frac{k}{a^2} - \frac{\dot{a}^2}{a^2}, \tag{4}$$

where $a(t)$ is the scale factor, $k=0,\pm 1$ the curvature index, Λ the cosmological constant, and $\kappa^2 = 8\pi G$ is the Einstein constant.

Using Eqs. (3), (4) one gets the Friedmann equation in the form

$$\frac{\dot{a}^2}{a^2} = \frac{\kappa^2}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}. \tag{5}$$

After imposing conservation law $\varrho a^{3\gamma} = (3/\kappa^2) C_\gamma = \text{const}$, one gets Eq. (5) as follows:

$$\frac{1}{a^2}\left(\frac{da}{dt}\right)^2 = \frac{C_\gamma}{a^{3\gamma}} - \frac{k}{a^2} + \frac{\Lambda}{3}. \tag{6}$$

The cases which involve all types of cosmological fluids, i.e., with $\gamma=4/3$ (radiation), $\gamma=1$ (dust), $\gamma=2/3$ (cosmic strings), $\gamma=1/3$ (domain walls), and $\gamma=0$ (cosmological constant) have been exactly integrated in terms of elliptic functions [28]. In fact, a large class of oscillating (nonsingular) solutions have been found.

Now we extend this discussion of the exact cosmological solutions into the case of phantom matter with $\gamma=-1/3$ (we will call it phantom) and $\gamma=-2/3$ (we will call it superphantom). One should emphasize that the "border" between "standard" and phantom models is given by the value of barotropic index

$$\gamma = 0 \quad (w=-1). \tag{7}$$

Due to the appearance of the higher than four powers of the scale factor in the Friedmann equation one cannot integrate a general case which involves phantom matter by elliptic functions. However, a lot of interesting special solutions can be found without using the elliptic functions and this is the matter of the present paper. We also include stiff-fluid matter with $\gamma=2$.

From the conservation law we have

$$\varrho \propto a^{-3\gamma} \quad \text{for} \quad \gamma > 0 \tag{8}$$

and

$$\varrho \propto a^{3|\gamma|} \quad \text{for} \quad \gamma < 0 \quad (\text{phantom}). \tag{9}$$

From Eq. (9) it is clear that big-bang and big-crunch singularities appear at infinite values of the scale factor for phantom cosmologies. This is also obvious after studying the simplest solution of Eq. (6) for $k=\Lambda=0$, which reads

$$a(t) \propto t^{2/3\gamma}, \tag{10}$$

so that

$$\varrho \propto t^{-2}. \tag{11}$$

In other words, taking $\gamma=-1/3$ (phantom) one has $a(t)\to\infty$ and $\varrho\to\infty$ if $t\to 0$, while $a(t)\to 0$ and $\varrho \to 0$ if $t\to\infty$. On the other hand, in a standard case $\gamma=1/3$, for example, one has $a(t)\to 0$ and $\varrho\to\infty$ if $t\to 0$, while $a(t)\to\infty$ and $\varrho\to 0$ if $t\to\infty$. Similarly, one can show that the curvature invariants are proportional to the energy density and so they are divergent wherever the density/scale factor diverges

$$R = \kappa^2(3\gamma-2)\varrho,$$

$$R_{\mu\nu}R^{\mu\nu} = \frac{\kappa^4}{4}\left[(3\gamma-2)^2 + \frac{1}{3}(3\gamma+2)^2\right]\varrho^2. \tag{12}$$

Another interesting remark can be extracted from Eqs. (3), (4) and (8), (9) if we admit shear anisotropy σ_0^2/a^6 ($\sigma_0 = \text{const}$) and consider nonisotropic Bianchi type IX models. Namely, for $\gamma<0$, the shear anisotropy cannot dominate over the phantom matter on the approach to a singularity when $a\to\infty$, i.e., we have

$$\varrho a^{3|\gamma|} > \frac{\sigma_0^2}{a^6} \quad \text{for } a\to\infty \tag{13}$$

and this prevents the appearance of chaotic behavior of the phantom cosmologies of the Bianchi type IX [40,41].

III. PHANTOM MODELS DUALITY

The system of equations (3), (4) can be presented in the form of the nonlinear oscillator

$$\ddot{X} - \frac{D^2}{3}\Lambda X + D(D-1)kX^{1-2/D} = 0, \tag{14}$$

after introducing the variables [42]

$$X = a^{D(w)}, \quad D(w) = \frac{3}{2}(1+w). \tag{15}$$

For flat $k=0$ models the oscillator (mathematical pendulum) is in the lower equilibrium position provided $\Lambda<0$ and in the upper equilibrium position provided $\Lambda>0$. It is also easy to notice that Eq. (14) preserves its form under the change

$$D \to -D, \tag{16}$$

or

$$\gamma \to -\gamma \quad [w \to -(w+2)]. \tag{17}$$

It appears that there is a duality (similar to the scale-factor duality in pre-big-bang models which is motivated by superstring theory duality symmetries [43,44]) between the scale factor and its inverse $a \to a^{-1}$ for standard matter ($\gamma > 0$) and phantom matter ($\gamma < 0$) models. Namely, the standard models for $w = \gamma - 1$ are dual to phantom models for $w' = -(2+w) = -\gamma - 1$ with respect to $\gamma = 0$ line. For example, the domain wall models $\gamma_w = 1/3$ are dual to phantom models $\gamma = -1/3$ and cosmic string models ($\gamma_{cs} = 2/3$) are dual to superphantom models. However, this simple duality is valid only for flat models (see also Refs. [45,46] where similar results for both flat and nonflat models have been obtained and in the nonflat case the analogy might perhaps be the non-Abelian duality [47]). In the former case we have (for walls, the phantom, and $\Lambda < 0$)

