Past attractor in inhomogeneous cosmology

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We present a general framework for analyzing spatially inhomogeneous cosmological dynamics. It employs Hubble-normalized scale-invariant variables which are defined within the orthonormal frame formalism, and leads to the formulation of Einstein's field equations with a perfect fluid matter source as an autonomous system of evolution equations and constraints. This framework incorporates spatially homogeneous dynamics in a natural way as a special case, thereby placing earlier work on spatially homogeneous cosmology in a broader context, and allows us to draw on experience gained in that field using dynamical systems methods. One of our goals is to provide a precise formulation of the approach to the spacelike initial singularity in cosmological models, described heuristically by Belinskiǐ, Khalatnikov and Lifshitz. Specifically, we construct an invariant set which we conjecture forms the local past attractor for the evolution equations. We anticipate that this new formulation will provide the basis for proving rigorous theorems concerning the asymptotic behavior of spatially inhomogeneous cosmological models.

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I. INTRODUCTION

Scales and scale invariance play a crucial role in practically all branches of physics, and general relativity (GR) and cosmology are no exceptions.[1] In these cases one is interested in self-gravitating systems, which in the cosmological context requires a matter model as well as a spacetime description. This in turn requires consideration of scales. In modern cosmology one assumes that (today) there exists a global scale—that of the particle horizon. The empirical data are usually interpreted as follows: on sufficiently large spatial scales, say a few percent of the particle horizon, everything looks statistically roughly the same in all directions. Combined with the Copernican principle ("we are not located at a special place"), this suggests that one can replace a very complicated matter distribution by a simple one: a smooth distribution that is spatially homogeneous and isotropic, obtained by averaging over sufficiently large spatial scales. Then it is further assumed that one can also approximate the geometry of the spacetime by a spatially homogeneous and isotropic geometry, i.e., one assumes that the geometrical features trace those of the matter and that possible "excited geometrical modes," such as gravitational waves, are negligible on these scales. This then leads to modeling the cosmological spacetime by a Robertson-Walker (RW) geometry.

The assumption of a RW geometry subsequently forces the summed matter content to take the form of a perfect fluid through Einstein's field equations (EFE). Although the mathematically simplest matter model is a single perfect fluid, we need more complex matter models to describe the real Universe. Indeed, matter in the Universe consists of many components: at least (i) radiation (photons), (ii) baryonic matter, (iii) neutrinos, (iv) dark matter, and (v) dark energy or quintessence (other components like cosmological magnetic fields are usually neglected). Once the matter content has been specified and equations of state, scalar field potentials, particle distribution functions, etc., have been chosen, the evolution of the model is determined by the EFE and the matter equations, e.g., the evolution equation for a scalar field. This then leads to a Friedmann-Lemaître (FL) model for the Universe.

The next step, aimed at describing the actual inhomogeneous Universe, is to perturb the FL model and describe the evolution of large-scale structures in the Universe, which appear at many scales—filaments and voids, superclusters of galaxies, galaxies, etc. But it is generally believed that linear perturbation theory can account for them all on these large scales. And, indeed, the FL scenario and the linear perturbations thereof ("almost-FL models") form a remarkably successful framework—it seems to consistently account for present observational evidence, at least over sufficiently large smoothing scales. Moreover, it forms an interpretational framework that encourages and steers further observations. These are currently focused on determining the various density contributions Ω_i (including Ω_Λ for the cosmological constant), and the spectrum and growth of density perturbations. This is the simplest scenario consistent with current observations.

Nevertheless, there are issues that need elucidation that by necessity lie outside the domain of the standard almost-FL picture. Here are some of them:

(a) To investigate the constraints observations impose on the spacetime geometry of the Universe requires investigating a hierarchy of more general models, perhaps characterized by assumed "priors" (where removing a prior necessar-

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[1]See, e.g., the recent Resource Letter by Wiesenfeld on scale invariance in physics and beyond [1].

ily involves looking beyond one's favorite model, with the hope of getting further support for it).

(b) To understand how special the FL models are and put them into a broader context requires looking beyond them.

(c) While the Universe is well described by almost-FL models at present, this may not always have been true, and it may not remain true in the far future; in particular, we would like to know the largest class of models that can look like a FL model at some stage of their history.

(d) Using a FL and a linearized FL scenario rules out possible nonlinear effects, but these may dominate; e.g., as structures become much more dense while aggregating to smaller scales.

(e) GR is a highly nonlinear theory; to prove that the linear theory is correct requires going beyond linear perturbations.

(f) The averaging and fitting procedures motivating the FL models do not *a priori* commute with the EFE, i.e., starting with an inhomogeneous model and smoothing it does not necessarily lead to the model that has been smoothed from the outset and then perturbed. This gives rise to a number of questions, e.g., can inhomogeneities affect the overall evolution? How do they affect observations?

(g) There are deep connections between GR, cosmology and thermodynamics, e.g., relating (gravitational) entropy and the arrow of time. To better understand such connections requires a state space picture describing the set of solutions, where one can examine coarse-graining and existence of attractors on this state space, toward which the evolving cosmological models move. Since entropy requires counting of *possible* states this requires looking at models beyond FL.

(h) What is the detailed nature of possible singularities? A better understanding of generic features of singularities and their dependence on matter content and initial data might shed light on how the real Universe evolved initially. There might also exist at least a local mathematical connection between the initial singularity and the singularities of gravitational collapse. To understand such a relationship, or its nonexistence, again requires an inspection of cosmological solutions beyond the restrictions imposed by RW geometries.

(i) A better classical understanding of singularities might help to produce gravitational theories with greater domains of validity; e.g., finding asymptotic symmetries of the field equations when approaching singularities may provide sufficient structure to asymptotically quantize the theory in a regime where quantum gravity is supposed to be of importance.

Thus there is ample motivation to probe a larger subset of the cosmological solution space of the EFE than just the almost-FL models. Our first goal in this paper is to develop a framework for this purpose.

In view of the above-mentioned importance of scale invariance in physics, we propose to introduce scale-invariant variables, and to describe the evolution of a cosmological model by an orbit in an infinite-dimensional dynamical state space, governed by first-order (in time) autonomous evolution equations derived from the EFE and the matter equations. The behavior of the model in the asymptotic regimes, i.e., near the initial singularity and at late times, can then possibly be described by a past attractor and a future attractor of the evolution equations.

The appropriate mathematical vehicle for implementing this proposal is the orthonormal frame formalism, since (i) it describes the essential degrees of freedom of the gravitational field in a coordinate independent manner, (ii) orthonormal frame vectors provide local reference scales and so it allows one to naturally introduce scale-invariant variables, and (iii) it leads directly to first-order (in time) autonomous evolution equations.

To be more specific, by a cosmological model we mean a four-dimensional spacetime manifold $\mathcal{M}$ endowed with a Lorentzian metric **g** which satisfies the EFE with an appropriate matter or energy distribution. We assume the existence of a local foliation of the spacetime manifold by a one-parameter family of spacelike 3-surfaces with a future-directed unit normal congruence **u**. We naturally choose this unit vector field to be the timelike vector field in the orthonormal frame. We assume that the cosmological model is *expanding*, i.e., the volume expansion rate[2] Θ of the normal congruence is positive. Because we are working in a cosmological setting we will replace Θ by the Hubble scalar[3] $H=\frac{1}{3}\Theta$.

In the orthonormal frame formalism, the frame vector components and the commutation functions are the basic dynamical variables for describing gravitational fields, and they each have physical dimension[4] $[\text{length}]^{-1}$. The Hubble scalar also has physical dimension $[\text{length}]^{-1}$, and constitutes the natural cosmological length scale through the Hubble radius H^{-1}. This fact motivates one of the key steps in our approach, namely, the introduction of *Hubble-normalized variables* by dividing the frame vector components and the commutation functions by H. Curvature quantities such as the matter density and the orthonormal frame components of the Weyl curvature tensor have physical dimension $[\text{length}]^{-2}$, and hence are normalized by dividing by H^2. This process of Hubble-normalization has two important consequences. First, dimensional variables are replaced by dimensionless ones, leaving H as the only variable carrying physical dimensions. Secondly, one is essentially factoring out the overall expansion of the Universe, thereby measuring the dynamical importance of physical quantities (e.g., the matter density) relative to the overall expansion (cf. Kristian and Sachs [2]). This choice also provides a link between mathematical analysis and observation, since key observational variables are Hubble-normalized. Earlier investigations of the asymptotic dynamics of cosmological models using scale-invariant variables dealt with *spatially homogeneous (SH) cosmologies*, i.e., models that admit a three-parameter group

[2]Here $\Theta=-(\text{tr}\ k)$, where Θ is the volume expansion rate of the normal congruence and $(\text{tr}\ k)$ is the trace of the extrinsic curvature of the spacelike 3-surfaces.

[3]When evaluated at the present epoch, the Hubble scalar equals the Hubble constant H_0, familiar from observational cosmology.

[4]We will use units such that Newton's gravitational constant G and the speed of light in vacuum c are given by $8\pi G/c^2=1$ and $c=1$.

of isometries acting on spacelike 3-surfaces [see Wainwright and Ellis (WE) [3] and other references therein], and the so-called G_2 *cosmologies*, which admit a two-parameter group of isometries acting on spacelike 2-surfaces (see Refs. [4] and [5]). The framework that we develop in this paper generalizes a program that has been extremely successful in a SH context to a completely general spatially inhomogeneous setting, i.e., to models that admit no isometries. Indeed, the SH and G_2 cosmologies will be incorporated in a natural way as invariant sets of the infinite-dimensional Hubble-normalized state space. For brevity, and to emphasize that they admit no isometries, we shall refer to the models under consideration as G_0 *cosmologies*.

The framework that we are developing can be used to study *both* asymptotic regimes in an ever-expanding cosmological model. In this paper we focus on the initial singularity. The problem of asymptotics in spacetimes that exhibit no isometries poses a formidable challenge. Nevertheless, concerning the existence of singularities, remarkable progress was made several decades ago by Penrose and Hawking, leading to their singularity theorems [6,7]. However, the singularity theorems do not tell us much about the nature of the singularities. Detailed asymptotic analysis, using the full EFE, is required for this purpose. To date rigorous results have, with few exceptions, been confined to cosmological models with isometries, in particular SH and G_2 cosmologies. We shall discuss these results in Sec. V. As regards initial singularities in G_0 cosmologies, heuristic results were obtained by Belinskiǐ, Khalatnikov and Lifshitz (BKL) [8,9] by making *ad hoc* metric assumptions that were subsequently inserted into the EFE with the purpose of showing that they were consistent. This analysis led to a remarkable, although heuristic, conjecture that has become part of the folklore of relativistic cosmology.

> *The BKL conjecture*. For almost all cosmological solutions of Einstein's field equations, a spacelike initial singularity is *vacuum-dominated, local and oscillatory*.

For cosmological models with a perfect fluid matter source, the phrase "vacuum-dominated," or, equivalently, "matter is not dynamically significant," is taken to mean that the Hubble-normalized matter density (i.e., the density parameter Ω) tends to zero at the initial singularity. The phrase "for almost all" is needed because there are a number of exceptional cases. First, if the perfect fluid has a stiff equation of state, the density parameter does not tend to zero (see Andersson and Rendall [10]). Secondly, there is a special type of initial singularity called an *isotropic initial singularity*, in the neighborhood of which the solution is approximated locally by a spatially flat FL model (see Goode and Wainwright [11]), with the result that the density parameter tends to the value 1. Isotropic initial singularities, however, only arise from initial data that form a set of measure zero.[5]

The word "local" in the BKL conjecture means heuristically that the evolution at different spatial points effectively decouples as the initial singularity is approached, with the result that geometrical information propagation is asymptotically eliminated. It is natural to describe this phenomenon as *asymptotic silence of the gravitational field dynamics*. We shall refer to the associated initial singularity as being a *silent initial singularity*.[6]

The word "oscillatory" in the BKL conjecture means that the evolution into the past along a typical timeline passes through an infinite sequence of Kasner states, generalizing the behavior first encountered in the so-called Mixmaster models (SH models of Bianchi type IX; see Misner [12]).

Our second main goal in this paper is to give a precise statement of the BKL conjecture, within the framework of the Hubble-normalized state space.

The plan of this paper is as follows. In Sec. II we derive the Hubble-normalized evolution equations and constraints for G_0 cosmologies that arise from the EFE and the matter equations. In Sec. III we make a choice of gauge and then describe some features of the Hubble-normalized state space, in particular the SH invariant set and the silent boundary. We then define the notion of a silent initial singularity. In Sec. IV, by analyzing the dynamics on the silent boundary, we are led to construct an invariant set which we conjecture is the local past attractor for G_0 cosmologies with a silent initial singularity, thereby making precise the notion of an oscillatory initial singularity. In Sec. V we consider various classes of cosmological models with isometries and use the past attractor to predict whether the initial singularity is oscillatory or not. We conclude in Sec. VI with a discussion of silent initial singularities and the BKL conjecture, and raise some issues for future study. Useful mathematical relations, such as the propagation laws for the constraints and expressions for the Hubble-normalized components of the Weyl curvature tensor, have been gathered in the Appendix.

II. EVOLUTION EQUATIONS AND CONSTRAINTS

In this paper, we consider spatially inhomogeneous cosmological models with a positive cosmological constant, Λ, and a perfect fluid matter source with a linear barotropic equation of state. We thus have

$$\tilde{p}(\tilde{\mu})=(\gamma-1)\tilde{\mu}, \tag{2.1}$$

where $\tilde{\mu}$ is the total energy density (assumed to be non-negative) and $\tilde{p}$ the isotropic pressure, in the rest 3-spaces associated with the fluid 4-velocity vector field $\tilde{\mathbf{u}}$, while γ is a constant parameter. The range

$$1 \leqslant \gamma \leqslant 2$$

is of particular physical interest, since it ensures that the perfect fluid satisfies the dominant and strong energy condi-

[5]There is in fact a wide variety of known special SH and G_2 solutions in which the initial singularity is matter-dominated, i.e., Ω does not tend to zero. Like the isotropic initial singularities, these singularities only arise from initial data that form a set of measure zero.

[6]For further discussions on the suppression of information propagation and asymptotic silence, see Ref. [4].

tions and the causality requirement that the speed of sound should be less than or equal to that of light. The values $\gamma=1$ and $\gamma=\frac{4}{3}$ correspond to incoherent pressure-free matter ("dust") and incoherent radiation, respectively. In cosmology it is natural to single out a future-directed timelike reference congruence $\mathbf{e}_0=\mathbf{u}$ of unit magnitude. This gives rise to a $(1+3)$-decomposition of the perfect fluid energy-momentum-stress tensor

$$T_{ab}=\mu u_a u_b+2q_{(a}u_{b)}+ph_{ab}+\pi_{ab}, \tag{2.2}$$

with

$$\mu=\Gamma^2 G_+\tilde{\mu},\quad p=G_+^{-1}\left[(\gamma-1)+\left(1-\frac{2}{3}\gamma\right)v^2\right]\mu,$$
$$q^a=\gamma G_+^{-1}\mu v^a,\quad \pi_{ab}=\gamma G_+^{-1}\mu v_{\langle a}v_{b\rangle}. \tag{2.3}$$

The vector field $\mathbf{v}$, which represents the peculiar velocity of the fluid relative to the rest 3-spaces of $\mathbf{e}_0$, is defined by

$$\tilde{u}^a:=\Gamma(u^a+v^a),\quad v_a u^a=0, \tag{2.4}$$

with the Lorentz factor given by

$$\Gamma:=\frac{1}{\sqrt{1-v^2}},\quad v^2:=v_a v^a. \tag{2.5}$$

The scalars $G_\pm$ (we shall require G_- later) are defined by

$$G_\pm:=1\pm(\gamma-1)v^2. \tag{2.6}$$

To obtain an orthonormal frame, $\{\mathbf{e}_a\}_{a=0,1,2,3}$, we supplement the timelike reference congruence $\mathbf{e}_0$ with an orthonormal spatial frame $\{\mathbf{e}_\alpha\}_{\alpha=1,2,3}$ in the rest 3-spaces of $\mathbf{e}_0$. The frame metric is then given by $\eta_{ab}=\text{diag}[-1,1,1,1]$. In the orthonormal frame formalism, introduced in relativistic cosmology, among others, by Ellis [13], the basic variables are the frame vector components, the commutation functions associated with the frame, and the matter variables, and the dynamical equations are provided by the EFE, the Jacobi identities and the contracted Bianchi identities (the latter, for a perfect fluid, corresponding to the relativistic extension of Euler's equations). We will make use of an extended version of this formalism given by van Elst and Uggla [14]. The dynamical equations consist of two sets, those containing the temporal frame derivative $\mathbf{e}_0$, which we refer to as *evolution equations*, and those not containing $\mathbf{e}_0$, which we refer to as *constraints*.

