Gauge equivalence in QCD: The Weyl and Coulomb gauges

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The Weyl-gauge ($A_0^a=0$) QCD Hamiltonian is unitarily transformed to a representation in which it is expressed entirely in terms of gauge-invariant quark and gluon fields. In a subspace of gauge-invariant states we have constructed that implement the non-Abelian Gauss's law, this unitarily transformed Weyl-gauge Hamiltonian can be further transformed and, under appropriate circumstances, can be identified with the QCD Hamiltonian in the Coulomb gauge. We demonstrate an isomorphism that materially facilitates the application of this Hamiltonian to a variety of physical processes, including the evaluation of *S*-matrix elements. This isomorphism relates the gauge-invariant representation of the Hamiltonian and the required set of gauge-invariant states to a Hamiltonian of the same functional form but dependent on ordinary unconstrained Weyl-gauge fields operating within a space of ''standard'' perturbative states. The fact that the gauge-invariant chromoelectric field is not Hermitian has important implications for the functional form of the Hamiltonian finally obtained. When this non-Hermiticity is taken into account, the ''extra'' vertices in the Christ-Lee' Coulomb-gauge Hamiltonian are natural outgrowths of the formalism. When this non-Hermiticity is neglected, the Hamiltonian used in the earlier work of Gribov and others results.

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I. INTRODUCTION

In earlier work on QCD in the Weyl gauge ($A_0^a=0$), we have constructed gauge-invariant operator-valued quark and gluon fields [1]; these include the gauge-invariant quark field

$$\psi_{\mathsf{GI}}(\mathbf{r})=V_{\mathcal{C}}(\mathbf{r})\psi(\mathbf{r}) \quad \text{and} \quad \psi^\dagger_{\mathsf{GI}}(\mathbf{r})=\psi^\dagger(\mathbf{r})V_{\mathcal{C}}^{-1}(\mathbf{r}), \tag{1}$$

where

$$V_{\mathcal{C}}(\mathbf{r})=\exp\left(-ig\overline{\mathcal{Y}^\alpha}(\mathbf{r})\frac{\lambda^\alpha}{2}\right)\exp\left(-ig\mathcal{X}^\beta(\mathbf{r})\frac{\lambda^\beta}{2}\right), \tag{2}$$

$$V_{\mathcal{C}}^{-1}(\mathbf{r})=\exp\left(ig\mathcal{X}^\beta(\mathbf{r})\frac{\lambda^\beta}{2}\right)\exp\left(ig\overline{\mathcal{Y}^\alpha}(\mathbf{r})\frac{\lambda^\alpha}{2}\right), \tag{3}$$

and where the λ^a designate the Gell-Mann matrices. In these expressions $\mathcal{X}^\alpha(\mathbf{r})=[(\partial_j/\partial^2)A_j^\alpha(\mathbf{r})]$, so that $\partial_i\mathcal{X}^\alpha(\mathbf{r})$ is the ith component of the longitudinal gauge field [2] and $\overline{\mathcal{Y}^\alpha}(\mathbf{r})$ is defined as $\overline{\mathcal{Y}^\alpha}(\mathbf{r})=[(\partial_j/\partial^2)\overline{\mathcal{A}_j^\alpha}(\mathbf{r})]$. $\overline{\mathcal{A}_j^\alpha}(\mathbf{r})$, which we refer to as the ''resolvent field,'' is an operator-valued functional of the gauge field, and is represented in Refs. [1] and [3] as the solution of an integral equation. Constructing a gauge-invariant quark field by attaching $V_{\mathcal{C}}(\mathbf{r})$ to the quark field ψ represents an extension, into the non-Abelian domain, of a method of creating gauge-invariant charged fields originated by Dirac for QED [4]; and, like Dirac's procedure, this non-Abelian construction is free of path-dependent integrals. An explicit demonstration that $\psi_{\mathsf{GI}}(\mathbf{r})$ is invariant to non-Abelian gauge transformations has been given by implementing gauge transformations with the generator $\exp\{-i\int d\mathbf{y}\hat{\mathcal{G}}^a(\mathbf{y})\omega^a(\mathbf{y})\}$, where $\hat{\mathcal{G}}^a$ is the non-Abelian ''Gauss's law operator''

$$\hat{\mathcal{G}}^a=\partial_i\Pi_i^a+gf^{abc}A_i^b\Pi_i^c+g\psi^\dagger\frac{\lambda^a}{2}\psi, \tag{4}$$

and ω^a is a number-valued gauge function. With the use of this generator, under which

$$\psi(\mathbf{r})\to\psi'(\mathbf{r})=\exp\left(-i\omega^\alpha(\mathbf{r})\frac{\lambda^\alpha}{2}\right)\psi(\mathbf{r}) \tag{5}$$

and

$$A_i^b(\mathbf{r})\frac{\lambda^b}{2}\to\exp\left(-i\omega^\alpha(\mathbf{r})\frac{\lambda^\alpha}{2}\right)\left(A_i^b(\mathbf{r})\frac{\lambda^b}{2}+\frac{i}{g}\partial_i\right)\times\exp\left(i\omega^\alpha(\mathbf{r})\frac{\lambda^\alpha}{2}\right), \tag{6}$$

it has been shown that $V_{\mathcal{C}}(\mathbf{r})$ also gauge transforms as

$$V_{\mathcal{C}}(\mathbf{r})\to V_{\mathcal{C}}(\mathbf{r})\exp\left(i\omega^\alpha(\mathbf{r})\frac{\lambda^\alpha}{2}\right),$$

$$V_{\mathcal{C}}^{-1}(\mathbf{r})\to\exp\left(-i\omega^\alpha(\mathbf{r})\frac{\lambda^\alpha}{2}\right)V_{\mathcal{C}}^{-1}(\mathbf{r}) \tag{7}$$

so that $\psi_{\mathsf{GI}}(\mathbf{r})$ remains gauge invariant [1]. The resolvent field $\overline{\mathcal{A}_j^b}$ also has an important role in the gauge-invariant gauge field

