Matching functions for heavy particles

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We introduce *matching functions* as a means of summing heavy-quark logarithms to any order. Our analysis is based on Witten's approach, where heavy quarks are decoupled one at a time in a mass-independent renormalization scheme. The outcome is a generalization of the matching conditions of Bernreuther and Wetzel: we show how to derive closed formulas for summed logarithms to any order, and present explicit expressions for leading order and next-to-leading order contributions. The decoupling of heavy particles in theories lacking asymptotic freedom is also considered.

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I. INTRODUCTION

Decoupling a heavy quark when the renormalization scheme is mass independent was originally discussed by Witten [1]. He showed that the results can be elegantly expressed in terms of a renormalization group (RG) invariant running coupling $\tilde{\alpha}_h$ associated with the mass m_h of the heavy quark h. Subsequently, Bernreuther and Wetzel [2–5] proposed a systematic method for dealing with the matching problem, i.e. the lack of explicit decoupling in mass independent schemes. They applied the Appelquist-Carrazone decoupling theorem [6] to the gluon coupling α_Q^{MO} in the momentum subtraction (MO) scheme, i.e. renormalized at spacelike momentum Q,

$$\alpha_Q^{\mathrm{MO}}|_{\text{with } h} = \alpha_Q^{\mathrm{MO}}|_{\text{no } h} + O(m_h^{-1}) \tag{1}$$

and compared calculations of α_Q^{MO} in the full $F = f + 1$ and effective f flavor $\overline{\mathrm{MS}}$ (modified minimal subtraction) theories. When $O(m_h^{-1})$ terms are neglected, the strong coupling $\alpha_F = g_F^2/(4\pi)$ for the F-flavor $\overline{\mathrm{MS}}$ theory is calculable as a power series in its f-flavor counterpart α_f and logarithms of m_h. Results for the first few loops of perturbation theory appear in the literature [2–5,7–9]. Bernreuther has constructed a similar matching procedure to deal with the effects of mass renormalization [4].

This paper arises from the observation [2,4] that the RG relates coefficients of perturbative mass logarithms $\sim \alpha_F^r \ln^s m_h$ in matching relations. This suggests that we seek an analogy with the behavior of Green's functions at large momenta q, where in general [10,11], each perturbative order in the Gell-Mann–Low function $\Psi(x)$ or the Callan-Symanzik β, γ, δ functions determines all coefficients to logarithmic order κ, i.e. all coefficients of momentum logarithms $\{\alpha_F^{r+\kappa} \ln^r q, r = 0, 1, 2, \ldots\}$.

We set up the formalism for mass logarithms by introducing *matching functions* $\mathcal{F}(\tilde{\alpha}_h)$ and $\mathcal{G}(\tilde{\alpha}_h)$ associated with coupling constant and mass renormalization. The coupling constant and masses are matched to all orders, with heavy quarks (in this paper) decoupled one by one ($F = f + 1$). For coupling constant renormalization, we derive in Sec. IV the following key equations,

$$\ln\frac{\bar{m}_h}{\bar{\mu}} = \int_{\alpha_F}^{\tilde{\alpha}_h} dx \, \frac{1}{\beta_F(x)} \tag{2}$$

$$\ln\frac{\bar{m}_h}{\bar{\mu}} = \int_{\alpha_f}^{\tilde{\alpha}_h} dx \, \frac{1}{\beta_f(x)} + \mathcal{F}_{F \to f}(\tilde{\alpha}_h) \tag{3}$$

where $\bar{m}_h$ is Witten's renormalization group invariant heavy-quark mass [1], $\beta_f(x)$ is the $\overline{\mathrm{MS}}$ β-function for the f-flavor theory, and

$$\bar{\mu} = \mu_{\mathrm{dim}}\sqrt{4\pi} e^{-\gamma/2}, \quad \gamma = 0.5772\ldots \tag{4}$$

is the $\overline{\mathrm{MS}}$ scale derived from the scale μ_{dim} used to define dimensional regularization and renormalization. The matching function $\mathcal{F}_{F \to f}$ is a series in $\tilde{\alpha}_h$ whose coefficients can be determined perturbatively by comparison with Eq. (1). The desired matching relation between α_F and α_f is the result of eliminating $\tilde{\alpha}_h$ from Eqs. (2) and (3):

$$\alpha_F = \alpha_F(\alpha_f, \ln(\bar{m}_h/\bar{\mu})). \tag{5}$$

Similar conditions for mass matching are presented in Sec. VII.

The role played by $\mathcal{F}$, $\mathcal{G}$ and the β, γ, δ functions in matching conditions is just like that of the β, γ, δ functions for large-momentum logarithms. Each order of perturbation theory for these functions determines the coefficients of a

68 096005-1

new tower of mass logarithms: leading order (LO), next-to-leading order (NLO), next-to-next-to-leading order (NNLO), and so on. Both $\mathcal{F}(\tilde{\alpha}_h)$ and $\mathcal{G}(\tilde{\alpha}_h)$ vanish for LO and NLO, but then there are contributions from successive terms in their power series in $\tilde{\alpha}_h$, starting with NNLO for $\mathcal{F}(\tilde{\alpha}_h)$ and NNNLO for $\mathcal{G}(\tilde{\alpha}_h)$.

We find, as for large-q logarithms, that results for coupling constants, running couplings and light masses can be most elegantly expressed as closed expressions or generating functions for summed towers of mass logarithms. Examples are the decoupling formulas for α_{f+1} and $\tilde{\alpha}_h$ quoted previously by us [12], which we derive in Sec. VI. Almost all of our results are for quantum chromodynamics (QCD) with three colors, but the technique can be applied to any renormalizable theory.

Sections II and III are brief summaries of Witten's treatment of heavy-quark decoupling in QCD and the matching procedure of Bernreuther and Wetzel for coupling constants. This lays the foundation for the RG analysis in Sec. IV, from which we are led to construct the matching function $\mathcal{F}$ for coupling constant renormalization. Perturbation theory for $\mathcal{F}$ is considered in Sec. V, with the result that the first nonzero term (NNLO) in $\mathcal{F}$ is obtained. In Sec. VI, we show that Eqs. (2) and (3) lead directly to closed expressions for heavy-quark logarithms to a given logarithmic order, and present explicit NLO expressions. Section VII is an extension of our RG analysis to deal with the matching problem for mass renormalization. It is here that we introduce the mass-matching function $\mathcal{G}$. In Sec. VIII, we decouple more than one heavy quark sequentially, for example $\ln(m_t/\bar{\mu}) \gg \ln(m_b/\bar{\mu}) \to \infty$, and derive the NLO closed formula for coupling constant renormalization in this limit.

Section IX suggests that the consistency of theories lacking asymptotic freedom, such as quantum electrodynamics (QED) with heavy leptons, be tested by imposing the physical requirement that all heavy particles decouple in the infinite-mass limit. Both ultraviolet and infrared stable fixed points enter the analysis.

Other applications of our technique are discussed in Sec. X.

II. WITTEN'S METHOD

This section summarizes some key points of Witten's procedure [1,12].

By convention, the same $\overline{\text{MS}}$ scale $\bar{\mu}$ is used for the initial F-flavor and all residual f-flavor theories. Whenever heavy quarks (masses m_h) are decoupled,

$$F \to f \quad \text{flavors}, \quad m_h \to \infty$$

all parameters of the *residual f*-flavor theory are held fixed: the scale $\bar{\mu}$, all momenta $\mathbf{p}$, the coupling α_f, and all light-quark masses m_{lf}. In any order of perturbation theory, amplitudes

$$\mathcal{A}_F = \mathcal{A}_F(\mathbf{p}, \bar{\mu}, \alpha_F, m_{lF}, m_h)$$

are power series in m_h^{-1} with each power modified by a polynomial in $\ln(m_h/\bar{\mu})$. We will consider the leading power $\tilde{\mathcal{A}}_F$:

$$\mathcal{A}_F = \tilde{\mathcal{A}}_F\{1 + O(m_h^{-1})\}. \tag{6}$$

The notation $O(m_h^{-1})$ refers to any subleading power, including its logarithmic modifications.

Logarithms in $\tilde{\mathcal{A}}_F$ for $m_h \sim \infty$ are generated by 1PI (one-particle irreducible) subgraphs with at least one heavy-quark propagator and with degree of divergence at least logarithmic. It is as if all contributing 1PI parts were shrunk to a point. All F-flavor amplitudes $\tilde{\mathcal{A}}_F$ tend to amplitudes $\mathcal{A}_f$ of the residual f-flavor theory, apart from m_h-dependent renormalizations of the coupling constant, light masses, and amplitudes [6]:

$$\tilde{\mathcal{A}}_F(\mathbf{p}, \bar{\mu}, \alpha_F, m_{lF}, m_h)$$
$$= \sum_{\mathcal{A}'} \mathcal{Z}_{\mathcal{A}\mathcal{A}'}(\alpha_F, m_h/\bar{\mu}) \mathcal{A}'_f(\mathbf{p}, \bar{\mu}, \alpha_f, m_{lf}) \tag{7}$$

$$\alpha_f = \alpha_f(\alpha_F, m_h/\bar{\mu}), \quad m_{lf} = m_{lF} D(\alpha_F, m_h/\bar{\mu}). \tag{8}$$

For practical applications, Eq. (8) has to be inverted, so that α_f and m_{lf} become the dependent variables instead of α_F and m_{lF}. That is because we hold α_f and m_{lf} fixed as $m_h \to \infty$.

