Zero mode in the time-dependent symmetry breaking of $\lambda\phi^4$ theory

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We apply the quartic exponential variational approximation to the symmetry breaking phenomena of a scalar field in three and four dimensions. We calculate the effective potential and effective action for the time-dependent system by separating the zero mode from other nonzero modes of the scalar field and treating the zero mode quantum mechanically. It is shown that the quantum mechanical properties of the zero mode play a nontrivial role in the symmetry breaking of the scalar $\lambda\phi^4$ theory.

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I. INTRODUCTION

The variational approach for scalar field theory using the Gaussian effective potential was well studied in Refs. [1,2], and references therein. The renormalizability and initial value problems for the Gaussian approximation are checked in [3,4]. Many authors have studied the symmetry breaking phase structures in the large N approximation [5]. The nonequilibrium dynamics of symmetry breaking [6] and the second order phase transition [7] have also been studied in the Gaussian framework.

On the other hand, it is known that the Gaussian approach is inappropriate to treat the time-dependent symmetry breaking of initially symmetric states because of their own limits [8] that they cannot be applied to a double well type potential for large dispersions. To overcome this difficulty, many approaches have attempted to go beyond Gaussian methods by using a quartic exponential ansatz [9], perturbative expansion around the Gaussian approximation [10], and variation including higher excited states of the Gaussian approximation [11].

It is therefore important to understand the limit of the approaches based on the Gaussian wave functional and to develop a consistent method going beyond the Gaussian approximation. Recently, the present authors have developed a new quartic exponential type variational approximation [12] which is suitable for systems with double well type potentials in the quantum mechanical context. In this paper, by applying the approximation to the zero mode of the scalar field, we found the renormalized equations of motion which can describe the symmetry breaking phenomena starting from the symmetric states of the scalar ϕ^4 theory in three and four dimensions. This kind of zero mode separation was considered in the previous papers [13] using the autonomous renormalization of ϕ^4 or speculating on the $p\to 0$ limit of spontaneous symmetry breaking.

The paper is organized as follows. First we separate the zero mode of the scalar field from other modes in Sec. II and then calculate the formal expression for the effective action. We then calculate the renormalized effective potentials and actions for the (2+1)-dimensional case in Sec. III, and the (3+1)-dimensional case in Sec. IV. Finally, we summarize our results and present some discussions in Sec. V.

II. MODE SEPARATION OF THE SELF-INTERACTING SCALAR FIELD

We first separate the zero-mode from other non-zero excitations and present the concept of effective action adapted in the present paper. The Lagrangian of the scalar ϕ^4 theory in $n+1$ dimensions is

$$L=\int d^n\mathbf{x}\left[\frac{1}{2}\partial^\mu\phi(x^\nu)\partial_\mu\phi(x^\nu) -\frac{1}{2}\mu^2(t)\phi^2(x^\nu)-\frac{\lambda}{4!}\phi^4(x^\nu)\right], \tag{1}$$

where we explicitly included the volume integral so that we can write the volume factor in the zero mode part of the Lagrangian and $\mu^2(t)$ asymptotically approaches to a negative value $\mu^2<0$ so that the system undergoes symmetry breaking. Let the theory be defined in a box of volume V. Then, its zero mode can be extracted by the following definitions:

$$\phi(t)=\frac{1}{V}\int d^n\mathbf{x}\phi(\mathbf{x},t), \qquad \psi(\mathbf{x},t)=\phi(\mathbf{x},t)-\phi(t). \tag{2}$$

The new field, ψ, satisfies $\int d^n\mathbf{x}\psi(\mathbf{x},t)=0$. The Lagrangian (1),

$$\begin{aligned}L=&V\left[\frac{1}{2}\dot\phi^2(t)-\frac{\mu^2(t)}{2}\phi^2(t)-\frac{\lambda}{4!}\phi^4(t)\right]\\&+\int d^n\mathbf{x}\left\{\frac{1}{2}\partial^\mu\psi(x^\nu)\partial_\mu\psi(x^\nu)\right.\\&-\frac{1}{2}\left[\mu^2(t)+\frac{\lambda}{2}\phi^2(t)\right]\psi^2(x^\nu)-\frac{\lambda}{4!}\psi^4(x^\nu)\\&\left.-\frac{\lambda}{6}\phi(t)\psi^3(x^\nu)\right\},\end{aligned} \tag{3}$$

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then, becomes a coupled Lagrangian of a quantum mechanical quartic oscillator ϕ and a scalar field ψ which has only nonzero mode excitations.