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$$\mathsf{A}_{\mathsf{GI}i}(\mathbf{r}) = \left[A^b_{\mathsf{GI}i}(\mathbf{r})\frac{\lambda^b}{2}\right] = V_{\mathcal{C}}(\mathbf{r})\left[A^b_i(\mathbf{r})\frac{\lambda^b}{2}\right]V_{\mathcal{C}}^{-1}(\mathbf{r}) + \frac{i}{g}V_{\mathcal{C}}(\mathbf{r})\partial_i V_{\mathcal{C}}^{-1}(\mathbf{r}), \tag{8}$$

which can be shown to be the transverse field [1]

$$A^b_{\mathsf{GI}i}(\mathbf{r}) = A^{b\mathsf{T}}_i(\mathbf{r}) + \left[\delta_{ij} - \frac{\partial_i\partial_j}{\partial^2}\right]\overline{\mathcal{A}^b_j}(\mathbf{r}) = \left[\delta_{ij} - \frac{\partial_i\partial_j}{\partial^2}\right][A^b_j(\mathbf{r}) + \overline{\mathcal{A}^b_j}(\mathbf{r})]. \tag{9}$$

Equation (8), as well as the fact that $A^b_{\mathsf{GI}i}(\mathbf{r})$ and $\hat{\mathcal{G}}^c(\mathbf{x})$ commute, demonstrate that $A^b_{\mathsf{GI}i}(\mathbf{r})$ is gauge invariant—more precisely, invariant to ‘‘small’’ gauge transformations. We can also define a gauge-invariant chromoelectric field $E^a_{\mathsf{GI}i} = -\Pi^a_{\mathsf{GI}i}$ [5]. A natural definition of $\Pi^a_{\mathsf{GI}i}$ in this formulation is

$$\Pi_{\mathsf{GI}i}(\mathbf{r}) = \left[\Pi^b_{\mathsf{GI}i}(\mathbf{r})\frac{\lambda^b}{2}\right] = V_{\mathcal{C}}(\mathbf{r})\frac{\lambda^b}{2}V_{\mathcal{C}}^{-1}(\mathbf{r})\Pi^b_i(\mathbf{r}) \tag{10}$$

or, equivalently,

$$\Pi^a_{\mathsf{GI}i} = R_{ab}\Pi^b_i \quad \text{where} \quad R_{ab} = \frac{1}{2}\mathsf{Tr}[\lambda^a V_{\mathcal{C}}\lambda^b V_{\mathcal{C}}^{-1}], \tag{11}$$

where Π^a_i is the momentum conjugate to the gauge field A^a_i in the Weyl gauge. With the use of the commutator

$$[\hat{\mathcal{G}}^c(\mathbf{x}), \mathcal{R}_{ab}(\mathbf{y})] = igf^{cbq}\mathcal{R}_{aq}(\mathbf{y})\delta(\mathbf{x}-\mathbf{y}), \tag{12}$$

obtained in Ref. [5], it is easy to verify that $\Pi^a_{\mathsf{GI}i}(\mathbf{y})$ commutes with $\hat{\mathcal{G}}^c(\mathbf{x})$ and therefore also is gauge invariant.

In this work we will use a representation, which we discuss in Sec. II, in which the Weyl-gauge QCD Hamiltonian is expressed entirely in terms of gauge-invariant fields. Since the gauge-invariant gauge field is transverse, it is of interest to relate this gauge-invariant formulation to the Coulomb gauge. We address this question in Sec. II B. In Sec. II we also show that the Weyl-gauge QCD Hamiltonian in this representation—in which all operator-valued fields are gauge-invariant—must be applied to a set of gauge-invariant states that are solutions of the non-Abelian Gauss's law. In Sec. III, we address the problem that these states, which solve Gauss's law in QCD, are complicated constructions that are difficult to use. We demonstrate an isomorphism in this section between this Hamiltonian, which operates on gauge-invariant states, and a corresponding Hamiltonian that is a functional of gauge-dependent Weyl-gauge fields and that operates on a set of ‘‘standard’’ perturbative states. Also, in Sec. III, we relate these Hamiltonians to those obtained from Coulomb-gauge formulations of QCD. We discuss the implications of our work in Sec. IV.

II. RELATION OF THE GAUGE-INVARIANT REPRESENTATION OF THE WEYL GAUGE TO THE COULOMB GAUGE

The QCD Hamiltonian in the Weyl gauge has been expressed in terms of gauge-invariant operator-valued fields [5,6]. In this work, extensive use has been made of the unitary equivalence of $\hat{\mathcal{G}}^a$—the ‘‘Gauss's law operator’’ given in Eq. (4), which imposes the non-Abelian Gauss's law—to the ‘‘pure glue’’ version of that operator

$$\mathcal{G}^a = \partial_i\Pi^a_i + gf^{abc}A^b_i\Pi^c_i \tag{13}$$

as shown by

$$\mathcal{G}^a = \mathcal{U}_C^{-1}\hat{\mathcal{G}}^a\mathcal{U}_C, \tag{14}$$

where

$$\mathcal{U}_C = \exp\left[i\int d\mathbf{r}\mathcal{X}^a(\mathbf{r})j^a_0(\mathbf{r})\right]\exp\left[i\int d\mathbf{r}'\overline{\mathcal{Y}^c}(\mathbf{r}')j^c_0(\mathbf{r}')\right]. \tag{15}$$

This unitary equivalence has been used to establish a new representation—the $\mathcal{N}$ representation in which $\mathcal{G}^a$ represents the complete Gauss's law operator $\hat{\mathcal{G}}^a$, and ψ represents the gauge-invariant quark field because it commutes with $\mathcal{G}^a$. The $\mathcal{N}$ representation is unitarily equivalent to the $\mathcal{C}$ representation in which $\hat{\mathcal{G}}^a$ and ψ_{GI} designate the Gauss's law operator and the gauge-invariant spinor (quark) field, respectively. In the $\mathcal{N}$ representation, $j^a_0(\mathbf{r}) = g\psi^\dagger(\mathbf{r})(\lambda^a/2)\psi(\mathbf{r})$ and $j^a_i(\mathbf{r}) = g\psi^\dagger(\mathbf{r})\alpha_i(\lambda^a/2)\psi(\mathbf{r})$ are the gauge-invariant quark color charge and quark color current densities, respectively.