For any number of flavors f (including F), let

$$\mathcal{D}_f = \bar{\mu} \frac{\partial}{\partial \bar{\mu}} + \beta_f(\alpha_f) \frac{\partial}{\partial \alpha_f} + \delta_f(\alpha_f) \sum_{k=1}^{f} m_{kf} \frac{\partial}{\partial m_{kf}} \tag{9}$$

be the corresponding Callan-Symanzik operator. Since $\mathcal{A}_F$ satisfies an F-flavor improved Callan-Symanzik equation [13], so also does its leading power:

$$\{\mathcal{D}_F + \gamma_F(\alpha_F)\} \tilde{\mathcal{A}}_F = 0. \tag{10}$$

In general, both γ_F and $\mathcal{Z} = (\mathcal{Z}_{\mathcal{A}\mathcal{A}'})$ are matrices.

If we substitute Eq. (7) in Eq. (10) and change variables

$$\mathcal{D}_F = \bar{\mu} \frac{\partial}{\partial \bar{\mu}} + (\mathcal{D}_F \alpha_f) \frac{\partial}{\partial \alpha_f} + \sum_{k=1}^{f} (\mathcal{D}_F m_{kf}) \frac{\partial}{\partial m_{kf}} \tag{11}$$

the result is an improved Callan-Symanzik equation for each residual amplitude,

$$\{\mathcal{D}_f + \gamma_f(\alpha_f)\} \mathcal{A}_f = 0 \tag{12}$$

where the functions [1,2]

$$\beta_f(\alpha_f) = \mathcal{D}_F \alpha_f \tag{13}$$

$$\delta_f(\alpha_f) = \mathcal{D}_F \ln m_{lf} \tag{14}$$

$$\gamma_f(\alpha_f) = \mathcal{Z}^{-1}[\gamma_F(\alpha_F) + \mathcal{D}_F] \mathcal{Z} \tag{15}$$

depend *solely* on α_f. The absence of m_l dependence in the renormalization factors in Eqs. (7) and (8) ensures mass-independent renormalization for the residual theory.

While these equations hold for any $f < F$, their solutions can be readily formulated in terms of running couplings only when the heavy quarks are decoupled one at a time. Indeed, Witten's running coupling

$$\tilde{\alpha}_h = \tilde{\alpha}_h(\alpha_F, \ln(m_h/\bar{\mu})) \qquad (16)$$

is defined for the case $F = f + 1$ where just one quark h is heavy, with $\overline{\mathrm{MS}}_F$ renormalized mass m_h. The definition of $\tilde{\alpha}_h$ is formulated implicitly [1]:

$$\ln(m_h/\bar{\mu}) = \int_{\alpha_F}^{\tilde{\alpha}_h} dx [1 - \delta_F(x)]/\beta_F(x). \qquad (17)$$

It satisfies the constraints

$$\tilde{\alpha}_h(\alpha_F, 0) = \alpha_F, \quad \tilde{\alpha}_h(\alpha_F, \infty) = 0 \qquad (18)$$

where the latter follows from the asymptotic freedom of the F-flavor theory ($F \leq 16$). Equations (9), (13), and (17) imply that $\tilde{\alpha}_h$ is RG invariant:

$$\mathcal{D}_F \tilde{\alpha}_h = 0. \qquad (19)$$

III. MATCHING COUPLING CONSTANTS

Generally, the solutions of Witten's equations depend on renormalized parameters α_F and m_{lF} of the original $F = f + 1$ flavor theory, whereas the limit $m_h \to \infty$ is to be taken with parameters α_f and m_{lf} of the *residual* theory held fixed. To complete the analysis, it is necessary to derive asymptotic series in $\ln(m_h/\bar{\mu})$ which relate the initial and residual parameters, i.e. to "match" α_F and m_{lF} with α_f and m_{lf}. As noted in Sec. I, Bernreuther and Wetzel [2–5] have set up a systematic procedure for this. This section is a brief account of their scheme for the case of coupling-constant matching.

The decoupling formula (1) works to any order of perturbation theory, so the task is to express the leading power of the RG-invariant gluon coupling α_Q^{MO} with and without the heavy-quark h as perturbative series in α_F and α_f respectively. Generally this involves gluon and other self-energy insertions and a vertex amplitude such as fermion-gluon [2,3] or ghost-gluon [4].

For one-loop contributions [14], vertex and propagator corrections cancel ($Z_1 = Z_2$), so only the gluon self-energy amplitude

$$\Pi_{f\mu\nu}^{ab} = i\delta^{ab}(g_{\mu\nu}q^2 - q_\mu q_\nu)\Pi_f(\sqrt{-q^2})$$

is needed. In that case, we can make the replacement

$$\alpha_Q^{\mathrm{MO}}|_{f \text{ flavors}} \to \alpha_f/[1 - \Pi_f(Q)]$$

in Eq. (1), with the result

$$\alpha_{f+1}^{-1} - \alpha_{f+1}^{-1}\Pi_{f+1}(Q) = \alpha_f^{-1} - \alpha_f^{-1}\Pi_f(Q) + O(m_h^{-1}, \alpha_f^2). \qquad (20)$$

Comparing the original and residual theories, we have

$$\Pi_{f+1}(Q) = \alpha_{f+1}\{\Gamma_{h\text{-loop}} + \Gamma_{\mathrm{other}}\} + O(\alpha_{f+1}^2)$$

$$\Pi_f(Q) = \alpha_f \Gamma_{\mathrm{other}} + O(\alpha_f^2) \qquad (21)$$

where

$$\Gamma_{h\text{-loop}} = \frac{1}{\pi}\int_0^1 ds\, s(1-s)\ln\left(\frac{m_h^2 + s(1-s)Q^2}{\bar{\mu}^2}\right) \qquad (22)$$

is the contribution of the heavy-quark loop, and Γ_{other} represents other one-loop terms.

The leading power contributed by the heavy-quark loop is

$$\Gamma_{h\text{-loop}} = C_{\mathrm{LO}}\ln\frac{m_h}{\mu} + C_{\mathrm{NLO}} + O(Q^2/m_h^2) \qquad (23)$$

with coefficients for leading and nonleading logarithmic orders given by

$$C_{\mathrm{LO}} = 1/(3\pi) \quad \text{and} \quad C_{\mathrm{NLO}} = 0. \qquad (24)$$

The vanishing of the NLO constant term is a well-known characteristic of the $\overline{\mathrm{MS}}$ gluon self-energy [14].

Eliminating Γ_{other} from Eq. (21) and combining the result with Eqs. (20) and (23), we recover the standard one-loop matching condition

$$\alpha_{f+1}^{-1} - \alpha_f^{-1} = \frac{1}{3\pi}\ln\frac{m_h}{\bar{\mu}} + O(\alpha_f, m_h^{-1}) \qquad (25)$$

or equivalently

$$\alpha_{f+1} = \alpha_f - \frac{\alpha_f^2}{3\pi}\ln\frac{m_h}{\bar{\mu}} + O(\alpha_f^3, m_h^{-1}). \qquad (26)$$

The two-loop analysis is much more complicated, so we simply quote the result [2–5], taking into account a subsequent correction [2,7,8]. We find it convenient to consider the inverse form where α_{f+1} is written as a series in α_f. For the special case of three colors, the result is

$$\alpha_{f+1} = \alpha_f - \frac{\alpha_f^2}{6\pi}\ln\frac{m_h^2}{\bar{\mu}^2} + \frac{\alpha_f^3}{36\pi^2}\ln^2\frac{m_h^2}{\bar{\mu}^2}$$

$$- \frac{11\alpha_f^3}{24\pi^2}\ln\frac{m_h^2}{\bar{\mu}^2} - \frac{11\alpha_f^3}{72\pi^2} + O(\alpha_f^4). \qquad (27)$$

The first three terms of the right-hand side belong to the leading order LO, i.e. they are proportional to α_f times a power of $\{\alpha_f \ln(m_h/\bar{\mu})\}$. Only the fourth term is NLO; there is no $O(\alpha_f^2)$ term independent of m_h because the NLO constant in Eq. (24) vanishes. The fifth term is $O(\alpha_f^3)$ and m_h-independent, so it is the first example of a NNLO term. The three-loop result, including the NNNLO constant term, is now known [8].

