Conserved cosmological perturbations

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A conserved cosmological perturbation is associated with each quantity whose local evolution is determined entirely by the local expansion of the Universe. It may be defined as the appropriately normalized perturbation of the quantity, defined using a slicing of spacetime such that the expansion between slices is spatially homogeneous. To first order, on superhorizon scales, the slicing with unperturbed intrinsic curvature has this property. A general construction is given for conserved quantities, yielding the curvature perturbation ζ as well as other more recently considered conserved perturbations. The construction may be extended to higher orders in perturbation theory and even into the non-perturbative regime.

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I. INTRODUCTION

Observation of the peak structure in the cosmic microwave background anisotropy has now confirmed that cosmological perturbations are present before the relevant scales enter the horizon, with an almost flat (scale-invariant) spectrum [1–3]. The only known explanation for this state of affairs is that the perturbations originate during an almost exponential inflation, from the vacuum fluctuation of one or more light scalar fields.[1] In the simplest case only one light field is responsible for the perturbations observed, either the inflaton or some other field.

According to this explanation, classical cosmological perturbations first come into existence a few Hubble times after horizon exit during inflation. At that stage the situation is very simple; each light field (defined as one with an effective mass much less than the Hubble parameter H) has a Gaussian perturbation with an almost flat spectrum, $(H/2\pi)^2$. The problem is to evolve this simple initial condition forward in time to the primordial nucleosynthesis epoch, in the face of our ignorance about the detailed evolution of the Universe before nucleosynthesis.

Fortunately, scales of cosmological interest are still far outside the horizon at nucleosynthesis. As a result there exist perturbations which are under suitable conditions conserved, and largely avoiding the need for more detailed information. One of these [8–11] is the "curvature perturbation" ζ, which is associated with the perturbation in the total energy density ρ.[2] In the usual case that ζ originates from the perturbation in the inflaton field, it is supposed to be conserved between the end of inflation and the primordial era, and in the alternative curvaton scenario [12,13] (see also [14,15]) ζ is supposed to be conserved after the curvaton decays.[3] Recently, further conserved quantities ζ_i and $\tilde{\zeta}_i$ have been considered that are associated with the perturbations in individual energy densities ρ_i [11] and number densities n_i of conserved quantities [19]. The conservation of the former is invoked in the curvaton scenario, during the era when the curvaton field is oscillating and ζ is growing. The latter are invoked when considering possible isocurvature components of the primordial density perturbation.

In this paper, we present a unified treatment of the conserved quantities ζ, ζ_i and $\tilde{\zeta}_i$, which is more complete than anything that has been given before. Taking the particular example of ζ as a starting point, we begin in Sec. II by showing how, to any order in cosmological perturbation theory, conserved quantities may be constructed from perturbations that are defined on a spacetime slicing of uniform integrated expansion. Here and throughout this paper we refer to the choice of temporal gauge, defining the spatial hypersurfaces of fixed coordinate time, as the spacetime *slicing* and the choice of spatial gauge, defining the worldlines of fixed spatial coordinates, as the *threading*. In Sec. III, we show that in the usual case of first-order perturbation theory, the spatially flat slicing is one of uniform expansion if the shear of the worldlines is negligible. In Sec. IV we consider the comoving shear, and show that it is expected to be negligible in the entire superhorizon regime. In Sec. V we generalize the construction and consider the conserved quantities ζ_i and $\tilde{\zeta}_i$. We conclude in Sec. VI. The appendices discuss some peripheral issues.

[1] A related hypothesis replaces inflation by an era of collapse ("pre-big-bang" [4,30,5] or "ekpyrotic" [31,29,6]), but there is so far no accepted theory of a bounce and therefore no firm prediction from collapsing cosmologies. In particular, there is so far no accepted string-theoretic description of a bounce [7].

[2] The quantity ζ defines the curvature perturbation on spacetime slices of uniform energy density. As we discuss in Sec. IV, on superhorizon scales it is practically the same as $\mathcal{R}$ which defines the curvature perturbation on slices orthogonal to comoving worldlines. The latter quantity is the $\phi_{\rm m}$ of [28].

[3] An analogous scenario has been proposed in the pre-big-bang scenario [16,17]. In this scenario though, the required scale-invariant curvaton field perturbations will be generated only if the curvaton has a non-trivial coupling and for particular initial conditions [18,5].

II. ENERGY CONSERVATION AND THE CURVATURE PERTURBATION

In this section we explain the general principle which allows us to construct conserved quantities. We focus on the particularly important example of the curvature perturbation ζ [8–11], after which it is clear how other conserved perturbations may be constructed. The curvature perturbation ζ is so called because it defines the curvature perturbation on slices of uniform energy density [11]. Equivalently though, via the gauge transformations of Sec. III, it defines the energy density perturbation on spatially flat slices, according to the formula [9]

$$\zeta = \frac{\delta\rho}{3(\rho+P)}. \tag{1}$$

This definition is the one that we shall use.