$$a_w = \left(\sin \frac{|D|^{1/2}}{\sqrt{3}} |\Lambda| t \right)^{1/2}, \tag{18}$$

$$a_{ph} = \left(\sin \frac{|D|^{1/2}}{\sqrt{3}} |\Lambda| t \right)^{-1/2}, \tag{19}$$

so that we have

$$a_w = a_{ph}^{-1}, \quad D_w = 1/2 = -D_{ph}. \tag{20}$$

From these we can conclude that standard fluids like dust $\gamma = 1$, radiation $\gamma = 4/3$, and stiff-fluid $\gamma = 2$ are dual to phantoms with $\gamma = -1$ ($p = -2\varrho$), $\gamma = 4/3$, and $\gamma = -2$ ($p = -3\varrho$), respectively.

It is important to notice that duality in scale factor does not lead to the avoidance of singularity in the energy density which results from Eqs. (8), (9) and shows that whatever the behavior of the scale factor, the density diverges leading to a big-bang singularity.

IV. PHANTOM COSMOLOGICAL MODELS

For the sake of a possible comparison with observational data we introduce dimensionless density parameters [38,48]

$$\Omega_{x0} = \frac{\kappa^2}{3H_0^2} \varrho_{x0}, \tag{21}$$

$$\Omega_{K0} = \frac{K}{H_0^2 a_0^2}, \tag{22}$$

$$\Omega_{\Lambda_0} = \frac{\Lambda_0}{3H_0^2}, \tag{23}$$

where $x \equiv sp, ph, w, cs, m, r, st$ (superphantom, phantom, domain walls, cosmic strings, dust, radiation, and stiff-fluid, respectively), $H = \dot{a}/a$, and $q = -\ddot{a}a/\dot{a}^2$, while [48]

$$\Omega_{K'0} = \Omega_{cs0} - \Omega_{K0}, \tag{24}$$

which means that the Friedmann equation (6) can be written down in the form

$$\Omega_{sp0} + \Omega_{ph0} + \Omega_{\Lambda_0} + \Omega_{w0} + \Omega_{K'0} + \Omega_{m0} + \Omega_{r0} + \Omega_{st0} = 1. \tag{25}$$

Now using the variables

$$y = \frac{a}{a_0}, \quad u = H_0 t, \tag{26}$$

one turns Eq. (6) into the form

$$\left(\frac{dy}{du}\right)^2 = \Omega_{sp0} y^4 + \Omega_{ph0} y^3 + \Omega_{\Lambda_0} y^2 + \Omega_{w0} y + \Omega_{K'0} + \Omega_{m0} y^{-1} + \Omega_{r0} y^{-2} + \Omega_{st0} y^{-4} \equiv Q(y). \tag{27}$$

A. Negative pressure fluids only phantom models

General solutions of Eq. (27) are given in terms of elliptic or hyperelliptic functions. Here we concentrate only on exact elementary solutions leaving a general discussion for a separate paper [49]. First, we assume nonvanishing negative pressure fluids only (except negative Λ-term which gives positive pressure), i.e., $\Omega_{m0} = \Omega_{r0} = \Omega_{st0} = 0$, so that we have

$$\left(\frac{dy}{du}\right)^2 = \Omega_{sp0} y^4 + \Omega_{ph0} y^3 + \Omega_{\Lambda 0} y^2 + \Omega_{w0} y + \Omega_{K'0}. \tag{28}$$

Note that after a change of variables

$$p \equiv \frac{1}{y}, \tag{29}$$

we have

$$\left(\frac{dp}{du}\right)^2 = \Omega_{sp0} + \Omega_{ph0} p + \Omega_{\Lambda 0} p^2 + \Omega_{w0} p^3 + \Omega_{K'0} p^4, \tag{30}$$

which may also be useful for exact integration.

Now we consider some special cases and refer to their mathematical discussion presented in the Appendix.

1. No-wall models with $\Omega_{w0} = \Omega_{K'0} = 0$

This case falls under a general classification given in the Appendix with $a_2 = \Omega_{\Lambda 0} \in \mathbf{R}$, $a_1 = \Omega_{ph0} \geqslant 0$, and $a_0 = \Omega_{sp0} \geqslant 0$ (we exclude the possibility of both phantoms lacking, i.e., $\Omega_{sp0} = \Omega_{ph0} = 0$). The possible solutions are thus (cf. the Appendix):

(1) $\Omega_{sp0} = 0$, $\Omega_{ph0} > 1$, and $\Omega_{\Lambda 0} < 0$ (case 1.3.2, cf. Fig. 7);
(2) $\Omega_{sp0} > 0$, $\Omega_{ph0} > 1 - \Omega_{sp0}$, and $\Omega_{\Lambda 0} < 0$ (case 1.3.3);
(3) $\Omega_{sp0} > 0$, $\Omega_{ph0} \in [0; 2\sqrt{\Omega_{sp0}}(1 - \sqrt{\Omega_{sp0}})]$, and $\Omega_{\Lambda 0} > 0$ (case 2.1);
(4) $\Omega_{sp0} < 1$, and $\Omega_{\Lambda 0} > 0$ (case 2.2.3);
(5) $\Omega_{sp0} = 0$, $0 < \Omega_{ph0} < 1$, and $\Omega_{\Lambda 0} > 0$ (case 2.3.4); and