The effective action for a time-dependent system is given by the functional integral

$$\Gamma = \int dt \langle \phi,\psi | i\partial_t - H | \phi,\psi \rangle, \tag{4}$$

where the state $|\phi,\psi\rangle$ is the one that extremizes Γ. The effective potential is given by the negative of its static limit. Before we consider the quantum mechanical corrections, we give the tree level analysis first. The effective mass of ϕ and ψ are given by

$$m_\phi^2(t) = \mu^2 + \frac{\lambda}{2V}\int d^n\mathbf{x}\langle \psi^2(\mathbf{x},t)\rangle,$$

$$m_\psi^2(t) = \mu^2 + \frac{\lambda}{2}\langle \phi^2(t)\rangle, \tag{5}$$

where $\langle A\rangle$ denotes the expectation value of A for a symmetric wave functional $\Phi(\phi,\psi)$ given below. $\langle \psi^2(x)\rangle$ usually become independent of x^i, the spatial coordinates, due to the translational invariance of Green's function and its spatial integral cancels the volume factor at the denominator in the first equation of (5). In the presence of symmetry breaking, we expect that the expectation value of the field $\langle \phi^2(x,t)\rangle$ increases to a larger value $\phi^2 \sim -6\mu^2/\lambda$. This allows the possibility of $\langle \phi^2(t)\rangle$ having large enough value so that $m_\phi^2 < 0$ and $m_\psi^2 > 0$. This case is exactly what we are interested in in this paper. Since the mass m_ψ^2 is positive definite, the $\langle \psi^2\rangle$ cannot grow to larger value and be confined to a near zero value. The stable equilibrium of ϕ is at $\pm\sqrt{(6/\lambda)[-\mu^2 - (\lambda/2V)\int d^n\mathbf{x}\langle\psi^2\rangle]}$. Therefore, ϕ stays in the degenerate ground state since the large volume factor V makes the potential wall between the positive and negative minima infinitely high.

These discussions justify the use of the Gaussian approximation for the ψ field. On the other hand, we use the quartic exponential approximation for the zero mode, ϕ, to include better quantum mechanical effects. To describe the evolution of the symmetric state we use the trial wave functional of the form

$$\begin{aligned}\Phi[\phi,\psi] = N \exp\Bigg\{ &-\frac{1}{2}\left[\frac{1}{2g^2(t)} + i\pi(t)\right]\phi^4 \\ &+ \left[\frac{x(t)}{g(t)} + ip(t)\right]\phi^2 - \int_{\mathbf{xy}} \psi(\mathbf{x}) \\ &\times \left[\frac{G^{-1}(\mathbf{x},\mathbf{y};t)}{4!} - i\Pi(\mathbf{x},\mathbf{y};t)\right]\psi(\mathbf{y})\Bigg\},\end{aligned} \tag{6}$$

where $\int_{\mathbf{x}} = \int d^n\mathbf{x}$ and we use the unit which makes $\hbar = 1$. The use of this symmetric trial wave functional simplifies the computation since the contribution from the interaction term, $\phi\psi^3$, in the Lagrangian (3) vanishes. Following Ref. [12] we define some physical quantities of the zero mode:

$$q^2(t) = \langle \phi^2(t)\rangle, \quad y(t) = \frac{\langle \phi^4(t)\rangle}{(\langle \phi^2(t)\rangle)^2}. \tag{7}$$

By introducing the integral [12]

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{g}}\int_{-\infty}^{\infty} dQ \exp\left(-\frac{Q^4}{2g^2} + \frac{2xQ^2}{g}\right) \\ &= \frac{\pi}{\sqrt{2}}|x|^{1/2} e^{x^2}[I_{-1/4}(x^2) + \mathrm{sgn}(x) I_{1/4}(x^2)], \end{aligned} \tag{8}$$

we get

$$q^2(t) = \frac{gf'}{2f}, \quad y(t) = \frac{1 + 2xf'/f}{f'^2/(2f^2)}. \tag{9}$$

We additionally introduce the notation

$$\frac{d\eta}{dy} \equiv D = \frac{1}{4}\sqrt{\frac{1+Y}{y(3-y)}}, \quad Y = \frac{2xf'(x)}{f(x)}$$

as in Ref. [12].

Using these, we write the effective action (4) as a functional of $q(t)$, $p(t)$, $y(t)$, $\pi(t)$, $\Pi(\mathbf{x},\mathbf{y};t)$, and $G(\mathbf{x},\mathbf{y};t)$:

$$\begin{aligned}\Gamma = \int dt \Bigg\{ &\frac{yq^4\dot{\pi}}{2} - q^2\dot{p} - 2y\left[y - \frac{y-3}{Y+1}\right]\frac{q^6\pi^2}{V} - 2\frac{q^2p^2}{V} \\ &+ yq^4\frac{\pi p}{V} - VV_{eff}(q,y) + \int_{\mathbf{xy}}\Pi\dot{G} - 2\int_{\mathbf{xyz}}\Pi G\Pi \\ &- \int_{\mathbf{x}}\left[\frac{1}{8}G^{-1}(\mathbf{x},\mathbf{x}) - \frac{1}{2}\nabla_{\mathbf{x}}^2 G(\mathbf{x},\mathbf{y};t)\Big|_{\mathbf{x}=\mathbf{y}} \right. \\ &\left. + \frac{1}{2}V^{(2)}G(\mathbf{x},\mathbf{x};t)\right] - \frac{\lambda}{8}\int_{\mathbf{x}} G(\mathbf{x},\mathbf{x};t)G(\mathbf{x},\mathbf{x};t)\Bigg\},\end{aligned} \tag{10}$$

where $V^{(2)} = \mu^2 + (\lambda/2)q^2(t)$ and $V_{eff}(q,y) = [V_F(y)/8V^2q^2] + (\mu^2/2)q^2 + (\lambda/4!)yq^4$. We do not need the explicit form of V_F, instead, it is enough to know that V_F becomes divergent at $y=1$. Therefore, in large V limit,

$$V_{eff} = \begin{cases} \frac{1}{2}\mu^2q^2 + \frac{\lambda}{4!}yq^4, & y \geq 1 \text{ and } q \geq 0, \\ \infty, & \text{otherwise.} \end{cases} \tag{11}$$

Note that V_{eff} is finite even in the limits of zero dispersion $q^2 \to 0$ or $y \to 1$.