Our starting point is the energy continuity equation. In an unperturbed Friedmann-Robertson-Walker (FRW) universe the continuity equation for the energy density ρ takes the form

$$\dot{\rho} = -3H(\rho+P), \tag{2}$$

where H is the Hubble expansion rate and P is the pressure. In the real perturbed Universe, the same Eq. (2) still holds along each comoving worldline, so long as the dot is taken to denote the derivative with respect to the proper time τ along the comoving worldline and we define H locally through the equation

$$H \equiv \frac{1}{3}\mathcal{V}^{-1}\frac{d\mathcal{V}}{d\tau}, \tag{3}$$

where $\mathcal{V}$ is an infinitesimal comoving volume. Equivalently, the local continuity equation may be written as

$$\mathcal{V}\frac{d\rho}{d\mathcal{V}} = -(\rho+P), \tag{4}$$

or

$$\frac{d\rho}{dN} = -3(\rho+P), \tag{5}$$

where N is the local logarithmic integrated expansion (the number of Hubble times) defined as

$$N \equiv \int H d\tau. \tag{6}$$

Our crucial assumption now is that the pressure perturbation is practically adiabatic. This assumption means that the local pressure P is a practically unique function of local energy density ρ, i.e.,

$$P = \overline{P}(\rho), \tag{7}$$

where $\overline{P}$ is the same function for all worldlines. This allows Eq. (5) to be integrated. Setting $N=0$ on an initial spacetime slice the integration gives ρ as a unique function of the local integrated expansion N up to an initial integration constant:

$$\rho = \overline{\rho}(N+\delta N), \tag{8}$$

where the integration constant for each worldline, δN, is determined by the actual density on the initial hypersurface, $\rho|_{N=0} = \overline{\rho}(\delta N)$.

Subsequent spacetime slices of fixed N correspond to a *uniform integrated expansion* slicing of the spacetime,[4] meaning that the integrated expansion going from one slice to another is spatially homogeneous. For linear perturbations about a FRW cosmology, there is an infinity of such uniform-N slicings, since we can start with any initial slice and propagate it by calculating N from that slice along each comoving worldline. In Secs. III and IV, we show that on superhorizon scales a particular uniform-N slicing is the uniform curvature slicing (i.e., the one with unperturbed intrinsic scalar curvature). In what follows we will restrict our attention to spatially flat FRW models and will refer to this as the spatially flat slicing.

Now we come to the crucial point. When evaluating the density ρ on any uniform-N slicing, the perturbation δN of the quantity appearing in Eq. (8) is time independent, by construction. This statement holds to any order in cosmological perturbation theory so long as one can construct a uniform-N slicing along the comoving worldlines.

Writing δN in terms of the density perturbation on spatially flat slices, to first order, one finds the conserved quantity

$$\delta N = \frac{dN}{d\rho}\delta\rho = \frac{\delta\rho}{\rho'(N)} = H\frac{\delta\rho}{\dot{\rho}} \tag{9}$$

which is $-\zeta$ defined in Eq. (1). This derivation is close in spirit to the analysis of Sasaki and Stewart [20] who studied multi-field inflation models and identified the curvature perturbation with the perturbed expansion with respect to an initially flat slice. The relation between their calculation of the curvature perturbation and ours is explained in Appendix A.

To arrive at the conserved quantity ζ, we considered the flat slicing. Were we instead to use some other uniform-expansion slicing, the conserved quantity defined by the right hand side of Eq. (9) would be different from ζ, but it would be related to ζ by the gauge transformation (25). Hence it would be conserved if and only if ζ is conserved, and we lose no generality by fixing the choice of the uniform-expansion slicing as the flat one.

The constancy of ζ (on sufficiently large scales and assuming that the pressure perturbation is adiabatic) was obtained several years ago [8] in the context of Einstein gravity. More recently, its constancy under the same condition was obtained directly from the local conservation of energy [11]

[4]Note that this is *not* the same as the *uniform Hubble* slicing introduced by Bardeen [28,8] which refers to the local expansion rate of the normals.

using a purely geometric argument equivalent to the one that we have given. In the present paper, we are going to show in Secs. III and IV that in this context all superhorizon scales are "sufficiently large;" in other words, we will show that the flat slicing is a uniform-expansion one on all superhorizon scales.

It is worth noting that the conservation of ζ can hold in even more general circumstances, because it comes from the generalized adiabatic condition, Eq. (8), which may hold even if the energy conservation equation (5) fails. Thus, ζ will be conserved even if there is an additional source term Q on the right-hand side of Eq. (5), so long as Q (the energy transfer per Hubble time) is itself a unique function of the local density for all worldlines, i.e., $Q=\overline{Q}(\rho)$, as reported in Ref. [21].

Going to second order, and again working on some uniform-N slicing, the conserved quantity is

$$\delta N = \frac{dN}{d\rho}\delta\rho + \frac{1}{2}\frac{d^2N}{d\rho^2}(\delta\rho)^2 \tag{10}$$

$$= \frac{\delta\rho}{\rho'} - \frac{1}{2}\frac{\rho''}{\rho'^3}(\delta\rho)^2. \tag{11}$$

This second-order extension of the conserved quantity ζ has not been given before.[5] It will be useful in propagating forward the evolution of second-order perturbations produced during inflation [23–25] through the end of inflation and relating them to observations. Also, we note that Sasaki and Tanaka [23] have shown that it is possible to use a uniform-N slicing to study non-linear field perturbations on large scales during inflation.

III. UNIFORM EXPANSION BETWEEN FLAT SLICES

The main goal of this section is to show that the local expansion of the Universe between spatially flat slices is uniform on sufficiently large scales where shear is negligible. This result is purely geometric, making no reference to the theory of gravity. Our treatment amplifies the original one in Ref. [11], and in particular we show for the first time that the result is independent of the spacetime threading that defines the expansion.

A. The metric perturbation

The unperturbed FRW metric and comoving coordinates are defined by the line element

$$ds^2 = -dt^2 + a^2(t)\,\delta_{ij}dx^i dx^j, \tag{12}$$

corresponding to metric components $g_{00}=-1$, $g_{0i}=0$ and $g_{ij}=\delta_{ij}a^2$.