Now we would like to know what the renormalization group implies for matching relations of this type. Some results for coefficients to a given order of perturbation theory already appear in [2–4]. Consider Eq. (3) of Ref. [2],

$$\frac{\alpha_f}{\pi} = \frac{\alpha_{f+1}}{\pi} + \sum_{k=1}^{\infty} \left(\frac{\alpha_{f+1}}{\pi}\right)^{k+1} C_k\left(\ln\frac{m_h^2}{\mu^2}\right) + O(m_h^{-1}) \tag{28}$$

where C_k is a polynomial of degree k, as noted below Eq. (8) of Ref. [2],

$$C_k = c_{k,k}[\ln(m_h^2/\mu^2)]^k + \cdots + c_{k,0}. \tag{29}$$

The constants $c_{1,0}, c_{2,0}, c_{3,0}, \ldots$ are the remainders left when all terms depending on $\ln(m_h^2/\mu^2)$ are subtracted from the leading-power functions $C_1, C_2, C_3, \ldots$. Then, if all coefficients and RG functions are known to $k-1$ loops, the RG determines all k-loop coefficients $c_{k,j}$ in C_k *except* for $c_{k,0}$. The latter *cannot* be deduced from the RG, to any number of loops; rather, $c_{k,0}$ must be calculated explicitly via a separate k-loop matching calculation. For example, the NNLO coefficient $-11/(72\pi^2)$ in Eq. (27) is just $-c_{2,0}$.

Instead of Eq. (28), we prefer to consider the inverse relation

$$\alpha_{f+1} = \alpha_f + \sum_{k=1}^{\infty} \alpha_f^{k+1} P_k\left(\ln\frac{m_h}{\bar\mu}\right) + O(m_h^{-1}) \tag{30}$$

because that is what is required in order to take $m_h \to \infty$ with α_f held fixed. The analogue of Eq. (29) is

$$P_k = p_{k,k}[\ln(m_h/\bar\mu)]^k + p_{k,k-1}[\ln(m_h/\bar\mu)]^{k-1} + \cdots + p_{k,0}. \tag{31}$$

An analysis in the style of Bernreuther and Wetzel produces identical conclusions for the remainder constants $p_{k,0}$: given $p_{1,0}, p_{2,0}, \ldots, p_{k-1,0}$, one can use the RG to deduce $p_{k,k}, \ldots, p_{k,1}$ but not $p_{k,0}$.

Most practical applications require that terms of the same logarithmic order be summed. This is straightforward for LO logarithms, because the LO coefficients $c_{k,k}$ in Eq. (29) obey a simple relationship [2]

$$c_{k,k} = (c_{1,1})^k \tag{32}$$

which makes the series geometric,

$$\alpha_f \underset{\mathrm{LO}}{=} \alpha_{f+1} \left/ \left(1 - \frac{\alpha_{f+1}}{3\pi}\ln\frac{m_h}{\bar\mu}\right)\right. . \tag{33}$$

This expression is LO with respect to powers of $\alpha_{f+1} = \alpha_F$ and $\ln(m_h/\bar\mu)$. Equation (33) implies that the term $O(\alpha_f, m_h^{-1})$ in Eq. (25) is NLO or higher order:

$$\alpha_{f+1}^{-1} - \alpha_f^{-1} \underset{\mathrm{LO}}{=} \frac{1}{3\pi}\ln\frac{m_h}{\bar\mu}. \tag{34}$$

This leads directly to the inverse of Eq. (33),

$$\alpha_{f+1} \underset{\mathrm{LO}}{=} \alpha_f \left/ \left(1 + \frac{\alpha_f}{3\pi}\ln\frac{m_h}{\bar\mu}\right)\right. \tag{35}$$

where now LO refers to powers of α_f and $\ln(m_h/\bar\mu)$. Note that the LO coefficients $p_{k,k}$ in Eq. (31) are given by

$$p_{k,k} = (-1/3\pi)^k. \tag{36}$$

Beyond LO, formulas for all the relevant coefficients become complicated, making order-by-order summation too cumbersome to be practical. The rest of this paper is concerned with a RG analysis which allows us to consider matching relations to a given logarithmic order without having to expand in perturbative order.

IV. MATCHING FUNCTION

Any RG analysis of decoupling involves at least *two* renormalization groups: one for the initial F-flavor theory, and one for each f-flavor theory produced as a heavy particle decouples. We append a flavor subscript to make the distinction, viz. RG_F or RG_f.

A key observation is that any quantity which is RG_F invariant must also be RG_f invariant ($f < F$). For example, Witten's RG_F invariant running coupling $\tilde\alpha_h$ must satisfy the condition

$$\mathcal{D}_f \tilde\alpha_h = \left(\bar\mu\frac{\partial}{\partial\bar\mu} + \beta\frac{\partial}{\partial\alpha}\right)_f \tilde\alpha_h = 0. \tag{37}$$

Generally, the substitution

$$\mathcal{D}_F \to \mathcal{D}_f \tag{38}$$

works when applied to any quantity which survives the limit $m_h \to \infty$. An example is the formula

$$\mathcal{D}_6 \alpha_5 = \mathcal{D}_5 \alpha_5 = \beta_5(\alpha_5). \tag{39}$$

which agrees with the general result (13). However, *the converse is not generally true*. For example, the top-quark mass m_t is $\mathrm{RG}_{f=5}$ invariant, but it is certainly *not* $\mathrm{RG}_{f=6}$ invariant. Therefore a study of the RG for the original F-flavor theory is both necessary and sufficient for the full implications of the RG to be understood.

Our starting point is Witten's definition (17) of the invariant running coupling $\tilde\alpha_h$. Let us regard this as a formula for $\ln(m_h/\bar\mu)$ in terms of $\tilde\alpha_h$ and α_F. Specifically, the right-hand side is an integral from α_F to $\tilde\alpha_h$ involving RG_F functions β_F and δ_F. Can a similar formula be constructed from RG_f functions such that this mass logarithm becomes a function of $\tilde\alpha_h$ and α_f?

If such a formula exists, it must be consistent with the requirements of the RG_F group for the *original* theory. However, mass renormalization produces an unwelcome F dependence in equations such as

$$\mathcal{D}_F \ln(m_h/\bar\mu) = \delta_F(\alpha_F) - 1. \tag{40}$$

So let us amend the proposal: instead of the $\overline{\mathrm{MS}}_F$ mass m_h, consider Witten's invariant mass[1]

$$\bar{m}_h = m_h \exp \int_{\alpha_F}^{\tilde{\alpha}_h} dx \, \delta_F(x) / \beta_F(x). \qquad (41)$$

Since $\bar{m}_h$ is RG_F invariant,

$$\mathcal{D}_F \bar{m}_h = 0 \qquad (42)$$

replacing m_h by $\bar{m}_h$ in Eq. (40) eliminates the unwanted dependence on δ_F:

$$\mathcal{D}_F \ln(\bar{m}_h/\bar{\mu}) = -1. \qquad (43)$$

Notice that formula (2) for $\ln(\bar{m}_h/\bar{\mu})$ is an immediate consequence of definitions (17) and (41) of $\tilde{\alpha}_h$ and $\bar{m}_h$. As a check, $\mathcal{D}_F$ can be applied to the right-hand side of Eq. (2) to give the result -1, in agreement with Eq. (43).

Now observe that, because of Eq. (13), the replacement $F \to f$ everywhere on the right-hand side of Eq. (2) produces a quantity which transforms in the same way under the RG_F of the *original* theory:

$$\mathcal{D}_F \int_{\alpha_f}^{\tilde{\alpha}_h} dx \frac{1}{\beta_f(x)} = -1. \qquad (44)$$

Comparing Eqs. (43) and (44), we see that a RG_F invariant quantity $\mathcal{F}$ can be defined as follows:

$$\ln \frac{\bar{m}_h}{\bar{\mu}} = \int_{\alpha_f}^{\tilde{\alpha}_h} dx \frac{1}{\beta_f(x)} + \mathcal{F} \text{ implies } \mathcal{D}_F \mathcal{F} = 0. \qquad (45)$$

Since $\mathcal{F}$ is dimensionless, it can depend on $\tilde{\alpha}_h$, but RG_F invariance forbids dependence on other dimensionless variables, such as α_F, α_f, $\bar{m}_h/m_h$, or $m_h/\bar{\mu}$. We call it the *matching function*:

$$\mathcal{F} = \mathcal{F}_{F \to f}(\tilde{\alpha}_h). \qquad (46)$$

This completes the derivation of the key equations (2) and (3):

$$\ln \frac{\bar{m}_h}{\bar{\mu}} = \int_{\alpha_F}^{\tilde{\alpha}_h} dx \frac{1}{\beta_F(x)}$$

$$\ln \frac{\bar{m}_h}{\bar{\mu}} = \int_{\alpha_f}^{\tilde{\alpha}_h} dx \frac{1}{\beta_f(x)} + \mathcal{F}_{F \to f}(\tilde{\alpha}_h).$$

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[1]See Eq. (16) of [1]. Similar effective masses have been invented for the cases of large momenta [13] and light quarks [15]. Their RG invariance makes them useful in phenomenology [1,15] and lattice calculations [16].