The Weyl-gauge QCD Hamiltonian can be transformed from its familiar $\mathcal{C}$ representation form

$$H = \int d\mathbf{r}\left\{\frac{1}{2}\Pi^a_i(\mathbf{r})\Pi^a_i(\mathbf{r}) + \frac{1}{4}F^a_{ij}(\mathbf{r})F^a_{ij}(\mathbf{r}) + \psi^\dagger(\mathbf{r})\left[\beta m - i\alpha_i\left(\partial_i - igA^a_i(\mathbf{r})\frac{\lambda^\alpha}{2}\right)\right]\psi(\mathbf{r})\right\} \tag{16}$$

to the $\mathcal{N}$ representation, as shown by

$$\hat{H}_{\mathsf{GI}} = \mathcal{U}_C^{-1}H\mathcal{U}_C. \tag{17}$$

This similarity transformation leaves the gauge field untransformed, but it transforms the quark field and the negative chromoelectric field as shown by [7]

$$\mathcal{U}_C^{-1}(\mathbf{x})\psi(\mathbf{x})\mathcal{U}_C(\mathbf{x}) = V_{\mathcal{C}}^{-1}(\mathbf{x})\psi(\mathbf{x}) \tag{18}$$

and

$$\mathcal{U}_C^{-1}(\mathbf{x})\Pi^a_i(\mathbf{x})\mathcal{U}_C(\mathbf{x}) = \Pi^a_i(\mathbf{x}) - R_{ba}(\mathbf{x})\partial^{(\mathbf{x})}_i \times \int d\mathbf{y}\mathcal{D}^{bc}(\mathbf{x},\mathbf{y})j^c_0(\mathbf{y}). \tag{19}$$

The transformed, $\mathcal{N}$-representation Hamiltonian $\hat{H}$ can be expressed entirely in terms of gauge-invariant variables by making use of the identities $R_{aq}R_{bq}=\delta_{ab}$, $f^{duv}R_{ua}R_{vb}=f^{abq}R_{dq}$, and $\partial_i R_{ba}=-f^{uvb}R_{ua}P_{vi}$ where $P_{vi}=-i\,\mathrm{Tr}[\lambda^v V_{\mathcal{C}}\partial_i V_{\mathcal{C}}^{-1}]$. The QCD Hamiltonian in the $\mathcal{N}$ representation, expressed in terms of gauge-invariant fields, is

$$\hat{H}_{\mathrm{GI}}=\int d\mathbf{r}\bigg[\frac{1}{2}\Pi^{a\dagger}_{\mathrm{GI}i}(\mathbf{r})\Pi^{a}_{\mathrm{GI}i}(\mathbf{r})+\frac{1}{4}F^{a}_{\mathrm{GI}ij}(\mathbf{r})F^{a}_{\mathrm{GI}ij}(\mathbf{r})-\psi^{\dagger}(\mathbf{r})\times(\beta m-i\alpha_i\partial_i)\psi(\mathbf{r})\bigg]+\frac{1}{2}\int d\mathbf{x}d\mathbf{y}[J^{a\dagger}_{0(\mathrm{GI})}(\mathbf{x})\times\mathcal{D}^{ab}(\mathbf{x},\mathbf{y})j^{b}_{0}(\mathbf{y})+j^{b}_{0}(\mathbf{y})\mathcal{D}^{ba}(\mathbf{y},\mathbf{x})J^{a}_{0(\mathrm{GI})}(\mathbf{x})]-\frac{1}{2}\int d\mathbf{r}d\mathbf{x}d\mathbf{y}j^{c}_{0}(\mathbf{y})\mathcal{D}^{ca}(\mathbf{y},\mathbf{r})\partial^{2}_{(\mathbf{r})}\mathcal{D}^{ab}(\mathbf{r},\mathbf{x})j^{b}_{0}(\mathbf{x})-\int d\mathbf{r}j^{a}_{i}(\mathbf{r})A^{a}_{\mathrm{GI}i}(\mathbf{r})+H_{\mathcal{G}}. \tag{20}$$

where

$$F^{a}_{\mathrm{GI}ij}(\mathbf{r})=\partial_j A^{a}_{\mathrm{GI}i}(\mathbf{r})-\partial_i A^{a}_{\mathrm{GI}j}(\mathbf{r})-gf^{abc}A^{b}_{\mathrm{GI}i}(\mathbf{r})A^{c}_{\mathrm{GI}j}(\mathbf{r}), \tag{21}$$

from which it follows that

$$F^{a}_{\mathrm{GI}ij}(\mathbf{r})=R_{aq}(\mathbf{r})F^{q}_{ij}(\mathbf{r}). \tag{22}$$

Because $\hat{H}_{\mathrm{GI}}$ is in the $\mathcal{N}$ representation, ψ and $\psi^{\dagger}$ denote the gauge-invariant quark fields. $\mathcal{D}^{ab}(\mathbf{x},\mathbf{y})$ is the inverse Faddeev-Popov operator, which we will discuss in Sec. II A, and $J^{a}_{0(\mathrm{GI})}(\mathbf{r})$ is the gauge-invariant gluon color charge density, defined as

$$J^{a}_{0(\mathrm{GI})}(\mathbf{r})=gf^{abc}A^{b}_{\mathrm{GI}i}(\mathbf{r})\Pi^{c}_{\mathrm{GI}i}(\mathbf{r}). \tag{23}$$