We are interested in the perturbed spacetime that is our Universe, which we assume can be described by linear perturbations about a FRW geometry. To define the perturbations one has to choose a coordinate system which reduces to Eq. (12) in the limit where the perturbations vanish. Such a coordinate system (gauge) defines a time slicing (the spatial hypersurfaces with constant time coordinate) and a threading (the worldlines with constant space coordinates) of the spacetime. Since the coordinate system is required to coincide with Eq. (12) in the limit where the perturbations vanish, the slicing and threading coincide with the unperturbed ones in that limit. We shall take this requirement for granted when referring to a "generic" slicing or threading.

Once the perturbations are defined, their evolution to first order may be described using the unperturbed coordinate system, and Fourier components with different wave vectors **k** decouple. The superhorizon regime is the regime $aH/k \gg 1$.

In this paper we are interested in the scalar mode of the perturbations in the metric (no gravitational waves or vorticity). In a generic gauge the Fourier components of the metric perturbation are specified by functions A, B, D and E,[6]

$$\tfrac{1}{2}\delta g_{00} \equiv -A \tag{13}$$

$$a^{-2}\delta g_{0i} \equiv -B_i \equiv i k_i B \tag{14}$$

$$\tfrac{1}{2}a^{-2}\delta g_{ij} \equiv \delta_{ij}D + P_{ij}E \tag{15}$$

$$= -\psi\delta_{ij} - \frac{k_i k_j}{k^2}E, \tag{16}$$

where P_{ij} projects out the traceless part;

$$P_{ij} \equiv -\frac{k_i k_j}{k^2} + \frac{1}{3}\delta_{ij} \tag{17}$$

and

$$-\psi \equiv D + \tfrac{1}{3}E. \tag{18}$$

Under the coordinate transformation $t \to t+\Delta t$ and $x^i \to x^i + \Delta x^i$, with the Fourier component of Δx^i of the form

$$\Delta x^i = -i\frac{k_i}{k}\Delta x, \tag{19}$$

the metric components transform according to

$$\Delta A = -\dot{\Delta t} \tag{20}$$

$$\Delta B = a\dot{\Delta x} + k\Delta t \tag{21}$$

$$\Delta D = -\frac{k}{3}\Delta x - H\Delta t \tag{22}$$

[5] It can be verified that the second-order part of δN coincides with the gauge-invariant quantity $-(\zeta_2/2)+\zeta_1^2$ defined in Ref. [22].

[6] These are the quantities defined in [26], and are equal respectively to the quantities A, $B^{(0)}$, H_L and $H_T^{(0)}$ of Bardeen [28]. The quantities Δx and Δt below are respectively the L and aT of Bardeen, and the quantity V in Eq. (45) is the $v_S^{(0)}$ of Bardeen.

$$\Delta E = k\Delta x \tag{23}$$

$$\Delta\psi = H\Delta t. \tag{24}$$

The perturbation E can be eliminated by a transformation of the spatial coordinates, while ψ depends only on the slicing. The latter defines the intrinsic scalar 3-curvature of the slicing, and is called its curvature perturbation.

The perturbation in a generic scalar quantity g defined in the unperturbed Universe, such as energy density, pressure or the value of a scalar field, has the transformation

$$\Delta g = -\dot{g}\Delta t. \tag{25}$$

Applied to the energy density, this transformation along with Eq. (24) leads to the gauge-invariant definition of the curvature perturbation ζ,

$$\zeta = -\psi - H\frac{\delta\rho}{\dot{\rho}}. \tag{26}$$

Evaluated on slices of uniform density it is indeed the curvature perturbation, but evaluated on flat slices it specifies instead the energy density perturbation through Eq. (1).

B. Shear and the expansion rate

A given threading of spacetime is associated with an expansion θ, a traceless symmetric shear σ_{ij} and an antisymmetric vorticity ω_{ij}. At a given spacetime point, in a locally inertial rest frame, these quantities are given by the standard decomposition [26,40],

$$\partial_i w_j = \tfrac{1}{3}\theta\delta_{ij} + \sigma_{ij} + \omega_{ij}, \tag{27}$$

where w_i is the three-velocity of the infinitesimally nearby threads. The expansion θ gives the rate of increase with respect to proper time τ of an infinitesimal volume $\mathcal{V}$ expanding with the threads,

$$\theta = \delta^{ij}\partial_i w_j = \frac{1}{\mathcal{V}}\frac{d\mathcal{V}}{d\tau}. \tag{28}$$

For the scalar perturbations that we are considering, the vorticity vanishes, and the shear takes the form

$$\sigma_{ij} = P_{ij}\sigma. \tag{29}$$

We shall call σ the shear as well.

In a given gauge, the metric perturbations determine the shear and expansion rate of the coordinate threads according to the expressions [27]

$$\sigma = \dot{E} \tag{30}$$

$$\delta\theta = 3\dot{D} - 3HA. \tag{31}$$

The second relation may be written as

$$\delta\theta = -3\dot{\psi} - \sigma - 3HA. \tag{32}$$

Going to a new threading corresponding to the transformation, Eq. (19), the change in the Fourier component of the local velocity field of the threads is

$$\Delta w_i = \Delta\dot{x}^i = -i\frac{k_i}{k}\Delta\dot{x}. \tag{33}$$

The corresponding changes in the shear and the expansion are equal and opposite,

$$\Delta(\delta\theta) = -\Delta\sigma = k\Delta\dot{x}, \tag{34}$$

while the perturbations ψ and A are unchanged. It follows that Eq. (32) is valid for any threading, not only for the coordinate threading. (In [11], Eq. (32) was given for the special case of the threading normal to the coordinate slicing.) Note also that $|\Delta(\delta\theta)| < k/a$ for $|\Delta v| = |a\Delta\dot{x}| < 1$ so the expansion of all worldlines becomes the same on large scales where $k \ll aH$.