To convert the dynamical equations of the orthonormal frame formalism to a system of partial differential equations (PDE), it is necessary to introduce a set of local coordinates $\{x^\mu\}_{\mu=0,1,2,3}=\{t,x^i\}_{i=1,2,3}$. We do so by adopting the standard $(3+1)$-approach (see, e.g., Refs. [15] and [16]). Here $\mathbf{e}_0$ is assumed to be *vorticity-free* and, thus, hypersurface-orthogonal. As is well known, this gives rise to a local foliation of the spacetime manifold $\mathcal{M}$ by a one-parameter family of spacelike 3-surfaces, $\mathcal{S}:\{t=\text{const}\}$. The $(3+1)$-approach leads to the following coordinate expressions for the frame vector fields (cf. Ref. [14]):

$$\mathbf{e}_0=N^{-1}(\partial_t-N^i\partial_i),\quad \mathbf{e}_\alpha=e_\alpha{}^i\,\partial_i, \tag{2.7}$$

where N and N^i are known as the lapse function and the shift vector field, respectively.

A. Dimensional equation system

We now present the dynamical equations as given in Ref. [14], simplified by the assumption that $\mathbf{e}_0$ is vorticity-free $(\omega^\alpha=0)$. We begin with the commutator equations, which serve to introduce the basic gravitational field variables, and which will later be used to derive some additional evolution equations and constraints.
Commutator equations:

$$[\mathbf{e}_0,\mathbf{e}_\alpha](f)=\dot{u}_\alpha\mathbf{e}_0(f)-(H\delta_\alpha{}^\beta+\sigma_\alpha{}^\beta-\epsilon_{\alpha\gamma}{}^\beta\Omega^\gamma)\mathbf{e}_\beta(f) \tag{2.8}$$

$$0=(C_{\text{com}})_{\alpha\beta}(f):=[\mathbf{e}_\alpha,\mathbf{e}_\beta](f)-(2a_{[\alpha}\delta_{\beta]}{}^\gamma+\epsilon_{\alpha\beta\delta}n^{\delta\gamma})\mathbf{e}_\gamma(f), \tag{2.9}$$

with f denoting an arbitrary real-valued spacetime scalar. Here H is the Hubble scalar which is related to the volume expansion rate Θ of $\mathbf{e}_0$ according to $H:=\frac{1}{3}\Theta$. The quantities $\dot{u}^\alpha$ and $\sigma_{\alpha\beta}$ are the acceleration and shear rate of $\mathbf{e}_0$, respectively, while Ω^α describes the angular velocity of the spatial frame $\{\mathbf{e}_\alpha\}$ along the integral curves of $\mathbf{e}_0$ relative to a Fermi-propagated one. The quantities a^α and $n_{\alpha\beta}$ determine the connection on the spacelike 3-surfaces $\mathcal{S}:\{t=\text{const}\}$.

Einstein's field equations, Jacobi identities and contracted Bianchi identities (Euler's equations):

Evolution equations:

$$\mathbf{e}_0(H)=-H^2-\frac{1}{3}(\sigma_{\alpha\beta}\sigma^{\alpha\beta})-\frac{1}{6}(\mu+3p)+\frac{1}{3}\Lambda+\frac{1}{3}(\mathbf{e}_\alpha+\dot{u}_\alpha-2a_\alpha)(\dot{u}^\alpha) \tag{2.10}$$

$$\mathbf{e}_0(a^\alpha)=-(H\delta^\alpha{}_\beta+\sigma^\alpha{}_\beta-\epsilon^\alpha{}_{\gamma\beta}\Omega^\gamma)a^\beta-\frac{1}{2}(\mathbf{e}_\beta+\dot{u}_\beta)(2H\delta^{\alpha\beta}-\sigma^{\alpha\beta}-\epsilon^{\alpha\beta}{}_\gamma\,\Omega^\gamma) \tag{2.11}$$

$$\begin{aligned}\mathbf{e}_0(\sigma^{\alpha\beta})=&-3H\sigma^{\alpha\beta}-2n^{\langle\alpha}{}_\gamma\,n^{\beta\rangle\gamma}+n_\gamma{}^\gamma n^{\langle\alpha\beta\rangle}-\delta^{\gamma\langle\alpha}\mathbf{e}_\gamma(a^{\beta\rangle})\\&+\epsilon^{\gamma\delta\langle\alpha}[(\mathbf{e}_\gamma+\dot{u}_\gamma-2a_\gamma)(n^{\beta\rangle}{}_\delta)+2\Omega_\gamma\sigma^{\beta\rangle}{}_\delta]\\&+\pi^{\alpha\beta}+(\delta^{\gamma\langle\alpha}\mathbf{e}_\gamma+\dot{u}^{\langle\alpha}+a^{\langle\alpha\rangle})(\dot{u}^{\beta\rangle})\end{aligned} \tag{2.12}$$

$$\begin{aligned}\mathbf{e}_0(n^{\alpha\beta})=&-(H\delta^{(\alpha}{}_\delta-2\sigma^{(\alpha}{}_\delta-2\epsilon^\gamma{}_\delta{}^{(\alpha}\,\Omega_\gamma)n^{\beta)\delta}\\&-(\mathbf{e}_\gamma+\dot{u}_\gamma)(\epsilon^{\gamma\delta(\alpha}\sigma^{\beta)}{}_\delta-\delta^{\gamma(\alpha}\Omega^{\beta)}+\delta^{\alpha\beta}\Omega^\gamma)\end{aligned} \tag{2.13}$$

$$\mathbf{e}_0(\mu)=-3H(\mu+p)-(\mathbf{e}_\alpha+2\dot{u}_\alpha-2a_\alpha)(q^\alpha)-(\sigma_{\alpha\beta}\pi^{\alpha\beta}) \tag{2.14}$$

$$\mathbf{e}_0(v^\alpha) = \frac{G_+}{\gamma G_- \mu}[-\gamma v^\alpha \mathbf{e}_0(\mu) + (G_- \delta^\alpha{}_\beta + 2(\gamma-1) v^\alpha v_\beta)\mathbf{e}_0(q^\beta)], \tag{2.15}$$

where

$$\mathbf{e}_0(q^\alpha) = -(4H\delta^\alpha{}_\beta + \sigma^\alpha{}_\beta - \epsilon^\alpha{}_{\gamma\beta}\Omega^\gamma)q^\beta - \delta^{\alpha\beta}\mathbf{e}_\beta(p) - (\mu+p)\dot{u}^\alpha - (\mathbf{e}_\beta + \dot{u}_\beta - 3a_\beta)(\pi^{\alpha\beta}) + \epsilon^{\alpha\beta\gamma} n_{\beta\delta}\pi_\gamma{}^\delta. \tag{2.16}$$

Constraints:

$$0 = (C_{\rm G}) := 2(2\mathbf{e}_\alpha - 3a_\alpha)(a^\alpha) - (n_{\alpha\beta}n^{\alpha\beta}) + \frac{1}{2}(n_\alpha{}^\alpha)^2 + 6H^2 - (\sigma_{\alpha\beta}\sigma^{\alpha\beta}) - 2\mu - 2\Lambda \tag{2.17}$$

$$0 = (C_{\rm C})^\alpha := -\mathbf{e}_\beta(2H\delta^{\alpha\beta} - \sigma^{\alpha\beta}) - 3a_\beta\sigma^{\alpha\beta} - \epsilon^{\alpha\beta\gamma}n_{\beta\delta}\sigma_\gamma{}^\delta + q^\alpha \tag{2.18}$$

$$0 = (C_{\rm J})^\alpha := \mathbf{e}_\beta(n^{\alpha\beta} + \epsilon^{\alpha\beta\gamma}a_\gamma) - 2a_\beta n^{\alpha\beta}. \tag{2.19}$$

Note that we are *not* provided with evolution equations for any of the[7] coordinate gauge source functions N and N^i (which reside in $\mathbf{e}_0$) or the frame gauge source functions $\dot{u}^\alpha$ and Ω^α. Note also that these ten gauge source functions do *not* appear in the constraints. Independent of a choice of gauge (to be discussed in Sec. III), the evolution equations (2.8) and (2.10)–(2.16) propagate the constraints (2.9) and (2.17)–(2.19) along the integral curves of $\mathbf{e}_0$ according to Eqs. (A1)–(A4) in the Appendix.

There are, in addition, two *gauge constraints* that restrict four of the gauge source functions, given by

$$0 = (C_\omega)^\alpha := [\epsilon^{\alpha\beta\gamma}(\mathbf{e}_\beta - a_\beta) - n^{\alpha\gamma}]\dot{u}_\gamma \tag{2.20}$$

$$0 = (C_{\dot{u}})_\alpha := N^{-1}\mathbf{e}_\alpha(N) - \dot{u}_\alpha. \tag{2.21}$$

The former is a consequence of assuming $\mathbf{e}_0$ to be vorticity-free, the latter follows from Eq. (2.8) upon substitution of Eq. (2.7). The propagation of the gauge constraints along the integral curves of $\mathbf{e}_0$ can be established once a choice of temporal gauge has been made [as this determines what the currently unknown frame derivatives $\mathbf{e}_0(\dot{u}^\alpha)$ and $\mathbf{e}_0(N)$ should be].

B. Scale-invariant equation system

We now introduce Hubble-normalized frame, connection and matter variables as follows:

$$\boldsymbol{\partial}_0 := \frac{1}{H}\mathbf{e}_0, \quad \boldsymbol{\partial}_\alpha := \frac{1}{H}\mathbf{e}_\alpha, \tag{2.22}$$

[7]Employing the terminology of Friedrich [17], Sec. 5.2.

$$\{\dot{U}^\alpha, \Sigma_{\alpha\beta}, A^\alpha, N_{\alpha\beta}, R^\alpha\} := \{\dot{u}^\alpha, \sigma_{\alpha\beta}, a^\alpha, n_{\alpha\beta}, \Omega^\alpha\}/H \tag{2.23}$$

$$\{\Omega, \Omega_\Lambda, P, Q^\alpha, \Pi_{\alpha\beta}\} := \{\mu, \Lambda, p, q^\alpha, \pi_{\alpha\beta}\}/(3H^2). \tag{2.24}$$

It follows from Eq. (2.3) that

$$P = G_+^{-1}\left[(\gamma-1) + \left(1 - \frac{2}{3}\gamma\right)v^2\right]\Omega, \quad Q^\alpha = \frac{\gamma}{G_+}\Omega v^\alpha, \quad \Pi_{\alpha\beta} = \frac{\gamma}{G_+}\Omega v_{\langle\alpha}v_{\beta\rangle}. \tag{2.25}$$

Expressing the Hubble-normalized frame derivatives $\boldsymbol{\partial}_0$ and $\boldsymbol{\partial}_\alpha$ with respect to the local coordinates introduced in Eq. (2.7) leads to

$$\boldsymbol{\partial}_0 = \mathcal{N}^{-1}(\partial_t - N^i\partial_i), \quad \boldsymbol{\partial}_\alpha = E_\alpha{}^i\partial_i, \tag{2.26}$$

where

$$\mathcal{N} := NH, \quad E_\alpha{}^i := \frac{e_\alpha{}^i}{H}. \tag{2.27}$$

In order to write the dimensional equation system in Hubble-normalized form, it is necessary to introduce the *deceleration parameter* q and the *spatial Hubble gradient* r_α, defined by

$$(q+1) := -\frac{1}{H}\boldsymbol{\partial}_0 H, \tag{2.28}$$

$$r_\alpha := -\frac{1}{H}\boldsymbol{\partial}_\alpha H. \tag{2.29}$$

The definition (2.28), together with Raychaudhuri's equation (2.10) and Eqs. (2.23) and (2.24), lead to the following key expression for q:

$$q = 2\Sigma^2 + \frac{1}{2}(\Omega + 3P) - \Omega_\Lambda - \frac{1}{3}(\boldsymbol{\partial}_\alpha - r_\alpha + \dot{U}_\alpha - 2A_\alpha)\dot{U}^\alpha, \tag{2.30}$$

where $\Sigma^2 := \frac{1}{6}(\Sigma_{\alpha\beta}\Sigma^{\alpha\beta})$.

We now use Eqs. (2.28) and (2.29) to write the commutator equations (2.8) and (2.9) in Hubble-normalized form. The result is

$$[\boldsymbol{\partial}_0, \boldsymbol{\partial}_\alpha]f = -(r_\alpha - \dot{U}_\alpha)\boldsymbol{\partial}_0 f + (q\delta_\alpha{}^\beta - \Sigma_\alpha{}^\beta + \epsilon_{\alpha\gamma}{}^\beta R^\gamma)\boldsymbol{\partial}_\beta f, \tag{2.31}$$

$$0 = (\mathcal{C}_{\rm com})_{\alpha\beta}(f) := [\boldsymbol{\partial}_\alpha, \boldsymbol{\partial}_\beta]f - [2(r_{[\alpha} + A_{[\alpha})\delta_{\beta]}{}^\gamma + \epsilon_{\alpha\beta\delta}N^{\delta\gamma}]\boldsymbol{\partial}_\gamma f. \tag{2.32}$$

We now write the evolution equations (2.10)–(2.16) and the constraints (2.17)–(2.19) in Hubble-normalized form.[8]

Evolution equations:

$$\boldsymbol{\partial}_0 A^\alpha = (q\,\delta^\alpha{}_\beta - \Sigma^\alpha{}_\beta + \epsilon^\alpha{}_{\gamma\beta} R^\gamma) A^\beta - \frac{1}{2}(\boldsymbol{\partial}_\beta - r_\beta + \dot{U}_\beta)(2\,\delta^{\alpha\beta} - \Sigma^{\alpha\beta} - \epsilon^{\alpha\beta}{}_\gamma R^\gamma), \tag{2.33}$$

$$\boldsymbol{\partial}_0 \Sigma^{\alpha\beta} = (q-2)\Sigma^{\alpha\beta} - 2N^{\langle\alpha}{}_\gamma N^{\beta\rangle\gamma} + N_\gamma{}^\gamma N^{\langle\alpha\beta\rangle} - \delta^{\gamma\langle\alpha}(\boldsymbol{\partial}_\gamma - r_\gamma)A^{\beta\rangle} + \epsilon^{\gamma\delta\langle\alpha}[(\boldsymbol{\partial}_\gamma - r_\gamma + \dot{U}_\gamma - 2A_\gamma)N^{\beta\rangle}{}_\delta + 2R_\gamma \Sigma^{\beta\rangle}{}_\delta] + 3\Pi^{\alpha\beta} + (\delta^{\gamma\langle\alpha}\boldsymbol{\partial}_\gamma - r^{\langle\alpha} + \dot{U}^{\langle\alpha} + A^{\langle\alpha})\dot{U}^{\beta\rangle}, \tag{2.34}$$

$$\boldsymbol{\partial}_0 N^{\alpha\beta} = (q\,\delta^{(\alpha}{}_\delta + 2\Sigma^{(\alpha}{}_\delta + 2\epsilon^\gamma{}_\delta{}^{(\alpha}R_\gamma)N^{\beta)\delta} - (\boldsymbol{\partial}_\gamma - r_\gamma + \dot{U}_\gamma)(\epsilon^{\gamma\delta(\alpha}\Sigma^{\beta)}{}_\delta - \delta^{\gamma(\alpha}R^{\beta)} + \delta^{\alpha\beta}R^\gamma), \tag{2.35}$$

$$\boldsymbol{\partial}_0 \Omega = (2q-1)\Omega - 3P - (\boldsymbol{\partial}_\alpha - 2r_\alpha + 2\dot{U}_\alpha - 2A_\alpha)Q^\alpha - (\Sigma_{\alpha\beta}\Pi^{\alpha\beta}), \tag{2.36}$$

$$\boldsymbol{\partial}_0 v^\alpha = \frac{G_+}{\gamma G_- \Omega}[-\gamma v^\alpha(\boldsymbol{\partial}_0 - 2q - 2)\Omega + (G_-\delta^\alpha{}_\beta + 2(\gamma-1)v^\alpha v_\beta)(\boldsymbol{\partial}_0 - 2q - 2)Q^\beta], \tag{2.37}$$

$$\boldsymbol{\partial}_0 \Omega_\Lambda = 2(q+1)\Omega_\Lambda, \tag{2.38}$$

where[9]

$$\boldsymbol{\partial}_0 Q^\alpha = [2(q-1)\delta^\alpha{}_\beta - \Sigma^\alpha{}_\beta + \epsilon^\alpha{}_{\gamma\beta}R^\gamma]Q^\beta - \delta^{\alpha\beta}(\boldsymbol{\partial}_\beta - 2r_\beta)P - (\Omega + P)\dot{U}^\alpha - (\boldsymbol{\partial}_\beta - 2r_\beta + \dot{U}_\beta - 3A_\beta)\Pi^{\alpha\beta} + \epsilon^{\alpha\beta\gamma}N_{\beta\delta}\Pi_\gamma{}^\delta. \tag{2.39}$$

Constraints:

$$0 = (\mathcal{C}_G) := 1 - \Omega_k - \Sigma^2 - \Omega - \Omega_\Lambda, \tag{2.40}$$

$$0 = (\mathcal{C}_C)^\alpha := \boldsymbol{\partial}_\beta \Sigma^{\alpha\beta} + (2\,\delta^\alpha{}_\beta - \Sigma^\alpha{}_\beta)r^\beta - 3A_\beta \Sigma^{\alpha\beta} - \epsilon^{\alpha\beta\gamma}N_{\beta\delta}\Sigma_\gamma{}^\delta + 3Q^\alpha, \tag{2.41}$$

$$0 = (\mathcal{C}_J)^\alpha := (\boldsymbol{\partial}_\beta - r_\beta)(N^{\alpha\beta} + \epsilon^{\alpha\beta\gamma}A_\gamma) - 2A_\beta N^{\alpha\beta}, \tag{2.42}$$

$$0 = (\mathcal{C}_\Lambda)_\alpha := (\boldsymbol{\partial}_\alpha - 2r_\alpha)\Omega_\Lambda, \tag{2.43}$$

where

$$\Omega_k := -\frac{1}{3}(2\boldsymbol{\partial}_\alpha - 2r_\alpha - 3A_\alpha)A^\alpha + \frac{1}{6}(N_{\alpha\beta}N^{\alpha\beta}) - \frac{1}{12}(N_\alpha{}^\alpha)^2. \tag{2.44}$$

We have also included an evolution equation and a constraint for Ω_Λ, which are a direct consequence of Eqs. (2.24), (2.28) and (2.29).

