Transport coefficients from the two particle irreducible effective action

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We show that the lowest nontrivial truncation of the two-particle irreducible (2PI) effective action correctly determines transport coefficients in a weak coupling or $1/N$ expansion at leading (logarithmic) order in several relativistic field theories. In particular, we consider a single real scalar field with cubic and quartic interactions in the loop expansion, the $O(N)$ model in the 2PI-$1/N$ expansion, and QED with single and many fermion fields. Therefore, these truncations will provide a correct description, to leading (logarithmic) order, of the long time behavior of these systems, i.e. the approach to equilibrium. This supports the promising results obtained for the dynamics of quantum fields out of equilibrium using 2PI effective action techniques.

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INTRODUCTION

Recent developments in heavy-ion collisions and cosmology have spurred the theoretical understanding of the dynamics of quantum fields out of equilibrium. In particular, the thermalization of quantum fields is a subject of both fundamental and practical relevance. For quantum fields far from equilibrium, promising results have been obtained from a systematic use of the two-particle irreducible (2PI) effective action [1], formulated along the Schwinger-Keldysh contour. While the basic formulation of this approach is well known [2] (see Refs. [3] for recent applications in equilibrium), the recent progress has been the numerical solution of the resulting evolution equations for the (one- and) two-point functions without any further approximation. This allows one to go far from equilibrium and describe, e.g., the emergence of quasiparticles in a completely self-consistent way. This program has been carried out for a single scalar field with quartic self-interactions using a three-loop expansion in 1+1 dimensions [4], for the $O(N)$ model using the 2PI–$1/N$ expansion [5] in 1+1 [6,7] and 3+1 dimensions [8], and recently also for a chirally invariant Yukawa model in 3+1 dimensions [9]. The (mostly numerical) results obtained so far suggest that the lowest nontrivial truncation beyond the mean-field approximation is necessary and sufficient to describe in one formalism both the dynamics far from equilibrium as well as the subsequent equilibration.

A necessary requirement for any method to successfully describe nonequilibrium field dynamics is that it encompasses the correct long-time behavior. Close to equilibrium, the evolution of the system on long time and length scales is characterized by transport coefficients, which have been computed at leading order in a weak coupling or $1/N$ expansion [10–12]. In order to assess the validity of truncations of the 2PI effective action, it is therefore crucial that transport coefficients obtained within the 2PI formalism agree, in the weak coupling limit, with those results.

In this paper we show how the calculation of transport coefficients is organized in the framework of the 2PI effective action. We then consider a variety of theories and show, by comparing with results obtained previously, that truncations currently used in far-from-equilibrium applications include in the weak coupling or large N limit the appropriate diagrams to yield the correct result for transport coefficients to leading (logarithmic) order. This result provides strong support for the applicability of truncations of the 2PI effective action to describe the equilibration of quantum fields.

2PI EFFECTIVE ACTION

The Kubo formula relates transport coefficients to the expectation value of appropriate composite operators in thermal equilibrium (to be precise, to the imaginary part of retarded correlators at zero momentum and vanishing frequency). The correlator we will be interested in here is therefore of the form

$$\langle O(x)O(y)\rangle-\langle O(x)\rangle\langle O(y)\rangle, \tag{1}$$

with O an operator bilinear in the fundamental fields. For example, for the shear viscosity $O(x)=\pi_{ij}(x)$, where for a real scalar field $\pi_{ij}=\partial_i\phi\partial_j\phi-\frac{1}{3}\delta_{ij}\partial_k\phi\partial_k\phi$, and for the electrical conductivity $O(x)=j^i(x)$, with $j^i=\bar{\psi}\gamma^i\psi$.

We first demonstrate that the 2PI effective action generates precisely correlators of the form (1). We consider the case of a real field ϕ, coupled to a bilinear source K (the extension to fermions is straightforward). The path integral is

$$Z[K]=e^{iW[K]}=\int \mathcal{D}\phi e^{i[S[\phi]+(1/2)\phi^i K_{ij}\phi^j]}, \tag{2}$$

with S the classical action. (We use a condensed notation where latin indices summarize space-time as well as internal indices, and integration and summation over repeated indices are understood.) For the application we discuss here it is not necessary to couple a source to ϕ itself and we assume throughout that $\langle\phi^i\rangle$ vanishes. The 2PI effective action is defined as the Legendre transform of W, $\Gamma[G]=W[K]$

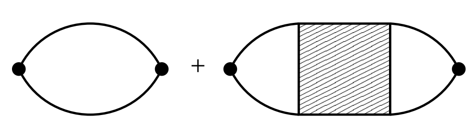

FIG. 1. Expectation value of bilinear operators (black dots) from the 2PI effective action. The shaded square denotes the connected 4-point function (with the external legs amputated).