When $\tilde{\alpha}_h$ is eliminated from these equations, the desired matching relation (5) is obtained:

$$\alpha_F = \alpha_F(\alpha_f, \ln(\bar{m}_h/\bar{\mu})), \quad F = f + 1.$$

V. PERTURBATION THEORY FOR THE MATCHING FUNCTION

Perturbative matching relations can be used to determine successive coefficients in the Taylor series of the matching function $\mathcal{F}_{F \to f}(\tilde{\alpha}_h)$. One needs to calculate the mass-independent terms $c_{k,0}$ in Eq. (29) or equivalently $p_{k,0}$ in Eq. (31). As noted in Sec. III, these constants cannot be deduced from the RG. This corresponds to the fact that $\mathcal{F}_{F \to f}$ cannot be deduced from the RG because it is RG_F invariant.

To illustrate the procedure, let us deduce the consequences for $\mathcal{F}_{f+1 \to f}$ of the perturbative matching relation (28). We need the two-loop β function for three colors:

$$\beta_f(x) = -\frac{x^2}{6\pi}(33 - 2f) - \frac{x^3}{12\pi^2}(153 - 19f) + O(x^4). \qquad (47)$$

Its reciprocal is

$$\{\beta_f(x)\}^{-1} = -\frac{6\pi}{33 - 2f} \left(\frac{1}{x^2} - \frac{b_f}{x} + b'_f \right) + O(x) \qquad (48)$$

where b_f stands for the constant

$$b_f = \frac{1}{2\pi} \frac{153 - 19f}{33 - 2f} \qquad (49)$$

and b'_f is another constant whose precise value is not of concern here.

Expansion (48) inserted into Eqs. (2) and (3) yields the following equations:

$$\frac{33 - 2(f+1)}{6\pi} \ln \frac{\bar{m}_h}{\bar{\mu}} = \tilde{\alpha}_h^{-1} - \alpha_{f+1}^{-1} + b_{f+1} \ln \frac{\tilde{\alpha}_h}{\alpha_{f+1}} + O(\alpha_f^2) \qquad (50)$$

$$\frac{33 - 2f}{6\pi} \ln \frac{\bar{m}_h}{\bar{\mu}} = \tilde{\alpha}_h^{-1} - \alpha_f^{-1} + b_f \ln \frac{\tilde{\alpha}_h}{\alpha_f} + O(\alpha_f^2)$$

$$+ \frac{33 - 2f}{6\pi} \mathcal{F}_{f+1 \to f}(\tilde{\alpha}_h). \qquad (51)$$

Note that the corrections are $O(\alpha_f^2)$: contributions to Eqs. (50) and (51) from the constant term in Eq. (48) are

$$b'_{f+1}(\tilde{\alpha}_h - \alpha_{f+1}) = O(\alpha_f^2) \text{ and } b'_f(\tilde{\alpha}_h - \alpha_f) = O(\alpha_f^2)$$

because $\tilde{\alpha}_h$, α_{f+1} and α_f differ by $O(\alpha_f^2)$.

The next step is to eliminate $\tilde{\alpha}_h$. This is partially achieved by subtracting Eq. (50) from Eq. (51):

$$\frac{1}{3\pi}\ln\frac{\bar{m}_h}{\bar{\mu}}=\alpha_{f+1}^{-1}-\alpha_f^{-1}+b_f\ln\frac{\tilde{\alpha}_h}{\alpha_f}-b_{f+1}\ln\frac{\tilde{\alpha}_h}{\alpha_{f+1}}$$

$$+\frac{33-2f}{6\pi}\mathcal{F}_{f+1\to f}(\tilde{\alpha}_h)+O(\alpha_f^2). \qquad (52)$$

Since $\ln(\tilde{\alpha}_h/\alpha_f)$ and $\ln(\tilde{\alpha}_h/\alpha_{f+1})$ are both $O(\alpha_f)$, Eqs. (25) and (52) imply

$$\mathcal{F}_{f+1\to f}(\tilde{\alpha}_h)=O(\alpha_f)$$

and so, from Eqs. (50) and (51), we conclude

$$\alpha_{f+1}/\tilde{\alpha}_h=1+\frac{\alpha_f}{6\pi}[33-2(f+1)]\ln\frac{m_h}{\bar{\mu}}+O(\alpha_f^2)$$

$$\alpha_f/\tilde{\alpha}_h=1+\frac{\alpha_f}{6\pi}(33-2f)\ln\frac{m_h}{\bar{\mu}}+O(\alpha_f^2). \qquad (53)$$

The logarithms of these expressions can then be substituted back into Eq. (52), with the result

$$\frac{1}{3\pi}\ln\frac{\bar{m}_h}{\bar{\mu}}=\alpha_{f+1}^{-1}-\alpha_f^{-1}-\frac{19\alpha_f}{12\pi^2}\ln\frac{m_h}{\bar{\mu}}$$

$$+\frac{33-2f}{6\pi}\mathcal{F}_{f+1\to f}(\tilde{\alpha}_h)+O(\alpha_f^2). \qquad (54)$$

The next step is to relate the logarithms of $\bar{m}_h$ and m_h. First substitute the three-color formula

$$\delta_f(x)=-\frac{2x}{\pi}+O(x^2) \qquad (55)$$

into definition (41) of $\bar{m}_h$,

$$\ln\frac{\bar{m}_h}{m_h}=\frac{12}{33-2(f+1)}\ln\frac{\tilde{\alpha}_h}{\alpha_{f+1}}+O(\alpha_f^2) \qquad (56)$$

and then substitute Eq. (53) for $\alpha_{f+1}/\tilde{\alpha}_h$. The result is

$$\ln\frac{\bar{m}_h}{\bar{\mu}}=\left(1-\frac{2\alpha_f}{\pi}\right)\ln\frac{m_h}{\bar{\mu}}+O(\alpha_f^2). \qquad (57)$$

Then the logarithm of $\bar{m}_h$ can be eliminated from Eqs. (54) and (57):

$$\alpha_{f+1}=\alpha_f-\frac{\alpha_f^2}{3\pi}\ln\frac{m_h}{\bar{\mu}}+\frac{\alpha_f^3}{9\pi^2}\ln^2\frac{m_h}{\bar{\mu}}-\frac{11\alpha_f^3}{12\pi^2}\ln\frac{m_h}{\bar{\mu}}$$

$$+\frac{\alpha_f^2}{6\pi}(33-2f)\mathcal{F}_{f+1\to f}(\tilde{\alpha}_h)+O(\alpha_f^4). \qquad (58)$$

Comparing this with the two-loop matching condition (27), we see that all mass logarithms are correctly reproduced, and

that the first nonzero term in the matching function can be deduced from the constant NNLO term in Eq. (27):

$$\mathcal{F}_{f+1\to f}(\tilde{\alpha}_h)=-\frac{11}{12\pi(33-2f)}\tilde{\alpha}_h+O(\tilde{\alpha}_h^2). \qquad (59)$$

The $O(\tilde{\alpha}_h^2)$ term in $\mathcal{F}$ can be found by substituting Eq. (59) back into Eq. (3) and repeating the above process using the three-loop β_f and two-loop δ_f functions. The answer follows by comparison with the known three-loop matching condition [8]. That is the limit of current calculations, but in principle, this strategy could be pursued to any order, with all mass logarithms correctly reproduced.

VI. CLOSED EXPRESSIONS FOR HEAVY-QUARK LOGARITHMS

The importance of the matching function $\mathcal{F}$ is that it allows us to work to a given logarithmic order without having to sum mass logarithms order by order in perturbation theory. Indeed, the role of $\mathcal{F}$ is essentially the same as that of the RG functions β, γ and δ: each term in the series for $\mathcal{F}$ corresponds to a particular logarithmic order. For LO and NLO, $\mathcal{F}$ does not contribute, but NNLO requires that the $O(\tilde{\alpha}_h)$ term in Eq. (59) be included, NNNLO requires the $O(\tilde{\alpha}_h^2)$ term, and so on.

To illustrate, let us derive the closed NLO formula for the matching relation between α_{f+1} and α_f which we announced in [12].