The role of the spatial Hubble gradient r_α requires comment. One can use the Codacci constraint (2.41) to express r_α in terms of Hubble-normalized variables. The resulting formula for r_α involves the inverse of the matrix $(2\,\delta^\alpha{}_\beta - \Sigma^\alpha{}_\beta)$, and in order to avoid this algebraic complication, we propose to treat r_α as a dependent variable. Choosing $f = H$ in the commutator equations (2.31) and (2.32), and making use of Eqs. (2.28) and (2.29), leads to both an evolution equation and a constraint for r_α:

$$\boldsymbol{\partial}_0 r_\alpha = (q\,\delta_\alpha{}^\beta - \Sigma_\alpha{}^\beta + \epsilon_{\alpha\gamma}{}^\beta R^\gamma)r_\beta + (\boldsymbol{\partial}_\alpha - r_\alpha + \dot{U}_\alpha)(q+1), \tag{2.45}$$

$$0 = (\mathcal{C}_r)^\alpha := [\epsilon^{\alpha\beta\gamma}(\boldsymbol{\partial}_\beta - A_\beta) - N^{\alpha\gamma}]r_\gamma. \tag{2.46}$$

These equations constitute integrability conditions for Eqs. (2.28) and (2.29).

When we write the evolution equations and constraints as PDE by expressing $\boldsymbol{\partial}_0$ and $\boldsymbol{\partial}_\alpha$ in terms of partial derivatives using Eq. (2.26), the frame components $E_\alpha{}^i$ enter into the equations as dependent variables. Successively choosing $f = x^i$, $i = 1,2,3$, in the commutator equations (2.31) and (2.32) leads to an evolution equation and a constraint for $E_\alpha{}^i$:

$$\boldsymbol{\partial}_0 E_\alpha{}^i = (q\,\delta_\alpha{}^\beta - \Sigma_\alpha{}^\beta + \epsilon_{\alpha\gamma}{}^\beta R^\gamma)E_\beta{}^i - \mathcal{N}^{-1}\boldsymbol{\partial}_\alpha N^i, \tag{2.47}$$

$$0 = (\mathcal{C}_{\rm com})^i{}_{\alpha\beta} := 2(\boldsymbol{\partial}_{[\alpha} - r_{[\alpha} - A_{[\alpha})E_{\beta]}{}^i - \epsilon_{\alpha\beta\delta}N^{\delta\gamma}E_\gamma{}^i. \tag{2.48}$$

Finally, we give the Hubble-normalized form of the gauge constraints (2.20) and (2.21):

$$0 = (\mathcal{C}_W)^\alpha := [\epsilon^{\alpha\beta\gamma}(\boldsymbol{\partial}_\beta - r_\beta - A_\beta) - N^{\alpha\gamma}]\dot{U}_\gamma, \tag{2.49}$$

$$0 = (\mathcal{C}_{\dot{U}})_\alpha := \boldsymbol{\partial}_\alpha \ln \mathcal{N} + (r_\alpha - \dot{U}_\alpha). \tag{2.50}$$

III. GAUGE FIXING AND THE HUBBLE-NORMALIZED STATE SPACE

In the previous section we presented a constrained system of coupled PDE that govern the evolution of G_0 cosmologies. The dependent variables are (i) the spatial frame vector field components $E_\alpha{}^i$, (ii) the spatial Hubble gradient

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[8] In explicit component form these equations are available online at the URL given in Ref. [18]. An earlier scale-invariant equation system (based on an orthonormal frame formulation), which employs the once-contracted second Bianchi identities and Weyl curvature variables, was derived by two of the authors (H.v.E. and C.U.) and given in Ref. [19], but no specific choice of temporal gauge or spatial frame was introduced then.

[9] We give the Hubble-normalized relativistic Euler equations, Eqs. (2.36) and (2.37), in explicit form in the Appendix; see Eqs. (A6) and (A7).

r_α, and (iii) the gravitational field and matter variables $\Sigma_{\alpha\beta}$, A^α, $N_{\alpha\beta}$, Ω, v^α, and Ω_Λ.

The system of PDE is underdetermined due to the presence of the gauge source functions

$$\mathcal{N}, \;\; N^i, \;\; \dot{U}^\alpha, \;\; R^\alpha,$$

which reflects the fact that there is freedom in the choice of the local coordinates and of the orthonormal frame. We now use this gauge freedom to specify the gauge source functions, and then proceed to describe some aspects of the Hubble-normalized state space.

A. Fixing the gauge

We begin by using the coordinate freedom to set the shift vector field in Eqs. (2.7) and (2.26) to zero:

$$N^i = 0. \tag{3.1}$$

We then choose the timelike reference congruence $\mathbf{e}_0$ so that

$$\boldsymbol{\partial}_\alpha \mathcal{N} = 0. \tag{3.2}$$

We are then free to specialize the time coordinate t so that

$$\mathcal{N} = 1. \tag{3.3}$$

The effect of these choices is that the dimensional lapse function in Eq. (2.7) is given by $N = H^{-1}$, as follows from Eq. (2.27). The gauge constraint (2.50), taken in conjunction with the above conditions, reduces to

$$0 = (\mathcal{C}_{\dot{U}})^{\mathrm{sv}}_\alpha = (r_\alpha - \dot{U}_\alpha) \Rightarrow \dot{U}_\alpha = r_\alpha, \tag{3.4}$$

thus determining the frame gauge source functions $\dot{U}_\alpha$. The advantage of making the choices (3.1) and (3.3) is that the temporal frame derivative $\boldsymbol{\partial}_0$, given by Eqs. (2.26), simplifies to a partial derivative,

$$\boldsymbol{\partial}_0 = \partial_t. \tag{3.5}$$

The combined gauge choices (3.1) and (3.3) have a simple geometrical interpretation in terms of the *volume density* $\mathcal{V}$ associated with the family of spacelike 3-surfaces $\mathcal{S}:\{t = \text{const}\}$, which is defined by

$$\mathcal{V}^{-1} := \det(e_\alpha{}^i). \tag{3.6}$$

Using Eq. (3.1), the commutator equations yield

$$\mathcal{N}^{-1}\frac{\partial_t \mathcal{V}}{\mathcal{V}} = 3, \quad E_\alpha{}^i \frac{\partial_i \mathcal{V}}{\mathcal{V}} = -2A_\alpha - \partial_i E_\alpha{}^i + r_\alpha. \tag{3.7}$$

It follows with Eq. (3.3) that

$$\mathcal{V} = \ell_0^3 e^{3t} \hat{m}, \tag{3.8}$$

where $\hat{m} = \hat{m}(x^i)$ is a freely specifiable positive real-valued function of x^i, which we consider given, and ℓ_0 is the unit of the physical dimension [length]. We thus refer to this gauge choice as the *separable volume gauge*. Note that the reduced gauge constraint (3.4) propagates along $\mathbf{e}_0$ according to Eq. (A5) in the Appendix. *This ensures the local existence (in time) of the separable volume gauge*.

Equations (3.7) and (3.8) subsequently yield the constraint

$$0 = (\mathcal{C}_A)_\alpha := A_\alpha + \frac{1}{2}(\partial_i E_\alpha{}^i - r_\alpha + E_\alpha{}^i \partial_i \ln \hat{m}). \tag{3.9}$$

Finally we use a time- and space-dependent rotation of the spatial frame to relate the frame gauge source functions R^α to the off-diagonal components of the shear rate tensor according to[10]

$$(R_1, R_2, R_3)^T = (\Sigma_{23}, \Sigma_{31}, \Sigma_{12})^T. \tag{3.10}$$

At this stage there is no freedom remaining in the choice of frame.[11] The coordinate freedom is

$$t' = t + \text{const}, \quad x^{i\prime} = f^i(x^j).$$

An important question, which we do not pursue at present, except in a footnote in Sec. V, is to what extent the analysis in this paper (in particular the construction of the past attractor) depends on the choice of temporal gauge. Here we use the separable volume gauge, as defined by Eqs. (3.1) and (3.3), which appears to be particularly well-adapted to Hubble-normalized variables. For G_2 cosmologies, which we shall refer to later, the usual and most convenient temporal gauge is the so-called separable area gauge (see, e.g., Ref. [4]).

B. Hubble-normalized state space

1. Overview

The Hubble-normalized state vector for G_0 cosmologies is given by

$$\mathbf{X} = (E_\alpha{}^i, r_\alpha, \Sigma_{\alpha\beta}, N_{\alpha\beta}, A^\alpha, \Omega, v^\alpha, \Omega_\Lambda)^T. \tag{3.11}$$

The evolution equations and constraints in the previous section can be written concisely in the form

$$\partial_t \mathbf{X} = \mathbf{F}(\mathbf{X}, \partial_i \mathbf{X}, \partial_i \partial_j \mathbf{X}), \tag{3.12}$$

$$0 = \mathbf{C}(\mathbf{X}, \partial_i \mathbf{X}), \tag{3.13}$$

[10] In contrast to the present frame choice, one can use the frame freedom to reduce the number of variables, e.g., by diagonalizing the shear rate tensor. However, the present choice leads to great simplification of the equations when it comes to analyzing the past attractor. There are other useful choices; in particular, when one has a preferred spatial direction induced by an isometry. In such a case it is often advantageous to choose the R^α-component associated with the preferred direction to have the opposite sign compared with the present choice.

[11] With the exception of the special cases when the shear rate tensor is locally rotationally symmetrical or zero; when the frame is uniquely determined, all the Hubble-normalized connection and curvature variables employed are scalar invariants.

with the spatial derivatives appearing linearly (apart from the evolution equation for r_α). A surprising feature of the evolution equations is that they contain second-order spatial derivatives; in this respect they are reminiscent of a system of quasilinear diffusion equations. The only second-order spatial derivatives in the evolution equations, however, are those of the spatial Hubble gradient, $\partial_i\partial_j r_\alpha$, and they appear only in the evolution equation for r_α itself. They arise due to the fact that in the separable volume gauge the deceleration parameter q contains the first spatial derivatives of r_α. In fact, in the separable volume gauge Eq. (2.30) for q assumes the form

$$q=2\Sigma^2+G_+^{-1}[(3\gamma-2)+(2-\gamma)v^2]\Omega-\Omega_\Lambda - \frac{1}{3}(\boldsymbol{\partial}_\alpha-2A_\alpha)r^\alpha. \tag{3.14}$$

The term $\boldsymbol{\partial}_\alpha q$ in the evolution equation for r_α, Eq. (2.45), thus contains $\partial_i\partial_j r_\alpha$.

A second noteworthy feature of the system of PDE (3.12) is that the evolution equation for $E_\alpha{}^i$ is *homogeneous*, which implies that the equation $E_\alpha{}^i=0$ defines an invariant set. We shall discuss the significance of this set later in this section. In order to clearly exhibit these aspects of the evolution equations, we now decompose the Hubble-normalized state vector (3.11) as follows:

$$\mathbf{X}=(E_\alpha{}^i,r_\alpha)^T\oplus\mathbf{Y}, \tag{3.15}$$

where

$$\mathbf{Y}=(\Sigma_{\alpha\beta},N_{\alpha\beta},A^\alpha,\Omega,v^\alpha,\Omega_\Lambda)^T. \tag{3.16}$$

We can now write the system (3.12) in a more explicit form as follows:

$$\partial_t E_\alpha{}^i=(q\delta_\alpha{}^\beta-\Sigma_\alpha{}^\beta+\epsilon_{\alpha\gamma}{}^\beta R^\gamma)E_\beta{}^i, \tag{3.17}$$

with q given by Eq. (3.14), and

$$\partial_t r_\alpha=\left[G_\alpha{}^\beta(\mathbf{Y})+\frac{2}{3}(\boldsymbol{\partial}_\alpha+r_\alpha)A^\beta\right]r_\beta+G_\alpha{}^{\beta\gamma}(\mathbf{Y})\boldsymbol{\partial}_\beta r_\gamma - \frac{1}{3}\boldsymbol{\partial}_\alpha(\boldsymbol{\partial}_\beta r^\beta)+\boldsymbol{\partial}_\alpha G(\mathbf{Y}), \tag{3.18}$$

$$\partial_t Y_A=F_A(\mathbf{Y})+F_A{}^{B\alpha}(\mathbf{Y})\boldsymbol{\partial}_\alpha Y_B+F_A{}^{\alpha\beta}(\mathbf{Y})\boldsymbol{\partial}_\alpha r_\beta + F_A{}^\alpha(\mathbf{Y})r_\alpha, \tag{3.19}$$

with

$$\boldsymbol{\partial}_\alpha=E_\alpha{}^i\partial_i.$$

The coefficients $G_\alpha{}^\beta$, $G_\alpha{}^{\beta\gamma}$, G, F_A, $F_A{}^{B\alpha}$, $F_A{}^{\alpha\beta}$ and $F_A{}^\alpha$ are functions of the components of $\mathbf{Y}$.

2. Spatially homogeneous cosmologies

We now discuss how the SH cosmologies are described within the G_0 framework. These are obtained by requiring that the spatial frame derivatives of the gravitational field and matter variables $\mathbf{Y}$, and of the normalization factor H, be zero, i.e.,

$$\boldsymbol{\partial}_\alpha\mathbf{Y}=\mathbf{0},\quad r_\alpha=0. \tag{3.20}$$

It then follows that all the dimensional commutation functions and matter variables are constant on the spacelike 3-surfaces $\mathcal{S}$:$\{t=\text{const}\}$, which are thus the orbits of a three-parameter group of isometries. The evolution equations (3.18) and (3.19) imply that the SH restrictions (3.20) define an invariant set of the full evolution equations, which we shall call the *SH invariant set*. Indeed, Eq. (3.18) is trivially satisfied, and Eq. (3.19) reduces to a system of ordinary differential equations, namely

$$\partial_t Y_A=F_A(\mathbf{Y}). \tag{3.21}$$

The nontrivial constraints defined by $(\mathcal{C}_{\rm G})$, $(\mathcal{C}_{\rm C})^\alpha$ and $(\mathcal{C}_{\rm J})^\alpha$ [cf. Eqs. (2.40)–(2.42)] become purely algebraical restrictions on $\mathbf{Y}$, which we write symbolically as

$$\mathcal{C}(\mathbf{Y})=0. \tag{3.22}$$

An important aspect of this process of specialization is that the evolution equation (3.17) for $E_\alpha{}^i$ decouples from the evolution equation for $\mathbf{Y}$, which means that *the dynamics of SH cosmologies can be analyzed using only Eqs. (3.21) and (3.22)* (cf. WE). In this context, one can think of the variables $\mathbf{Y}$ as defining a *reduced Hubble-normalized state space*, of finite dimension, for the SH cosmologies.