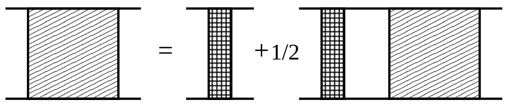

FIG. 2. Integral equation for the 4-point function, derived from the 2PI effective action.

$-\frac{1}{2}G^{ij}K_{ij}$, with $\delta W[K]/\delta K_{ij}=\frac{1}{2}G^{ij}$, and $\delta\Gamma[G]/\delta G^{ij}=-\frac{1}{2}K_{ij}$. The two-point function $G^{ij}=\langle T_{\mathcal{C}}\phi^i\phi^j\rangle$ is the (connected) time-ordered two-point function along the contour $\mathcal{C}$ in the complex-time plane. Our discussion will be general and we do not need to specify the contour here; at any moment one may specialize to the Matsubara contour or the Schwinger-Keldysh contour.

In order to obtain a correlator of the form (1), we differentiate W twice with respect to K:

$$\frac{\delta^2 W[K]}{\delta K_{ij}\delta K_{kl}}=\frac{i}{4}[\langle T_{\mathcal{C}}\phi^i\phi^j\phi^k\phi^l\rangle-G^{ij}G^{kl}]$$

$$=\frac{i}{4}[G_c^{ij;kl}+G^{ik}G^{jl}+G^{il}G^{jk}], \tag{3}$$

where $G_c^{ij;kl}$ is the usual connected 4-point function (semicolons separate indices with a different origin). We note that W does not generate connected Green functions, instead it generates a 4-point function which, after identifying i,j with x and k,l with y, is precisely of the form (1) (see Fig. 1). To proceed further we remove the external legs, $G_c^{ij;kl}=G^{ii'}G^{jj'}G^{kk'}G^{ll'}\Gamma^{(4)}_{i'j';k'l'}$, and concentrate on the 4-point vertex function $\Gamma^{(4)}$. Note that in a theory with cubic interactions $\Gamma^{(4)}$ is not 1-particle irreducible. The vertex function obeys an integral equation that can be obtained using standard functional relations [1]. It reads (see Fig. 2)

$$\Gamma^{(4)}_{ij;kl}=\Lambda_{ij;kl}+\frac{1}{2}\Lambda_{ij;mn}G^{mm'}G^{nn'}\Gamma^{(4)}_{m'n';kl}, \tag{4}$$

where the 4-point kernel follows from

$$\Lambda_{ij;kl}=2\frac{\delta\Sigma_{ij}[G]}{\delta G^{kl}},\quad \Sigma_{ij}=2i\frac{\delta\Gamma_2[G]}{\delta G^{ij}}, \tag{5}$$

when the effective action is written as [1]

$$\Gamma[G]=\frac{i}{2}\mathrm{Tr}\ln G^{-1}+\frac{i}{2}\mathrm{Tr}\,G_0^{-1}(G-G_0)+\Gamma_2[G]. \tag{6}$$

Here, iG_0^{-1} is the free inverse propagator and $\Gamma_2[G]$ is the sum of all 2PI diagrams with no external legs and exact propagators on the internal lines. We emphasize that Eq. (4) is exact and applies out of equilibrium. The first derivative of $\Gamma[G]$ determines the gap equation $G^{-1}=G_0^{-1}-\Sigma$ in the absence of sources.

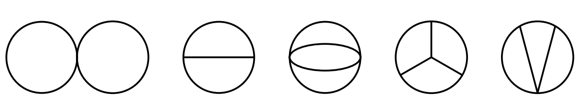

FIG. 3. Contributions to the 2PI effective action in the loop expansion up to three loops in a theory with cubic and quartic interactions.