Although $\hat{H}_{\mathrm{GI}}$ is Hermitian, $\Pi^{a}_{\mathrm{GI}i}$ is not, because, as can be seen from Eq. (11), $\Pi^{a\dagger}_{\mathrm{GI}i}=\Pi^{b}_{i}R_{ab}$, and Π^{b}_{i} does not commute with R_{ab}. Similarly, $J^{a}_{0(\mathrm{GI})}$ is not Hermitian, and $J^{a\dagger}_{0(\mathrm{GI})}=gf^{abc}\Pi^{c\dagger}_{\mathrm{GI}i}A^{b}_{\mathrm{GI}i}$. The last part of the QCD Hamiltonian is

$$H_{\mathcal{G}}=-\frac{1}{2}\int d\mathbf{x}d\mathbf{y}[\mathcal{G}^{a}_{\mathrm{GI}}(\mathbf{x})\mathcal{D}^{ab}(\mathbf{x},\mathbf{y})j^{b}_{0}(\mathbf{y})+j^{b}_{0}(\mathbf{y})\mathcal{D}^{ba}(\mathbf{y},\mathbf{x})\mathcal{G}^{a}_{\mathrm{GI}}(\mathbf{x})] \tag{24}$$

where $\mathcal{G}^{a}_{\mathrm{GI}}$ is the gauge-invariant Gauss's law operator [5]

$$\mathcal{G}^{a}_{\mathrm{GI}}=\partial_i\Pi^{a}_{\mathrm{GI}i}+gf^{abc}A^{b}_{\mathrm{GI}i}\Pi^{c}_{\mathrm{GI}i}=R_{ab}\mathcal{G}^{b}$$

which consists solely of gauge-invariant fields, every one of which commutes with $\mathcal{G}^{a}$, the Gauss's law operator in the $\mathcal{N}$ representation; $\mathcal{G}^{a}_{\mathrm{GI}}$ is Hermitian because R_{ab} and $\mathcal{G}^{b}$ commute [5].

Equation (20) resembles the QCD Hamiltonian in the Coulomb gauge. The only direct interaction between color currents j^{a}_{i} and the gauge field involve the transverse current only. The other interactions in which quarks participate are nonlocal, involve the quark color-charge density j^{a}_{0}, and are mediated by Green's functions that are the non-Abelian generalizations of the Abelian ∂^{-2}. These interactions still involve the longitudinal component of the gauge-invariant chromoelectric field, but we will show how this can be eliminated in Sec. II B.

A. The inverse Faddeev-Popov operator

The Faddeev-Popov operator in the gauge-invariant representation of the Weyl gauge is

$$\partial\cdot D^{ab}_{(\mathbf{x})}=\frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_i}\delta_{ab}+gf^{aqb}A^{q}_{\mathrm{GI}i}(\mathbf{x})\right)=\left(\frac{\partial}{\partial x_i}\delta_{ab}+gf^{aqb}A^{q}_{\mathrm{GI}i}(\mathbf{x})\right)\frac{\partial}{\partial x_i}; \tag{25}$$

∂_i and D_i commute because $A^{q}_{\mathrm{GI}i}$ is transverse. The Faddeev-Popov operator has a formal inverse, which can be represented as the series

$$\mathcal{D}^{bh}(\mathbf{y},\mathbf{x})=\sum_{n=0}^{\infty}f^{\vec{\delta}bh}_{(n)}(-1)^{n+1}g^{n}\frac{1}{\partial^{2}}[\mathcal{T}^{\vec{\delta}}_{(n)}(\mathbf{y})\delta(\mathbf{y}-\mathbf{x})], \tag{26}$$

where $f^{\vec{\alpha}bh}_{(n)}$ represents the chain of SU(3) structure constants

$$f^{\vec{\alpha}bh}_{(n)}=f^{\alpha_1 bs_1}f^{s_1\alpha_2 s_2}f^{s_2\alpha_3 s_3}\cdots f^{s_{(n-2)}\alpha_{(n-1)}s_{(n-1)}}f^{s_{(n-1)}\alpha_n h}, \tag{27}$$

and where repeated superscripted indices are summed from $1\rightarrow 8$; for $n=1$, the chain reduces to $f^{\vec{\alpha}bh}_{1}=f^{\alpha bh}$; and for $n=0$, $f^{\vec{\alpha}bh}_{0}=-\delta_{bh}$. $\mathcal{T}^{\vec{\alpha}}_{(n)}(\mathbf{r})j^{h}_{0}(\mathbf{r})$ is a special case of a general form $\mathcal{T}^{\vec{\alpha}}_{(n)}(\mathbf{r})\varphi^{h}(\mathbf{r})$ for an arbitrary $\varphi^{h}(\mathbf{r})$ given by

$$\mathcal{T}^{\vec{\alpha}}_{(n)}(\mathbf{r})\varphi^{h}(\mathbf{r})=A^{\alpha(1)}_{\mathrm{GI}j(1)}(\mathbf{r})\frac{\partial_{j(1)}}{\partial^{2}}\bigg\{A^{\alpha(2)}_{\mathrm{GI}j(2)}(\mathbf{r})\frac{\partial_{j(2)}}{\partial^{2}}\times\bigg[\cdots\bigg(A^{\alpha(n)}_{\mathrm{GI}j(n)}(\mathbf{r})\frac{\partial_{j(n)}}{\partial^{2}}(\varphi^{h}(\mathbf{r}))\bigg)\bigg]\bigg\}, \tag{28}$$

with

$$\mathcal{T}^{\vec{\alpha}}_{(0)}(\mathbf{r})\varphi^{h}(\mathbf{r})=\varphi^{h}(\mathbf{r})\quad\text{and}\quad\mathcal{T}^{\vec{\alpha}}_{(1)}(\mathbf{r})\varphi^{h}(\mathbf{r})=A^{\alpha}_{\mathrm{GI}i}(\mathbf{r})\frac{\partial_i}{\partial^{2}}\varphi^{h}(\mathbf{r}). \tag{29}$$