In the SH context the restriction $v^\alpha=0$ defines an invariant subset, giving the so-called nontilted SH cosmologies, and the Bianchi classification of the isometry group leads to a hierarchy of invariant subsets, some of which have been analyzed in detail in the literature. For example, the conditions

$$v^\alpha=0,\quad A^\alpha=0,\quad N_{\alpha\beta}=0\ \ (\alpha\neq\beta),\quad R^\alpha=0,\quad \Omega_\Lambda=0, \tag{3.23}$$

give the nontilted SH perfect fluid cosmologies of class A in the canonical frame (see WE, Chap. 6, but with some differences in notation).

Specializing further, by requiring the shear rate to be zero,

$$\Sigma_{\alpha\beta}=0, \tag{3.24}$$

in addition to conditions (3.20), we obtain the *FL invariant set*, which describes the familiar Friedmann-Lemaître cosmologies. Equations (3.20) and (3.24) imply that $v^\alpha=0$ and $\mathcal{S}_{\alpha\beta}=0$, where $\mathcal{S}_{\alpha\beta}$ is the tracefree part of the 3-Ricci curvature (see Appendix 3), and hence that the spacelike 3-surfaces $\mathcal{S}$:$\{t=\text{const}\}$ are of constant curvature. In addition, the electric and magnetic parts of the Weyl curvature (see Appendix 3) are zero, $0=\mathcal{E}_{\alpha\beta}=\mathcal{H}_{\alpha\beta}$. The deceleration parameter simplifies to

$$q=\frac{1}{2}(3\gamma-2)\Omega-\Omega_\Lambda.$$

3. Silent boundary

We noted earlier that, because the evolution equation (3.17) for $E_\alpha{}^i$ is homogeneous, the equation

$$E_\alpha{}^i = 0 \tag{3.25}$$

defines an invariant set of the full evolution equations.[12] In the Introduction we discussed the notion of a *silent initial singularity*, which was introduced heuristically as an initial singularity with the property that the evolution along neighboring timelines decouples as the singularity is approached. In Sec. IV we shall make a formal definition of a silent initial singularity, but for now we note that a key requirement for an initial singularity to be silent is

$$\lim_{t\to-\infty} E_\alpha{}^i = 0, \tag{3.26}$$

i.e., the orbit that describes the evolution of the model is past asymptotic to the invariant set $E_\alpha{}^i = 0$. We will thus refer to this invariant set as the *silent boundary*.

On the silent boundary, the evolution equation (3.18) for r_α simplifies to the homogeneous form

$$\partial_t r_\alpha = \left[G_\alpha{}^\beta(\mathbf{Y}) + \frac{2}{3} r_\alpha A^\beta \right] r_\beta . \tag{3.27}$$

It follows that the equation

$$r_\alpha = 0 \tag{3.28}$$

defines an invariant subset of the silent boundary. On this invariant subset the remaining evolution equation (3.19) reduces to

$$\partial_t Y_A = F_A(\mathbf{Y}), \tag{3.29}$$

which coincides with the evolution equation (3.21) for the SH cosmologies. The remaining constraints are purely algebraical, and can be written symbolically as

$$\mathcal{C}(\mathbf{Y}) = 0. \tag{3.30}$$

One thus obtains a representation of the SH dynamics on the invariant set

$$E_\alpha{}^i = 0, \quad r_\alpha = 0, \tag{3.31}$$

i.e., within the silent boundary. Since $E_\alpha{}^i = 0$, however, the spatial dependence of the Hubble-normalized variables $\mathbf{Y}$ is completely unrestricted, and hence these solutions of the evolution equations and constraints do *not* in general correspond to exact solutions of the EFE.

[12] Note that this does not necessarily imply $\lim_{t\to-\infty} E_\alpha{}^i = 0$.

IV. SILENT INITIAL SINGULARITIES AND THE GENERALIZED MIXMASTER ATTRACTOR

In this section, we formalize the notion of a silent initial singularity, which was introduced heuristically in Sec. I.[13] We then construct an invariant set in the silent boundary that we conjecture is the local past attractor for G_0 cosmologies with a silent initial singularity. The detailed structure of the past attractor in turn relies heavily on our knowledge of the asymptotic dynamics near the initial singularity in SH cosmologies.

A. Silent initial singularities

In terms of Hubble-normalized variables and the separable volume gauge, the spacelike initial singularity in a G_0 cosmology is approached as $t \to -\infty$. We now define a *silent initial singularity* to be one which satisfies

$$\lim_{t\to-\infty} E_\alpha{}^i = 0, \tag{4.1}$$

$$\lim_{t\to-\infty} r_\alpha = 0, \tag{4.2}$$

and

$$\lim_{t\to-\infty} \boldsymbol{\partial}_\alpha \mathbf{Y} = \mathbf{0}, \tag{4.3}$$

where the $E_\alpha{}^i$ are the Hubble-normalized components of the spatial frame vectors [see Eq. (2.26)], r_α is the spatial Hubble gradient [see Eq. (2.29)] and $\mathbf{Y}$ represents the Hubble-normalized gravitational field and matter variables [see Eq. (3.16)]. More precisely, we require that Eqs. (4.1)–(4.3) are satisfied *along typical timelines of* $\mathbf{e}_0$.

One might initially think that the condition (4.3) is a consequence of Eq. (4.1), since

$$\boldsymbol{\partial}_\alpha \mathbf{Y} = E_\alpha{}^i \frac{\partial \mathbf{Y}}{\partial x^i}.$$

However, the analysis of Gowdy solutions with so-called spikes (see Refs. [20], [21] and [22]) shows that the partial derivatives $\partial \mathbf{Y}/\partial x^i$ can diverge as $t \to -\infty$. Thus the requirement (4.3) demands that the $E_\alpha{}^i$ tend to zero sufficiently fast.

We now present some evidence to justify proposing the above definition. First, for SH cosmologies, which we have seen satisfy the restrictions (3.20), the evolution equation for the $E_\alpha{}^i$ decouples from the equation for $\mathbf{Y}$. This evolution equation, in conjunction with the known results about the asymptotic behavior of the variables $\mathbf{Y}$ (see WE, Chaps. 5 and 6, and Ringström [23]), provides strong evidence that typical solutions satisfy[14] the remaining requirement (4.1) for a silent initial singularity. An example of an exceptional class

[13] The concepts we propose for classifying an initial singularity as ''silent'' can be applied analogously to final singularities.

[14] We are indebted to Hans Ringström for helpful discussions on this matter.

of SH solutions, i.e., solutions for which the initial singularity is not silent, are those that are past asymptotic to the flat Kasner solution (the Taub form of Minkowski spacetime), given by

$$ds^2 = -dT^2 + T^2 dx^2 + \ell_0^2(dy^2+dz^2),$$

where T is clock time. In terms of the dimensionless separable volume time $t=\ln(T/\ell_0)$, this line element reads

$$\ell_0^{-2} ds^2 = e^{2t}(-dt^2+dx^2)+dy^2+dz^2,$$

from which it follows that

$$\lim_{t\to-\infty} E_\alpha{}^i = \mathrm{diag}(3,0,0).$$

Secondly, further evidence is provided by recent research on G_2 cosmologies, although the situation is clouded by the fact that an area time gauge rather than the separable volume gauge is used (but see the footnote in the next section about the gauge issue). Indeed, one can use the asymptotic analysis of vacuum orthogonally transitive G_2 cosmologies (the so-called Gowdy solutions [24,25]; in the present context see in particular Ref. [26]) to show that the conditions (4.1) and (4.3) are satisfied along typical timelines, even when spikes occur. However, in general G_2 cosmologies the situation is more complicated and further studies are needed to establish if condition (4.3) holds or if it is violated along exceptional timelines due to the presence of spikes.[15]

These results suggest that the notion of a silent initial singularity may be of importance as regards the description of generic spacelike initial singularities. Further support is provided by heuristic arguments of a physical nature, as follows. We anticipate that generic spacelike initial singularities are associated with increasingly strong gravitational fields, gradually approaching local curvature radii of Planck-scale order, which will lead to the formation of *particle horizons* (see, e.g., Rindler [27,28]). The existence of particle horizons is governed by null geodesics, which satisfy

$$1 = \delta_{\alpha\beta}\left(E^\alpha{}_i \frac{dx^i}{dt}\right)\left(E^\beta{}_j \frac{dx^j}{dt}\right) \tag{4.4}$$

[see Eq. (A25) in the Appendix], where $E^\alpha{}_i$ are the components of the Hubble-normalized 1-forms associated with the orthonormal frame:

$$E^\alpha{}_i E_\beta{}^i = \delta^\alpha{}_\beta. \tag{4.5}$$

If particle horizons form, we expect that the past-directed null geodesics emanating from a chosen point P will satisfy $x^i(t)\to x^i_H$ (const) and $dx^i/dt\to 0$, as $t\to-\infty$. It follows from Eq. (4.4) that

$$E^\alpha{}_i \frac{dx^i}{dt} \to b^\alpha,$$

with $\delta_{\alpha\beta} b^\alpha b^\beta = 1$, and, hence, that

$$\left(\lim_{t\to-\infty} E_\alpha{}^i\right) b^\alpha = 0.$$

Since this must hold for all null geodesics emanating from P, b^α is arbitrary, implying that the limit (4.1) holds. In other words, we expect that the increasingly strong gravitational field associated with a typical spacelike initial singularity will lead to the first condition in the proposed definition of a silent initial singularity.

We now show heuristically that condition (4.3) restricts the scale of spatial inhomogeneities as the initial singularity is approached. If $E_\alpha{}^i$ tends to zero at an exponentially bounded rate as $t\to-\infty$ (as in SH and G_2 cosmologies), the coordinate distance to the particle horizon in a direction b^α will also tend to zero at an exponentially bounded rate:

$$\Delta x^i_H \approx b^\alpha E_\alpha{}^i \quad \text{as} \quad t\to-\infty.$$

The change $\mathbf{\Delta Y}$ in the Hubble-normalized variables $\mathbf{Y}$ corresponding to a change Δx^i_H is approximated by

$$\mathbf{\Delta Y} \approx \frac{\partial \mathbf{Y}}{\partial x^i}\Delta x^i_H \approx b^\alpha \boldsymbol{\partial}_\alpha \mathbf{Y}.$$

It thus follows from the limit (4.3) that $\mathbf{\Delta Y}\to 0$ as $t\to-\infty$. In other words, *the physical significance of the limit (4.3) is that spatial inhomogeneities have superhorizon scale asymptotically as* $t\to-\infty$, and, hence, up to the particle horizon scale a solution is asymptotically SH.

With the preceding discussion as motivation we now make our first conjecture.

> *Conjecture 1*. For almost all cosmological solutions of Einstein's field equations, a spacelike initial singularity is silent.

Proving this conjecture entails establishing the limits (4.1)–(4.3).

B. Stable subset into the past

We think of the evolution of the Hubble-normalized state vector $\mathbf{X}(t,x^i)$, *for fixed* x^i, as being described by an orbit in a *finite-dimensional* Hubble-normalized state space. As $t\to-\infty$, this orbit will be asymptotic to a *past attractor*, which, in accordance with the definition of a silent initial singularity [see Eqs. (4.1)–(4.3)], will be contained in the subset of the silent boundary defined by

$$E_\alpha{}^i = 0, \quad r_\alpha = 0. \tag{4.6}$$

The evolution of a spatially inhomogeneous model is described by infinitely many such orbits, each of which is asymptotic to the past attractor. The details of the approach to the past attractor, however, will depend on spatial position x^i, thereby reflecting the spatial inhomogeneity of the model. On the other hand, the evolution of an SH model will be described by a single orbit. The essential point is that *the dynamics in the invariant set* (4.6), *which govern the*

[15]Woei Chet Lim (private communication).

asymptotic dynamics of both classes of models, is determined by the SH evolution equations and constraints, as shown in Sec. III B 3.

The next step in constructing the putative past attractor is to partition the Hubble-normalized state vector **X** into stable and unstable variables, as regards evolution into the past. First, within our framework, the BKL conjecture means that

$$\lim_{t\to-\infty}\Omega=0,\quad \lim_{t\to-\infty}\Omega_\Lambda=0 \qquad (4.7)$$

(i.e., the initial singularity is vacuum-dominated). Secondly, asymptotic analysis and numerical experiments for SH cosmologies and G_2 cosmologies suggest that

$$\lim_{t\to-\infty}A^\alpha=0,\quad \lim_{t\to-\infty}N_{\alpha\beta}=0(\alpha\neq\beta), \qquad (4.8)$$

along a typical orbit. It is thus convenient to decompose the Hubble-normalized state vector **X** as follows:

$$\mathbf{X}=\mathbf{X}_{\rm s}\oplus\mathbf{X}_{\rm u},$$

where

$$\mathbf{X}_{\rm s}=(E_\alpha{}^i,r_\alpha,A^\alpha,N_{\alpha\beta}(\alpha\neq\beta),\Omega,\Omega_\Lambda)^T \qquad (4.9)$$

and

$$\mathbf{X}_{\rm u}=(\Sigma_\alpha,R^\alpha,N_\alpha,v^\alpha)^T. \qquad (4.10)$$

Here, for brevity, we have written[16]

$$\Sigma_\alpha:=\Sigma_{\alpha\alpha},\quad N_\alpha:=N_{\alpha\alpha}.$$

In terms of this notation, our conjectures (4.1), (4.2), (4.7) and (4.8) can be written

$$\lim_{t\to-\infty}\mathbf{X}_{\rm s}=\mathbf{0}. \qquad (4.11)$$

We shall refer to the variables $\mathbf{X}_{\rm s}$ as the *stable variables*, and the remaining variables $\mathbf{X}_{\rm u}$ in Eq. (4.10) as the *unstable variables*. We shall provide evidence that the variables $\mathbf{X}_{\rm u}$ remain bounded as $t\to-\infty$, but that their limits do not exist. We note in passing that further justification for the terminology "stable" and "unstable" in this context will be provided shortly, when we show that the variables in $\mathbf{X}_{\rm s}$ are stable on the Kasner circles, while the variables in $\mathbf{X}_{\rm u}$ are unstable.

We now list the evolution equations on the subset $\mathbf{X}_{\rm s}=0$. First, the variables Σ_α, R^α and N_α satisfy

$$\partial_t\Sigma_1=2(1-\Sigma^2)\Sigma_1+2(R_2^2-R_3^2)-3S_1, \qquad (4.12)$$

$$\partial_tR_1=[-2(1-\Sigma^2)+\Sigma_2-\Sigma_3]R_1, \qquad (4.13)$$

$$\partial_tN_1=2(\Sigma^2+\Sigma_1)N_1, \qquad (4.14)$$

where

$$S_1:=\frac{2}{9}N_1^2-\frac{1}{3}N_1(N_2+N_3)-\frac{1}{9}(N_2-N_3)^2, \qquad (4.15)$$

and cycle on (1,2,3). These variables are restricted by the Gauß constraint (2.40), which now reads

$$1=\Sigma^2+\frac{1}{6}(N_1^2+N_2^2+N_3^2)-\frac{1}{12}(N_1+N_2+N_3)^2, \qquad (4.16)$$

with

$$\Sigma^2=\frac{1}{6}(\Sigma_1^2+\Sigma_2^2+\Sigma_3^2+2R_1^2+2R_2^2+2R_3^2). \qquad (4.17)$$

Secondly, the evolution equation for v^α now reads

$$\partial_tv^\alpha=\frac{1}{G_-}[(3\gamma-4)(1-v^2)+(2-\gamma)(\Sigma_{\beta\gamma}v^\beta v^\gamma)]v^\alpha$$
$$-[\Sigma^\alpha{}_\beta-\epsilon^\alpha{}_{\gamma\beta}(R^\gamma+N^\gamma{}_\delta v^\delta)]v^\beta, \qquad (4.18)$$

where it is convenient to retain the index notation. We note for future use that Eq. (4.18) implies

$$\partial_tv^2=\frac{2}{G_-}(1-v^2)[(3\gamma-4)v^2-(\Sigma_{\alpha\beta}v^\alpha v^\beta)]. \qquad (4.19)$$

Although the variables $\mathbf{X}_{\rm u}$ are unstable into the past, it turns out that certain combinations of these unstable variables are in fact stable. First, the limit (4.11), in conjunction with the equation for $\partial_tN_{\alpha\beta}(\alpha\neq\beta)$ and the Codacci constraint, leads to the following limits:

$$\lim_{t\to-\infty}R_\alpha N_\beta=0,\quad \alpha\neq\beta. \qquad (4.20)$$

As a result, the subset of the Hubble-normalized state space defined by

$$\mathbf{X}_{\rm s}=\mathbf{0} \qquad (4.21)$$

is an invariant set only if the following restrictions hold:

$$R_\alpha N_\beta=0,\quad \alpha\neq\beta. \qquad (4.22)$$

The essential point is that the products $R_\alpha N_\beta$ $(\alpha\neq\beta)$ are stable into the past.