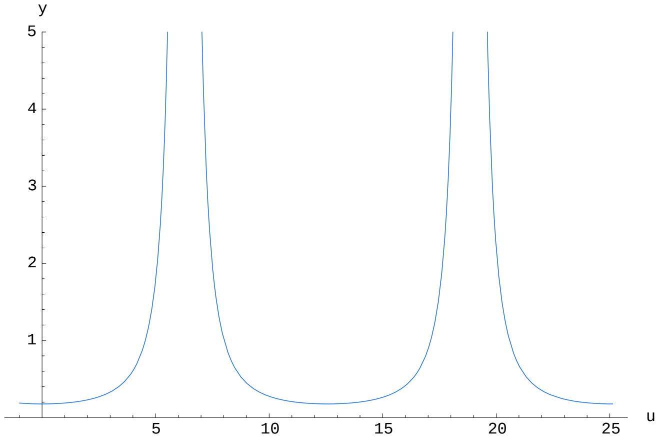

FIG. 1. The "bounce" solution (31). The big-bang/big-crunch singularities appear for $y \to \infty$ since $\varrho \propto y \to \infty$ at these points so that this "bounce" in scale factor does not lead to singularity avoidance.

(6) $\Omega_{sp0} > 0$, $0 < \Omega_{ph0} < 1 - \Omega_{sp0}$, and $\Omega_{\Lambda 0} > 0$ (case 2.3.5).

Most of these solutions [cases (3), (4), (5), (6)] pass through zero in x, so there arise infinite values of y for finite times. Bearing in mind Eq. (9) one can notice that this corresponds to the energy density approaching zero. However, in cases (1) and (2) one obtains cyclic "bounce" in scale factor y solution (which is also a "bounce" in the energy density ϱ but not singularity avoidance since $\varrho \to \infty$ at finite u) of the form

$$y = \frac{2\Omega_{\Lambda 0}}{\sqrt{\Omega_{ph0}^2 - 4\Omega_{sp0}\Omega_{\Lambda 0}} \sin[\sqrt{|\Omega_{\Lambda 0}|}(u-u_0)] - \Omega_{ph0}} \tag{31}$$

which is shown in Fig. 1. In fact, there is a competition between the positive pressure of a negative cosmological term and the negative pressure of a phantom, but one is not able to avoid singularity on the same basis as with negative cosmological constant and domain walls [28].

Similar behavior is present in case (3), though now there is only one infinity—either the scale factor collapses from infinite into a finite size in finite time, and continues to shrink asymptotically to zero, or it expands from zero at $u = -\infty$, and reaches infinite size in a finite time. This is shown in Fig. 2. In fact, this behavior is similar to what one has in pre-big-bang cosmology [44] with y_- as a pre-big-bang branch and y_+ as a post-big-bang branch.

Cases (4), (5), and (6) are analogous to case (3). The appropriate formulas are obtained from those of Sec. 2.3 in the Appendix, but the graphs are practically like those in Fig. 2.

2. Phantom only models: $\Omega_{\Lambda 0} = \Omega_{w0} = \Omega_{K'0} = 0$

If both phantoms only are present in Eq. (28) we have

$$y = \frac{1}{\Omega_{ph0}\left(\frac{u-u_0}{2}\right)^2 - \frac{\Omega_{sp0}}{\Omega_{ph0}}} \tag{32}$$

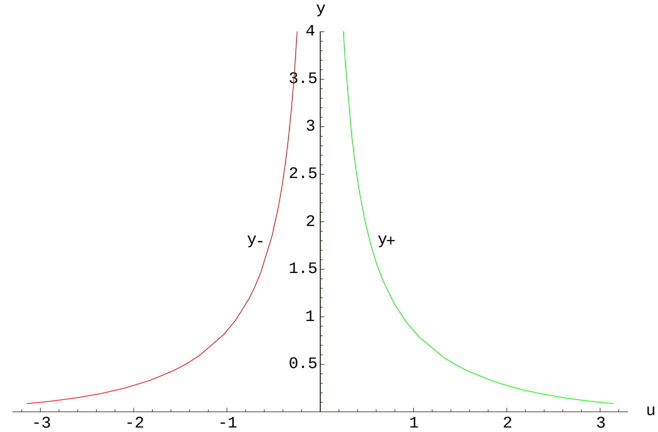

FIG. 2. Phantom with negative $\Omega_{\Lambda 0}$ models [case (2) of Sec. IV A 1] with two branches separated by curvature singularity (compare [44]).

for $\Omega_{ph0} + \Omega_{sp0} = 1$ and $\Omega_{ph0} \neq 0$ (see Fig. 3) while for $\Omega_{ph0} = 0$ we have

$$y_{\pm} = \pm \frac{1}{u - u_0}, \tag{33}$$

and the behavior of the model is as in Fig. 2.