Following [11] we consider, instead of θ, the expansion $\tilde{\theta}$ with respect to coordinate time. Its perturbation is

$$\delta\tilde{\theta} = -3\dot{\psi} - \sigma. \tag{35}$$

For the flat slicing $(\psi = 0)$ this becomes

$$\delta\tilde{\theta} = -\sigma. \tag{36}$$

This result is true for any choice of the threading that defines the expansion rate, and has been derived without any reference to a theory of gravity.

The expressions given so far are valid quite generally. We are interested though in superhorizon scales. On sufficiently large scales the shear must become negligible compared with the Hubble parameter, so that we recover an unperturbed FRW universe. It follows that $\delta\tilde{\theta}$ is negligible on sufficiently large scales, or in other words that the expansion between successive flat slices becomes unperturbed. This means that on sufficiently large scales we can make the approximation

$$\tilde{\theta}(\mathbf{x},t) = 3H(t), \tag{37}$$

where $H(t)$ is the usual unperturbed quantity and $\tilde{\theta}$ is the expansion with respect to coordinate time of a generic threading. (Remember that we consider only threadings which coincide with the unperturbed one in the limit of zero perturbation.)

Using Eq. (37), we can combine the results of Secs. II and III to derive the following general result for cosmological perturbations.

Consider a monotonically increasing or decreasing quantity f, defined in some region of spacetime, and its first-order perturbation δf defined on the spatially flat slicing. Consider also some threading of spacetime, defining an infinitesimal volume element $\mathcal{V}$. If f satisfies a local conservation equation of the form

$$\mathcal{V}\frac{df}{d\mathcal{V}} = y(f), \tag{38}$$

then the rate of change of the perturbation

$$X_f \equiv -H\frac{\delta f}{\dot{f}} \tag{39}$$

is

$$\dot{X}_f = \tfrac{1}{3}\sigma \tag{40}$$

where σ is the shear of the threading. As a result, X_f is conserved on sufficiently large scales, where the shear is negligible.

C. The variation of ζ

Taking the derivative of Eq. (26) and using the local conservation of energy along comoving worldlines one finds

$$\dot{\zeta} = -\frac{H}{\rho+P}\delta P_{\rm nad} - \sigma, \tag{41}$$

where σ is the shear of the comoving worldlines and the non-adiabatic part of the pressure perturbation is

$$\delta P_{\rm nad} \equiv \delta P - \frac{\dot{P}}{\dot{\rho}}\delta\rho. \tag{42}$$

(In Appendix C we comment on the apparent contradiction between the adiabatic condition for the pressure perturbation and the pressure perturbation defined on comoving slices during single-field inflation.)

This result was derived by essentially the above method in [11]. It was first derived (by a different method, and actually for the curvature perturbation $\mathcal{R}$) in [28], Eqs. (5.19)–(5.21). In the particular case that the Universe consists entirely of matter and radiation, ζ is given by Eq. (61). One easily checks [19] that this expression is compatible with Eq. (41).

IV. SHEAR ON SUPERHORIZON SCALES

According to Eq. (40), the quantity X_f is conserved on scales which are sufficiently large that σ is negligible. In this section we argue that any superhorizon scale is sufficiently large in this context. To be more precise, we argue that the shear satisfies

$$|\sigma|/H \ll (k/aH), \tag{43}$$

which ensures that in one Hubble time the change in X_f is less than k/aH.[7]

In making the argument, we shall invoke the Einstein field equations. This is not much of a restriction as we do not necessarily have to specify any physical origin for the stress-energy tensor, but simply equate it with a fixed multiple of the Einstein tensor derived from the metric. This leads to a purely geometrical definition of, e.g., the comoving density perturbation, which need have nothing to do with the motion of particles. However at relatively late cosmic times, it may be safe to assume that the Einstein field equations are satisfied with the stress-energy tensor related to particle physics content in the usual way.

The other assumption we make is that anisotropic stress is negligible on superhorizon scales. As noted by Bardeen in 1980 [28], significant anisotropic stress on superhorizon scales would generate shear and the curvature perturbation, but there is no known mechanism for generating such stress.

We first note that any spatial gauge transformation, Eq. (19), corresponds to a change in the local physical velocity $\Delta v_i \equiv a\Delta w_i$ [Eq. (33)]. This generates a change in the shear $|\Delta\sigma|/H = vk/aH$. Since we are dealing with small perturbations, $v \ll 1$ so that $|\Delta\sigma|/H \ll k/aH$.

It follows that we need only establish Eq. (43) for the shear of the comoving threading, which σ shall denote from now on. Generalizing the discussion of Bardeen [28] to include the case where P/ρ may vary, we shall show that in fact

$$|\sigma|/H \ll (k/aH)^2. \tag{44}$$

The comoving shear is related to two commonly used gauge-invariant variables, namely the curvature perturbation $-\mathcal{R}$ of slices orthogonal to comoving worldlines (comoving slices) and the curvature perturbation Φ of zero-shear hypersurfaces (the Bardeen potential):[8]

$$\frac{\sigma}{H} = \frac{k}{aH}V = \left(\frac{k}{aH}\right)^2(\mathcal{R}+\Phi). \tag{45}$$

(The quantity V defines the velocity v_i of the comoving worldlines relative to the zero-shear threading, through the relation $v_i = -i(k_i/k)V$ [26].) The curvature perturbation $\mathcal{R}$ is closely related to the curvature of uniform-density slices, ζ:

$$\mathcal{R} = \zeta - \frac{H\delta\rho_{\rm com}}{\dot{\rho}}, \tag{46}$$

where the subscript "com" denotes the comoving slicing.