By expanding $\mathcal{D}^{bh}(\mathbf{y},\mathbf{x})$ and combining terms of the same order in g, it can be observed that, as will be proven in Appendix A,

$$\partial\cdot D^{ah}_{(\mathbf{y})}\mathcal{D}^{hb}(\mathbf{y},\mathbf{x})=\delta_{ab}\delta(\mathbf{y}-\mathbf{x}) \tag{30}$$

where $D^{ah}_{i}=\partial_i\delta_{ah}+gf^{a\gamma h}A^{\gamma}_{\mathrm{GI}i}$ and that

$$\mathcal{D}^{bh}(\mathbf{y},\mathbf{x})\partial\cdot\overleftarrow{D^{ha}}_{(\mathbf{x})}=\delta_{ba}\delta(\mathbf{y}-\mathbf{x}), \tag{31}$$

where

$$\overleftarrow{D}_i^{hb}=(\overleftarrow{\partial}_i\delta_{hb}-gf^{hqb}A^q_{\mathsf{GI}i}) \tag{32}$$

and

$$\partial\cdot\overleftarrow{D^{hb}}=(\overleftarrow{\partial}^2\delta_{hb}-gf^{hqb}\overleftarrow{\partial}_iA^q_{\mathsf{GI}i}) \tag{33}$$

and the $\leftarrow$ symbol indicates that ∂^2 and ∂_i differentiate to the left. In demonstrating Eqs. (30) and (31), it can be helpful to use the expanded form of the nth order term of the inverse Faddeev-Popov operator series

$$\mathcal{D}^{ah}_{(n)}(\mathbf{y},\mathbf{x})=g^nf^{\delta_1as_1}f^{s_1\delta_2s_2}\cdots f^{s_{(n-1)}\delta_nh}\times\int\frac{d\mathbf{z}(1)}{4\pi|\mathbf{y}-\mathbf{z}(1)|}A^{\delta_1}_{\mathsf{GI}l_1}(\mathbf{z}(1))\frac{\partial}{\partial z(1)_{l_1}}\times\int\frac{d\mathbf{z}(2)}{4\pi|\mathbf{z}(1)-\mathbf{z}(2)|}A^{\delta_2}_{\mathsf{GI}l_2}(\mathbf{z}(2))\frac{\partial}{\partial z(2)_{l_2}}\cdots\times\int\frac{d\mathbf{z}(n)}{4\pi|\mathbf{z}(n-1)-\mathbf{z}(n)|}A^{\delta_n}_{\mathsf{GI}l_n}(\mathbf{z}(n))\frac{\partial}{\partial z(n)_{l_n}}\times\frac{1}{4\pi|\mathbf{z}(n)-\mathbf{x}|} \tag{34}$$

with

$$\mathcal{D}^{ah}_{(0)}(\mathbf{y},\mathbf{x})=\frac{-\delta_{ah}}{4\pi|\mathbf{y}-\mathbf{x}|} \tag{35}$$

and

$$\mathcal{D}^{ah}_{(1)}(\mathbf{y},\mathbf{x})=gf^{\delta_1ah}\int\frac{d\mathbf{z}}{4\pi|\mathbf{y}-\mathbf{z}|}A^{\delta_1}_{\mathsf{GI}k}(\mathbf{z})\frac{\partial}{\partial z_k}\left(\frac{1}{4\pi|\mathbf{z}-\mathbf{x}|}\right). \tag{36}$$

Integration by parts with respect to the $\mathbf{z}(i)$ and the identity $f^{\vec{\alpha}ah}_{(n)}=(-1)^nf^{\vec{\alpha}ha}_{(n)}$ demonstrate that

$$\mathcal{D}^{ah}(\mathbf{y},\mathbf{x})=\mathcal{D}^{ha}(\mathbf{x},\mathbf{y}). \tag{37}$$

It is apparent from Eqs. (34)–(36) that $\mathcal{D}^{bh}(\mathbf{y},\mathbf{x})$ obeys the integral equation [8]

$$\mathcal{D}^{bh}(\mathbf{y},\mathbf{x})=-\left(\frac{\delta_{bh}}{4\pi|\mathbf{y}-\mathbf{x}|}+gf^{\delta bs}\times\int\frac{d\mathbf{z}}{4\pi|\mathbf{y}-\mathbf{z}|}A^{\delta}_{\mathsf{GI}k}(\mathbf{z})\frac{\partial}{\partial z_k}\mathcal{D}^{sh}(\mathbf{z},\mathbf{x})\right), \tag{38}$$

which has these equations as an iterative solution.

Equation (26) enables us to express the commutator of the gauge-invariant gauge field and the negative gauge-invariant chromoelectric field as

$$[\Pi^b_{\mathsf{GI}j}(\mathbf{y}),A^a_{\mathsf{GI}i}(\mathbf{x})]=-i\left(\delta_{ab}\delta_{ij}\delta(\mathbf{x}-\mathbf{y})+\frac{\partial}{\partial y_j}\mathcal{D}^{bh}(\mathbf{y},\mathbf{x})\overleftarrow{D_i^{ha}}(\mathbf{x})\right). \tag{39}$$

Equation (39) and the commutator, obtained in Ref. [5],

$$[\Pi^a_{\mathsf{GI}i}(\mathbf{x}),\Pi^b_{\mathsf{GI}j}(\mathbf{y})]=ig\left\{\frac{\partial}{\partial x_i}\mathcal{D}^{ah}(\mathbf{x},\mathbf{y})f^{hcb}\Pi^c_{\mathsf{GI}j}(\mathbf{y})-\frac{\partial}{\partial y_j}\mathcal{D}^{bh}(\mathbf{y},\mathbf{x})f^{hca}\Pi^c_{\mathsf{GI}i}(\mathbf{x})\right\}, \tag{40}$$

are in agreement with those given by Schwinger for the Coulomb gauge [9], except for some differences in operator order. This fact suggests that the gauge-invariant Weyl-gauge field and the Coulomb-gauge field discussed by Schwinger are very similar. The differences in operator-order should be expected because, in Ref. [9], ambiguities in operator order in the Coulomb gauge are resolved by symmetrizing noncommuting operator-valued quantities so that Coulomb-gauge operators are kept Hermitian. In our work in the gauge-invariant formulation of the Weyl gauge, ambiguities in operator order do not arise. When, because of a nonsymmetric ordering of gauge fields and chromoelectric fields, some gauge-invariant operator-valued quantities turn out not to be Hermitian, we leave them that way in order to avoid *ad hoc* changes in operator order.