B. Even powers of the polynomial $Q(y)$ only models

In this section we consider only the even powers of the polynomial $Q(y)$ in Eq. (27), i.e.,

$$\left(\frac{dy}{du}\right)^2 = \Omega_{sp0}y^4 + \Omega_{\Lambda 0}y^2 + \Omega_{K'0} + \Omega_{r0}y^{-2} + \Omega_{st0}y^{-4} \tag{34}$$

and make the substitution

$$z \equiv \frac{1}{y^2}, \quad d\eta \equiv \frac{2du}{y}, \tag{35}$$

so that Eq. (34) reads

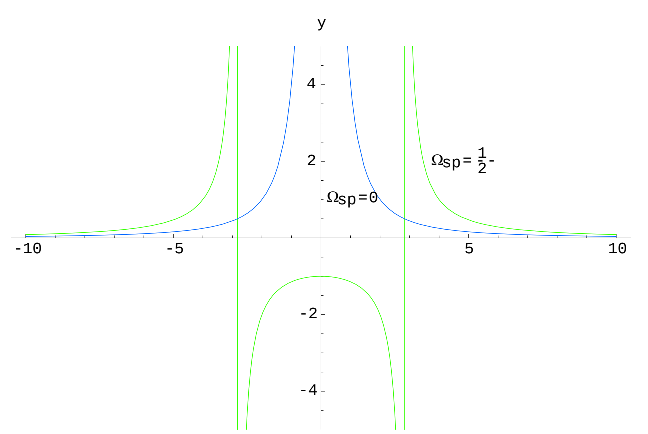

FIG. 3. The model (32) for two phantoms only. Two cases are shown: $\Omega_{sp0} = 0$, $\Omega_{ph0} = 1$ and $\Omega_{sp0} = \Omega_{ph0} = 1/2$.

$$\left(\frac{dz}{d\eta}\right)^2 = \Omega_{sp0} + \Omega_{\Lambda 0} z + \Omega_{K'0} z^2 + \Omega_{r0} z^3 + \Omega_{st0} z^4. \quad (36)$$

From now on let us assume that there is no radiation and stiff-fluid in Eq. (34), i.e., $\Omega_{r0} = \Omega_{st0} = 0$. The polynomial coefficients are then (cf. the Appendix): $a_2 = \Omega_{K'0}$, $a_1 = \Omega_{\Lambda 0}$, and $a_0 = \Omega_{sp0}$. The possible solutions are

(1′) $\Omega_{K'0} < 0$, $\Omega_{\Lambda 0} > 1 - \Omega_{sp0}$ (case 1.3.3);

(2′) $\Omega_{K'0} > 0$, $\Omega_{\Lambda 0} \in [-2\sqrt{\Omega_{sp0}}(1+\sqrt{\Omega_{sp0}})$; ΩΛ0∈[−2Ωsp0(1+Ωsp0);$2\sqrt{\Omega_{sp0}}(1-\sqrt{\Omega_{sp0}})]$ (case 2.1);

(3′) $\Omega_{K'0} > 0$, $\Omega_{\Lambda 0} = -2\sqrt{\Omega_{sp0}}(1+\sqrt{\Omega_{sp0}})$ when $\Omega_{sp0} < 1$, and $\Omega_{\Lambda 0} = -2\sqrt{\Omega_{sp0}}(\sqrt{\Omega_{sp0}} \pm 1)$ when $\Omega_{sp0} > 1$ (case 2.2.1);

(4′) $\Omega_{K'0} > 0$, $\Omega_{sp0} < 1$ and $\Omega_{\Lambda 0} = 2\sqrt{\Omega_{sp0}}(1-\sqrt{\Omega_{sp0}})$ (case 2.2.3);

(5′) $\Omega_{K'0} > 0$, $\Omega_{\Lambda 0} < 0 \wedge \Omega_{\Lambda 0} \notin [-2\sqrt{\Omega_{sp0}}(1+\sqrt{\Omega_{sp0}})$; $2\sqrt{\Omega_{sp0}}(1-\sqrt{\Omega_{sp0}})]$ (case 2.3.1); and

(6′) $\Omega_{K'0} > 0$, $0 < \Omega_{\Lambda 0} < 1 - \Omega_{sp0}$ (case 2.3.5).

All of these contain cases where z passes through zero, and that requires a closer analysis. First, because the solutions themselves change, we are interested in $y(u) = 1/\sqrt{z[\eta(u)]}$, which, like before, may become infinite. Second, we are now working in a modified conformal time η, and to analyze the solution in the cosmological time $t = u/H_0$, we need to consider the convergence of the integral

$$u = \int \frac{d\eta}{2\sqrt{z}}.$$

Fortunately, all the solutions are of trigonometric or exponential form, so that the integral reduces to a convergent elliptic one. In other words, the values of u are finite for finite values of η. Moreover, in some cases the integral converges with $\eta \to \infty$. The details are given in particular cases.

Although the relation between y and z is different than in Sec. IV A, the qualitative properties remain the same, that is to say, where the same classes of solutions apply.

Thus, case (1′) is again a "bounce" shown in Fig. 1.

Case (2′) is a peculiar model in which the aforementioned integrals converge, and u has finite values for both the point in which the size is infinite and when it is zero ($\eta \to \infty$). This is depicted in Fig. 4.

For the solutions of case (3′) the shape of the function $z(\eta)$ introduced in Eq. (35) is given in Fig. 9 with x replaced by z and η replaced by u in the graph. However, we also present these solutions in terms of the scale factor $y(u)$ in Fig. 5. From the diagram we can see that there is a static solution $y = \text{const}$ which corresponds to a double root of Eq. (34) with $\Omega_{r0} = \Omega_{st0} = 0$. Note that the static model falls outside the classification given in the Appendix since the Hubble parameter is equal to zero in this case [which is the result of the rescaled definition of the time parameter u in Eq. (26)]. Apart from the static model we have four asymptotic solutions: two of them asymptotically approach the static model in future infinity while the other two start asymptotically with the static model at past infinity.