The combined energy and momentum constraints of Einstein's equations relate the comoving density perturbation to the Bardeen potential:

$$\frac{H\delta\rho_{\rm com}}{\dot{\rho}} = \frac{2}{9(1+w)}\left(\frac{k}{aH}\right)^2\Phi, \tag{47}$$

where $w \equiv P/\rho$. Thus we can rewrite Eq. (45) for the comoving shear in terms of ζ and the Bardeen potential, giving

[7]On the left hand side of this expression, σ is the typical magnitude of the shear on scale k, defined for instance as [26] the square root of its spectrum $\mathcal{P}_\sigma(k)$.

[8]We are defining $-\mathcal{R} \equiv \psi$ with the right hand side evaluated on comoving slices, which corresponds to established conventions.

$$\frac{\sigma}{H}=\left(\frac{k}{aH}\right)^2\zeta+\left(\frac{k}{aH}\right)^2\left[1-\frac{2}{9(1+w)}\left(\frac{k}{aH}\right)^2\right]\Phi. \quad (48)$$

Equation (48) ensures that the comoving shear will be small ($\sigma/H\ll\zeta$) on superhorizon scales so long as the Bardeen potential Φ remains of the same order as ζ. Notice also that if the Φ remains finite on large scales, then the comoving density perturbation, Eq. (47), also vanishes on large scales, and $\mathcal{R}$ and ζ are equal (and constant) on large scales. (The vanishing of the density perturbation and the perturbation in the local Hubble rate on comoving slices on large scales is discussed in Appendix B.) However, it has been argued [28] that cosmological perturbation theory can still be valid even if certain curvature perturbations, in particular Φ, become formally bigger than one.

The Bardeen potential is not uniquely determined by the value of ζ, but the Einstein's equations give a first-order evolution equation [26] (in the absence of anisotropic stress [28])

$$H^{-1}\dot{\Phi}+\left[\frac{5+3w}{2}-\frac{1}{3}\left(\frac{k}{aH}\right)^2\right]\Phi=-\frac{3}{2}(1+w)\zeta. \quad (49)$$

During conventional slow-roll inflation (with $w\sim-1$ and $\dot{\Phi}\ll H\Phi$), the Bardeen potential is indeed small, $\Phi\sim-3(1+w)\zeta/2$ on superhorizon scales. But we wish to eliminate the possibility that the Bardeen potential subsequently becomes large on superhorizon scales.

We will restrict the rest of our discussion in this section to strictly adiabatic matter perturbations, with $\delta P_{\rm nad}=0$, but consider non-adiabatic perturbations in Appendix D. For adiabatic perturbations, Eqs. (41), (48) and (49) yield coupled first-order equations for the evolution of ζ and Φ, which to lowest order in k/aH gives

$$H^{-1}\dot{\zeta}\simeq\left(\frac{k}{aH}\right)^2\Phi, \quad (50)$$

$$H^{-1}\dot{\Phi}+\frac{5+3w}{2}\Phi=-\frac{3}{2}(1+w)\zeta, \quad (51)$$

which yield two independent long-wavelength solutions, which are represented by

$$\zeta\simeq C_+, \quad (52)$$

$$\Phi\simeq C_-e^{-(5+3\tilde{w})N/2}, \quad (53)$$

where $\tilde{w}N=\int w dN$. The first of these solutions, with constant ζ on large scales, remains the "growing mode" solution so long as

$$H^{-1}\dot{\zeta}\propto C_-k^2e^{3(2w-\tilde{w}-1)N/2}, \quad (54)$$

for the "decaying mode" on large scales, approaches zero. This is always true in an expanding universe ($N\to+\infty$) so long as $w\to w_\infty<1$, i.e., $P<\rho$. This is easily interpreted as the condition for the decay of the shear relative to the Hubble rate (σ/H) in an expanding universe.

Using these same equations, we can understand the superhorizon evolution of the shear and the curvature perturbations in a collapsing Universe ($N\to-\infty$). For $w<1$, the shear grows relative to the Hubble rate, and ζ does not remain constant. The critical case $w=1$ (maximally stiff fluid) occurs if the energy density is dominated by scalar fields with negligible potential. This is supposed to happen in the pre-big-bang scenario, and in the late stages of the second version of the ekpyrotic scenario [29] where the bounce is supposed to be singular from the four-dimensional viewpoint. For $w=1$, ζ on superhorizon scales grows logarithmically with respect to cosmic time and has a strongly scale-dependent spectrum [30]. (It is however [30] still small at the string epoch, which in the pre-big-bang scenario is supposed to be the bounce epoch.)

In the first version of the ekpyrotic scenario [31], where the bounce is supposed to be non-singular from the four-dimensional viewpoint, collapse is driven by a scalar field with a steep negative potential which violates the dominant energy condition and gives $w\gg1$. The same is supposed to happen in the second version of the ekpyrotic scenario [29] at early times. In these cases, the shear rapidly decreases [32,33] and ζ is constant on large scales, with a strongly scale-dependent spectrum $\mathcal{P}_\zeta^{1/2}\propto k^2$. The Bardeen potential Φ, related to ζ by Eq. (50), grows rapidly and has a flat spectrum [34,29,35] $\mathcal{P}_\Phi^{1/2}\propto k^0$. But Eq. (40) shows that it is only the comoving shear that affects ζ, and the shear is related to spatial gradients of the Bardeen potential, Eq. (48). A scale-invariant Bardeen potential ($\Phi\propto k^0$) corresponds to a strongly tilted blue spectrum for the shear ($\sigma\propto k^2$).

V. OTHER CONSERVED QUANTITIES

Generalizing from the construction of the conserved quantity ζ given in Sec. II it is clear that for any monotonically increasing or decreasing quantity, satisfying a local conservation equation of the form

$$\mathcal{V}\frac{\partial f}{\partial\mathcal{V}}=y(f), \quad (55)$$

we can construct a conserved perturbation that is given to first order as

$$X_f\equiv-H\frac{\delta f}{\dot{f}}, \quad (56)$$

with δf evaluated on some uniform-N slicing which we will take to be the spatially flat one. This construction gives $\zeta\equiv X_\rho$ as a special case, and we shall now see how it gives the other conserved quantities ζ_i and $\tilde{\zeta}_i$ [11,19].