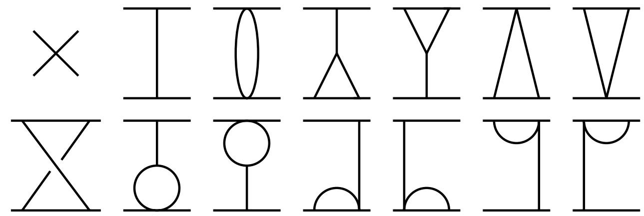

FIG. 4. Zero- and one-loop skeleton kernel from the contributions to the 2PI effective action shown in Fig. 3.

To apply the Kubo formula, we specialize to a system in thermal equilibrium that is invariant under space-time translations. In momentum space, Eq. (4) then reads

$$\Gamma^{(4)}(p,k)=\Lambda(p,k)+\frac{1}{2}\int_q\Lambda(p,q)G^2(q)\Gamma^{(4)}(q,k). \tag{7}$$

The importance of using Eq. (7) in the renormalization of the gap equation has been emphasized recently [13,14]. Equation (7) is valid both in the imaginary time as well as in the real-time formalism, where the 4-point function and kernel have a more complicated tensor structure [15].

We note that the 4-point function appears in the integral equation with a particular momentum configuration: the momentum p (k) enters and leaves on the left (right) and the two intermediate propagators carry the same momentum q. This configuration suffers therefore from pinching poles: when the loop momentum q is nearly on-shell, the product of propagators is potentially very large and all terms in the ladder series may be equally important. This situation is precisely the one that appears in the diagrammatic evaluation of transport coefficients [10]: computing transport coefficients diagrammatically amounts to studying an integral equation of the type (7), specialized to the case that the external momenta p and k as well as the internal momentum q are on-shell [10,15].

LOOP EXPANSION

As a first example, we consider a real scalar field with cubic and quartic interactions,

$$S=\int_x\left[\frac{1}{2}(\partial_\mu\phi)^2-\frac{1}{2}m^2\phi^2-\frac{g}{3!}\phi^3-\frac{\lambda}{4!}\phi^4\right]. \tag{8}$$

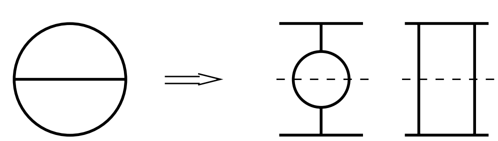

FIG. 5. One-loop perturbative kernel from the 2PI effective action at two-loop order in a theory with cubic interactions. The dashed line indicates how to cut the diagrams to make the connection with the scattering kernel in kinetic theory.

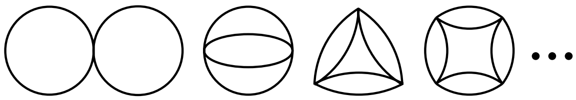

FIG. 6. Contributions to the 2PI effective action in the $O(N)$ model in the 2PI–$1/N$ expansion at LO and NLO. Only the first few diagrams at NLO are shown.

In the loop expansion for $\Gamma_2[G]$, we include terms up to three loops (see Fig. 3). The self-energy Σ is obtained by cutting the diagrams once. Cutting the result once more yields the kernel Λ in the integral equation for the 4-point function (see Fig. 4).

We now show that this truncation includes the physics relevant for the leading-order result of the shear viscosity in the weak-coupling limit, $\lambda \sim (g/m)^2 \ll 1$. In order to do this, it is sufficient to show that Eq. (7) includes all diagrams in the appropriate kinematic configuration which are known to contribute at leading order [10]. We therefore specialize to on-shell momenta p, k, and q, where the leading pinching-pole limit arises. One must then proceed to a careful analysis of all the possible perturbative contributions to the integral equation from the kernel $\Lambda(p,k)$. For example, the single rung with a bare propagator does not contribute straightforwardly due to kinematics, but it does when either it is iterated in the integral equation to get a one-loop kernel or a one-loop self-energy correction is included, as depicted in Fig. 5 (for detailed power-counting arguments, see Ref. [10]). In general, Eq. (7) will contain contributions that are subleading in the weak-coupling limit. For instance, the four final rungs in Fig. 4 can be seen as vertex corrections to the single rung and contribute at subleading order only. When the result of the power-counting analysis of Ref. [10] is carried over to this case, one finds that the leading-order contribution to the kernel can be written as an integral over an effective scattering kernel $|\lambda + g^2[G_R(s)+G_R(t)+G_R(u)]|^2$, where we write s,t,u to indicate the contributions from the three scattering channels and G_R denotes the retarded Green's function. This kernel is the square of the sum of all 2-to-2 processes in a theory with cubic and quartic interactions, which establishes the connection with kinetic theory [10]. It is instructive to see how this result is put together. Consider for a moment only the two-loop diagram with cubic interactions (see Fig. 5). The scattering kernel that arises from this diagram reads explicitly (see, e.g. [15]): $g^4[|G_R(s)|^2+|G_R(t)|^2+|G_R(u)|^2]$, i.e., the sum of the squares of the matrix elements, however, without interference terms. Indeed, it is easy to see that the interference terms originate from the 3-loop diagrams.