After solving the p and π equations, and using the translational invariance of G equation of motion we get

$$\Gamma[q,y;G,\Pi]=\int dtd^n\mathbf{x}\Bigg\{\bigg[\frac{q^2\dot{\eta}^2}{2}+\frac{\dot{q}^2}{2}-V_{eff}(q,y)\bigg]+\int_{\mathbf{k}}\bigg[\Pi(\mathbf{k},t)\dot{G}(\mathbf{k},t)-2\Pi^2(\mathbf{k},t)G(\mathbf{k},t)-\frac{1}{8}G^{-1}(\mathbf{k},t)-\frac{1}{2}(\mathbf{k}^2+V^{(2)})G(\mathbf{k},t)-\frac{\lambda}{8}G(\mathbf{k},t)\int_{\mathbf{q}}G(\mathbf{q},t)\bigg]\Bigg\}, \tag{12}$$

where $G(\mathbf{k},t)$ and $\Pi(\mathbf{k},t)$ are the Fourier transforms of $G(\mathbf{x,y};t)$ and $\Pi(\mathbf{x,y};t)$:

$$G(\mathbf{x,y};t)=\int_{\mathbf{k}}G(\mathbf{k},t)e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})},$$

$$\Pi(\mathbf{x,y};t)=\int_{\mathbf{k}}\Pi(\mathbf{k},t)e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}, \tag{13}$$

where $\int_{\mathbf{k}}=\int d^n\mathbf{k}/(2\pi)^n$.

The time-dependent variational equations are given by

$$\frac{d}{dt}(q^2D\dot{y})+\frac{\lambda}{4D}q^4=0, \tag{14}$$

$$\ddot{q}+[m^2(t)-\dot{\eta}^2(t)]q+\frac{\lambda}{6}(y-3)q^3=0, \tag{15}$$

$$\ddot{G}(\mathbf{k},t)=\frac{1}{2}G^{-1}(\mathbf{k},t)+\frac{1}{2}G^{-1}(\mathbf{k},t)\dot{G}^2(\mathbf{k},t)-2m^2(t)G(\mathbf{k},t), \tag{16}$$

where

$$m^2(t)=\mu^2(t)+\frac{\lambda}{2}q^2(t)+\frac{\lambda}{2}\int_{\mathbf{k}}G(\mathbf{k},t). \tag{17}$$

The potential divergences come from the integral $\int_{\mathbf{k}}G(\mathbf{k},t)$. $D(y)$ becomes divergent at $y=1$ such that $D(y)\sim(1/4)(1/\sqrt{y-1})$ as $y\to1$.

III. EFFECTIVE ACTION IN (2+1) DIMENSIONS

Since the Gaussian effective potential for the scalar ϕ^4 field theory in (2+1) dimensions shows symmetry breaking, (2+1)-dimensional theory is the best place to test our approximation method that uses zero-mode separation and the quartic Gaussian trial wave functional. During these calculations we assume $m_\psi^2\geqslant0$ always. Therefore, the nonzero modes stay always in symmetry restored states with the zero expectation value, $\langle\psi(\mathbf{x},t)\rangle=0$, and the zero mode is in a symmetry broken state.

A. Effective potential

In this section we shall briefly discuss how the static effective potential is renormalized in the quartic exponential treatment of the zero mode. The definition of an effective potential in this paper is slightly different from the standard one. In this paper we separate the field into the zero mode and nonzero mode parts, then we integrate out the nonzero mode [by solving the gap equation (16) of the nonzero modes] to get the effective potential for the zero mode. This definition gives a slightly different effective potential, since the zero mode is treated quantum mechanically in this paper, while the traditional method treats it as a background classical field. In spite of these conceptual differences the calculational procedure is similar to that of the Gaussian approximation.