Equation (40) leads to the commutation rule for the transverse parts of $\Pi^b_{\mathsf{GI}j}(\mathbf{y})$ [10],

$$[\Pi^{a\mathsf{T}}_{\mathsf{GI}i}(\mathbf{x}),\Pi^{b\mathsf{T}}_{\mathsf{GI}j}(\mathbf{y})]=0. \tag{41}$$

Equation (39) leads to the commutator of the transverse part of $\Pi^b_{\mathsf{GI}j}(\mathbf{y})$ and $A^a_{\mathsf{GI}i}(\mathbf{x})$ (which is transverse)

$$[\Pi^{b\mathsf{T}}_{\mathsf{GI}j}(\mathbf{y}),A^a_{\mathsf{GI}i}(\mathbf{x})]=-i\,\delta_{ab}\left(\delta_{ij}-\frac{\partial_i\partial_j}{\partial^2}\right)\delta(\mathbf{x}-\mathbf{y}). \tag{42}$$

Equation (39) can be shown to be consistent with $\partial_iA^a_{\mathsf{GI}i}=0$ because

$$\frac{\partial}{\partial x_i}[\Pi^b_{\mathsf{GI}j}(\mathbf{y}),A^a_{\mathsf{GI}i}(\mathbf{x})]=-i\left(\delta_{ab}\delta_{ij}\frac{\partial}{\partial x_j}\delta(\mathbf{x}-\mathbf{y})+\frac{\partial}{\partial y_j}\mathcal{D}^{bh}(\mathbf{y},\mathbf{x})\partial\cdot\overleftarrow{D^{ha}_{(\mathbf{x})}}\right)=-i\,\delta_{ab}\delta_{ij}\left(\frac{\partial}{\partial x_j}\delta(\mathbf{x}-\mathbf{y})+\frac{\partial}{\partial y_j}\delta(\mathbf{x}-\mathbf{y})\right)=0, \tag{43}$$

and with $D_j\Pi^b_{\mathsf{GI}j}\approx0$ because

$$D_j^{bc}(\mathbf{y})[\Pi^c_{\mathsf{GI}j}(\mathbf{y}),A^a_{\mathsf{GI}i}(\mathbf{x})]=0 \tag{44}$$

trivially.

The Faddeev-Popov operator has a well-documented importance in non-Abelian gauge theories. Gribov has shown that gauge fields that have been gauge fixed to have a vanishing divergence can differ from each other [11,12], and that the Faddeev-Popov operator does not have a unique inverse. In that same work, Gribov makes the suggestion that the zeros of the Faddeev-Popov operator $\partial^2\delta_{ac}+gf^{abc}A_i^b\partial_i$ might so intensify the interaction between color charges that the effect could account for confinement. Subsequent authors have reiterated this suggestion [13,14], and connections between the zeros of the Faddeev-Popov operator and color confinement have been discussed by other authors as well [15–17].

Equations (30) and (31) are based on a series representation of the operator-valued $\mathcal{D}^{bh}(\mathbf{y},\mathbf{x})$; they are obtained by combining all terms of equal order in g and noting cancellations within each order. They do not, however, establish that $\mathcal{D}^{bh}(\mathbf{y},\mathbf{x})$ is the unique inverse of the Faddeev-Popov operator. Questions about uniqueness can readily be formulated about number-valued functions, but are very difficult to address for operator-valued quantities. Equations (30) and (31) establish that $\mathcal{D}^{bh}(\mathbf{y},\mathbf{x})$ is an operator-valued inverse of $\partial\cdot D_{(\mathbf{y})}^{ah}$ (acting on the left) and of $\partial\cdot\overleftarrow{D_{(\mathbf{x})}^{ha}}$ (acting on the right) without addressing the question of its uniqueness. However, although $A_{\mathsf{GI}i}^a$ is an operator-valued quantity, the SU(2) versions of its constituents—the Weyl-gauge field A_i^a and the resolvent field $\overline{\mathcal{A}_i^a}$—can be, and often have been, represented by number-valued realizations as functions of spatial variables. Such realizations have been used extensively to study the topology of gauge fields [11,12,18]. When the integral equation for the resolvent field referred to in Sec. I is expressed in terms of a number-valued hedgehog representation, it can be transformed into a nonlinear differential equation that was shown to have multiple solutions [3]. Moreover, this nonlinear differential equation was shown to be very nearly identical in form to the one used by Gribov as a specific illustration of the fact that the Faddeev-Popov operator for the transverse SU(2) gauge field does not have a unique inverse. With this number-valued realization we were able to establish that the gauge-invariant field, which is transverse, has a Gribov ambiguity [3], even though there are no Gribov copies of the gauge-dependent Weyl-gauge field [19–21].

In the context of the quantized theory—for example, in $\hat{H}_{\mathsf{GI}}$—we will represent $\mathcal{D}^{bh}(\mathbf{y},\mathbf{x})$ as the operator-valued series described in Eqs. (26) and (34). Since each term in this series has unambiguous and self-consistent commutation relations with all other operator-valued quantities, the series representation of $\mathcal{D}^{bh}(\mathbf{y},\mathbf{x})$ is entirely satisfactory for determining the commutators of $\hat{H}_{\mathsf{GI}}$ with other gauge-invariant operators—and therefore determining their time dependence—even though number-valued realizations of the gauge-invariant gauge field lead to nonlinear integral equations that do not have unique solutions.