A. Separately conserved energy densities

Suppose the total energy density ρ of the Universe is a sum of components ρ_i, each one of them either radiation or matter, and with no energy transfer between the components. In that case the pressure of each component is a unique function of its energy ($P_i=\rho_i/3$ for radiation and $P_i=0$ for matter) and each component satisfies its own separate energy conservation equation[9] (since there is no energy transfer)

$$\mathcal{V}\frac{\partial \rho_i}{\partial \mathcal{V}} = -(\rho_i + P_i). \quad (57)$$

As a result there are the separately conserved perturbations

$$\zeta_i \equiv X_{\rho_i} \quad (58)$$

$$= -H\frac{\delta\rho_i}{\dot{\rho}_i} \quad (59)$$

$$= \frac{1}{3}\frac{\delta\rho_i}{\rho_i + P_i}, \quad (60)$$

where $\delta\rho_i$ is evaluated on the flat slicing. This is another result of [11].

One can express the total density perturbation ζ as a weighted sum of the separate ζ_i;

$$\zeta = \frac{\sum \dot{\rho}_i \zeta_i}{\sum \dot{\rho}_i}. \quad (61)$$

If the ζ_i are all equal, then $\zeta = \zeta_i$ which is constant. Otherwise ζ may have some variation, determined by the conserved isocurvature perturbations defined by

$$\mathcal{S}_{ij} \equiv 3(\zeta_i - \zeta_j). \quad (62)$$

The condition that the ζ_i are equal is just the adiabatic condition, that all of the separate energy densities (and hence the total pressure) are uniform on slices of uniform total energy density.

There are two eras in the early Universe where separately conserved ζ_i have been invoked. One is the comparatively late era, beginning when the temperature falls below 1 MeV and ending when cosmological scales start to approach the horizon.[10]

The energy density during this era has four components,

$$\rho = \rho_{\rm CDM} + \rho_B + \rho_\nu + \rho_\gamma, \quad (63)$$

with the radiation (photons and neutrinos) dominating the matter (cold dark matter and baryonic matter). The values of the four conserved quantities $\zeta_{\rm CDM}$, ζ_B, ζ_ν, and ζ_γ determine the evolution of the entire set of cosmological perturbations after horizon entry, and can therefore be determined by observation. The three isocurvature perturbations (conventionally defined relative to the photon density to be $\mathcal{S}_{\rm CDM} \equiv \mathcal{S}_{{\rm CDM}\gamma}$, $\mathcal{S}_B \equiv \mathcal{S}_{B\gamma}$ and $\mathcal{S}_\nu \equiv \mathcal{S}_{\nu\gamma}$) are found by observation to be at most of order ζ [36–38,2]. Since radiation dominates, one deduces from Eq. (61) that ζ is constant on large scales to high accuracy during this era.

[9] In this expression, $\mathcal{V}$ is the volume which is comoving with the flow of ρ_i. This is not strictly the same as the volume which is comoving with the flow of total energy density, but we have shown in Sec. III that on superhorizon scales the expansion of all comoving volumes become equivalent. See Eq. (34).

[10] Recall that electron-positron annihilation and neutrino decoupling both take place when the temperature is around 1 MeV.

The other era, which occurs in the recently proposed curvaton scenario [12,13] (see also [14,15]), is the era (after inflation, but before primordial nucleosynthesis) when the massive curvaton field σ oscillates ($P_\sigma = 0$) in a radiation background ($P_r = \rho_r/3$),

$$\rho = \rho_\sigma + \rho_{\rm r}. \quad (64)$$

Here, $\zeta_{\rm r}$ is supposed to be negligible so that the total curvature perturbation ζ is given by Eq. (61) as

$$\zeta(t) = \frac{3\rho_\sigma}{4\rho_{\rm r} + 3\rho_\sigma}\zeta_\sigma. \quad (65)$$

Well before the curvaton decays, the radiation is supposed to dominate so that ζ grows like $\rho_\sigma/\rho_{\rm r} \propto a(t)$, providing an example where the total ζ is not conserved on superhorizon scales after inflation.

B. Conserved number densities

If n_i is a conserved number density, then n_i is inversely proportional to the volume.[11] A conservation law of the form given in Eq. (38) is satisfied with $y(f) = -f$ and $f = n_i$, leading to the conservation of the first-order perturbation [19]

$$\tilde{\zeta}_i \equiv X_{n_i} = \frac{1}{3}\frac{\delta n_i}{n_i}. \quad (66)$$

These conserved quantities find an application [19,39] in connection with the three isocurvature perturbations $\mathcal{S}_{\rm cdm}$, $\mathcal{S}_B$ and $\mathcal{S}_\nu$. Before $T \sim 1$ MeV, these quantities are not the appropriate ones to consider, because the separate energy density perturbations ζ_i may vary with time or be simply undefined. (The latter is the case for ζ_B before the quark-hadron transition.) One can however consider instead the number density $n_{\rm cdm}$ of cold dark matter particles, the density n_B of baryon number and the density n_L of lepton number. Each of these number densities corresponds to a conserved quantity after some epoch, which may be regarded as the epoch when the quantity originates. The corresponding perturbations $\tilde{\zeta}_i$ are thus conserved, and after the temperature falls below 1 MeV they determine the three isocurvature perturbations according to the formulas [19]

$$\tfrac{1}{3}\mathcal{S}_{\rm CDM} \equiv \tilde{\zeta}_{\rm CDM} - \zeta \quad (67)$$

$$\tfrac{1}{3}\mathcal{S}_B \equiv \tilde{\zeta}_B - \zeta \quad (68)$$

[11] The volume should be the one comoving with the flow of the conserved quantity, but as already stated we are going to show in Secs. III and IV that the choice of comoving volume is irrelevant on superhorizon scales. This irrelevance is assumed implicitly when the conservation of $\tilde{\zeta}_i$ is discussed in [19].