Secondly, we can make use of known results about SH models to motivate another limit, in addition to Eq. (4.20). We introduce the function

$$\Delta_N:=(N_1N_2)^2+(N_2N_3)^2+(N_3N_1)^2. \qquad (4.23)$$

If $\Delta_N\neq0$, i.e., if more than one N_α is nonzero, then Eq. (4.22) implies $R^\alpha=0$. Then the evolution equations (4.12)–(4.14) reduce to the evolution equations for vacuum SH models of class A. It has been shown[17] that solutions of these evolution equations satisfy

[16]Not to be confused with the notation used in Ref. [30], where Σ_1 was defined to be equal to Σ_{23}, and cycle on (1,2,3).

[17]See Ringström [23] for the case where the N_α have the same sign (Bianchi type IX case). Numerical simulations suggest that this result is also true in the Bianchi type VIII case.

$$\lim_{t\to-\infty} \Delta_N = 0. \qquad (4.24)$$

It is thus plausible that if Eqs. (4.11) and (4.20) hold, then so does Eq. (4.24). We shall refer to the invariant set defined by

$$\mathbf{X}_s = \mathbf{0}, \quad R_\alpha N_\beta = 0 \;\; (\alpha \neq \beta), \quad \Delta_N = 0, \qquad (4.25)$$

as the *stable subset* into the past and make the following conjecture.

Conjecture 2. The local past attractor $\mathcal{A}^-$ for G_0 cosmologies with a silent initial singularity is a subset of the stable subset.

Proving this conjecture entails proving the limits (4.11), (4.20) and (4.24), assuming the validity of Eqs. (4.1)–(4.3).

We believe that this conjecture can be strengthened, however. In order to do this, we need to describe how the Kasner vacuum solutions are represented within the present framework.

C. Kasner circles

The line element for the Kasner vacuum solutions is

$$\ell_0^{-2} ds^2 = -dT^2 + T^{2p_1}dx^2 + T^{2p_2}dy^2 + T^{2p_3}dz^2,$$

where the Kasner exponents p_1, p_2 and p_3 are constants that satisfy

$$p_1 + p_2 + p_3 = 1, \quad p_1^2 + p_2^2 + p_3^2 = 1,$$

and $\ell_0 T$ is clock time. The Kasner exponents can take values that are described by the inequalities $-\frac{1}{3} \leqslant p_1 \leqslant 0 \leqslant p_2 \leqslant \frac{2}{3} \leqslant p_3 \leqslant 1$ (or permutations thereof); see Ref. [31], p. 196. Relative to the natural orthonormal frame associated with this line element, the Hubble-normalized connection variables are all zero except for the shear rate tensor, which is diagonal and given by

$$\Sigma_{\alpha\beta} = \mathrm{diag}(3p_1 - 1, 3p_2 - 1, 3p_3 - 1).$$

One can also represent the Kasner solutions relative to a spatial frame that is not Fermi-propagated, as is the spatial frame specified by Eq. (3.10). Some of these alternative representations are important in what follows.

Within our formulation, all possible representations of the Kasner solutions are given by

$$0 = A^\alpha = N_{\alpha\beta} = \Omega = v^\alpha = \Omega_\Lambda, \qquad (4.26)$$

$$r_\alpha = 0, \quad \boldsymbol{\partial}_\alpha \Sigma_{\beta\gamma} = 0, \quad \boldsymbol{\partial}_{[\alpha} E_{\beta]}{}^i = 0, \qquad (4.27)$$

with the R^α given by Eq. (3.10). The Gauß constraint (2.40), together with Eqs. (4.16) and (3.14), implies that

$$\Sigma^2 = 1, \quad q = 2. \qquad (4.28)$$

The evolution of the nonzero variables $E_\alpha{}^i$ and $\Sigma_{\alpha\beta}$ is governed by

$$\partial_t E_\alpha{}^i = (2\,\delta_\alpha{}^\beta - \Sigma_\alpha{}^\beta + \epsilon_{\alpha\gamma}{}^\beta R^\gamma) E_\beta{}^i \qquad (4.29)$$

$$\partial_t \Sigma^{\alpha\beta} = 2\,\epsilon^{\gamma\delta\langle\alpha} R_\gamma \Sigma^{\beta\rangle}{}_\delta, \qquad (4.30)$$

as follows from Eqs. (3.17) and (2.34).

In the physical region of the Hubble-normalized state space, i.e., $\det(E_\alpha{}^i) \neq 0$, Eqs. (4.27) imply that $\Sigma_{\alpha\beta} = \Sigma_{\alpha\beta}(t)$, and that the spatial coordinate freedom can be used to obtain $E_\alpha{}^i = E_\alpha{}^i(t)$, confirming that the Kasner solutions are SH and of Bianchi type I. On the silent boundary $(E_\alpha{}^i = 0)$, however, Eqs. (4.27) become trivial, with the result that *the spatial dependence of* $\Sigma_{\alpha\beta}$ *is unrestricted*. One thus obtains a representation of the Kasner dynamics locally on the silent boundary, even though the line element, as given by Eq. (A25) in the Appendix, is singular. Indeed, the Kasner dynamics on the silent boundary is described by the orbits that satisfy $\mathbf{X}_s = \mathbf{0}$ and the additional restriction $N_\alpha = 0$, as follows from Eqs. (4.9) and (4.26). We shall refer to this subset, defined by

$$\mathbf{X}_s = \mathbf{0}, \quad N_\alpha = 0, \qquad (4.31)$$

as the *Kasner set on the silent boundary*.

The evolution equations on the Kasner set are obtained by setting $\Sigma^2 = 1$ and $S_\alpha = 0$ in Eqs. (4.12) and (4.13), which yields

$$\partial_t \Sigma_1 = 2(R_2^2 - R_3^2), \qquad (4.32)$$

$$\partial_t R_1 = (\Sigma_2 - \Sigma_3) R_1, \qquad (4.33)$$

and cycle on (1,2,3). Note that the evolution equations for Σ_α and R^α decouple from that of v^α, discussed below.

It is important to note that if the spatial frame is *not* Fermi-propagated $(R^\alpha \neq 0)$, the $\Sigma_{\alpha\beta}$ evolve in time, with $\Sigma^2 = 1$, both on and off the silent boundary. On the other hand, if the spatial frame *is* Fermi-propagated $(R^\alpha = 0)$, then $\Sigma_{\alpha\beta}$ is constant in time by Eq. (4.30), and diagonal:

$$\Sigma_{\alpha\beta} = \mathrm{diag}(\Sigma_1, \Sigma_2, \Sigma_3), \qquad (4.34)$$

with $-2 \leqslant \Sigma_1 \leqslant -1 \leqslant \Sigma_2 \leqslant 1 \leqslant \Sigma_3 \leqslant 2$ (or permutations thereof). Thus, if the spatial frame is Fermi-propagated, the Kasner orbits on the silent boundary are equilibrium points of the shear evolution equations. Since $\Sigma_{\alpha\beta}$ is tracefree and satisfies $\Sigma^2 = 1$, we obtain

$$\Sigma_1 + \Sigma_2 + \Sigma_3 = 0 \qquad (4.35)$$

and

$$\Sigma_1^2 + \Sigma_2^2 + \Sigma_3^2 = 6. \qquad (4.36)$$

The dynamics on the Kasner set also includes the evolution equations (4.18) for v^α (with $N_{\alpha\beta} = 0$). It follows that if the $\Sigma_{\alpha\beta}$ satisfy Eqs. (4.34)–(4.36), then the evolution equations (4.18) and (4.19) for v^α admit the equilibrium sets

$$v^\alpha = 0 \quad \text{or} \quad v^2 = 1,$$

where the latter condition is also to be supplemented with one of the six choices for $\mathbf{v} = (v_1, v_2, v_3)^T$, namely

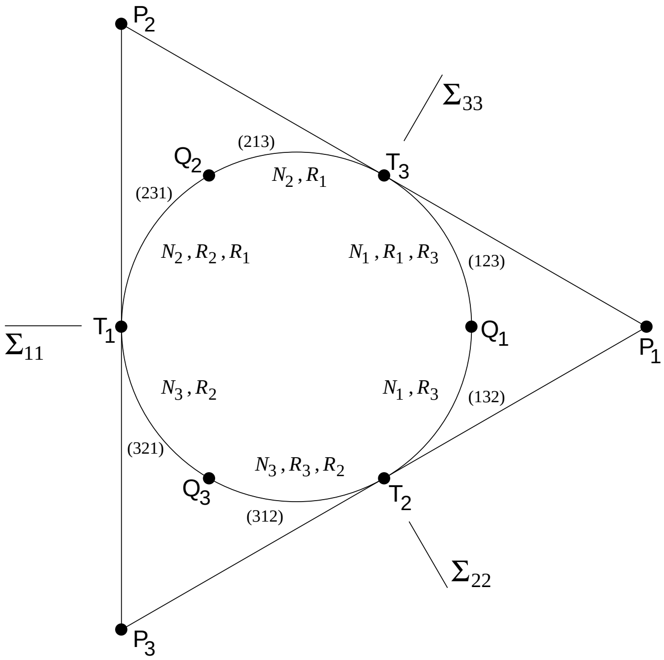

FIG. 1. A Kasner circle showing the six equivalent sectors and the variables that are unstable into the past in each sector.

$$\mathbf{v} = \pm \mathbf{E}_\alpha, \quad \alpha = 1,2,3, \tag{4.37}$$

where $\mathbf{E}_1 = (1,0,0)^T$, etc. Thus, there exist *seven* sets of equilibrium points forming circles in $\Sigma_{\alpha\beta}$-space, which we shall call *Kasner circles* [the intersection of the plane (4.35) with the sphere (4.36)], depending on the value of $\mathbf{v}$ in Eq. (4.37), which we denote by

$$\mathcal{K}, \quad \mathcal{K}_{\pm\alpha}.$$

In addition, it follows from Eq. (4.18) that for specific values of Σ_α subject to Eqs. (4.35) and (4.36), there are six additional *lines of equilibrium points*, given by

$$\Sigma_1 = 3\gamma - 4, \quad v_1 > 0 \quad \text{or} \quad v_1 < 0, \quad v_2 = v_3 = 0, \tag{4.38}$$

and cycle on (1,2,3), which join the various Kasner circles.

At this stage, we digress to describe the symmetry properties of the Kasner circles. Each circle is divided into six equivalent sectors, which we will label according to the ordering of the diagonal shear components Σ_α, which satisfy Eqs. (4.35) and (4.36). For example, in sector (123) these parameters satisfy $\Sigma_1 < \Sigma_2 < \Sigma_3$, etc. The sectors meet at points where two of the Σ_α are equal. These points are of two types, conventionally labeled T_α (the "Taub points") and Q_α, given by[18]

$$T_1: \ (\Sigma_1, \Sigma_2, \Sigma_3) = (2, -1, -1),$$

$$Q_1: \ (\Sigma_1, \Sigma_2, \Sigma_3) = (-2,1,1), \tag{4.39}$$

and cycle on (1,2,3). Figure 1 represents the plane $\Sigma_1 + \Sigma_2 + \Sigma_3 = 0$ in $\Sigma_{\alpha\beta}$-space, containing a Kasner circle, and showing the six sectors and the points T_α and Q_α. The figure also shows three additional points labeled P_α, which lie outside a Kasner circle, forming an equilateral triangle whose sides are tangential to the circle. These points, which are given by

$$P_1: \ (\Sigma_1, \Sigma_2, \Sigma_3) = (-4,2,2),$$

and cycle on (1,2,3), will be used to describe the so-called curvature transition sets.

D. Transition sets

The dynamics in the stable subset, defined by Eq. (4.25), is essentially determined by the fact that each Kasner equilibrium point is a saddle point, with at least two of the nine variables N_α, R^α and v^α being unstable into the past. Which of these variables are unstable at a particular Kasner point can be quickly determined by *linearizing* Eqs. (4.14), (4.13) and (4.18) in the neighborhood of such a point. On the Kasner circle $\mathcal{K}$ this yields

$$\partial_t N_1 = 2(1 + \Sigma_1) N_1, \tag{4.40}$$

$$\partial_t R_1 = (\Sigma_2 - \Sigma_3) R_1, \tag{4.41}$$

$$\partial_t v_1 = (3\gamma - 4 - \Sigma_1) v_1, \tag{4.42}$$

and similarly for indices 2 and 3. On the Kasner circles $\mathcal{K}_{\pm 1}$ (where, nearby, $v_1 = \pm 1 \mp \delta v_1$, $\delta v_1 > 0$), the linearized equations for N_1 and R_1 remain unchanged, while Eq. (4.18) yields

$$\partial_t \delta v_1 = -2 \frac{(3\gamma - 4 - \Sigma_1)}{(2 - \gamma)} \delta v_1, \tag{4.43}$$

$$\partial_t v_2 = (\Sigma_1 - \Sigma_2) v_2, \tag{4.44}$$

$$\partial_t v_3 = (\Sigma_1 - \Sigma_3) v_3, \tag{4.45}$$

and similarly for indices 2 and 3 on $\mathcal{K}_{\pm 2}$ and $\mathcal{K}_{\pm 3}$, respectively. It follows that

$$N_1 \ \text{is unstable into the past} \Leftrightarrow 1 + \Sigma_1 < 0,$$

$$R_1 \ \text{is unstable into the past} \Leftrightarrow \Sigma_2 - \Sigma_3 < 0,$$

$$v_1 \ \text{is unstable into the past on} \ \mathcal{K} \Leftrightarrow 3\gamma - 4 - \Sigma_1 < 0,$$

$$v_1 \ \text{is unstable into the past on} \ \mathcal{K}_{\pm 1} \Leftrightarrow 3\gamma - 4 - \Sigma_1 > 0,$$

$$v_2 \ \text{is unstable into the past on} \ \mathcal{K}_{\pm 1} \Leftrightarrow \Sigma_1 - \Sigma_2 < 0,$$

$$v_3 \ \text{is unstable into the past on} \ \mathcal{K}_{\pm 1} \Leftrightarrow \Sigma_1 - \Sigma_3 < 0,$$

and similarly for indices 2 and 3. The arcs of the Kasner circles on which the variables N_α and R^α are unstable are shown in Fig. 1, and those on which the variables v^α are unstable are shown in Figs. 2 and 3. A given Kasner equilib-

[18]The T_α correspond to the Taub form for Minkowski spacetime in the exact Kasner solution, and the Q_α correspond to the locally rotationally symmetrical nonflat Kasner solution.

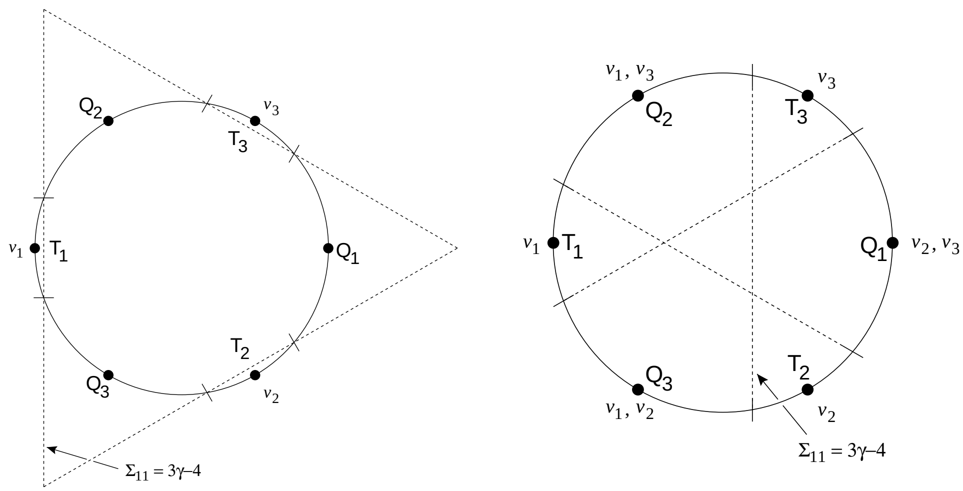

FIG. 2. The Kasner circle $\mathcal{K}$ showing the arcs on which the variables v^α are unstable into the past. Figure 2(a) shows the case $\frac{5}{3}<\gamma<2$ and Fig. 2(b) the case $1<\gamma<\frac{5}{3}$.

rium point P thus has an unstable manifold (into the past).[19] The key property of this unstable manifold is that *each of its orbits join P to some other Kasner point*. The simplest such orbits are those on which only one of the nine variables N_α, R^α and v^α is nonzero, or those on which two v^α are nonzero but with extreme tilt ($v^2=1$). These special orbits, which we shall refer to collectively as *transition sets*, are listed by name and symbol in Table I. In this table the subscripted letter on the $\mathcal{T}$ indicates the "excited" variable. We now discuss these transition sets in turn.