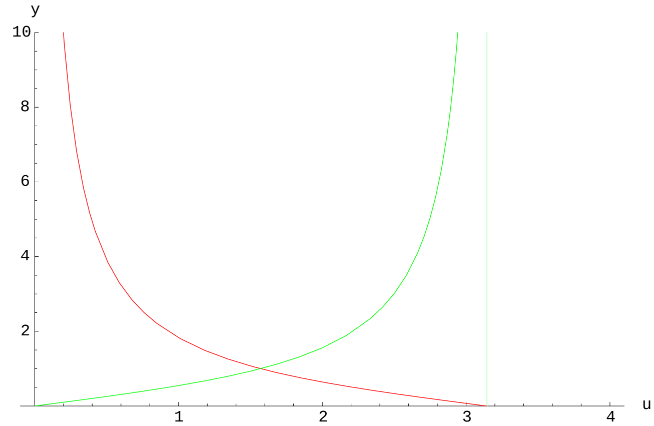

FIG. 4. The model with superphantom, cosmological constant and $\Omega_{K'0} \neq 0$ [solution (2′)].

Cases (4′) and (6′) are similar to case (2’) shown in Fig. 4.

For the solutions of case (5′), the shape of the function $z(\eta)$ introduced in Eq. (35) is given in Fig. 10 with x replaced by z and η replaced by u in the graph. In terms of the scale factor $y(u)$ these solutions are given in Fig. 6. One of them is similar to that of Fig. 1 (a cycle from $y \propto \varrho \to \infty$ to a minimum in y and ϱ and again to $y \propto \varrho \to \infty$) and the second describes a cycle in y and ϱ from zero to a maximum and again to zero.

C. Phantom, walls, and Λ-term models only

This is, in fact, a less general case than case (1), so it is a straightforward task to apply those solutions here. We have

$$\left(\frac{dy}{du}\right)^2 = \Omega_{ph0} y^3 + \Omega_{\Lambda 0} y^2 + \Omega_{w0} y + \Omega_{K'0} + \Omega_{m0} y^{-1}, \quad (37)$$

which after the change of variables

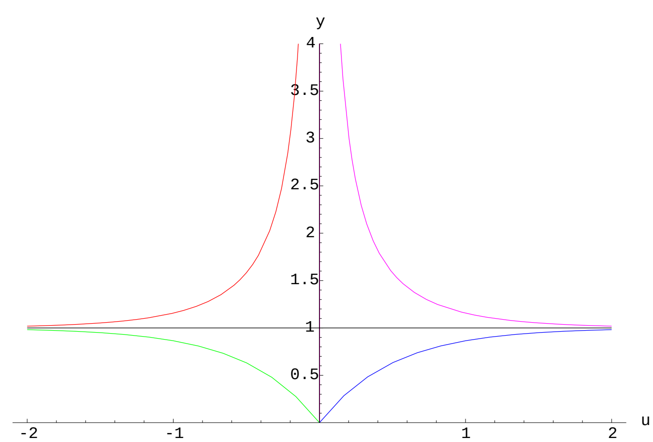

FIG. 5. The solution of case (3′). Apart from the static model (straight line in the middle) there are four asymptotic solutions: two of them asymptotically approach the static model in future infinity while the other two start asymptotically with the static model at past infinity.

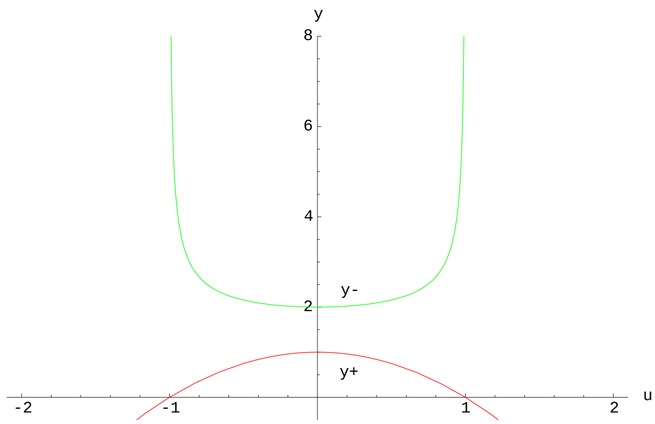

FIG. 6. The solutions of case (5′). The first is a cyclic solution from $y\propto\varrho\rightarrow\infty$ to a minimum in y and ϱ and again to $y\propto\varrho\rightarrow\infty$, and the second is a cycle in y and ϱ from zero to a maximum and again to zero.

$$p\equiv\frac{1}{y},\quad d\eta\equiv\frac{du}{\sqrt{y}} \tag{38}$$

gives

$$\left(\frac{dy}{d\eta}\right)^2=\Omega_{ph0}+\Omega_{\Lambda 0}p+\Omega_{w0}p^2+\Omega_{K'0}p^3+\Omega_{m0}p^4, \tag{39}$$

and this equation is integrable in terms of elliptic functions.

D. Radiation and superphantom only models

Leaving only the radiation pressure Ω_{r0} and superphantom Ω_{sp0} nonvanishing in Eq. (27) and making the substitution

$$r\equiv y^6,\quad d\eta\equiv y^4 du, \tag{40}$$

we get the simple equation

$$\left(\frac{dr}{d\eta}\right)^2=\Omega_{sp0}r+\Omega_{r0}, \tag{41}$$

which solves by

$$r=\frac{\left[\frac{1}{2}\Omega_{sp0}(\eta-\eta_0)\right]^2-1+\Omega_{sp0}}{\Omega_{sp0}}, \tag{42}$$

and we have used the condition that $\Omega_{r0}=1-\Omega_{sp0}$.