The effective potential per unit volume in the present approximation is given by

$$V_{eff}(y,q;G(q))=V_{eff}(y,q)+\frac{1}{4}\int_{\mathbf{k}}G^{-1}(\mathbf{k})-\frac{\lambda}{8}\bigg[\int_{\mathbf{k}}G(\mathbf{k})\bigg]^2. \tag{18}$$

The potential is minimized at $y=1$ and G is the solution of the gap equation

$$\frac{1}{4}G^{-2}(\mathbf{k})=\mathbf{k}^2+m^2=\mathbf{k}^2+\mu^2+\frac{\lambda}{2}q^2+\frac{\lambda}{2}\int_{\mathbf{k}}G(\mathbf{k}), \tag{19}$$

which gives $G(q)$. The possible source of divergence comes from the integral

$$\int_{k}G(\mathbf{k})=I_0(m)=\int_{\mathbf{k}}\frac{1}{2\sqrt{\mathbf{k}^2+m^2}}. \tag{20}$$

The present form of G is the same as that of the Gaussian approximation of Ref. [1] with a slight modification that q^2 is now the expectation value of the dispersion of zero mode. This divergence in $I_0(m)$ is absorbed in μ^2 by defining the renormalized mass

$$m_R^2=\mu^2+\frac{\lambda}{2}\int_{\mathbf{k}}G(\mathbf{k})\bigg|_{q=0}, \tag{21}$$

with the help of the formula

$$I_0(m)-I_0(m_R)=-\frac{m-m_R}{4\pi}. \tag{22}$$

Then, the gap equation (19) can be written to give the ratio of the mass with respect to the renormalized mass for general q

$$\sqrt{x}\equiv\frac{m}{m_R}=-\frac{\lambda}{16\pi m_R}+\bigg[\bigg(\frac{\lambda}{16\pi m_R}+1\bigg)^2+\frac{\lambda}{2m_R^2}q^2\bigg]^{1/2}. \tag{23}$$

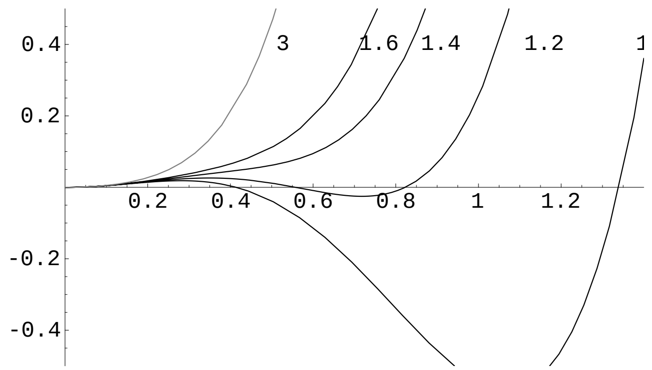

FIG. 1. The effective potential V/m_R^3 as a function of $\overline{q} = q/\sqrt{m_R}$. In this figure we set $\lambda = 4m_R$, $q_R = 0$, and the curves represent the cases of $y = 1$, 1.2, 1.4, 1.6, 3.

The effective potential in (2+1) dimensions can then be calculated to be

$$V(q,y) = \frac{1}{2}m_R^2 q^2 + \frac{\lambda}{4!}yq^4 - \frac{m_R^3}{24\pi}(\sqrt{x}-1)^2 \times\left[2\sqrt{x}+1+\frac{3\lambda}{16\pi m_R}\right], \quad q \geqslant 0, \tag{24}$$

which exactly reproduces the result of Ref. [1] if $y=1$. Note also that the potential for $\lambda \to \infty$ becomes

$$\frac{V(q,y)}{m_R^3} \simeq -\frac{\lambda}{24m_R^3}(3-y)q^4 + \frac{q^2}{2m_R} + \frac{2\pi q^4}{m_R^2} + \frac{8\pi q^6}{3m_R^3} + O(1/\lambda). \tag{25}$$

This form shows two things: First is that there is symmetry breaking for $y<3$ for large enough λ/m_R always. Therefore we have a critical value $\lambda/m_R = \hat{\lambda}_c$ such that if $\lambda/m_R > \widehat{\lambda_c}$ there is symmetry breaking for $y<3$. Second is that the potential for q at $y=3$ does not have the symmetry breaking form for any large value of λ/m_R. This fact shows that there is a critical value $y=y_c(\lambda)$ such that for $y<y_c<3$ and $\lambda/m_R > \hat{\lambda}_c$ symmetry breaking occurs. We draw the effective potential for various y values as a function of q in Fig. 1. From Fig. 1 we can deduce the dynamics of a zero-mode wave packet with the initial values $(q_0, y_0=3)$. The dynamics of the system is described by two processes. First is the dynamics until y decreases to $y=y_c$. During this process, the ground state for q is at $q=0$. The dispersion q^2 will oscillates around zero. The second process begins when y decreases to a value below y_c, so that the potential has symmetry breaking form. From this time, q increases until it takes a stable equilibrium value and y oscillates around $y=1$. During these processes, the zero mode continually interacts with the non-zero modes and may loose much of its energy. If one wants to understand these dynamical processes better, we need to calculate the effective action.