As in the previous section, we insert expansion (48) into Eqs. (2) and (3), but this time we omit the NNLO term $\mathcal{F}_{f+1\to f}$:

$$\frac{33-2(f+1)}{6\pi}\ln\frac{\bar{m}_h}{\bar{\mu}}\underset{\text{NLO}}{=}\tilde{\alpha}_h^{-1}-\alpha_{f+1}^{-1}+b_{f+1}\ln\frac{\tilde{\alpha}_h}{\alpha_{f+1}} \qquad (60)$$

$$\frac{33-2f}{6\pi}\ln\frac{\bar{m}_h}{\bar{\mu}}\underset{\text{NLO}}{=}\tilde{\alpha}_h^{-1}-\alpha_f^{-1}+b_f\ln\frac{\tilde{\alpha}_h}{\alpha_f}. \qquad (61)$$

The difference of these two equations is

$$\frac{1}{3\pi}\ln\frac{\bar{m}_h}{\bar{\mu}}\underset{\text{NLO}}{=}\alpha_{f+1}^{-1}-\alpha_f^{-1}-(b_f-b_{f+1})\ln\frac{\alpha_f}{\tilde{\alpha}_h}$$

$$-b_{f+1}\ln\frac{\alpha_f}{\alpha_{f+1}}. \qquad (62)$$

The logarithms on the right-hand side are already NLO, so we can use the LO parts of Eqs. (61) and (62) to approximate their arguments:

$$\alpha_f/\tilde{\alpha}_h\underset{\text{LO}}{=}1+\frac{\alpha_f}{6\pi}(33-2f)\ln\frac{m_h}{\bar{\mu}} \qquad (63)$$

$$\alpha_f/\alpha_{f+1} \underset{\text{LO}}{=} 1 + \frac{\alpha_f}{3\pi} \ln\frac{m_h}{\bar\mu}. \tag{64}$$

The result is a NLO generalization of Eq. (35):

$$\alpha_{f+1} \underset{\text{NLO}}{=} \alpha_f \Bigg/ \left\{ 1 + \frac{\alpha_f}{3\pi} \ln\frac{\bar m_h}{\bar\mu} + \alpha_f b_{f+1} \ln\left(1 + \frac{\alpha_f}{3\pi} \ln\frac{m_h}{\bar\mu}\right) \right.$$
$$\left. + \alpha_f(b_f - b_{f+1}) \ln\left[1 + \frac{\alpha_f}{6\pi}(33-2f)\ln\frac{m_h}{\bar\mu}\right] \right\}. \tag{65}$$

If desired, $\ln(\bar m_h/\bar\mu)$ can be eliminated in favor of $\ln(m_h/\bar\mu)$. The leading NLO effects of mass renormalization are due to the one-loop term of δ_f given by Eq. (55). When this term is substituted into the definition (41) of $\bar m_h$, keeping all logarithms to this order, we find an expression

$$\ln\frac{\bar m_h}{\bar\mu} \underset{\text{NLO}}{=} \ln\frac{m_h}{\bar\mu} + \frac{12}{31-2f}\left(\ln\frac{\alpha_f}{\alpha_{f+1}} - \ln\frac{\alpha_f}{\tilde\alpha_h}\right) \tag{66}$$

to which Eqs. (63) and (64) can be readily applied:

$$\ln\frac{\bar m_h}{\bar\mu} \underset{\text{NLO}}{=} \ln\frac{m_h}{\bar\mu} + \frac{12}{31-2f}\ln\left(1 + \frac{\alpha_f}{3\pi}\ln\frac{m_h}{\bar\mu}\right)$$
$$- \frac{12}{31-2f}\ln\left(1 + \frac{\alpha_f}{6\pi}(33-2f)\ln\frac{m_h}{\bar\mu}\right). \tag{67}$$

So, by combining Eqs. (65) and (67), we arrive at a complete NLO formula for the matching condition:

$$\alpha_{f+1}^{-1} \underset{\text{NLO}}{=} \alpha_f^{-1} + \frac{1}{3\pi}\ln\frac{m_h}{\bar\mu} + c_f \ln\left[1 + \frac{\alpha_f}{3\pi}\ln\frac{m_h}{\bar\mu}\right]$$
$$+ d_f \ln\left[1 + \frac{\alpha_f}{6\pi}(33-2f)\ln\frac{m_h}{\bar\mu}\right],$$
$$c_f = \frac{142-19f}{2\pi(31-2f)}, \quad d_f = \frac{57+16f}{2\pi(33-2f)(31-2f)}. \tag{68}$$

The same equations can also be used to obtain equations for the RG invariant $\tilde\alpha_h$ (also announced in [12]). Equations (49), (61), and (63) imply the NLO formula

$$\tilde\alpha_h^{-1} \underset{\text{NLO}}{=} \alpha_f^{-1} + \frac{1}{6\pi}(33-2f)\ln\frac{\bar m_h}{\bar\mu}$$
$$+ \frac{153-19f}{2\pi(33-2f)}\ln\left[1 + \frac{\alpha_f}{6\pi}(33-2f)\ln\frac{m_h}{\bar\mu}\right]. \tag{69}$$

Again, Eq. (67) can be used to write $\ln(\bar m_h/\bar\mu)$ in terms of $\ln(m_h/\bar\mu)$. This leads to the following asymptotic formula for $\tilde\alpha_h$ as $m_h\to\infty$:

$$\tilde\alpha_h \sim 6\pi \Bigg/ \left\{ (33-2f)\ln\frac{m_h}{\bar\mu} + K_f \ln\ln\frac{m_h}{\bar\mu} + O(1)\right\},$$

$$K_f = \frac{3(153-19f)}{33-2f} - \frac{12(33-2f)}{31-2f}. \tag{70}$$

These results show that we have complete control over the matching process. Once closed expressions such as Eqs. (68) and (69) have been obtained, RG invariance can be maintained for each logarithmic order, and so there is no need to truncate to a given order of perturbation theory.

VII. MASS-MATCHING FUNCTION

Most of the analysis above is restricted to the case of just one heavy quark h, but it can be readily extended to include *sequential* decoupling, where heavy quarks are decoupled *one at a time*. The new feature which arises is the need to match the mass of the second heavy quark. For example, suppose that, having decoupled the t quark in $F=6$ flavor QCD, we would like to decouple the b quark as well:

$$m_t\to\infty \quad \text{first, then} \quad m_b\to\infty. \tag{71}$$

Then it will be necessary to match the six-flavor definition $m_{b6}=m_b$ of the bottom quark mass to its five-flavor definition m_{b5}.

As for the matching of couplings, the key is to find a RG invariant definition of mass to which the Appelquist-Carrazone theorem [6] can be applied. This problem was solved by Bernreuther [4], again by recourse to the MO scheme.

Let $A_\ell(p^2)$ and $B_\ell(p^2)$ denote the form factors for the unrenormalized 1PI light-quark self-energy amplitude $-i(\not p A_\ell - m_{0\ell}B_\ell)$. This corresponds to the unrenormalized quark propagator

$$S(p) = \frac{i}{\not p(1-A_\ell) - m_{0\ell}(1-B_\ell)}$$
$$= (1-A_\ell)^{-1}\frac{i}{\not p - m_{0\ell}(1-B_\ell)/(1-A_\ell)}. \tag{72}$$

Define MO light-quark masses $M_\ell^{\text{MO}}(Q)$ at a fixed space-like point $p^2 = -Q^2$:

$$M_\ell^{\mathrm{MO}}(Q)=m_{0\ell}(1-B_\ell(-Q^2))/(1-A_\ell(-Q^2)). \quad (73)$$

This mass is RG invariant because, expressed in terms of renormalized quantities, it is finite. The choice of spacelike subtraction point $-Q^2$ means that mass renormalization, as well as coupling constant and wave-function renormalization, is performed off-shell:[2]

$$iS^{-1}(p)|_{p^2=-Q^2}=\not{p}-M_\ell^{\mathrm{MO}}(Q)\neq 0 \quad \text{at} \quad p^2=-Q^2. \quad (74)$$

This avoids problems with Bloch-Nordsieck logarithms produced by n-loop perturbation theory at the on-shell point $p^2\sim m^2$ [6]:

$$iS^{-1}(p)\sim(p^2-m^2)^{-1}[\ln(p^2-m^2)]^n.$$

A complication familiar to many authors [19,20] is that, unlike an on-shell renormalized mass, $M_\ell^{\mathrm{MO}}(Q)$ is gauge dependent. Despite this, the resulting mass-matching relation between $m_{\ell f}$ and $m_{\ell(f+1)}$ is gauge invariant [4] because the relations between $\overline{\mathrm{MS}}$ masses and their bare counterparts are gauge invariant. In two-loop perturbation theory, the result [4] for QCD with three colors and f light flavors is

$$\frac{m_{\ell f}}{m_{\ell(f+1)}}=1+\frac{\alpha_{f+1}^2}{12\pi^2}\left(\ln^2\frac{m_h^2}{\bar\mu^2}+\frac{5}{3}\ln\frac{m_h^2}{\bar\mu^2}+\frac{89}{36}\right)+O(\alpha_{f+1}^3). \quad (75)$$

We would like to extend this result to include all terms of the same *logarithmic* order. This is achieved by introducing our second matching function, $\mathcal{G}$—the matching function for mass renormalization. On order to reduce notational complexity, we consider the special case $f=5$ mentioned at the beginning of this section.

Consider the RG equation

$$\mathcal{D}_6\ln\frac{m_{b6}}{m_{b5}}=\delta_6(\alpha_6)-\delta_5(\alpha_5) \quad (76)$$

The leading power in a large-m_t expansion of m_{b6}/m_{b5} is a function of α_5 and $\ln(m_t/\bar\mu)$ but does not depend on light-quark masses, so the general solution of Eq. (76) is

$$\ln\frac{m_{b6}}{m_{b5}}=\int_{\alpha_5}^{\tilde\alpha_t}dx\frac{\delta_5(x)}{\beta_5(x)}-\int_{\alpha_6}^{\tilde\alpha_t}dx\frac{\delta_6(x)}{\beta_6(x)}+\mathcal{G}_{6\to5}(\tilde\alpha_t)+O(m_t^{-1}) \quad (77)$$

where $\mathcal{G}_{6\to5}(\tilde\alpha_t)$ is the *mass-matching function*. Like $\mathcal{F}_{6\to5}(\tilde\alpha_t)$ in Eq. (45), $\mathcal{G}_{6\to5}$ arises as an integration constant

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[2]A similar MO definition for heavy-quark masses M_h [17] yields a β function $\beta(g,M_h/Q)$ [18] with smooth threshold behavior at $Q\sim M_h$.