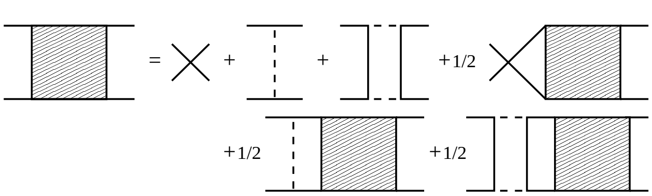

FIG. 7. Integral equation for the 4-point function in the 2PI–$1/N$ expansion of the $O(N)$ model at NLO.

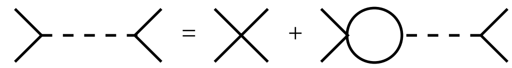

FIG. 8. Integral equation for the auxiliary correlator D.

We conclude therefore that the 3-loop approximation of the 2PI effective action is necessary and sufficient to yield the leading-order result for the shear viscosity in a weakly coupled scalar theory [16].

$O(N)$ MODEL

We now consider a real scalar N-component quantum field ϕ_a $(a=1,\ldots,N)$ with a classical $O(N)$-invariant action and an interaction term $(\lambda/4!N)(\phi_a\phi_a)^2$. Instead of the loop expansion, we will use the 2PI–$1/N$ expansion to next-to-leading order (NLO), which is discussed in detail in Refs. [5,6]. The 2PI part of the effective action can be written as $\Gamma_2[G]=\Gamma_2^{\rm LO}[G]+\Gamma_2^{\rm NLO}[G]+\ldots$, with (see Fig. 6)

$$\Gamma_2^{\rm LO}[G] = -\frac{\lambda}{4!N}\int_x G_{aa}(x,x)G_{bb}(x,x), \tag{9}$$

$$\Gamma_2^{\rm NLO}[G] = \frac{i}{2}\,{\rm Tr}\ln \mathbf{B}(G), \tag{10}$$

where $\mathbf{B}(x,y;G)=\delta_C(x-y)+(i\lambda/6N)G_{ab}(x,y)G_{ab}(x,y)$ sums bubbles (which can be seen by re-expanding the logarithm). In this case the kernel reads (see Fig. 7)

$$\Lambda^{\rm LO}_{ab;cd}(p,k) = -\frac{i\lambda}{3N}\delta_{ab}\delta_{cd}, \tag{11}$$

$$\Lambda^{\rm NLO}_{ab;cd}(p,k) = -[\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc}]D(p-k) + 2\int_q G_{ab}(p-q)D^2(q)G_{cd}(k-q), \tag{12}$$

where we used the auxiliary correlator $D=(i\lambda/3N)\mathbf{B}^{-1}$ to sum the chain of bubbles (see Fig. 8): $D(p)=(i\lambda/3N)[1+\Pi(p)D(p)]$, with $\Pi(p)=-\frac{1}{2}\int_q G_{ab}(q)G_{ab}(p+q)$ [5].

The advantage of the large N expansion employed here is that it allows for a computation of the shear viscosity to first nontrivial order in the $1/N$ expansion *without* a restriction to small λ. This is especially pressing for applications of the $O(4)$ model to QCD phenomenology in which the coupling constant λ has to be taken large. As far as we know, such a calculation of the shear viscosity has not yet been performed.

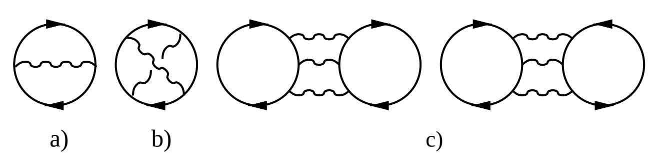

FIG. 9. Contributions to the 2PI effective action in QED in the loop expansion with (a) 2 and (b) 3 loops or in the 2PI-$1/N$ expansion at (a) NLO and (b,c) NNLO.