B. Renormalization of the effective action

It is known in Ref. [4] that not all states are allowed as the initial states in the Gaussian variational approximation. What we consider in this paper is that the evolution of symmetric initial states. Even though we treat the zero mode separately, our method uses the Gaussian approximation for the nonzero modes. This implies that the renormalizability condition for the time-dependent variational equations in Ref. [4] applies also to the present case. Thus we simply outline the initial condition and then write down the equations of motion. We choose the initial state of nonzero modes as a Gaussian with $G(x,y;0) = G(x,y)$, and $\Pi(x,y;0) = \Pi(x,y)$, and the initial state of the zero mode as a quartic exponential with $q = q_0$, $y = y_0$, $p(0)=0$, $\pi(0)=0$ in Eq. (6). Here we assume that there is no $\psi(x,t) - \phi(t)$ correlation in the initial state for simplicity. Then the nonzero overlapping condition for the initial states with the vacuum, prescribed in Ref. [4], are given by

$$\lim_{k\to\infty} G(k,0) = \frac{1}{2k}\left(1 + \frac{g\cos\alpha(k)}{k} + \frac{\overline{m}^2}{k^2}\right),$$
$$\lim_{k\to\infty} \dot{G}(\mathbf{k},0) = \frac{B\cos\beta(k)}{k} + \frac{A}{k^2}, \tag{26}$$

where α and β are nonoscillatory and g, A, B, and $\overline{m}$ are k independent constants.

The renormalization condition (21) can be generalized to the time-dependent case

$$m_R^2(t) = \mu^2(t) + \frac{\lambda}{2}\int_{\mathbf{k}} G_V(\mathbf{k},t)\bigg|_{q=0}, \tag{27}$$

where

$$G_V(\mathbf{k},t) = \frac{1}{2\left[\mathbf{k}^2 + \mu^2(t) + \frac{\lambda}{2}q^2 + \frac{\lambda}{2}\int_{\mathbf{k}} G_V\right]^{1/2}}$$

is the value of G in the instantaneous vacuum for a given q. The finite time-dependent mass is

$$m^2(t) = \left[\mu^2(t) + \frac{\lambda}{2}q^2(t) + \frac{\lambda}{2}\int_{\mathbf{k}} G_V(\mathbf{k},t)\right] + \frac{\lambda}{2}\int_{\mathbf{k}}[G(\mathbf{k},t) - G_V(\mathbf{k},t)] = m_V^2(t) + \tilde{m}^2(t), \tag{28}$$

where

$$m_V^2(t) \equiv \mu^2(t) + \frac{\lambda}{2}q^2(t) + \frac{\lambda}{2}\int_{\mathbf{k}} \frac{1}{2[\mathbf{k}^2 + m_V^2(t)]^{1/2}} = m_R^2(t) + \frac{\lambda}{2}q^2(t) + \frac{\lambda}{2}[I_0(m_V) - I_0(m_R)]. \tag{29}$$

Using Eq. (22) we get

$$m_V(t) = -\frac{\lambda}{16\pi} + \left[\left(\frac{\lambda}{16\pi}\right)^2 + m_R^2(t) + \frac{\lambda}{8\pi} m_R(t) + \frac{\lambda}{2} q^2(t)\right]^{1/2}. \tag{30}$$

One can show that the difference of mass-squared $\tilde{m}^2 = m^2 - m_V^2$,

$$\tilde{m}^2(t) \equiv \frac{\lambda}{2}\int_{\mathbf{k}} [G(\mathbf{k},t) - G_V(\mathbf{k},t)], \tag{31}$$

is finite by using the explicit asymptotic form of G in Eq. (26) at $t=0$. Now we need to show that $\tilde{m}^2(t)$ is finite for all t. This can be done by solving the renormalized equation of motion for G. We do not write this calculation explicitly here since it is a mere repetition of Eq. (4.11) of Ref. [4].

The renormalized equations of motion for q and $G(\mathbf{k},t)$ become

$$\ddot{q} + [m_V^2(t) + \tilde{m}^2(t)]q + \frac{\lambda}{6} y q^3 = \dot{\eta}^2 q, \tag{32}$$

$$\ddot{G}(\mathbf{k},t) = \frac{1}{2}G^{-1}(\mathbf{k},t) + \frac{1}{2}G^{-1}(\mathbf{k},t)\dot{G}^2(\mathbf{k},t) - 2[\mathbf{k}^2 + m_V^2(t) + \tilde{m}^2(t)]G(\mathbf{k},t). \tag{33}$$

Note that near $y=1$, $D \sim 1/(4\sqrt{y-1})$ diverges. Therefore $q^2\dot{\eta}$ becomes almost constant in this region. Then, the driving term of q in the right-hand side of Eq. (32) becomes divergent as $1/q^3$ for small q. This may enable q to take over the low potential wall located between $q=0$ and the true vacuum in the effective potential in Fig. 1.

We have thus shown that for the initial states, which belong to the Fock space built on the vacuum, the time-dependent variational equation is made finite by the static renormalization used in the vacuum sector in (2+1) dimensions. The present method naturally incorporates the quantum mechanical correction by y to the classical potential which is given by $y=1$. A notable conceptual difference coming from the quantum treatment of the zero mode of the (2+1)-dimensional scalar field theory is that a Gaussian wave packet is not a minimum energy packet even for an unbroken nondegenerated vacuum. From Eq. (14), we see that y is stable only at $y=1$, the two deltalike packets limit. In Eq. (32) y plays a dual role as a potential which rolls down q for $y>y_c$, then, as an external force which boosts q to a larger value for $y<y_c$. This quantum correction from y contributes to G only indirectly through q.