FIG. 1. One-loop correction to mass renormalization.

of a RG equation, so it can depend only on the RG invariant $\tilde\alpha_t$. Also like $\mathcal{F}$, it cannot be deduced from the RG and must be calculated separately.

At one-loop order, there are no top-quark corrections (Fig. 1), so the mass-matching condition is trivial [4]:

$$M_b^{\mathrm{MO}}=m_{b,6}[1+\alpha_6\{1\text{-loop}\}]+O(\alpha_6^2)$$
$$=m_{b,5}[1+\alpha_5\{1\text{-loop}\}]+O(\alpha_5^2). \quad (78)$$

Here $\alpha_f\{1\text{-loop}\}$ denotes the self-energy amplitude derived from Fig. 1. Eliminating M_b^{MO}, we find

$$\ln\frac{m_{b6}}{m_{b5}}=(\alpha_5-\alpha_6)\{1\text{-loop}\}+O(\alpha_5^2)=O(\alpha_5^2). \quad (79)$$

At this point, we need to specify what is LO, NLO, and so on. If we were talking only about corrections to mass, we might consider terms $\sim(\alpha_f\ln m_t/\bar\mu)^n$ as LO, but in general, it is more convenient to regard them as NLO. That is because mass renormalization does not contribute to physical amplitudes in LO. With this convention, Eqs. (48), (53), and (55) imply

$$\int_{\alpha_f}^{\tilde\alpha_t}dx\frac{\delta_f(x)}{\beta_f(x)}\underset{\mathrm{NLO}}{=}\frac{6}{\dfrac{33}{2}-f}\ln\frac{\tilde\alpha_t}{\alpha_f}$$

$$\underset{\mathrm{NLO}}{=}-\frac{6}{\dfrac{33}{2}-f}\ln\left[1+\frac{\alpha_f}{3\pi}\left(\frac{33}{2}-f\right)\ln\frac{m_t}{\bar\mu}\right]. \quad (80)$$

In lowest-order perturbation theory, we have

$$\int_{\alpha_5}^{\tilde\alpha_t}dx\frac{\delta_5(x)}{\beta_5(x)}-\int_{\alpha_6}^{\tilde\alpha_t}dx\frac{\delta_6(x)}{\beta_6(x)}=\frac{2}{\pi}(\alpha_6-\alpha_5)\ln\frac{m_t}{\bar\mu}+O(\alpha_5^2)$$
$$=O(\alpha_5^2) \quad (81)$$

since the couplings α_6 and α_5 differ by $O(\alpha_5^2)$. From Eqs. (77), (79), and (81), we conclude

$$\mathcal{G}_{6\to5}(\tilde\alpha_t)=O(\tilde\alpha_t^2). \quad (82)$$

Having established that $\mathcal{G}$ is irrelevant at NLO, we neglect it in Eq. (77) and substitute Eq. (80). This yields the complete NLO expression:
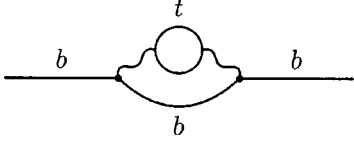
FIG. 2. Two-loop heavy-quark contribution to mass renormalization of the bottom quark.

$$\ln\frac{m_{b6}}{m_{b5}}\bigg|_{\mathrm{NLO}} = \frac{6}{\frac{33}{2}-6}\ln\left[1+\frac{\alpha_6}{3\pi}\left(\frac{33}{2}-6\right)\ln\frac{m_t}{\mu}\right]$$

$$-\frac{6}{\frac{33}{2}-5}\ln\left[1+\frac{\alpha_5}{3\pi}\left(\frac{33}{2}-5\right)\ln\frac{m_t}{\mu}\right]. \quad (83)$$

If desired, α_6 can be eliminated in favor of α_5 via Eq. (35). Note that the $O(\alpha_5^2)$ NLO term is a double logarithm which arises from the diagram with a t-loop inserted in the gluon propagator (Fig. 2):

$$\ln\frac{m_{b6}}{m_{b5}}\bigg|_{\mathrm{NLO}} = \frac{2}{\pi}(\alpha_6-\alpha_5)\ln\frac{m_t}{\mu} - \frac{\alpha_6^2}{3\pi^2}\left(\frac{33}{2}-6\right)\ln^2\frac{m_t}{\mu}$$

$$+\frac{\alpha_5^2}{3\pi^2}\left(\frac{33}{2}-5\right)\ln^2\frac{m_t}{\mu}+O(\alpha_5^3)$$

$$\underset{\mathrm{NLO}}{=} -\frac{1}{3}\left(\frac{\alpha_5}{\pi}\right)^2\ln^2\frac{m_t}{\mu}+O(\alpha_5^3). \quad (84)$$

This reproduces the NLO term of Bernreuther's result (75) for[3] m_f/m_{f+1}.

Equation (83) generates the complete set of NLO logarithms. They correspond to diagrams with a string of one-loop t-quark bubbles inserted in the gluon propagator of the one-loop b-quark self-energy amplitude.

The constant term $89\alpha_f^2/(432\pi^2)$ in Eq. (75) corresponds to the first nonzero contribution to the matching function $\mathcal{G}$. We state the result for any value of f:

$$\mathcal{G}_{f+1\to f}(\tilde\alpha_h) = -\frac{89}{432\pi^2}\tilde\alpha_h^2+O(\tilde\alpha_h^3). \quad (85)$$

This term is required if NNNLO corrections are being calculated.

VIII. APPLICATION TO SEQUENTIAL DECOUPLING

When decoupling the b quark, it is natural to define five-flavor quantities $\tilde\alpha_{b5}$ and $\bar m_{b5}$ by analogy with the six-flavor running coupling $\tilde\alpha_t$ and mass $\bar m_t$ for the top quark:

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[3]Note that Bernreuther expands in α_6 instead of α_5, but to this order the coefficient is the same.

$$\ln\frac{m_{b5}}{\mu} = \int_{\alpha_5}^{\tilde\alpha_{b5}}dx\,\frac{1-\delta_5(x)}{\beta_5(x)} \quad (86)$$

$$\ln\frac{\bar m_{b5}}{m_{b5}} = \int_{\alpha_5}^{\tilde\alpha_{b5}}dx\,\frac{\delta_5(x)}{\beta_5(x)}. \quad (87)$$

Clearly, both $\tilde\alpha_{b5}$ and $\bar m_{b5}$ are $\mathrm{RG}_{f=5}$ invariant,

$$\mathcal{D}_5\tilde\alpha_{b5}=0, \quad \mathcal{D}_5\bar m_{b5}=0 \quad (88)$$

but in fact, they are also $\mathrm{RG}_{f=6}$ invariant as a consequence of Eqs. (13) and (14):

$$\mathcal{D}_6\tilde\alpha_{b5}=0, \quad \mathcal{D}_6\bar m_{b5}=0. \quad (89)$$

This means that $\tilde\alpha_{b5}$ and $\bar m_{b5}$ can be expressed in terms of invariants of the original six-flavor theory.[4] To see this, first combine Eqs. (87) and (41), and then (77):

$$\ln\frac{\bar m_t}{m_t} - \ln\frac{\bar m_{b5}}{m_{b5}} = \int_{\alpha_6}^{\tilde\alpha_t}dx\,\frac{\delta_6(x)}{\beta_6(x)} - \int_{\alpha_5}^{\tilde\alpha_{b5}}dx\,\frac{\delta_5(x)}{\beta_5(x)}$$

$$= -\ln\frac{m_{b6}}{m_{b5}} + \int_{\tilde\alpha_{b5}}^{\tilde\alpha_t}dx\,\frac{\delta_5(x)}{\beta_5(x)}$$

$$+\mathcal{G}_{6\to5}(\tilde\alpha_t)+O(m_t^{-1}). \quad (90)$$

This equation simplifies to

$$\ln\frac{\bar m_t}{\bar m_{b5}} = \ln\left(\frac{m_t}{m_b}\right)_6 + \int_{\tilde\alpha_{b5}}^{\tilde\alpha_t}dx\,\frac{\delta_5(x)}{\beta_5(x)}+\mathcal{G}_{6\to5}(\tilde\alpha_t)+O(m_t^{-1}). \quad (91)$$

The sum of Eqs. (86) and (87),

$$\ln\frac{\bar m_{b5}}{\mu} = \int_{\alpha_5}^{\tilde\alpha_{b5}}dx\,\frac{1}{\beta_5(x)} \quad (92)$$

can be subtracted from Eq. (3), with the result

$$\ln\frac{\bar m_t}{\bar m_{b5}} = \int_{\tilde\alpha_{b5}}^{\tilde\alpha_t}dx\,\frac{1}{\beta_5(x)}+\mathcal{F}_{6\to5}(\tilde\alpha_t)+O(m_t^{-1}). \quad (93)$$