First, there are the *curvature transition sets*, $\mathcal{T}_{N_\alpha}$, $\alpha=1,2,3$, which are shown in Fig. 4. In this and all subsequent figures, *orbits are directed toward the past*. For $\alpha=1$ the curvature transition orbits are given by

$$(1-k)(2-\Sigma_2)=(1+k)(2-\Sigma_3), \tag{4.46}$$

where k is a parameter that satisfies $-1\leqslant k\leqslant 1$. This relation follows from Eqs. (4.12) and (4.14) with $0=N_2=N_3=R^\alpha$. In the spatially homogeneous setting these orbits describe the Taub vacuum Bianchi type II solutions, and determine the past attractor for vacuum and nontilted SH models of Bianchi type VIII and type IX (in a Fermi-propagated frame; see WE, Fig. 6.6 on p. 138, and pp. 143–7, but note differences in labeling). These curvature transitions link different "Kasner epochs," according to a transition law for the Kasner exponents first found by BKL (see Ref. [8], pp. 535–7; also WE, p. 236). We derive this transition law in Appendix 4.

Secondly, there are the *frame transition sets*, $\mathcal{T}_{R_\alpha}$, which are shown in Fig. 5. For $\alpha=1$ they are given by

$$\Sigma_1=k,$$

where k is a parameter that satisfies $-2<k<2$. In the spatially homogeneous setting these transition sets map a Kasner solution into a physically equivalent Kasner solution through rotation of the spatial frame by $\pi/2$ about one of its axes. For example, the transition sets $\mathcal{T}_{R_1}$ result in the interchange $\Sigma_2<\Sigma_3\rightarrow\Sigma_3<\Sigma_2$, as can be seen by comparing Figs. 1 and 5(a).

Thirdly, there are the *tilt transition sets*, $\mathcal{T}_{v_\alpha}$, which are shown in Fig. 6. They are simply lines with Σ_α constant and one of the v^α nonzero. Whether the orbits join a point on $\mathcal{K}$ to a point on one of the Kasner circles $\mathcal{K}_{\pm\alpha}$, or vice versa, depends on the values of the Σ_α (see Figs. 2 and 3). The reversal of direction of these orbits is governed by the six lines of equilibrium points given by Eq. (4.38).

Finally, there are the *extreme-tilt transition sets*, $\mathcal{T}_{v_\alpha v_\beta}$. Let us consider the subset $v^2=1$ (with fixed Σ_α). Then $v^\alpha=e^\alpha$, where e^α is a unit vector, which may be parametrized according to

$$e^\alpha=(\cos\vartheta,\sin\vartheta\cos\varphi,\sin\vartheta\sin\varphi)^T,\quad 0\leqslant\vartheta\leqslant\pi,$$
$$0\leqslant\varphi\leqslant 2\pi. \tag{4.47}$$

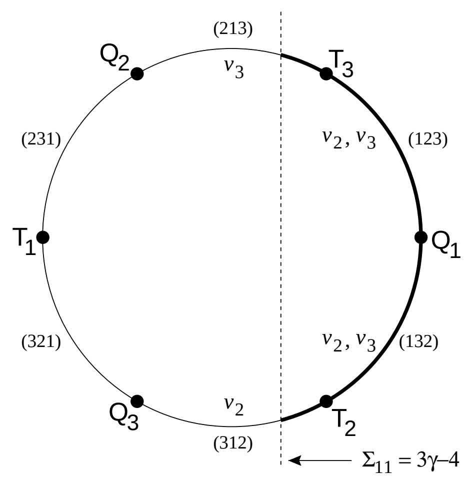

FIG. 3. The Kasner circles $\mathcal{K}_{\pm1}$ showing the arcs on which the variables v^α are unstable into the past. The variable v_1 is unstable on the boldface arc to the right of the line $\Sigma_{11}=3\gamma-4$.

[19]As Figs. 1–3 show, the unstable manifold is at most five-dimensional.

TABLE I. The transition sets.

Name	*Symbol*	*Transitions*
curvature	$\mathcal{T}_{N_\alpha}$	$\mathcal{K}\to\mathcal{K}$ $(v^\alpha=0)$ or $\mathcal{K}_{\pm\alpha}\to\mathcal{K}_{\pm\alpha}$ $(v^2=1)$
frame	$\mathcal{T}_{R_\alpha}$	$\mathcal{K}\to\mathcal{K}$ $(v^\alpha=0)$ or $\mathcal{K}_{\pm\alpha}\to\mathcal{K}_{\pm\alpha}$ $(v^2=1)$
tilt	$\mathcal{T}_{v_\alpha}$	$\mathcal{K}\to\mathcal{K}_{\pm\alpha}$ or vice versa ($\Sigma_{\alpha\beta}$ fixed)
extreme tilt	$\mathcal{T}_{v_\alpha v_\beta}$ $(\alpha\neq\beta)$	$\mathcal{K}_{\pm\alpha}\to\mathcal{K}_{\pm\beta}$ ($\Sigma_{\alpha\beta}$ fixed, $v^2=1$)

We then obtain from Eq. (4.18) a simple dynamical system for the polar angles $\{\vartheta, \varphi\}$, given by

$$\partial_t\vartheta=-a^2\sin 2\vartheta,\quad a^2:=\frac{3}{2}(-p_1+p_2\cos^2\varphi+p_3\sin^2\varphi),$$

$$\partial_t\varphi=-b^2\sin 2\varphi,\quad b^2:=-\frac{3}{2}(p_2-p_3). \tag{4.48}$$

It is easily seen that if $\Sigma_1<\Sigma_2<\Sigma_3$ (or equivalently $p_1<p_2<p_3$), then the past attractor of this dynamical system is given by $\{\vartheta,\varphi\}=\{\frac{1}{2}\pi,\frac{1}{2}\pi\}$ and $\{\vartheta,\varphi\}=\{\frac{1}{2}\pi,\frac{3}{2}\pi\}$, and, hence,

$$\lim_{t\to-\infty} e^\alpha=\pm\mathbf{E}_3, \tag{4.49}$$

and similarly for other orderings of Σ_α. Thus extreme-tilt transition sets are orbits that lie on the extreme-tilt sphere $v^2=1$ in v^α-space, with $0=N_\alpha=R^\alpha$, and Σ_α fixed. Figure 7 shows the direction corresponding to $\Sigma_1<\Sigma_2<\Sigma_3$, i.e., the arc (123) on the Kasner circles. The other cases can be obtained by interchanging 1, 2 and 3. The points $B_{\pm\alpha}$ in Fig. 7 correspond to the points on the Kasner circles $\mathcal{K}_{\pm\alpha}$ determined by the values of the Σ_α. The directions of the orbits joining the points $B_{\pm\alpha}$ depend on the ordering of the Σ_α.

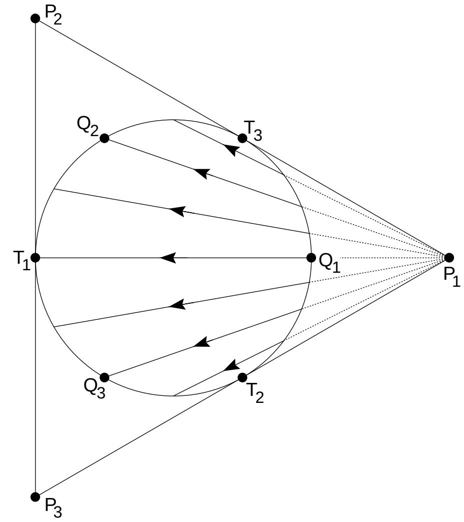

FIG. 4. The curvature transition set $\mathcal{T}_{N_1}$. The sets $\mathcal{T}_{N_2}$ and $\mathcal{T}_{N_3}$ are obtained by cycling on (1,2,3).

E. Structure of the past attractor

We are now in a position to state our conjecture concerning the local past attractor of the G_0 evolution equations for generic initial data satisfying the constraints. We introduce the following notation:

$$\mathcal{T}_N:=\cup\mathcal{T}_{N_\alpha},\quad \mathcal{T}_R:=\cup\mathcal{T}_{R_\alpha},\quad \mathcal{T}_{\rm tilt}:=\cup\mathcal{T}_{v_\alpha},$$

$$\mathcal{T}_{\rm extreme}:=\cup\mathcal{T}_{v_\alpha v_\beta},\quad \mathcal{K}_{\rm extreme}:=\cup\mathcal{K}_{\pm\alpha}. \tag{4.50}$$

Thus, $\mathcal{T}_N$ is the union of all curvature transition sets, $\mathcal{T}_R$ is the union of all frame transition sets, etc. (see Table I for the complete list of transition sets).

We now make the following conjecture concerning the local past attractor $\mathcal{A}^-$.

Conjecture 3. The local past attractor $\mathcal{A}^-$ for G_0 cosmologies with a silent initial singularity is given by

$$\mathcal{A}^-=\mathcal{K}\cup\mathcal{K}_{\rm extreme}\cup\mathcal{T}_N\cup\mathcal{T}_R\cup\mathcal{T}_{\rm tilt}\cup\mathcal{T}_{\rm extreme}. \tag{4.51}$$

The essential property of the various transition sets is that they define so-called *infinite heteroclinic sequences* on the past attractor, i.e., infinite sequences of Kasner equilibrium points joined by transition sets, oriented into the past. In particular, a typical Kasner point will be the starting point for

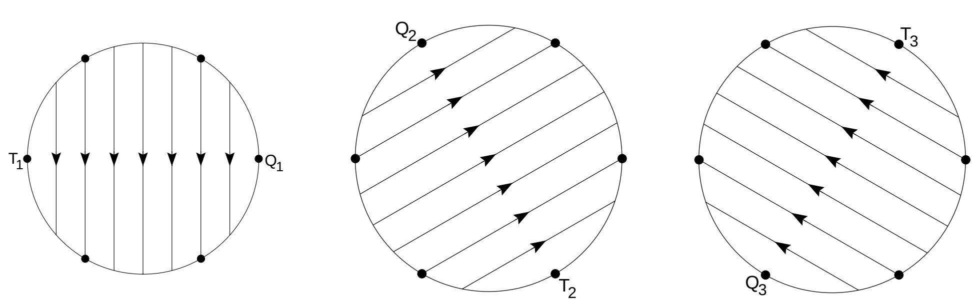

FIG. 5. The frame transitions sets $\mathcal{T}_{R_1}$ (a), $\mathcal{T}_{R_2}$ (b) and $\mathcal{T}_{R_3}$ (c).

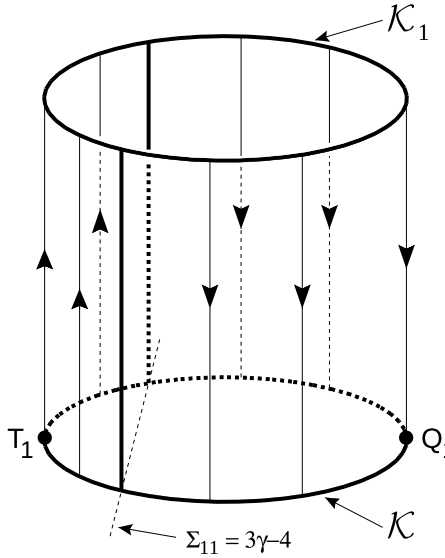

FIG. 6. The tilt transition set $\mathcal{T}_{v_1}$.

infinitely many heteroclinic sequences (infinitely many, because at least two transition sets emanate from each Kasner point). The significance of the heteroclinic sequences is that an orbit that is asymptotic to the past attractor (for given values of the x^i) will shadow a heteroclinic sequence, and hence the cosmological model will be approximated locally by a sequence of Kasner states.

The conjectured past attractor, $\mathcal{A}^-$, is a proper subset of the stable subset, defined by Eq. (4.25). Thus, in addition to the stable variables (4.9), various expressions involving the unstable variables N_α, R^α and v^α will tend to zero on the attractor, even though the limits of these variables as $t \to -\infty$ do not exist. The desired expressions depend on the following quantities:

$$N^2 := N_\alpha N^\alpha, \quad R^2 := R_\alpha R^\alpha, \quad v^2 := v_\alpha v^\alpha,$$

and

$$\Delta_N := (N_1 N_2)^2 + (N_2 N_3)^2 + (N_3 N_1)^2,$$

together with analogous quantities Δ_R and Δ_v for R^α and v^α. The past attractor is then characterized by the following limits [in addition to Eqs. (4.1)–(4.3) and (4.11)]:

$$\lim_{t\to-\infty} N^2 R^2 = 0, \tag{4.52}$$

$$\lim_{t\to-\infty} v^2(1-v^2)(N^2+R^2) = 0, \tag{4.53}$$

$$\lim_{t\to-\infty} (\Delta_N, \Delta_R) = (0,0), \tag{4.54}$$

$$\lim_{t\to-\infty} (\Delta_v(N^2+R^2), \Delta_v(1-v^2)) = (0,0), \tag{4.55}$$

$$\lim_{t\to-\infty} v_1 v_2 v_3 = 0. \tag{4.56}$$

The limits (4.52) and (4.53) imply that if N_α (R^α, respectively) is active, than R^α (N_α, respectively) must be close to zero, and likewise either v^2 or $1-v^2$ must be close to zero, as on the $\mathcal{T}_{N_\alpha}(\mathcal{T}_{R_\alpha})$ transition sets, respectively. The limit (4.53) also implies that if v^2 is not close to 0 or 1, then N_α and R^α must be close to zero, as on the tilt transition sets $\mathcal{T}_{v_\alpha}$. The limit (4.54) implies that at most one N_α and at most one R^α can be active simultaneously, as on the $\mathcal{T}_{N_\alpha}$ and $\mathcal{T}_{R_\alpha}$ transition sets. The limit (4.55) implies that if two v^α are active simultaneously, (i.e., $\Delta_v \neq 0$), then N_α and R^α must be close to zero and v^2 must be close to 1, as on the extreme-tilt transition sets $\mathcal{T}_{v_\alpha v_\beta}$.

The conjectured structure of the past attractor $\mathcal{A}^-$ in terms of Kasner equilibrium points and transition sets, as given by Eq. (4.51), or as described by the limits (4.1)–(4.3), (4.11) and (4.52)–(4.56), embodies the notion that as one approaches the attractor into the past along an orbit, the probability that more than one of the nine unstable variables N_α, R^α and v^α is active during any one transition (except for the extreme-tilt transition sets on which two v^α are nonzero, but are constrained by $v^2=1$) tends to zero. Evidence that the probability of multiple transitions involving pairs such as (R_1, N_1) or (R_1, R_2) tends to zero as $t \to -\infty$ is provided by numerical simulations for nontilted SH models of Bianchi type $\mathrm{VI}^*_{-1/9}$, reported by Hewitt *et al.* [30].

We shall refer to the local past attractor $\mathcal{A}^-$, defined by Eq. (4.51), as the *generalized Mixmaster attractor*, since it generalizes the past attractor for the so-called Mixmaster models (SH models of Bianchi type IX; see WE, p. 146, and Table II to follow), making precise the heuristic notion of an oscillatory initial singularity.

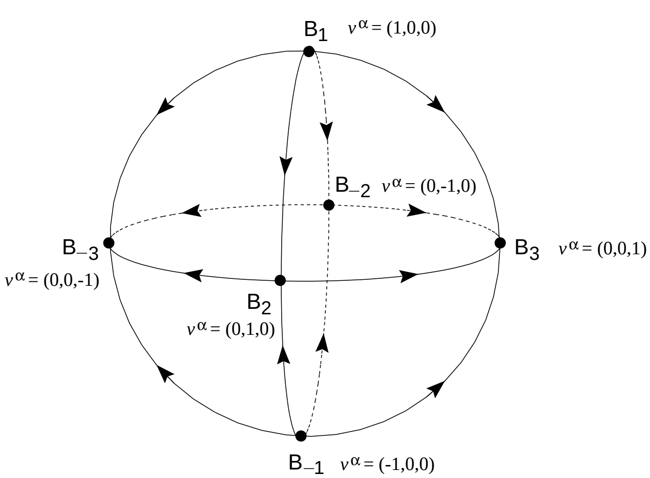

FIG. 7. The extreme-tilt transition sets on the extreme-tilt sphere $v^2=1$, $0=N_\alpha=R^\alpha$, with Σ_α fixed and $\Sigma_1<\Sigma_2<\Sigma_3$.