IV. EFFECTIVE ACTION IN (3+1) DIMENSIONS

In (3+1) dimensions, the situation is quite different from the (2+1)-dimensional case. This is because of the renormalization condition, which demands $\lambda \to 0_-$ which in turn leads to the conclusion that there is no phase transition in the Gaussian effective potential of the four-dimensional scalar ϕ^4 theory. At the present case, if $\lambda \to 0_-$, the effective potential loses the y dependent term and y dynamics decouples from the rest of the effective potential. If we start from the effective action, however, the y dynamics leads to a repulsive potential proportional to $1/q^2$, which may give rise to the symmetry breaking form of the potential even though some interpretational difficulties remain.

A. Effective potential in the standard method

Let us consider the effective potential first. Since the present form for G in Eq. (19) is the same as that of the Gaussian approximation with a slight modification that q^2 is now the expectation value of dispersion, the renormalized quantities are similarly defined as in Ref. [4]:

$$\frac{\mu_R^2}{\lambda_R} = \frac{\mu^2}{\lambda} + \frac{1}{2}I_1, \quad I_1 \equiv \int_k \frac{1}{2k}, \tag{34}$$

$$\frac{1}{\lambda_R} = \frac{1}{\lambda} + \frac{1}{2}I_2(M),$$

$$I_2(M) \equiv \frac{1}{M^2}\int_k \left[\frac{1}{2k} - \frac{1}{2(k^2+M^2)^{1/2}}\right], \tag{35}$$

where M is an arbitrary mass scale, at which the renormalization is performed. The gap equation (19) becomes a kind of mass renormalization formula,

$$m^2 = \mu_R^2 + \frac{\lambda_R}{2}q^2 + \frac{\lambda_R}{2}m^2[I_2(M) - I_2(m)] = \mu_R^2 + \frac{\lambda_R}{2}q^2 + \frac{\lambda_R}{32\pi^2}m^2 \ln\frac{m^2}{M^2}. \tag{36}$$

We obtain a finite expression of the effective potential:

$$V(y,q) = \frac{m^4 - \mu_R^4}{2\lambda_R} + \frac{m^4}{64\pi^2}\left(\ln\frac{M^2}{m^2} - \frac{1}{2}\right) + \frac{y-2}{24}\lambda q^4, \tag{37}$$

where we have adjusted a constant $-\mu^4/2\lambda$ in the limit of infinite cutoff, $\lambda \to 0_-$. At $y=1$, this effective potential (37) and the gap equation (36) reproduce the results of Pi and Samiullah [4] and Stevenson [1], where it was shown that the scalar ϕ^4 model does not have symmetry breaking in (3+1) dimensions in the Gaussian approximation. As the coupling λ goes to zero, the y dependence in the quartic term disappears. We interpret this potential as the effective potential for q. The explicit form of the effective potential with respect to q can be obtained by replacing m^2 in Eq. (37) by using Eq. (36).

B. Renormalization of the effective action in 3+1 dimensions

The conditions for the initial states [4], in (3+1) dimensions, are given by

$$\lim_{k\to\infty} G(k,0)=\frac{1}{2k}\left(1-\frac{\overline{m}^2}{2k^2}+\frac{g\cos\alpha(k)}{k^2}\right),$$

$$\lim_{k\to\infty}\dot{G}(k,0)=\frac{A+B\cos\beta(k)}{k^2}, \tag{38}$$

with nonoscillatory α and β and k-independent constants g, A, B, and $\overline{m}$, the last being a mass parameter that we shall specify shortly.

We generalize the renormalization condition to the time dependent μ^2 as follows:

$$\frac{\mu_R^2(t)}{\lambda_R}=\frac{\mu^2(t)}{\lambda}+\frac{1}{2}I_1, \tag{39}$$

$$\frac{1}{\lambda_R}=\frac{1}{\lambda}+\frac{1}{2}I_2(M). \tag{40}$$

This renormalization condition implies $\lambda\to 0_-$. Therefore, the first equation in (14) for y gives $q^2\dot{\eta}=C$, a constant of motion. With this condition, we remove the y dependence and write the second equation of (14) as

$$\ddot{q}+m^2(t)q-\frac{C^2}{q^3}=0. \tag{41}$$

Then, the effective action (12) we are to solve becomes

$$\Gamma'[q,G,\Pi]=\int dtd^3x\Bigg\{\left[\frac{\dot{q}^2}{2}-\frac{C^2}{2q^2}-\frac{1}{2}\mu^2q^2\right]+\int_{\mathbf{k}}\Bigg[\Pi(\mathbf{k},t)\dot{G}(\mathbf{k},t)-2\Pi^2(\mathbf{k},t)G(\mathbf{k},t)-\frac{1}{8}G^{-1}(\mathbf{k},t)-\frac{1}{2}(\mathbf{k}^2+V^{(2)})G(\mathbf{k},t)-\frac{\lambda}{8}G(\mathbf{k},t)\int_{\mathbf{q}}G(\mathbf{q},t)\Bigg]\Bigg\}. \tag{42}$$