The typical evolution as considered in terms of the scale factor y is similar to that given in Fig. 4. Despite the fact that the scale factor y reaches either zero or infinity in both situations there are singularities, for $y\rightarrow 0$ it is dominated by the radiation so $\varrho\rightarrow\infty$ while for $y\rightarrow\infty$ it is dominated by superphantom and $\varrho\rightarrow\infty$, too.

E. Dust and phantom only models

Although this case is not elementary, we present it as an important result in which we do not abandon the basic component of the universe. Taking all the other terms in Eq. (27) equal to zero except for dust Ω_{m0} and phantom Ω_{ph0} and making the change of the time coordinate u as follows

$$d\eta=\sqrt{\Omega_{ph0}}\frac{du}{\sqrt{y}}, \tag{43}$$

we have the equation

$$\left(\frac{dy}{d\eta}\right)^2=y^4+\frac{\Omega_{m0}}{\Omega_{ph0}}, \tag{44}$$

which solves by

$$y^2=\frac{4\Omega_{m0}\mathcal{P}(\eta)}{4\Omega_{ph0}\mathcal{P}^2(\eta)-\Omega_{m0}}, \tag{45}$$

with $\mathcal{P}(\eta)$ the Weierstrass elliptic function [28].

Despite this solution being elliptic, the evolution is also as in Fig. 4. The first singularity appears when $y\rightarrow 0$ and it is dominated by dust while the second singularity appears when $y\rightarrow\infty$ and it is dominated by the phantom.

V. CONCLUSION

We have studied cosmological models with phantom $p<-\varrho$ matter which violates the weak energy condition. We have shown that phantom matter characterized by negative barotropic index $\gamma<0$ allows the curvature singularity for infinite values of the scale factor both in the past and in the future. We have also shown that the singularity cannot be dominated by shear so that unlike for standard $\gamma>0$ cosmologies in a class of nonisotropic Bianchi type IX phantom models there should be no chaotic behavior on the approach to a singularity. We have discussed a simple model of duality of the scale factor for phantom $(\gamma<0)$ and standard $(\gamma>0)$ matter solutions which is like the scale factor duality of pre-big-bang cosmology motivated by superstring theory duality symmetries.

We have presented a series of exact phantom cosmologies which integrate elementary, leaving the investigation of the full class of models which can be integrated in terms of elliptic functions for a future paper [49]. Apart from phantom matter we have admitted most of the known types of matter content in the Universe with discrete barotropic index γ in the equation of state such as dust, radiation, stiff-fluid, domain walls, cosmic strings, and cosmological constant.

One interesting class of solutions we obtained contains models which start with a singularity in which the scale factor is infinite $(a\rightarrow\infty)$, then decreases, reaching some minimum $a=a_{\min}$, finally expanding again into another singularity where $(a\rightarrow\infty)$. In the standard picture for $\gamma>0$ matter, the Universe follows the sequence: expansion–maximum–recollapse. On the other hand, for phantom $\gamma<0$ models one has the opposite sequence: collapse–minimum–expansion.

Another interesting class of solutions is characterized by either an expansion from zero energy density to a future curvature singularity, or a collapse from the infinite value of the scale factor into a state of zero energy density. There exists also static solutions (generalized Einstein static universes) together with asymptotic solutions though asymptotic solutions approach the static model either from singularity of the energy density (in which the scale factor $a \to \infty$), or from the state of zero energy density (in which the scale factor $a \to 0$).

Finally, in the models which contain a mixture of phantom and standard matter (e.g., dust, radiation) there exist two types of singularities (with the energy density $\varrho \to \infty$): one is dominated by the phantom with the scale factor $y \to \infty$ and another is dominated by standard matter with the scale factor $y \to 0$.

We remark that phantom cosmologies form an interesting set of the models of the universe which do not contradict observational data of supernovae type Ia and that their mathematical and physical properties deserve further study.

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APPENDIX: GENERAL CLASSIFICATION OF SOLUTIONS TO THE CANONICAL EQUATION

We take into account the equation

$$\left(\frac{dx}{du}\right)^2 = a_2 x^2 + a_1 x + a_0 = W(x) \qquad \text{(A1)}$$

together with the constraint

$$a_2 + a_1 + a_0 = 1, \qquad \text{(A2)}$$

and seek its solutions that might depict the evolution of the Universe, that is, such that $x \geqslant 0$. We also define e_1 and e_2 to be the roots of the equation $W(x) = 0$, and $\Delta = a_1^2 - 4a_0 a_2$. The main constraint on y is the non-negativity of the polynomial $W(x)$ implied by Eq. (A1).

The general solution of Eq. (A1) is

$$\frac{2a_2 x + a_1}{\sqrt{\Delta}} = \cosh[\sqrt{a_2}(u - u_0)] \qquad \text{(A3)}$$

where u_0 can, in general, be complex. The graphs depicting the particular cases were drawn with $u_0 = 0$, except for case 2.3.1 whose solution requires $u_0 = i\pi/2\sqrt{a_2}$. It should be kept in mind that the solutions may be arbitrarily shifted in the direction of the u axis.