V. COSMOLOGIES WITH ISOMETRIES

In Sec. IV, we proposed a detailed structure for the past attractor for G_0 cosmologies with a silent initial singularity [see Eq. (4.51)]. Classes of cosmologies that admit an isometry of some sort are described by invariant sets of the Hubble-normalized state space. In this section we exploit this fact to predict the structure of the past attractor for these more specialized models, thereby providing a link to much recent research.

For models with an isometry it is possible that one or

TABLE II. Perfect fluid SH cosmologies with oscillatory initial singularity.

Class of models	*Nonzero unstable variables*	*Past attractor*
nontilted type VIII and type IX (WE, p. 146, Ringström [23])	N_1, N_2, N_3	$\mathcal{K}\cup\mathcal{T}_N$
nontilted type $\mathrm{VI}^*_{-1/9}$ (Hewitt *et al.* [30])	N_3, R_1, R_3	$\mathcal{K}\cup\mathcal{T}_{N_3}\cup\mathcal{T}_{R_1}\cup\mathcal{T}_{R_3}$
tilted type II (Hewitt *et al.* [29])	N_3, R_1, R_3, v_1	$\mathcal{K}\cup\mathcal{K}_1\cup\mathcal{T}_{N_3}\cup\mathcal{T}_{R_1}\cup\mathcal{T}_{R_3}\cup\mathcal{T}_{v_1}$

more of the nine unstable variables N_α, R^α or v^α is required to be zero, leading to two possibilities.

(i) The initial singularity is *oscillatory*.

This possibility occurs if each arc of the various Kasner circles has at least one unstable variable (refer to Figs. 1–3). The attractor will then include all available transition sets, and the evolution into the past along a typical timeline will be described by an infinite sequence of Kasner states, possibly of a more specialized nature than for G_0 cosmologies.

(ii) The initial singularity is *Kasner-like*.

This possibility occurs if at least one arc on one of the Kasner circles has no unstable variables. The arc(s) in question then form the past attractor, and the evolution into the past along a typical timeline will be described by a specific Kasner state. A cosmological solution with this type of singularity is also referred to as being *asymptotically velocity(-term)-dominated*, a term that has its origins in the work of Eardley, Liang and Sachs [32] and Isenberg and Moncrief [33].

We now present various classes of cosmologies with isometries, whose initial singularities have been discussed in the literature, and give the conjectured past attractor in terms of our formulation. The specific nature of the isometry determines whether the initial singularity is oscillatory or Kasner-like.

First, we consider SH cosmologies, which, as we have seen in Sec. III B 2, can be described by the finite-dimensional reduced state space defined by the variables $\mathbf{Y}$. Since the definition of the generalized Mixmaster attractor $\mathcal{A}^-$ involves only the variables $\mathbf{Y}$, the attractor also exists as an invariant set in this reduced state space. Indeed, we conjecture that in this context *$\mathcal{A}^-$ is the past attractor for the general class of SH cosmologies*, and that it will thus contain the past attractors for the three special classes of SH cosmologies with an oscillatory singularity that have been analyzed in detail to date. In Table II we give these three special classes of SH cosmologies and list the key references. These papers use Hubble-normalized variables, but there are some differences in the labeling of variables compared to the present paper. It is notoriously difficult to prove rigorous results about oscillatory singularities and little progress has been made until recently, when Ringström [23], in a remarkable piece of mathematical analysis, rigorously established the existence of the past attractor for the class of nontilted Bianchi type IX cosmologies, by proving the required limits, conjectured earlier by WE (see p. 146–7).

Secondly, we consider spatially inhomogeneous cosmologies. Most recent research on the initial singularity has been restricted in two ways:

(i) the spacetime is assumed to have compact spatial sections,

(ii) the energy-momentum-stress tensor is assumed to be zero (vacuum solutions).

The first restriction is made because it enables one to prove results about the global existence of solutions. It does not, however, affect the structure of the past attractor, since it is determined by the dynamics along individual timelines. In view of the BKL conjecture, namely that matter is not significant dynamically as the initial singularity is approached, one might believe that the second restriction can be made without loss of generality when determining the past attractor. This conclusion is not valid, however. Our analysis leads to the conjecture that the past attractor for vacuum G_0 models is in fact the much simpler set given by

$$\mathcal{A}^-_{\mathrm{vac}}=\mathcal{K}\cup\mathcal{T}_N\cup\mathcal{T}_R, \tag{5.1}$$

since the Hubble-normalized state vector for vacuum models does not contain the peculiar velocity variable v^α, which implies that the extreme Kasner circles $\mathcal{K}_{\mathrm{extreme}}$ and the transition sets $\mathcal{T}_{\mathrm{tilt}}$ and $\mathcal{T}_{\mathrm{extreme}}$ cannot be part of the past attractor [see Eq. (4.51)]. Nevertheless, determining the vacuum past attractor is an important first step in determining the past attractor for nonvacuum models.

In Table III we list the classes of vacuum spatially inhomogeneous cosmologies whose initial singularity has been studied. In each case we can predict immediately whether the singularity will be oscillatory or Kasner-like. In the table we give the past attractor for each class, which is a subset of the general vacuum attractor $\mathcal{A}^-_{\mathrm{vac}}$.[20]

[20]At this stage the reader might be concerned with the fact that these models have not been studied in the separable volume gauge. However, we believe that, due to asymptotic silence, our discussion is "gauge robust," i.e., that the local asymptotic temporal behavior is not affected by the choice of temporal gauge. To make this more substantial we note that the choice $\mathcal{N}=1$ and $N^i=0$ was not necessary for obtaining our picture of the past attractor. Any sufficiently smooth choice $\mathcal{N}(\mathbf{X})$ such that $\mathcal{N}$ is positive and bounded on the attractor does not change the flow on the past attractor and thus one would obtain the same results as the choice $\mathcal{N}=1$ yields; N^i can be similarly generalized. We also note that these are not necessary conditions, and that even wider sets of gauge choices are allowed if one takes into account the detailed structure of the EFE.

TABLE III. Past attractor for vacuum spatially inhomogeneous cosmologies with isometries. In the first three cases the initial singularity is Kasner-like, and in the last two cases the initial singularity is oscillatory. In all cases we have adopted the convention of aligning the frame vector field $\mathbf{e_3}$ with (one of) the Killing vector field(s).

Class of models	*Nonzero unstable variables*	*Vacuum past attractor*
Polarized Gowdy≡ diagonal G_2 (Isenberg and Moncrief [33])	all zero	$\mathcal{K}$
Unpolarized Gowdy≡ OT G_2 (Kichenassamy and Rendall [26])	N_3, R_1	$\mathrm{arc}(T_2Q_1)\subset\mathcal{K}$
Polarized T_2-symmetric $\equiv G_2$ with one HO KVF (Isenberg and Kichenassamy [35])	R_3	$\mathrm{arc}(T_3Q_2T_1Q_3)\subset\mathcal{K}$
T_2-symmetric≡ generic G_2 (Berger *et al.* [38])	N_3, R_1, R_3	$\mathcal{K}\cup\mathcal{T}_{N_3}\cup\mathcal{T}_{R_1}\cup\mathcal{T}_{R_3}$
$U(1)$-symmetric≡ generic G_1 (Berger and Moncrief [36])	all nonzero	$\mathcal{K}\cup\mathcal{T}_N\cup\mathcal{T}_R$

One class of vacuum models is not included in Table III, the so-called polarized $U(1)$-symmetric models (Berger and Moncrief [34]). These are G_1 cosmologies for which the single spacelike Killing vector field is hypersurface-orthogonal. The reason for this exclusion is that the dynamical consequences of the hypersurface-orthogonality condition are not compatible with our choice of spatial frame, given by Eq. (3.10). These models could be incorporated by making a different choice of spatial frame, as discussed in footnote [10], but we will not pursue this matter here.

It should be noted that in the papers listed in Table III the conclusions about the dynamics near the initial singularity are not expressed in terms of a past attractor: we have reformulated their results within our dynamical systems framework, and at this stage most of the results about the past attractor have not been rigorously established. The papers referred to use a metric-based approach[21] instead of the orthonormal frame approach. Some of them make use of the so-called Fuchsian algorithm to establish the asymptotics at a Kasner-like initial singularity (see Refs. [26] and [35]) while others rely on a Hamiltonian formalism and the so-called method of consistent potentials to predict whether the initial singularity will be oscillatory or not (see Refs. [36], [37] and [38]). In this approach, the transitions between Kasner states are described heuristically as bounces off potential walls determined by the Hamiltonian. Some of these papers also describe numerical simulations that display a finite number of Mixmaster oscillations.

VI. CONCLUDING REMARKS

In this paper we have developed a mathematical framework for analyzing the dynamics of G_0 cosmologies, and in particular the BKL conjecture discussed in the Introduction. A key step was the introduction of scale-invariant variables, using the Hubble scalar defined by a timelike reference congruence as the normalization factor. One of the principal advantages of Hubble-normalization lies in the behavior of the dynamical variables as the initial singularity is approached: the dimensional variables diverge, while, for at least a generic family of solutions, *the Hubble-normalized variables remain bounded*.

The structure of the evolution equations and constraints led to the introduction of the *silent boundary* in the Hubble-normalized state space and enabled us to define a *silent initial singularity*. The next step was to construct the *generalized Mixmaster attractor*, which makes precise the notion of an oscillatory initial singularity in a G_0 cosmology, while having a simple geometrical structure (see Figs. 1–7), and allowed three precise conjectures on cosmological dynamics at early times to be formulated (Conjecture 1 in Sec. IV A, Conjecture 2 in Sec. IV B, and Conjecture 3 in Sec. IV E). The construction of the past attractor also highlights and clarifies the important role of SH dynamics in the G_0 context. Indeed, there is now considerable evidence, both numerical and analytical, that SH dynamics influences the asymptotic dynamics of spatially inhomogeneous cosmologies near the initial singularity in a significant way. Our formulation places this relationship on a sound footing: *the local past attractor for G_0 cosmologies with a silent initial singularity is the past attractor for SH cosmologies*. We are now in a position to restate the BKL conjecture in a precise form:

> For almost all cosmological solutions of Einstein's field equations, a spacelike initial singularity is *silent, vacuum-dominated* and *oscillatory*.

Proving this conjecture entails establishing all the limits associated with Conjectures 1, 2, and 3 in Sec. IV. As a first step, one would have to complete the proof for the SH models, begun by Ringström [23]. A natural second step would be to consider the simplest class of spatially inhomogeneous

[21] In a recent paper [39], however, the Gowdy models are analyzed using scale-invariant variables introduced in Refs. [5] and [4].

models with an oscillatory initial singularity, namely the generic G_2 models (see Table III), restricting consideration to vacuum solutions for simplicity. Analyzing the role of the spatial derivatives in a neighborhood of the silent boundary will be a major step in this analysis, and will clearly present a formidable challenge.

This unifying statement incorporates certain fundamental physical ideas about singularities, partially supported by known examples and theorems. It is useful to revisit the conjectured physical behavior in a way that highlights various aspects of the situation:

(i) The generic cosmological initial singularity is a strong-gravity phenomenon, and so should be linked to trapped surfaces, which intuitively capture the notion of ultrastrong gravitational fields (and thus also to the standard singularity theorems).

(ii) The generic cosmological initial singularity is likely to be a spacelike curvature singularity because a null singularity will be very special and timelike singularities will by their nature intersect relatively few worldlines of matter (but confirming this will depend on implementing a good measure on the space of cosmological models, which is needed in any case in order to put on a firm footing all talk about probabilities).

(iii) The generic spacelike curvature singularity is a scalar curvature singularity, since nonscalar curvature singularities require fine tuning of initial data.

(iv) If the energy conditions are strictly obeyed, the curvature singularity is generically Weyl curvature dominated, at least when vorticity in the matter fluid is not significant (this is not the case if the energy conditions are just marginally satisfied, as exemplified by stiff perfect fluids, but these are not physically likely states).

(v) The strong-gravity regime associated with the initial state leads to particle horizons, and spatial inhomogeneities are constrained to have superhorizon scale as the initial singularity is approached.

(vi) Increased strength of the gravitational field and the collapse of particle horizons lead to asymptotic silence, and on the scale of the particle horizon solutions therefore are asymptotically SH.

(vii) The past attractor describing asymptotic spatially inhomogeneous dynamics is thus given by the generalized Mixmaster attractor.

(viii) Final singularities are in essence the time reverse of initial singularities, and so we expect the above ideas to apply there too, and conversely that ideas from gravitational collapse can throw some light on cosmological initial singularities. In particular, in generic gravitational collapse, angular momentum plays a fundamental physical role, so this should also be true in many time-reversed cases, i.e., at the initial singularity; but when this is true, matter is dynamically important, in contrast to the cases considered above.

Each of these issues needs to be investigated and given a precise mathematical statement; e.g., the last may relate to the hypothesis that the tilt transition sets can be interpreted as physical or dynamical effects of vorticity in the matter fluid and associated transverse peculiar velocity components. In each case we wish to link our results and conjectures to physical ideas. The challenge is to explain the difference between the past attractor for vacuum G_0 models and the past attractor for perfect fluid G_0 models [compare Eqs. (4.51) and (5.1)] using physical principles. It is often stated that ''matter (energy) does not matter'' in the approach to the initial singularity—this view is reflected in our past asymptotic limits for the matter variables Ω and Ω_Λ. But, perhaps—and this is rather heuristic and speculative at this stage—''matter linear momentum does matter'' and/or ''matter angular momentum does matter.''

In the end the major physical statements are:

(a) Ultrastrong gravitational fields will occur in the early Universe, associated with local restrictions on causality.

(b) Propagating gravitational waves are not important in the cosmological context, but tidal forces are, and indeed are often more important than the gravitational fields caused directly by the matter.

(c) The relation between tidal forces and vorticity in the matter fluid is unclear and may contain some of the most interesting physics.

The relation between them is that—if our conjectures are correct—in the early Universe, energy and information mainly propagate along timelike world lines rather than along null rays. When matter moves relative to the irrotational timelike reference congruence, as must be the case when vorticity in the matter fluid is dynamically important, then the energy and information will flow with the matter. The primary effect of the gravitational field is in determining the motion of the matter through Coulomb-like effects; on the other hand, the effect of the matter on the gravitational field is primarily through concentrating that field into small regions, while conserving the constraints which embody the Gauß law underlying the Coulomb-like behavior. The effect of spatial curvature is to generate oscillatory behavior in tidal forces as this concentration takes place, as seems to be characteristic of generic cosmological initial singularities; but this is not wavelike in the sense of conveying information to different regions, it is just a localized oscillation.

It is issues such as these that need to be investigated when further developing the themes studied here.