In the present approach, $q^2(t)=\langle\phi^2(t)\rangle$ cannot vanish if $C\neq 0$, and should be dynamical. This fact gives a slight difference to the renormalization procedure from that of Ref. [4]. The finite time-dependent mass becomes

$$m^2(t)=\left[\mu^2(t)+\frac{\lambda}{2}q^2(t)+\frac{\lambda}{2}I_0(m_V(t))\right]+\frac{\lambda}{2}\int_k[G(k,t)-G_V(k,t)]=m_V^2(t)+\tilde{m}^2(t), \tag{43}$$

therefore

$$m_V^2(t)\equiv\mu^2(t)+\frac{\lambda}{2}q^2(t)+\frac{\lambda}{2}\int_k\frac{1}{2(k^2+m_V^2(t))^{1/2}}=\mu_R^2(t)+\frac{\lambda_R}{2}q^2(t)+\frac{\lambda_R}{32\pi^2}m_V^2(t)\ln\frac{m_V^2(t)}{M^2}, \tag{44}$$

where

$$\tilde{m}^2(t)\equiv\frac{\lambda}{2}\int_k[G(k,t)-G_V(k,t)]=\frac{\lambda_R}{2}\int_k[G(k,t)-G_V(k,t)]+\frac{\lambda_R}{2}\tilde{m}^2(t)I_2(M), \tag{45}$$

with $\mu_R(t)$ becoming independent of time after the symmetry breaking process is over. After rearranging these terms we get

$$m^2(t)=\mu_R^2(t)+\frac{\lambda_R}{2}u^2(t)+\frac{\lambda_R}{2}q^2(t),$$

$$u^2(t)=\frac{1}{16\pi^2}m_V^2(t)\ln\frac{m_V^2(t)}{M^2}+\int_k[G(k,t)-G_V(k,t)]+\tilde{m}^2(t)I_2(M). \tag{46}$$

The finiteness of $u^2(t)$ at $t=0$ gives $\overline{m^2}=m^2(0)=m_V^2(0)+\tilde{m}^2(0)$, where we use the asymptotic form (38) for G at $t=0$. In the time-independent limit, $u^2=(1/16\pi^2)m^2\ln(m^2/M^2)$. The renormalized set of equations of motion is given by

$$\ddot{q}+\left[\mu_R^2(t)+\frac{\lambda_R}{2}u^2(t)\right]q+\frac{\lambda_R}{2}q^3-\frac{C^2}{q^3}=0, \tag{47}$$

$$\ddot{G}=\frac{1}{2}G^{-1}+\frac{1}{2}G^{-1}\dot{G}^2-2\left[k^2+\mu_R^2(t)+\frac{\lambda_R}{2}u^2(t)+\frac{\lambda_R}{2}q^2(t)\right]G. \tag{48}$$

From the equation of motion (47) one finds that the minimum of the potential is given at the point q_v determined by

$$0=\left.\frac{\partial V'}{\partial q}\right|_{q=q_v}=q_v\left[\mu_R^2+\frac{\lambda_R}{2}u^2+\frac{\lambda_R}{2}q_v^2-\frac{C^2}{q_v^4}\right], \tag{49}$$

which signals that $q_v=0$ cannot be a minimum of the potential if $C\neq 0$.

Finally, let us examine the effective potential again after integrating out the equation of motion for y with nonzero C. Let us assume that the system is quasi-static so that G is described by its vacuum value. Since the potential is diver-

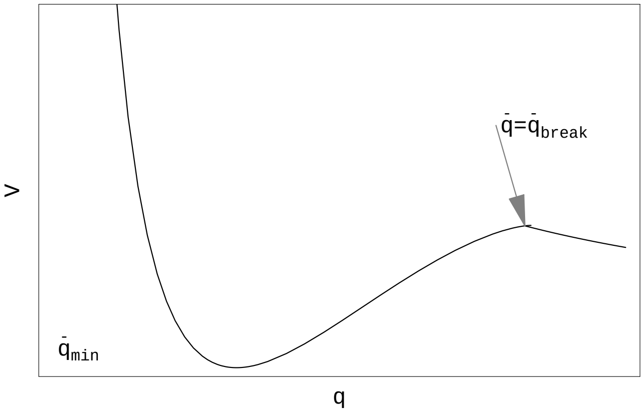

FIG. 2. The effective potential as a function of $\overline{q}$. Details of the analysis for $\overline{q}_{break}$ are the same as [1] and $\overline{q}_{min}=-\Phi_v^2$. At this point $q^2=0$.

gent at $q=0$, we define the renormalized mass at $q=q_v$ where the potential takes the minimum

$$m_R^2=\mu_R^2+\lambda_R/2\, q_v^2+\frac{\lambda_R}{32\pi^2}m_R^2\ln\frac{m_R^2}{M^2}. \tag{50}$$