It may seem surprising that, starting in the Weyl gauge and expressing the QCD Hamiltonian in that gauge in terms of gauge-invariant variables can lead to a form of the Hamiltonian that, while never actually having been gauge-transformed, has the same dynamical effect as the QCD Hamiltonian in the Coulomb gauge. But a remarkably similar state of affairs obtains in QED. When QED is formulated in the temporal gauge, and a unitary transformation is carried out that is the Abelian analog of the one that leads to the Hamiltonian described in Eqs. (20)–(24), the following result is obtained [22,23]: The QED Hamiltonian in the temporal gauge, unitarily transformed by $\exp[i\int(1/\partial^2)\partial_iA_i(\mathbf{r})j_0(\mathbf{r})d\mathbf{r}]$—the Abelian analog of the transformation $\mathcal{U}_C$ described in Eq. (15)—takes the form

$$\hat{H}_{QED}=\int d\mathbf{r}\left[\frac{1}{2}\Pi_i(\mathbf{r})\Pi_i(\mathbf{r})+\frac{1}{4}F_{ij}(\mathbf{r})F_{ij}(\mathbf{r})+\psi^\dagger(\mathbf{r})\times(\beta m-i\alpha_i\partial_i)\psi(\mathbf{r})\right]-\int d\mathbf{r}j_i(\mathbf{r})A_i^{\mathsf{T}}(\mathbf{r})+\int d\mathbf{r}d\mathbf{r}'\,\frac{j_0(\mathbf{r})j_0(\mathbf{r}')}{8\pi|\mathbf{r}-\mathbf{r}'|}+H_g. \tag{45}$$

A_i^{T} designates the transverse Abelian gauge field—which, in Abelian theories, is also the gauge-invariant field—and H_g can be expressed as

$$H_g=-\frac{1}{2}\int d\mathbf{r}\left(\partial_i\Pi_i(\mathbf{r})\frac{1}{\partial^2}j_0(\mathbf{r})+j_0(\mathbf{r})\frac{1}{\partial^2}\partial_i\Pi_i(\mathbf{r})\right). \tag{46}$$

H_g is the Abelian analog of $H_{\mathcal{G}}$, described in Eq. (24). The Abelian Gauss's law operator, $\hat{\mathcal{G}}=\partial_i\Pi_i+j_0$, transforms into $\partial_i\Pi_i$ in the representation in which ψ represents the gauge-invariant electron field; and the states that implement Gauss's law, which originally are selected by $\mathcal{G}(\mathbf{r})|\Psi\rangle=0$, are given by $\partial_i\Pi_i(\mathbf{r})|\Phi\rangle=0$ in the transformed representation [or, as is more appropriate for Abelian gauge theories, by $\mathcal{G}^{(+)}(\mathbf{r})|\Psi\rangle=0$ and $\partial_i\Pi_i^{(+)}(\mathbf{r})|\Phi\rangle=0$ respectively, where the superscript "(+)" designates the positive-frequency parts of operators] [22,24] As can be seen, $\hat{H}_{QED}$ also consists of two parts: the Hamiltonian for QED in the Coulomb gauge, and H_g, which has no effect on the time evolution of states that implement Gauss's law, but which "remembers" the fact that $\hat{H}_{QED}$ is the transformed Weyl-gauge Hamiltonian by preserving the field equations for that gauge. An identical transformation applies to covariant-gauge QED, the sole difference being in the form of the H_g produced by the transformation.

As we can see from Eqs. (20), (24), (45) and (46), and as will become even more evident in Eq. (60), QCD and QED are strikingly similar in the relation between their Hamiltonians in different gauges when these are represented in terms of gauge-invariant fields. Nevertheless, there are important differences between QED and QCD in the significance of this relationship. One such difference is that, in QED, we may safely use the original untransformed Weyl gauge or covariant-gauge Hamiltonian in a space of perturbative states when evaluating S-matrix elements, even though these gauge-dependent perturbative states fail to implement

Gauss's law. This means that, for perturbative calculations in QED, we can safely use the Lagrangian

$$\mathcal{L}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-j^{\mu}A_{\mu}+\bar{\psi}(i\gamma^{\mu}\partial_{\mu}-m)\psi+\mathcal{L}_g \tag{47}$$

with $\mathcal{L}_g=-A_0G$ for the Weyl gauge and $\mathcal{L}_g=-G\partial_{\mu}A^{\mu}+\frac{1}{2}(1-\gamma)G^2$ for the covariant gauge, without paying any attention to Gauss's law whatsoever. A corresponding practice in Weyl-gauge QCD is the use of the Weyl-gauge Hamiltonian H in a Fock space of perturbative states that are not annihilated by $\mathcal{G}^a$. There is, however, the following important difference between QED and QCD. The use of perturbative states in QED without implementing Gauss's law is permissible because, in QED, a unitary equivalence can be established between $\partial_i\Pi_i$ and $\partial_i\Pi_i+j_0$, so that $\partial_i\Pi_i$ can be interpreted as $\partial_i\Pi_i+j_0$ in a new representation [22,23]. In this way, it can be shown that perturbative states that implement only $\partial_i\Pi_i(\mathbf{r})\approx 0$ instead of $\partial_i\Pi_i+j_0(\mathbf{r})\approx 0$ may be used when evaluating S-matrix elements in QED; the only effect on S-matrix elements from this substitution consists of changes to the renormalization constants, which are unobservable [25]. But this dispensation to ignore Gauss's law in perturbative calculations has not been shown to extend to QCD, because $D_i\Pi_i^a+j_0(\mathbf{r})$ is unitarily equivalent only to $D_i\Pi_i^a$, but *not* to $\partial_i\Pi_i^a$; and states that implement the Gauss's law $D_i\Pi_i^a\approx 0$ cannot be perturbative states. In particular, the use of $\hat{H}_{\mathrm{GI}}$ for perturbative calculations using a space of perturbative states does not enjoy the same protection that the corresponding practice has in QED. In Sec. III we will establish an isomorphism between the gauge-invariant states that implement the non-Abelian Gauss's law and perturbative states. This isomorphism enables us to substitute "standard" calculations with perturbative states for prohibitively difficult ones with gauge-invariant states. By this means, we provide for QCD a substitution rule, similar to the one available in QED, that permits the use of perturbative Fock states in scattering calculations with the assurance that the results of these calculations will agree with what would have been obtained if gauge-invariant operators and states had been used.