If Eq. (93) is now combined with Eq. (91), we find that $\tilde\alpha_{b5}$ and hence $\bar m_{b5}$ can be expressed in terms of RG invariants of the original six-flavor theory, viz. $\tilde\alpha_t$ and the ratio $(m_t/m_b)_6$:

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[4]This property is essential for any generalization of the analysis to simultaneous decoupling, where all couplings and masses, running or otherwise, will have to be defined *only* in terms of the initial theory ($F=6$) or the residual theory ($f=4$ if just the t and b are being decoupled), with no reference to five-flavor couplings or masses.

$$\ln\left(\frac{m_t}{m_b}\right)_6 = \int_{\tilde{\alpha}_{b5}}^{\tilde{\alpha}_t} dx \, \frac{1-\delta_5(x)}{\beta_5(x)} + \mathcal{F}_{6\to5}(\tilde{\alpha}_t)$$

$$-\mathcal{G}_{6\to5}(\tilde{\alpha}_t) + O(m_t^{-1}). \tag{94}$$

The sequential decoupling of the t and b quarks refers to the limiting procedure

$$\ln(m_t/\bar{\mu}) \gg \ln(m_b/\bar{\mu}) \to \infty \tag{95}$$

where we choose a six-flavor definition for m_b as well as m_t. Leading-power six-flavor amplitudes are represented by logarithmic expansions for t-quark decoupling

$$\tilde{\mathcal{A}}_6 \sim \sum_{p \geqslant 0} \tilde{\mathcal{C}}_{p5} \ln^p(m_t/\bar{\mu}) \tag{96}$$

where each five-flavor coefficient $\tilde{\mathcal{C}}_{p5}$ is a leading-power asymptotic expansion for b-quark decoupling:

$$\tilde{\mathcal{C}}_{p5} \sim \sum_{q \geqslant 0} \mathcal{C}_{pq4} \ln^q(m_b/\bar{\mu}) \tag{97}$$

The last decoupling (that of the b quark) is carried out with α_4 held fixed. Therefore we seek formulas for couplings such as α_6 and $\tilde{\alpha}_t$ in terms of m_t, m_b and α_4.

As always, the key step in the derivation of NLO formulas is the neglect of some matching functions. In this case, we neglect the NNLO functions $\mathcal{F}_{6\to5} = O(\tilde{\alpha}_t)$ and $\mathcal{F}_{5\to4} = O(\tilde{\alpha}_{b5})$ for matching α_6 to α_5 and α_5 to α_4, and the NNNLO function $\mathcal{G}_{6\to5} = O(\tilde{\alpha}_t^2)$ for m_{b5} to be matched to $m_{b6} = m_b$.

We start with the NLO formula (83) for the five-flavor mass m_{b5}. To this order, all dependence on α_5 and α_6 can be eliminated via LO formulas derived from Eq. (35),

$$\alpha_5 \underset{\text{LO}}{=} \alpha_4 \left/ \left(1 + \frac{\alpha_4}{3\pi} \ln\frac{m_b}{\bar{\mu}}\right)\right. \tag{98}$$

$$\alpha_6 \underset{\text{LO}}{=} \alpha_4 \left/ \left\{1 + \frac{\alpha_4}{3\pi}\left(\ln\frac{m_t}{\bar{\mu}} + \ln\frac{m_b}{\bar{\mu}}\right)\right\}\right. \tag{99}$$

where (again to this order) m_{b5} may be replaced by m_b on the right-hand side. Then Eq. (83) becomes

$$\ln\frac{m_{b5}}{\bar{\mu}} \underset{\text{NLO}}{=} \ln\frac{m_b}{\bar{\mu}} - \frac{12}{23}\ln\left(1 + \frac{\alpha_4}{3\pi}\ln\frac{m_b}{\bar{\mu}}\right)$$

$$-\frac{8}{161}\ln\left(1 + \frac{23\alpha_4}{6\pi}\ln\frac{m_t}{\bar{\mu}} + \frac{\alpha_4}{3\pi}\ln\frac{m_b}{\bar{\mu}}\right)$$

$$+\frac{4}{7}\ln\left(1 + \frac{\alpha_4}{3\pi}\ln\frac{m_b}{\bar{\mu}} + \frac{\alpha_4}{3\pi}\ln\frac{m_t}{\bar{\mu}}\right). \tag{100}$$

Similarly, consider the NLO relation (68) between α_{f+1} and α_f. For the case $f=4$, the heavy-quark mass m_h in Eq. (68) is m_{b5}, but we can use Eq. (100) to eliminate m_{b5} in favor of m_b,

$$\alpha_5^{-1} \underset{\text{NLO}}{=} \alpha_4^{-1} + \frac{1}{3\pi}\ln\frac{m_{b5}}{\bar{\mu}}\bigg|_{\text{NLO}} + c_4 \ln\left[1 + \frac{\alpha_4}{3\pi}\ln\frac{m_b}{\bar{\mu}}\right]$$

$$+ d_4 \ln\left[1 + \frac{25\alpha_4}{6\pi}\ln\frac{m_b}{\bar{\mu}}\right] \tag{101}$$

where the constants c_4 and d_4 are given by

$$c_4 = 33/(23\pi), \quad d_4 = 121/(1150\pi). \tag{102}$$

For $f=5$, m_h in Eq. (68) is the six-flavor mass m_t. Any α_5 dependence can be removed via Eq. (98) or Eq. (101):

$$\alpha_6^{-1} \underset{\text{NLO}}{=} \alpha_5^{-1}\big|_{\text{NLO}} + \frac{1}{3\pi}\ln\frac{m_t}{\bar{\mu}} + c_5 \ln\left[1 + \frac{\alpha_5}{3\pi}\bigg|_{\text{LO}}\ln\frac{m_t}{\bar{\mu}}\right]$$

$$+ d_5 \ln\left[1 + \frac{23\alpha_5}{6\pi}\bigg|_{\text{LO}}\ln\frac{m_t}{\bar{\mu}}\right] \tag{103}$$

The coefficients c_5 and d_5 have numerical values

$$c_5 = 47/(42\pi), \quad d_5 = 137/(966\pi). \tag{104}$$

The result of combining these formulas is

$$\alpha_6^{-1} \underset{\text{NLO}}{=} \alpha_4^{-1} + \frac{1}{3\pi}\left(\ln\frac{m_b}{\bar{\mu}} + \ln\frac{m_t}{\bar{\mu}}\right)$$

$$+ \frac{55}{42\pi}\ln\left(1 + \frac{\alpha_4}{3\pi}\ln\frac{m_b}{\bar{\mu}} + \frac{\alpha_4}{3\pi}\ln\frac{m_t}{\bar{\mu}}\right)$$

$$+ \frac{121}{966\pi}\ln\left(1 + \frac{23\alpha_4}{6\pi}\ln\frac{m_t}{\bar{\mu}} + \frac{\alpha_4}{3\pi}\ln\frac{m_b}{\bar{\mu}}\right)$$

$$+ \frac{121}{1150\pi}\ln\left(1 + \frac{25\alpha_4}{6\pi}\ln\frac{m_b}{\bar{\mu}}\right). \tag{105}$$

The same procedure can be applied to the NLO formula (67) for the RG invariant mass $\bar{m}_h$. For $f=4$, the result is

$$\ln\frac{\bar{m}_{b5}}{\bar{\mu}} \underset{\text{NLO}}{=} \ln\frac{m_b}{\bar{\mu}} - \frac{12}{23}\ln\left(1 + \frac{25\alpha_4}{6\pi}\ln\frac{m_b}{\bar{\mu}}\right)$$

$$-\frac{8}{161}\ln\left(1 + \frac{23\alpha_4}{6\pi}\ln\frac{m_t}{\bar{\mu}} + \frac{\alpha_4}{3\pi}\ln\frac{m_b}{\bar{\mu}}\right)$$

$$+\frac{4}{7}\ln\left(1 + \frac{\alpha_4}{3\pi}\ln\frac{m_b}{\bar{\mu}} + \frac{\alpha_4}{3\pi}\ln\frac{m_t}{\bar{\mu}}\right) \tag{106}$$

after substitution of Eq. (100) for m_{b5}. For $f=5$, Eq. (67) expresses $\bar{m}_t$ in terms of α_5, for which the LO formula (98) may be substituted, with the result

$$\ln\frac{\bar{m}_t}{\bar{\mu}}\underset{\text{NLO}}{=}\ln\frac{m_t}{\bar{\mu}}+\frac{4}{7}\ln\left(1+\frac{\alpha_4}{3\pi}\ln\frac{m_t}{\bar{\mu}}+\frac{\alpha_4}{3\pi}\ln\frac{m_b}{\bar{\mu}}\right)$$

$$-\frac{4}{7}\ln\left(1+\frac{23\alpha_4}{6\pi}\ln\frac{m_t}{\bar{\mu}}+\frac{\alpha_4}{3\pi}\ln\frac{m_b}{\bar{\mu}}\right). \quad (107)$$

If desired, inverses of Eqs. (106) and (107) can be constructed and used to express quantities such as α_6 in terms of the invariant masses $\bar{m}_{b5}$ and $\bar{m}_t$ instead of m_b and m_t.