1. $a_2 < 0$

1.1. $\Delta < 0$

This makes evolution impossible, as $W(x) < 0$ for all real values of y. Writing $\Delta = a_1^2 + 4a_0 a_1 + 4a_0(a_0 - 1)$, we get its discriminant $\tilde{\Delta} = 16a_0$, and the roots $a_{1\mp} = -2\sqrt{a_0}(\sqrt{a_0} \pm 1)$. Clearly, $a_0 < 0$ would lead to $\Delta > 0$, and $a_0 = 0$ implies $a_2 = 0$ which contradicts Eq. (A2). For $a_0 > 0$, we would need $a_1 \in (a_{1-}; a_{1+})$, but this does not hold together with Eq. (A2).

Such a case is impossible with the assumptions made.

1.2. $\Delta = 0$

As above we could have $\tilde{\Delta} = 0$, but this would lead to $a_0 = a_1 = 0$, and break Eq. (A2). If $\tilde{\Delta} > 0$ the only possibilities, namely $a_1 = a_{1\mp}$, are ruled out by the same condition.

1.3. $\Delta > 0$

Here we could have $a_0 > 0$, and $a_1 > a_{1+}$ or $a_1 < a_{1-}$ which, reasoning as above, gives in the end $a_1 = 1 - a_2 - a_0 > 1 - a_0$. Alternatively, $a_0 \leqslant 0$, leads directly to $a_1 > 1 - a_0$.

1.3.1. $e_1 > e_2 > 0$

This requires $a_1 > 0, a_0 < 0$ which can hold together with $a_1 > 1 - a_0$. The solution is periodic in the form:

$$x = \frac{\sqrt{\Delta}\cos[\sqrt{|a_2|}(u - u_0)] - a_1}{2a_2}. \qquad \text{(A4)}$$

1.3.2. $e_1 > e_2 = 0$

Necessarily $a_0 = 0$ and $a_1 = 1 - a_2 > 1$. The solution is simply

$$y = \frac{a_1}{2a_2}(\cos[\sqrt{|a_2|}(u - u_0)] - 1). \qquad \text{(A5)}$$

It passes through zero, unlike the previous one.

1.3.3. $e_1 > 0 > e_2$

The conditions here are $a_0 > 0$ and $a_1 > 1 - a_0$. The solution now passes through zero twice, which means it is no longer periodic, but depicts a single "cycle." The formula is identical to Eq. (A4).

1.3.4. $0 \geqslant e_1 > e_2$

This case would require $a_0 \leqslant 0$ and $a_1 < 0$, which cannot hold together with Eq. (A2). The plots of 1.3.1, 1.3.2, and 1.3.3 are given in Fig. 7.

2. $a_2 > 0$

2.1. $\Delta < 0$

This is only possible with $\tilde{\Delta} > 0 \Rightarrow a_0 > 0$, $a_1 \in (a_{1-}; a_{1+})$, which is now a stronger restriction than $a_1 = 1 - a_2 - a_0 < 1 - a_0$. There are two solutions now, depending on the choice of the derivative sign in Eq. (A1):

$$x_\pm = \frac{\pm\sqrt{|\Delta|}\sinh[\sqrt{a_2}(u - u_0)] - a_1}{2a_2}. \qquad \text{(A6)}$$

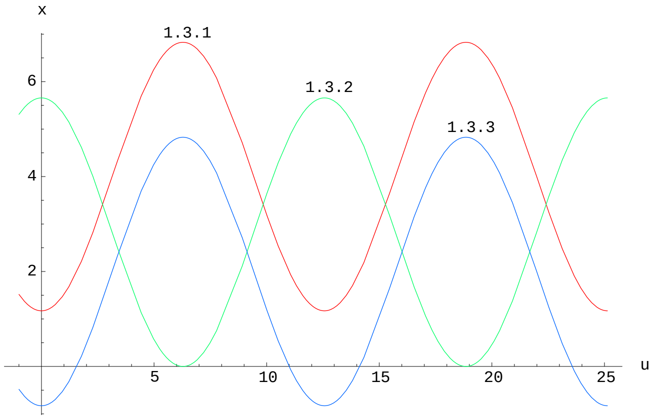

FIG. 7. Cases 1.3.1, 1.3.2, and 1.3.3.

These are, respectively, monotonic expansion (from zero size to infinity in infinite time) or monotonic collapse (in finite time from finite size). This is plotted in Fig. 8.

2.2. $\Delta=0$

Note that $a_0<0$ would lead to $\tilde{\Delta}<0$ and hence $\Delta>0$, and this is not the case.

2.2.1. $e_1=e_2>0$

$a_0>0$, so $a_1=a_{1-}<0$, and for $a_0>1$ we also have $a_1=a_{1+}<0$. For $x>e_1=-a_1/2a_2$ the evolution is either an unbounded expansion, or an asymptotical collapse.

$$x_{\pm}=\frac{-a_1}{2a_2}(1+e^{\pm\sqrt{a_2}(u-u_0)}). \tag{A7}$$

When $0\leqslant x<e_1$, the expansion is asymptotical, and collapse occurs in finite time.

$$x_{\pm}=\frac{-a_1}{2a_2}(1-e^{\pm\sqrt{a_2}(u-u_0)}). \tag{A8}$$

Also, $x=e_1$ is a static solution which is unstable due to nearby asymptotic solutions. The plot of 2.2.1 is given in Fig. 9.