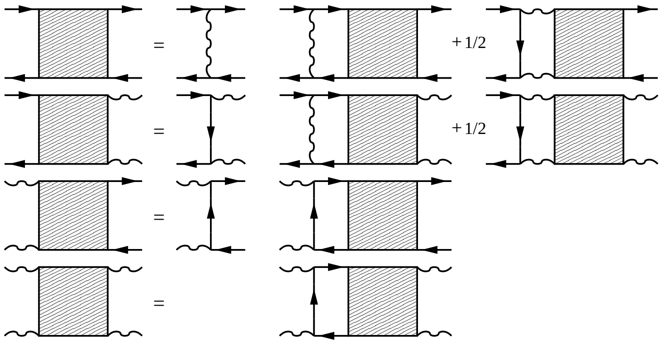

FIG. 10. Integral equations for the 4-point functions in QED at 2-loop order in the loop or at NLO in the $1/N$ expansion.

The integral equation derived here provides a convenient starting point for that and work in this direction is currently underway [17].

QED

As a last example we consider QED. Although for gauge theories there are a number of issues related to the gauge symmetry (dependence on the gauge-fixing parameter [18], Ward identities [19]) which have not been resolved completely, we think it is nevertheless worth analyzing to what accuracy transport coefficients can be expected to be computed within a specific truncation of the 2PI effective action.

In the loop expansion (see Fig. 9), we restrict the discussion to the 2-loop approximation. The corresponding integral equations are shown in Fig. 10. In order to analyze what processes are included in the weak-coupling limit, we follow the strategy of the scalar theory and rewrite the integral equation such that the external momenta are on-shell. The possible one-loop contributions to the kernel are shown in Fig. 11. This kernel contains the sum of the squares of all 2-to-2 processes, but not the interference terms. Indeed, it is the 3-loop diagram that is responsible for interference. Using the results from the diagrammatic analysis [20], we find that, in order to determine the shear viscosity and the electrical conductivity to leading-logarithmic order, only the first two rungs in Fig. 11 need to be considered, with the momentum flowing through the rung being soft and below the light cone. We conclude that the integral equations in the 2-loop approximation sum the necessary diagrams [20,21] to obtain the shear viscosity and electrical conductivity in QED to leading-logarithmic order.

Finally, we consider QED with N identical fermions and a rescaled coupling $e^2 \rightarrow e^2/N$ in the large N limit (see Fig. 9). We consider only the NLO contribution. In this case the kernel is dominated by the first and the fourth diagrams in Fig. 11, the other contributions are suppressed by $1/N$. The photon propagator still resums fermion bubbles and is in fact similar to the auxiliary correlator in the $O(N)$ model. Cutting the first and fourth diagrams as in Fig. 5 shows that they correspond to Coulomb scattering in all three channels, which are indeed [12] the dominant processes in a kinetic theory with many fermions. We conclude that the 2PI$-1/N$ expansion in QED with N fermions at NLO would yield the correct leading-order result for the shear viscosity and electrical conductivity.

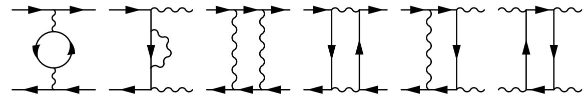

FIG. 11. One-loop perturbative kernel in the 2-loop approximation in QED. In the $1/N$ expansion, the first and the fourth diagrams dominate.

SUMMARY

We have shown how the calculation of transport coefficients is organized in the framework of the 2PI effective action. For a variety of models, we have discussed the first nontrivial truncations in a weak coupling or $1/N$ expansion and found that these truncations yield ladder integral equations with the particular kinematic configuration and rungs appropriate to obtain transport coefficients at leading (logarithmic) order in the weak coupling or $1/N$ expansion. This formulation offers therefore a systematic starting point for the derivation of these integral equations. Methods for the actual solution of the integral equations (a topic not discussed here) can be found in the literature.

In the wider context of nonequilibrium quantum field theory, our findings provide theoretical support for the successful description of quantum fields out of equilibrium using truncations of the 2PI effective action, which are currently actively under investigation. We would like to emphasize that any scheme which aspires to properly describe the nonequilibrium evolution and ensuing thermalization of a system has to render transport coefficients correctly.

ACKNOWLEDGMENTS

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