Finally, we extract NLO formulas for the invariant running couplings for sequential t,b decoupling from Eq. (69) for $\tilde{\alpha}_h$. For the case $f=4$, Eqs. (69) and (106) imply

$$\tilde{\alpha}_{b5}^{-1}\underset{\text{NLO}}{=}\alpha_4^{-1}+\frac{25}{6\pi}\ln\frac{m_b}{\bar{\mu}}-\frac{729}{1150\pi}\ln\left(1+\frac{25\alpha_4}{6\pi}\ln\frac{m_b}{\bar{\mu}}\right)$$

$$-\frac{100}{483\pi}\ln\left(1+\frac{23\alpha_4}{6\pi}\ln\frac{m_t}{\bar{\mu}}+\frac{\alpha_4}{3\pi}\ln\frac{m_b}{\bar{\mu}}\right)$$

$$+\frac{50}{21\pi}\ln\left(1+\frac{\alpha_4}{3\pi}\ln\frac{m_b}{\bar{\mu}}+\frac{\alpha_4}{3\pi}\ln\frac{m_t}{\bar{\mu}}\right). \quad (108)$$

For $f=5$, it is necessary to combine Eq. (69) with Eqs. (100), (101), and (107):

$$\tilde{\alpha}_t^{-1}\underset{\text{NLO}}{=}\alpha_4^{-1}+\frac{23}{6\pi}\ln\frac{m_t}{\bar{\mu}}+\frac{1}{3\pi}\ln\frac{m_b}{\bar{\mu}}$$

$$-\frac{457}{483\pi}\ln\left(1+\frac{23\alpha_4}{6\pi}\ln\frac{m_t}{\bar{\mu}}+\frac{\alpha_4}{3\pi}\ln\frac{m_b}{\bar{\mu}}\right)$$

$$+\frac{50}{21\pi}\ln\left(1+\frac{\alpha_4}{3\pi}\ln\frac{m_b}{\bar{\mu}}+\frac{\alpha_4}{3\pi}\ln\frac{m_t}{\bar{\mu}}\right)$$

$$+\frac{121}{1150\pi}\ln\left(1+\frac{25\alpha_4}{6\pi}\ln\frac{m_b}{\bar{\mu}}\right). \quad (109)$$

A useful check of the formalism can be obtained by showing that the difference

$$\tilde{\alpha}_t^{-1}-\tilde{\alpha}_{b5}^{-1}$$

is correctly given by Eq. (94) in NLO.

IX. THEORIES LACKING ASYMPTOTIC FREEDOM

So far, we have limited the discussion to heavy fermions in a gauge theory such as QCD and used asymptotic freedom to obtain decoupling in the infinite-mass limit.

Does decoupling work if asymptotic freedom is not present, as in QED or scalar field theory with $\lambda\phi^4$ interac-

tion in four dimensions? Perhaps this is part of the wider debate [21] about whether such theories are inconsistent or trivial, particularly for the continuum limit of the lattice approximation.

We start from the premise that a theory makes sense only if its heavy particles decouple in the infinite-mass limit to produce another consistent theory. Questions about how the nonperturbative theory could depend on details of regulators and their removal are not considered. We simply assume that some means of producing a fully interacting cutoff-independent theory has been found, e.g. for QED_k with k species of equal-charge leptons, and apply our premise.

The notation is similar to that used above for QCD. Let $\alpha_k=e_k^2/(4\pi)$ be the $\overline{\text{MS}}$ renormalized fine structure constant for QED_k, where the charged leptons have $\overline{\text{MS}}$ masses $m_j, j=1,\ldots,k$, and let β_k and δ_k be the Callan-Symanzik functions for charge and mass renormalization. Denote by $\tilde{\alpha}_H$ and $\bar{m}_H$ Witten's invariant versions of the running fine-structure constant and heavy-lepton mass. Then, repeating the arguments of Secs. IV and VII, we can construct coupling-constant and mass matching functions $\mathcal{F}$ and $\mathcal{G}$ for the decoupling of one species of lepton:

$$QED_{k+1}\rightarrow QED_k, \quad k\geqslant 1 \quad (110)$$

The free-photon theory QED_0 lacks a β function so it is a special case.

First consider the analogue of Eq. (2), including the case $k=0$:

$$\ln\frac{\bar{m}_H}{\bar{\mu}}=\int_{\alpha_{k+1}}^{\tilde{\alpha}_H}dx\,\frac{1}{\beta_{k+1}(x)} \quad (111)$$

The decoupling condition

$$\alpha_{k+1}\rightarrow 0 \quad\text{as}\quad m_H\rightarrow\infty \quad\text{for fixed}\quad \alpha_k \quad (112)$$

involves the $x=0$ solution of the equation $\beta_{k+1}(x)=0$, but it produces an infinity of the wrong sign because this fixed point is *infrared* stable. For consistency, $\tilde{\alpha}_H$ must approach a singularity of the integral sufficient to reverse the effect of the $x=0$ contribution. This could arise from an ultraviolet fixed point,

$$\tilde{\alpha}_H\rightarrow\alpha_{k+1,\infty} \quad (113)$$

or else $\tilde{\alpha}_H$ approaches $x=\infty$, in which case we must suppose that $1/\beta_{k+1}(x)$ is not integrable.

Next define $\mathcal{F}$ as in Eq. (3), with $k\geqslant 1$:

$$\ln\frac{\bar{m}_H}{\bar{\mu}}=\int_{\alpha_k}^{\tilde{\alpha}_H}dx\,\frac{1}{\beta_k(x)}+\mathcal{F}_{k+1\rightarrow k}(\tilde{\alpha}_H) \quad (114)$$

In the decoupling limit (112), the singularity on the left-hand side is generated *entirely* from the running of $\tilde{\alpha}_H$. We can conclude that this is due to a QED_k fixed point $\alpha_{k,\infty}$ *only if* it coincides with that of QED_{k+1}:

$$\alpha_{k+1,\infty}=\alpha_{k,\infty} \quad (115)$$

Similarly, a nonintegrable singularity at $x = \infty$ can be the sole cause only if this happens for both QED_k and QED_{k+1}. Otherwise, we must suppose that the matching function has a singularity at $\alpha_{k,\infty}$ or ∞.

The nonperturbative theory of QED of Johnson, Baker, Willey and Adler [22] is an example of case (115). Indeed, if the arguments of Adler for an infinite-order zero at the fixed point are applied to a many-species theory (all with the same charge), the result is clear: there is no dependence on the number of species.

Notice that these conclusions are driven by the lack of asymptotic freedom of the initial theory. For example, consider what happens in QCD when a heavy quark decouples from the nonasymptotically free 17-flavor theory to produce the 16-flavor theory with asymptotic freedom. Equation (2) for $F = 17$ implies that $\tilde{\alpha}_h$ increases. Thus in Eq. (3) with $f = 16$, $\tilde{\alpha}_h$ is driven towards a nonperturbative infrared region of the residual theory.

X. CONCLUDING REMARKS

The introduction of the matching functions $\mathcal{F}$ and $\mathcal{G}$ for coupling constants and masses [Eqs. (3) and (77)] completes the theoretical structure needed for a systematic application of the RG to the decoupling of heavy particles. We have considered just QCD and QED, but the field-theoretic principles are much the same for any theory. The main case still to be checked is a full RG analysis of the decoupling of heavy gauge bosons whose masses are induced by the Higgs mechanism.

In this paper, we decoupled only one particle at a time (for simplicity) and concentrated on field-theoretic aspects of the subject. Actually, our work on matching functions arises from a need to consider the simultaneous decoupling of more than one heavy quark in phenomenological applications. The challenge is to keep track of dependence on large logarithms such as

$$\ln \frac{m_t}{m_b} = \ln \frac{m_t}{\bar{\mu}} - \ln \frac{m_b}{\bar{\mu}} \qquad (116)$$

Since these logarithms do *not* depend on the $\overline{\text{MS}}$ scale $\bar{\mu}$, the conventional tactic used in phenomenology for single heavy quarks fails: there is no way of making such logarithms small by choosing $\bar{\mu} = O(m_h)$.

As indicated in our work [12] on NLO heavy-quark effects in axial charges of nucleons, the analysis can be generalized to include simultaneous decoupling of several heavy particles. This includes the introduction of matching functions of several variables, one for each heavy particle. We will present this extension of the theory in a forthcoming publication.

One can also anticipate generalizations to situations where momentum and mass logarithms grow large together. Examples from the literature occur in collider physics [23], Higgs boson and supersymmetric particle production [24], and deep-inelastic scattering through thresholds for heavy-particle production [25].

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