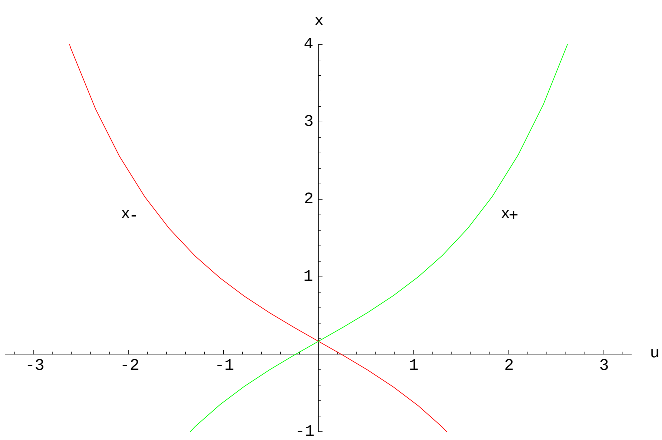

FIG. 8. Case 2.1.

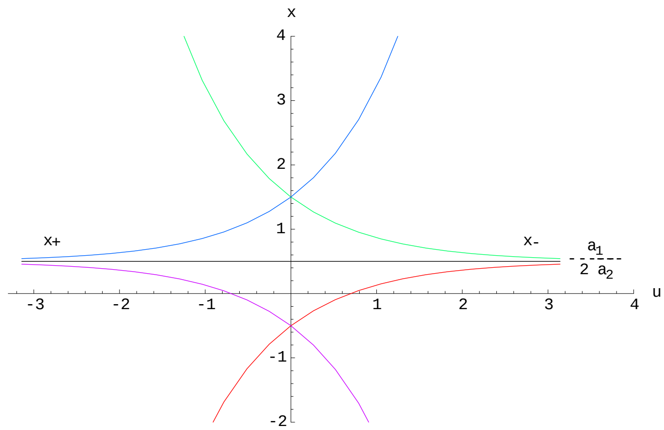

FIG. 9. Case 2.2.1. Case 2.2.2 may be pictured if $a_1=0$; case 2.2.3 when $-a_1/2a_2<0$.

2.2.2. $e_1=e_2=0$

We immediately get $a_1=a_0=0$ which yields the following solution:

$$x_{\pm}=e^{\pm\sqrt{a_2}(u-u_0)}. \tag{A9}$$

2.2.3. $e_1=e_2<0$

$a_1=a_{1+}>0$, requiring $a_0<1$. This case is almost identical to 2.2.1, only the root is now negative, eliminating the solutions $x\leqslant e_1$. The only possibility for collapse is for it to happen in finite time now.

2.3 $\Delta>0$

The condition $a_1<1-a_0$ applies to all the subcases here. Depending on a_0 there are also further restrictions:

(1) $a_0>0 \Rightarrow \tilde{\Delta}>0$, so $a_1\notin[a_{1-};a_{1+}]$.
(2) $a_0=0 \Rightarrow a_1\neq 0$.
(3) $a_0<0 \Rightarrow$ no other conditions.

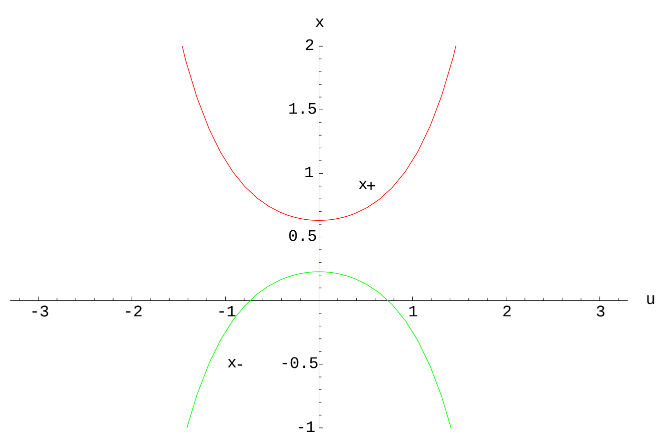

FIG. 10. Case 2.3.1. Other cases 2.3.2, 2.3.3, 2.3.4, and 2.3.5 are obtained when the graph is shifted down in the direction x so that x_- passes through zero, $x_-<0$ and $x_+>0$, x_+ passes through zero, and x_+ intersects the abscissa at two points, respectively.

In general, the situation resembles that in 1.3. All the graphs look the same, the only difference being a shift in the y direction.

2.3.1. $e_1 > e_2 > 0$

$a_0 > 0$ and $0 > a_1$ give two admissible regions of evolution. For $x \geqslant e_1$, we have a "bounce" (infinite size is reached in infinite time, though). When $0 \leqslant x \leqslant e_1$, the evolution is at most one "cycle" similar to that of 1.3.3. To be exact:

$$x_{\pm} = \frac{\pm\sqrt{\Delta}\cosh[\sqrt{a_2}(u-u_0)] - a_1}{2a_2}. \qquad \text{(A10)}$$

The plot of 2.3.1 is given in Fig. 10.

2.3.2. $e_1 > e_2 = 0$

This is almost identical to the previous case, only now x_- passes through zero, which is its only non-negative value, thus giving rise to a static universe of zero size. x_+ behaves as above. Here, we have $a_0 = 0$ and $a_1 < 0$.

2.3.3. $e_1 > 0 > e_2$

$a_0 < 0$ and $a_1 < 1 - a_0$ now. Again, it is like 2.3.1 with x_- discarded.

2.3.4. $e_1 = 0 > e_2$

$a_0 = 0$ and $0 < a_1 < 1$. This is another "bounce" solution which passes through a possible singularity at $x = 0$.

2.3.5. $0 > e_1 > e_2$

$a_0 > 0$ and $0 < a_1 < 1 - a_0$. The solution is separated into two independent ones, describing expansion or collapse, both going through zero.

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