By setting the arbitrary mass scale $M=m_R$, we get $m_R^2=\mu_R^2+\frac{\lambda_R}{2}q_v^2$. Following the calculation of Stevenson, the mass-square ratio $x=m^2/m_R^2$ is given by

$$\kappa\left(x-1-\frac{\overline{q}}{\kappa}\right)=x\ln x, \tag{51}$$

where $\kappa=32\pi^2/\lambda_R$ and $\overline{q}=16\pi^2[(q^2-q_v^2)/m_R^2]$. The effective potential is, then, given by

$$V'(x,\Phi)=\frac{64\pi^2V_{eff}}{m_R^4}(x,q)=\frac{c^2}{\Phi^2}+\kappa(x^2-1)-x^2\left(\ln x+\frac{1}{2}\right), \tag{52}$$

where $\Phi^2=16\pi^2q^2/m_R^2$ and $c=32\pi^2C/m_R^3$. Unlike the previous result (37), this potential in Fig. 2 has the symmetry breaking form, due to the C^2/Φ^2 term, which comes from y dynamics.

The presence of a symmetry breaking potential due to y dynamics appears to be surprising. However, many authors have been searching for a second order phase transition in the scalar ϕ^4 theory in (3+1) dimensions through the study of nonequilibrium dynamics. For example, the spinodal instability leads to a better understanding of the increase in the correlation length leading to the possibility of symmetry breaking [7,14] or the autonomous renormalization with the saddle-point approximation for the zero mode leads to symmetry breaking [13]. In this sense, the presence of the symmetry breaking potential is not surprising, but more careful study is needed. The reason is that our wave-functional ansatz (6) cannot describe regions with $y>3$. Therefore, we cannot predict anything for $y>3$. In this sense, more careful study with the trial wave functional, which includes the region with $y>3$, is needed to clarify the issue of the existence of symmetry breaking in the scalar ϕ^4 theory in (3+1) dimensions. One of methods to consider, $y>3$, is to include the excited states of (6) as suggested in Ref. [12].

V. SUMMARY AND DISCUSSIONS

We have calculated the effective action of a self-interacting scalar field in three and four dimensions with the use of a quartic exponential wave-functional ansatz for the zero mode. In (2+1) dimensions, we have calculated the renormalized effective potential and action. It is shown that the symmetry breaking occurs for $\lambda/m_R>\hat{\lambda}_c$ as in the Gaussian case. The effective potential is dependent on the shape, y, of the wave function of the zero mode, so that there is a critical value, y_c, such that there is no symmetry breaking if $y>y_c$. Especially, if $y=3$ there is no symmetry breaking for any value of λ/m_R. The shape of the symmetry breaking potential has a double well type of the first order transition. If the renormalized mass is defined at finite q_R,

$$m_R^2=\mu^2+\frac{\lambda}{2}q_R^2+\frac{\lambda}{2}\int_{\mathbf{k}}G(\mathbf{k})\bigg|_{q=q_R}, \tag{53}$$

then the ratio of the mass for general q and the renormalized mass is given by

$$\sqrt{x}\equiv\frac{m}{m_R}=-\frac{\lambda}{16\pi m_R}+\left[\left(\frac{\lambda}{16\pi m_R}+1\right)^2+\frac{\lambda}{2m_R^2}(q^2-q_R^2)\right]^{1/2}. \tag{54}$$

This definition allows the possibility that $m^2|_{q=0}\leqslant m_R^2$, and $m^2|_{q=q_{min}}=0$ at

$$\frac{q_{\min}^2}{m_R}=\frac{q_R^2}{m_R}-\frac{2m_R}{\lambda}-\frac{1}{4\pi}. \tag{55}$$

If we set $q_{\min}=0$, we get the second order transition and $(\lambda/2)q_R^2=m_R^2+\lambda m_R/(8\pi)$. The effective potential in (2+1) dimensions in this case becomes

$$V(q,y)=-\frac{\lambda m_R}{16\pi}q^2+\frac{\lambda}{4!}yq^4-\frac{m_R^3}{24\pi}(\sqrt{x}-1)^2\times\left[2\sqrt{x}+1+\frac{3\lambda}{16\pi m_R}\right]. \tag{56}$$

This is a new possibility for the (2+1) dimensional ϕ^4 scalar field theory since the center $q=0$ of this potential is an unstable maxima now.

We also have calculated the renormalized effective action $\Gamma[q(t),y(t);G,\Pi]$ and then the effective potential $V(q,y)$ as its static limit in (3+1) dimensions. The y-dependent term in $V(q,y)$ disappears after renormalization due to the renormalization condition $\lambda \rightarrow 0_-$, and it leads to the result that the symmetry breaking does not occur in four-dimensional ϕ^4 theory as in the case of the Gaussian approximation.

Since y dependence decouples from the rest during the renormalization process, one can integrate out the y dynamics from the effective action $\Gamma[q(t),y(t);G,\Pi]$ to obtain a new effective action $\Gamma'[q(t);G,\Pi]$ before one takes the static limit. The new effective potential $V'(q)$ obtained by the static limit of $\Gamma'[q(t);G,\Pi]$ has the symmetry breaking form. To take this result seriously, however, we need more careful study since in our trial wave functional (6) y cannot take values larger than 3.

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