$$\frac{1}{3}\mathcal{S}_{\nu} \equiv \frac{45}{7}\left(\frac{\xi}{\pi}\right)^{2}(\tilde{\zeta}_L-\zeta). \tag{69}$$

In the last formula, ξ is the lepton asymmetry which must satisfy the nucleosynthesis constraint $|\xi|<0.07$. In this way, the three isocurvature perturbations can be calculated (or shown to vanish) given a model of the early Universe.

VI. CONCLUSIONS

In this paper we have shown how local conservation laws (e.g., energy conservation or baryon number conservation) can lead to conserved perturbations in cosmology. Whenever we have a local continuity equation of the form given in Eq. (38), then we can construct a cosmological perturbation which is conserved after uniform expansion along comoving worldlines.

In particular we have shown that in linear perturbation theory the integrated expansion along comoving worldlines between spatially flat slices is just given by the comoving shear. Thus on sufficiently large scales (where the shear is negligible) the quantity X_f defined in Eq. (39) derived from the conservation equation (38) is conserved. The choice of spatially flat slices gives a gauge-invariant definition of the conserved quantity. This is a purely geometrical result, whose derivation does not require any gravitational field equations. We only require the gravitational field equations in order to estimate the actual comoving shear, finding it to be negligible on superhorizon scales.

The best known example is the curvature perturbation $\zeta \equiv X_\rho$, which specifies the total density perturbation on spatially flat slices or equivalently the curvature perturbation on uniform-density slices. ζ is constant on sufficiently large scales (where the comoving shear is negligible) for adiabatic density perturbations, for which the local pressure is a unique function of the local density and hence the total energy conservation is of the form required in Eq. (38). We have shown that on superhorizon scales, ζ coincides with the comoving curvature perturbation.

It is also possible to construct other perturbed quantities, such as the separate curvature perturbation $\zeta_i \equiv X_{\rho_i}$ for any perfect fluid whose energy is separately conserved [11], or $\tilde{\zeta}_i \equiv X_{n_i}$, for any conserved number density n_i obeying a local conservation equation of the form $\dot{n}_i = -3 n_i$ [19].

The general argument we have given for the existence of conserved quantities is not restricted to linear perturbation theory. As an example we give an expression for the conserved quantity to second-order in the density perturbation. It is necessary to understand the evolution of second order density perturbations in order, for example, to make any estimate of the primordial non-Gaussianity expected in density perturbations produced from inflation.

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APPENDIX A: THE SASAKI-STEWART EXPRESSION FOR ζ

On superhorizon scales, Eq. (35) becomes

$$\delta\tilde{\theta} = -3\dot{\psi}. \tag{A1}$$

This is valid in any gauge. Choose now a gauge whose slicing is flat at time t_1, and uniform density at time t. Integrating from t_1 to t, and using $\zeta=-\psi$ on the slice at t, we find that on superhorizon scales

$$\zeta(\mathbf{x},t) = \frac{1}{3}\int_{t_1}^{t}\tilde{\theta}dt = \delta N_{\rm SS}(\mathbf{x},t) \tag{A2}$$

where $N_{\rm SS}(\mathbf{x},t_1,t)$ is the integrated expansion from the flat slice at time t_1, to the uniform-density slice at time t. This is the expression of Sasaki and Stewart [20], used by them to calculate the curvature perturbation at the end of multi-field inflation. It is practically independent of the threading that defines the expansion, by virtue of the fact that we are dealing with superhorizon scales.

Our expression, Eq. (9), reads $\zeta=-\delta N$. In contrast with the Sasaki-Stewart expression, this one is valid only during an era when ζ is constant, corresponding to an adiabatic pressure perturbation. To understand the relation with the Sasaki-Stewart expression, we can integrate Eq. (5) from a uniform-density slice at time t to a flat slice at time t_1, both times being within the era when the pressure perturbation is adiabatic. Using $3\zeta=\delta\rho/(\rho+P)$ on the slice at t_1 we get the time-independent result

$$\zeta = -\delta N, \tag{A3}$$

where $N=-N_{\rm SS}$ is the integrated expansion *from t to* t_1. We see that the "integration constant" δN introduced in Sec. II can be interpreted as a perturbation in the integrated expansion between two slices, and that it is equal (as it must be) to $-\delta N_{\rm SS}$.

APPENDIX B: UNIFORM HUBBLE PARAMETER ON COMOVING SLICES

In a given gauge, the perturbation in the expansion $\tilde{\theta}$ with respect to coordinate time is given by Eq. (35), which for the flat slicing becomes Eq. (36),

$$\delta\tilde{\theta} = -\sigma. \tag{B1}$$

On superhorizon scales this gives $|\delta\tilde{\theta}|/H \ll 1$, valid for any threading. In other words, the expansion with respect to coordinate time is practically unperturbed on flat slices.