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APPENDIX

1. Propagation of constraints

Propagation of dimensional constraints:

$$\mathbf{e}_0[(C_{\rm com})_{\alpha\beta}(f)]=(C_{\rm com})_{\alpha\beta}[\mathbf{e}_0(f)]-2H(C_{\rm com})_{\alpha\beta}(f)+2\sigma_{[\alpha}{}^{\gamma}(C_{\rm com})_{\beta]\gamma}(f)-2\epsilon_{\gamma\delta[\alpha}\Omega^{\gamma}(C_{\rm com})_{\beta]}{}^{\delta}(f), \tag{A1}$$

$$\mathbf{e}_0(C_{\rm G})=-2H(C_{\rm G})+2(\mathbf{e}_\alpha+2\dot{u}_\alpha-2a_\alpha)(C_{\rm C})^{\alpha}+\epsilon^{\alpha\beta\gamma}(C_{\rm com})_{\alpha\beta}(\Omega_\gamma), \tag{A2}$$

$$\begin{aligned}\mathbf{e}_0(C_{\rm C})^{\alpha}=&-[4H\delta^{\alpha}{}_{\beta}+\sigma^{\alpha}{}_{\beta}-\epsilon^{\alpha}{}_{\gamma\beta}\Omega^{\gamma}](C_{\rm C})^{\beta}-\frac{1}{6}(\delta^{\alpha\beta}\mathbf{e}_\beta-2\dot{u}^{\alpha})(C_{\rm G})+\frac{1}{2}\epsilon^{\alpha\beta\gamma}(\mathbf{e}_\beta+\dot{u}_\beta-3a_\beta)(C_{\rm J})_{\gamma}\\&-\frac{3}{2}n^{\alpha}{}_{\beta}(C_{\rm J})^{\beta}+\frac{1}{2}n_{\beta}{}^{\beta}(C_{\rm J})^{\alpha}-(C_{\rm com})^{\alpha}{}_{\beta}(\dot{u}^{\beta}-a^{\beta})-\frac{1}{2}\epsilon^{\alpha\beta\gamma}(C_{\rm com})_{\beta\delta}(n_{\gamma}{}^{\delta})\\&+\frac{1}{4}\epsilon^{\beta\gamma\delta}(C_{\rm com})_{\beta\gamma}(n^{\alpha}{}_{\delta}),\end{aligned} \tag{A3}$$

$$\begin{aligned}\mathbf{e}_0(C_{\rm J})^{\alpha}=&-[2H\delta^{\alpha}{}_{\beta}-\sigma^{\alpha}{}_{\beta}-\epsilon^{\alpha}{}_{\gamma\beta}\Omega^{\gamma}](C_{\rm J})^{\beta}-\frac{1}{2}\epsilon^{\alpha\beta\gamma}(C_{\rm com})_{\beta\gamma}(H)+\frac{1}{2}\epsilon^{\alpha\beta\gamma}(C_{\rm com})_{\beta\delta}(\sigma_{\gamma}{}^{\delta})\\&-\frac{1}{4}\epsilon^{\beta\gamma\delta}(C_{\rm com})_{\beta\gamma}(\sigma^{\alpha}{}_{\delta})-(C_{\rm com})^{\alpha}{}_{\beta}(\Omega^{\beta}).\end{aligned} \tag{A4}$$

Propagation of dimensionless gauge fixing condition:

$$\partial_t(\mathcal{C}_{\dot{U}})^{\rm sv}_{\alpha}=-(\delta_{\alpha}{}^{\beta}+\Sigma_{\alpha}{}^{\beta}-\epsilon_{\alpha\gamma}{}^{\beta}R_{\gamma})(\mathcal{C}_{\dot{U}})^{\rm sv}_{\beta}. \tag{A5}$$

2. Hubble-normalized relativistic Euler equations

Upon substitution of the matter variables (2.25), Eqs. (2.36) and (2.37) assume the explicit form

$$\boldsymbol{\partial}_0\Omega=-\frac{\gamma}{G_+}v^{\alpha}\boldsymbol{\partial}_\alpha\Omega+G_+^{-1}[2G_+q-(3\gamma-2)-(2-\gamma)v^2-\gamma(\Sigma_{\alpha\beta}v^{\alpha}v^{\beta})-\gamma(\boldsymbol{\partial}_\alpha-2r_\alpha+2\dot{U}_\alpha-2A_\alpha)v^{\alpha}+\gamma v^{\alpha}\boldsymbol{\partial}_\alpha\ln G_+]\Omega, \tag{A6}$$

$$\begin{aligned}\boldsymbol{\partial}_0v^{\alpha}=&-v^{\beta}\boldsymbol{\partial}_\beta v^{\alpha}+\delta^{\alpha\beta}\boldsymbol{\partial}_\beta\ln G_+-\frac{(\gamma-1)}{\gamma}(1-v^2)\delta^{\alpha\beta}(\boldsymbol{\partial}_\beta\ln\Omega-2r_\beta)+G_-^{-1}\Big[(\gamma-1)(1-v^2)(\boldsymbol{\partial}_\beta v^{\beta})-(2-\gamma)v^{\beta}\boldsymbol{\partial}_\beta\ln G_+\\&+\frac{(\gamma-1)}{\gamma}(2-\gamma)(1-v^2)v^{\beta}(\boldsymbol{\partial}_\beta\ln\Omega-2r_\beta)+(3\gamma-4)(1-v^2)+(2-\gamma)(\Sigma_{\beta\gamma}v^{\beta}v^{\gamma})+G_-(\dot{U}_\beta v^{\beta})\\&+[G_+-2(\gamma-1)](A_\beta v^{\beta})\Big]v^{\alpha}-\Sigma^{\alpha}{}_{\beta}v^{\beta}+\epsilon^{\alpha}{}_{\beta\gamma}R^{\beta}v^{\gamma}-\dot{U}^{\alpha}-v^2A^{\alpha}+\epsilon^{\alpha\beta\gamma}N_{\beta\delta}v_{\gamma}v^{\delta}.\end{aligned} \tag{A7}$$

Using $v^{\alpha}=ve^{\alpha}$, $e_\alpha e^{\alpha}=1$, we easily obtain from Eq. (A7)

$$\begin{aligned}\boldsymbol{\partial}_0v^2=&-v^{\alpha}\boldsymbol{\partial}_\alpha v^2+\frac{2}{G_-}(1-v^2)\Big[v^{\alpha}\boldsymbol{\partial}_\alpha\ln G_++(\gamma-1)v^2(\boldsymbol{\partial}_\alpha v^{\alpha})-\frac{(\gamma-1)}{\gamma}(1-v^2)v^{\alpha}(\boldsymbol{\partial}_\alpha\ln\Omega-2r_\alpha)+(3\gamma-4)v^2\\&-(\Sigma_{\alpha\beta}v^{\alpha}v^{\beta})-G_-(\dot{U}_\alpha v^{\alpha})-2(\gamma-1)v^2(A_\alpha v^{\alpha})\Big],\end{aligned} \tag{A8}$$

$$\begin{aligned}\boldsymbol{\partial}_0e^{\alpha}=&-ve^{\beta}\boldsymbol{\partial}_\beta e^{\alpha}+\frac{1}{v}p^{\alpha\beta}\boldsymbol{\partial}_\beta\ln G_+-\frac{1}{v}\frac{(\gamma-1)}{\gamma}(1-v^2)p^{\alpha\beta}(\boldsymbol{\partial}_\beta\ln\Omega-2r_\beta)-p^{\alpha}{}_{\beta}\Sigma^{\beta}{}_{\gamma}e^{\gamma}+s^{\alpha}{}_{\beta}R^{\beta}-\frac{1}{v}p^{\alpha}{}_{\beta}\dot{U}^{\beta}-vp^{\alpha}{}_{\beta}A^{\beta}\\&+vs^{\alpha}{}_{\beta}N^{\beta}{}_{\gamma}e^{\gamma}.\end{aligned} \tag{A9}$$

Here $p^{\alpha}{}_{\beta}:=\delta^{\alpha}{}_{\beta}-e^{\alpha}e_{\beta}$ and $s^{\alpha}{}_{\beta}:=\epsilon^{\alpha}{}_{\beta\gamma}e^{\gamma}$.

3. Hubble-normalized curvature variables

The Hubble-normalized 3-Ricci curvature of a spacelike 3-surface $\mathcal{S}:\{t=\text{const}\}$ is defined through the symmetric-tracefree and trace parts. The trace part Ω_k was given in Eq. (2.44), while the tracefree part is given by

$$\mathcal{S}_{\alpha\beta}:=-\frac{1}{3}\epsilon^{\gamma\delta}{}_{\langle\alpha}(\boldsymbol{\partial}_{|\gamma|}-r_{|\gamma|}-2A_{|\gamma|})N_{\beta\rangle\delta}+\frac{1}{3}(\boldsymbol{\partial}_{\langle\alpha}-r_{\langle\alpha})A_{\beta\rangle}+\frac{2}{3}N_{\langle\alpha}{}^{\gamma}N_{\beta\rangle\gamma}-\frac{1}{3}N_{\gamma}{}^{\gamma}N_{\langle\alpha\beta\rangle}. \quad \text{(A10)}$$

The quantities $\mathcal{S}_{\alpha\beta}$ and Ω_k satisfy the Hubble-normalized twice-contracted 3-Bianchi identity given by

$$0\equiv(\boldsymbol{\partial}_{\beta}-2r_{\beta}-3A_{\beta})\mathcal{S}^{\alpha\beta}-\epsilon^{\alpha\beta\gamma}N_{\beta\delta}\mathcal{S}_{\gamma}{}^{\delta}+\frac{1}{3}\delta^{\alpha\beta}(\boldsymbol{\partial}_{\beta}-2r_{\beta})\Omega_k. \quad \text{(A11)}$$

Employing Eq. (A10), we can write the evolution equation for $\Sigma_{\alpha\beta}$ in the alternative form

$$\boldsymbol{\partial}_0\Sigma^{\alpha\beta}=(q-2)\Sigma^{\alpha\beta}-3(\mathcal{S}^{\alpha\beta}-\Pi^{\alpha\beta})+\epsilon^{\gamma\delta\langle\alpha}[2R_{\gamma}\Sigma^{\beta\rangle}{}_{\delta}-N^{\beta\rangle}{}_{\gamma}\dot{U}_{\delta}]+(\delta^{\gamma\rangle\alpha}\boldsymbol{\partial}_{\gamma}-r^{\langle\alpha}+\dot{U}^{\langle\alpha}+A^{\langle\alpha\rangle})\dot{U}^{\beta\rangle}. \quad \text{(A12)}$$

The conformal curvature properties of a spacelike 3-surface $\mathcal{S}:\{t=\text{const}\}$ are encoded in the Hubble-normalized 3-Cotton-York tensor

$$\mathcal{C}_{\alpha\beta}:=\epsilon^{\gamma\delta}{}_{\langle\alpha}(\boldsymbol{\partial}_{|\gamma|}-2r_{|\gamma|}-A_{|\gamma|})\mathcal{S}_{\beta\rangle\delta}-3N_{\langle\alpha}{}^{\gamma}\mathcal{S}_{\beta\rangle\gamma}+\frac{1}{2}N_{\gamma}{}^{\gamma}\mathcal{S}_{\alpha\beta}. \quad \text{(A13)}$$

The Hubble-normalized Weyl curvature variables take the explicit form

$$\mathcal{E}_{\alpha\beta}=\mathcal{S}_{\alpha\beta}+\frac{1}{3}\Sigma_{\alpha\beta}-\frac{1}{3}\Sigma_{\langle\alpha}{}^{\gamma}\Sigma_{\beta\rangle\gamma}-\frac{1}{2}\Pi_{\alpha\beta}, \quad \text{(A14)}$$

$$\mathcal{H}_{\alpha\beta}=\frac{1}{3}\epsilon^{\gamma\delta}{}_{\langle\alpha}(\boldsymbol{\partial}_{|\gamma|}-r_{|\gamma|}-A_{|\gamma|})\Sigma_{\beta\rangle\delta}-N_{\langle\alpha}{}^{\gamma}\Sigma_{\beta\rangle\gamma}+\frac{1}{6}N_{\gamma}{}^{\gamma}\Sigma_{\alpha\beta}, \quad \text{(A15)}$$

with $\mathcal{S}_{\alpha\beta}$ defined in Eq. (A10).

4. Curvature transitions

Although the relation (4.46) implicitly gives the rule for the relationship between two Kasner epochs, k is not particularly suitable for explicitly describing the ‘‘Kasner transformation law’’ for curvature transitions. However, that law can be elegantly obtained in the present dynamical systems framework as follows. The solutions on the $\mathcal{T}_{N_1}$ subset are determined by

$$\frac{1}{12}N_1^2=1-\Sigma^2,$$

and

$$\Sigma_1=-4+3Z,\quad \Sigma_2=2-\frac{3}{2}r_+Z,\quad \Sigma_3=2-\frac{3}{2}r_-Z,$$

$$\partial_t Z=-2(1-\Sigma^2)Z,$$

where $r_\pm:=1\pm\sqrt{1-\alpha^2}$, $\alpha\in[0,1]$, is a constant, and where $1-\Sigma^2=\frac{3}{4}[(\alpha-2)Z+2][(\alpha+2)Z-2]$. An orbit starts (with time direction reversed toward the past) from a Kasner point where $Z=Z_-=2/(2+\alpha)$, and ends at a Kasner point where $Z=Z_+=2/(2-\alpha)$. That is, Z is a parameter on the individual orbits that increases monotonically from Z_- to Z_+ toward the past, while α labels the different orbits. The value $Z=0$ determines the point P_1 outside the Kasner circle in Fig. 1. It is possible to express the constant α, and, hence, $r_\pm$, in terms of the standard Kasner parameter $u\geqslant 1$ (see, e.g., BKL [8], p. 528), where we assume that we are considering orbits that originate from sector (123) and thus that $p_1<p_2<p_3$. Then $(r_+,\alpha,r_-)=2(u^2,u,1)/(1+u^2)$, and thus α might be viewed as a compactified Kasner parameter. The above formulae directly yield the standard transformation laws from the initial Kasner exponents $\{p_\alpha\}$ to the final Kasner exponents $\{p'_\alpha\}$ for orbits originating from sector (123):

$$p'_1=\frac{-p_1}{1+2p_1},\quad p'_2=\frac{p_2+2p_1}{1+2p_1},\quad p'_3=\frac{p_3+2p_1}{1+2p_1}. \quad \text{(A16)}$$

Curvature transitions from other sectors than (123) are easily obtained through permutations.

5. Inverting the commutator equations

The commutator equations can be used to solve for the connection variables in terms of the frame variables. Let us introduce the *dual frame variables* $e^{\alpha}{}_i$ of $e_{\alpha}{}^i$, and their Hubble-normalized counterparts $E^{\alpha}{}_i:=H\,e^{\alpha}{}_i$. The dual frame variables can be used to conveniently describe the line element, giving

$$ds^2=H^{-2}[-\mathcal{N}^2dt^2+\delta_{\alpha\beta}E^{\alpha}{}_iE^{\beta}{}_j(N^idt+dx^i)(N^jdt+dx^j)]. \quad \text{(A17)}$$

As $E_{\alpha}{}^i$ and $E^{\alpha}{}_i$ satisfy the relations $E^{\alpha}{}_iE_{\beta}{}^i=\delta^{\alpha}{}_{\beta}$ and $E^{\alpha}{}_iE_{\alpha}{}^j=\delta_i{}^j$, one finds with Eq. (2.47) that the $E^{\alpha}{}_i$ evolve according to

$$\boldsymbol{\partial}_0E^{\alpha}{}_i=-(q\,\delta^{\alpha}{}_{\beta}-\Sigma^{\alpha}{}_{\beta}-\epsilon^{\alpha}{}_{\gamma\beta}R^{\gamma})E^{\beta}{}_i+\mathcal{N}^{-1}E^{\alpha}{}_j\partial_iN^j. \quad \text{(A18)}$$

On the other hand, from appropriately inverting Eqs. (2.47), (2.48) and (2.50) we obtain the explicit expressions

$$q = \frac{1}{3} E^{\alpha}{}_i \boldsymbol{\partial}_0 E_{\alpha}{}^i + \frac{1}{3} \mathcal{N}^{-1} \partial_i N^i, \tag{A19}$$

$$\Sigma_{\alpha\beta} = -\delta_{\gamma\langle\alpha} E^{\gamma}{}_i \boldsymbol{\partial}_0 E_{\beta\rangle}{}^i - \mathcal{N}^{-1} \delta_{\gamma\langle\alpha} E^{\gamma}{}_i \boldsymbol{\partial}_{\beta\rangle} N^i, \tag{A20}$$

$$R^{\alpha} = \frac{1}{2} \epsilon^{\alpha}{}_{\beta}{}^{\gamma} E^{\beta}{}_i \boldsymbol{\partial}_0{}^i E_{\gamma}{}^i + \frac{1}{2} \mathcal{N}^{-1} \epsilon^{\alpha}{}_{\beta}{}^{\gamma} E^{\beta}{}_i \boldsymbol{\partial}_{\gamma} N^i, \tag{A21}$$

$$A_{\alpha} + r_{\alpha} = \frac{1}{2} E^{\beta}{}_i \boldsymbol{\partial}_{\alpha} E_{\beta}{}^i - \frac{1}{2} \partial_i E_{\alpha}{}^i, \tag{A22}$$

$$N^{\alpha\beta} = E^{(\alpha}{}_i \epsilon^{\beta)\gamma\delta} \boldsymbol{\partial}_{\gamma} E_{\delta}{}^i, \tag{A23}$$

$$\dot{U}_{\alpha} - r_{\alpha} = \boldsymbol{\partial}_{\alpha} \ln \mathcal{N}. \tag{A24}$$

When working in the *separable volume gauge*, determined by conditions (3.3) and (3.1), the line element takes the form

$$ds^2 = H^{-2}[-dt^2 + \delta_{\alpha\beta} E^{\alpha}{}_i E^{\beta}{}_j dx^i dx^j]. \tag{A25}$$

Combining the above result for $A_{\alpha} + r_{\alpha}$ with the constraint (3.9) makes it then possible to solve for r_{α} and A_{α} separately:

$$r_{\alpha} = \frac{1}{3} E^{\beta}{}_i E_{\alpha}{}^j \partial_j E_{\beta}{}^i + \frac{1}{3} E_{\alpha}{}^i \partial_i \ln \hat{m}, \tag{A26}$$

$$A_{\alpha} = \frac{1}{6} E^{\beta}{}_i E_{\alpha}{}^j \partial_j E_{\beta}{}^i - \frac{1}{2} \partial_i E_{\alpha}{}^i - \frac{1}{3} E_{\alpha}{}^i \partial_i \ln \hat{m}. \tag{A27}$$

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