We could instead consider the perturbation in the expansion with respect to proper time, given by Eq. (32). On the comoving slices one has (Eqs. (5.20) and (5.21) of Bardeen [28]; see also [26])

$$\dot{\mathcal{R}} \equiv -\dot{\psi}_{\rm com} = HA_{\rm com}, \tag{B2}$$

and therefore

$$(\delta\theta)_{\rm com} = -\sigma. \tag{B3}$$

Like Eq. (B1), this expression is valid for any choice of the threading that defines the expansion, and on superhorizon scales it gives $|\delta\theta|/H \ll 1$ practically independently of the threading. In other words, the expansion with respect to proper time is practically unperturbed on comoving slices, for any choice of the threading. In particular the comoving expansion is practically unperturbed on such slices, $\delta H/H \ll 1$.

We argued in Sec. V (assuming that anisotropic stress is negligible) that on super-horizon scales $|\Phi| \ll 1$. From Eqs. (44) and (47) this implies that on comoving slices

$$|\delta H/H| = O\left(\left(\frac{k}{aH}\right)^2\right) \tag{B4}$$

$$|\delta\rho_{\rm com}/\rho| = O\left(\left(\frac{k}{aH}\right)^2\right). \tag{B5}$$

To summarize, on comoving slices, the perturbations in the locally defined Hubble parameter and in the energy density are both negligible in the superhorizon regime. The only significant perturbations on comoving slices are therefore curvature perturbation $\mathcal{R}$, and the pressure perturbation if it is not adiabatic. The statements of the previous paragraph remain true if we replace the comoving slicing by the uniform-density slicing, since we argued in Sec. IV that these slicings practically coincide on superhorizon scales.

APPENDIX C: THE ADIABATIC CONDITION ON THE PRESSURE PERTURBATION

In the text we defined the adiabatic condition on the pressure perturbation as the condition that the local pressure is a practically unique function of the local energy density. Taking it to be absolutely unique, we obtain the familiar adiabatic condition

$$\delta P = (\dot{P}/\dot{\rho})\,\delta\rho. \tag{C1}$$

However, on the comoving slicing where $\delta\rho$ is anomalously small, and on the uniform-density slicing where it vanishes, it is too strong to require that this expression is valid; there is no reason why the slices of uniform pressure should exactly coincide with the slices of uniform energy density even if the local pressure is a "practically" unique function of the local energy density.

An example is provided by single-field slow-roll inflation. During slow-roll the locally-defined inflaton field is a practically unique function of proper time, $\phi(\tau)$, up to the choice of origin for τ. On superhorizon scales, where spatial gradients are practically negligible, this gives practically unique functions $\rho(\tau)$ and $P(\tau)$, making P a practically unique function of ρ. In other words, the adiabatic condition for the pressure perturbation is satisfied on super-horizon scales during single-field slow-roll inflation (and even afterwards provided that no other field plays a significant role). However, on comoving slices the potential $V(\phi)$ is uniform, and as a result

$$\delta P_{\rm com} = \delta\rho_{\rm com}. \tag{C2}$$

This is not in accordance with the strict definition, Eq. (C1), of an isocurvature pressure perturbation. In particular, during slow-roll inflaton $\rho \simeq -P(\simeq V)$ which means that the adiabatic condition for the pressure perturbation is

$$\delta P \simeq -\delta\rho. \tag{C3}$$

On a generic slicing this is well satisfied, but on the comoving slicing it is at variance with Eq. (C2). All that matters, though, is that both the pressure perturbation and the energy density perturbation are both very small on the comoving slices. Or, to put it differently, that the pressure perturbation is very small on uniform-density slices. This is enough to ensure that the local pressure is a practically unique function of the energy density, leading to the conclusion that ζ is constant during single-field inflation.

APPENDIX D: SHEAR ON SUPERHORIZON SCALES FOR NON-ADIABATIC PERTURBATIONS

If we relax the assumption that the matter perturbations are adiabatic used in the final part of Sec. IV to calculate the shear σ on superhorizon scales (but still assuming no anisotropic stress) then we no longer have a closed system of equations for ζ and Φ. However, we can still estimate Φ in the long-wavelength regime given ζ and integrating Eq. (51),

$$\frac{2}{3}H^{-1}\dot{\Phi} + \frac{5+3w}{3}\Phi \simeq -(1+w)\zeta. \tag{D1}$$

We also need an initial condition a few Hubble times after horizon exit during slow-roll inflation. Starting with the vacuum fluctuation, direct calculation shows that ζ at this stage is either practically constant (single-field inflation) or only varying slowly on the Hubble time scale (multi-field inflation). Through Eq. (D1) this gives $\Phi \simeq -(3/2)(1+w)\zeta$, and hence $|\Phi| \ll |\zeta|$.

We are now going to argue that Eq. (D1) will keep $|\Phi| \lesssim |\zeta|$ throughout the super-horizon era. A rough argument is the following. Suppose that instead $|\Phi| \gg |\zeta|$ in some super-horizon regime. Then Eq. (D1) becomes

$$\frac{2}{3}H^{-1}(\ln\Phi)\dot{} \simeq -\frac{5+3w(t)}{3} < -(2/3), \tag{D2}$$

where we used the energy condition $w > -1$ which is always satisfied in scalar field theory with a positive kinetic energy. This equation shows that $|\Phi|$ would always be decreasing in any expanding universe where $|\Phi| \gg |\zeta|$, suggesting that such a regime cannot actually be reached starting from the initial condition $|\Phi| \ll |\zeta|$.

A more direct argument is to integrate Eq. (D1), giving

$$F\Phi = -\frac{3}{2}\int_{\ln a_1}^{\ln a}[1+w(a')]F(a')\zeta(a')d(\ln a'), \quad \text{(D3)}$$

where

$$\ln F = \int_{\ln a_1}^{\ln a}\left(\frac{5+3w(a')}{2}\right)d(\ln a'). \quad \text{(D4)}$$

Assuming that ζ is never very much bigger than its primordial value, this will give $|\Phi| \lesssim |\zeta|$ for any reasonable behavior of $w(a)$.

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