Another difference between QCD and QED is related to the fact that states that obey the condition

$$\mathcal{G}^a(\mathbf{r})|\Psi\rangle=0 \tag{48}$$

are not normalizable. We can see this easily by constructing, for example, the commutator of $\mathcal{G}^8(\mathbf{r})$ and an integral operator $\mathcal{I}=\int d\mathbf{r}'A_j^8(\mathbf{r}')\chi_j(\mathbf{r}')$ where $\chi_j(\mathbf{r}')$ is an arbitrary c-number-valued function. Since

$$[\mathcal{I},\mathcal{G}^8(\mathbf{r})]=i\partial_i\chi_i(\mathbf{r})$$

and

$$\langle\Psi|[\mathcal{I},\mathcal{G}^8(\mathbf{r})]|\Psi\rangle=i\partial_i\chi_i(\mathbf{r})\langle\Psi|\Psi\rangle,$$

and since $\mathcal{G}^8(\mathbf{r})$ is Hermitian so that $\langle\Psi|\mathcal{G}^8(\mathbf{r})=0$ as well as $\mathcal{G}^8(\mathbf{r})|\Psi\rangle=0$, this leads to $\langle\Psi|\Psi\rangle=0$, in contradiction to the assumption that $|\Psi\rangle$ is normalizable. This argument is a simple extension of one that was applied to the Fermi subsidiary condition for QED [26]. In the case of QED, however, this difficulty can be remedied because the non-normalizability of the states that are annihilated by the Abelian Gauss's law operator is entirely caused by the unobservable longitudinal nonpropagating photon "ghost" modes, which coincide exactly with the pure gauge degrees of freedom, and which can be kept separate from the gauge-invariant transversely polarized propagating photons in a variety of ways. In QCD, however, transverse modes can be pure gauge, and we do not know of a similarly satisfactory resolution of the non-normalizability of the state vectors that satisfy Eq. (48) [27,28]. The previously mentioned isomorphism, which will be demonstrated in Sec. III, mitigates this difficulty by establishing an equivalence between matrix elements evaluated with gauge-invariant states that are not normalizable, and corresponding ones evaluated with perturbative states.

B. Relation to QCD in the Coulomb gauge

Unlike the Weyl-gauge formulation of QCD, in which one can simply set $A_0^a=0$ and impose canonical quantization rules on the remaining fields [29,30], the quantization of Coulomb-gauge QCD requires that constraints be explicitly taken into account. In constrained quantization—one procedure for implementing consistency with constraints—this consistency is maintained by means of the so-called "Dirac-brackets," which replace the canonical equal-time commutation rules. When constrained quantization, such as the Dirac-Bergmann procedure [31], is applied to the Coulomb gauge, the generator of infinitesimal gauge transformations becomes a constraint; it then must commute with all fields, which therefore are invariant to small gauge transformations. Under these circumstances, the gauge field would automatically be invariant to small gauge transformations, although it might have discrete numbers of gauge copies.

However, carrying out the constrained quantization of QCD in the Coulomb gauge is problematical; one impediment stems from operator-ordering ambiguities of multilinear operator products. For example, in constrained quantization, the matrix of constraint commutators must be inverted. There are noncommuting operators in that matrix, and it is at best problematical to keep track of operator order in the process of finding this inverse. As a result, the Dirac brackets of some operators are not unambiguously specified. Because of the difficulties associated with the quantization of QCD in the Coulomb gauge, a number of workers have avoided the direct quantization of Coulomb-gauge QCD, and have proceeded by treating the $A_0^a=0$ gauge fields as a set of Cartesian coordinates and the Coulomb-gauge fields as a set of curvilinear coordinates, and have transformed from the former to the latter by using the familiar apparatus for such coordinate transformations [32–35].

In our work, we transform from the Weyl gauge to a representation in terms of gauge-invariant operator-valued fields. Our purpose is to implement gauge invariance, not to carry out a gauge transformation. We do not impose transversality on the gauge-invariant $A_{\mathrm{GI}\,i}^b$; in our work, $A_{\mathrm{GI}\,i}^b$ is transverse, but the transversality is not imposed as a condition—it emerges as a consequence of its gauge invariance. And the Gauss's law operator $\mathcal{G}^a$ does not vanish identically; in our work, Gauss's law is a condition on a set of states (the implementation of Gauss's law by imposing it on a set of states is also discussed in Refs. [32,34,36,37]).

Because our formulation of QCD in terms of gauge-invariant fields differs significantly from those whose purpose is to construct the QCD Hamiltonian in the Coulomb gauge, it is of interest to inquire how closely the resulting Hamiltonians resemble each other. In order to examine this question further, we will make some additional transformations of $\hat{H}_{\rm GI}$ that assume that the Hamiltonian acts only on states that implement Gauss's law. When $\hat{H}_{\rm GI}$ appears in a matrix element between two states $|\Psi_\alpha\rangle$ and $\langle\Psi_\beta|$ that obey $\mathcal{G}^c(\mathbf{x})|\Psi_\alpha\rangle=0$ and $\langle\Psi_\beta|\mathcal{G}^c(\mathbf{x})=0$, further transformations that eliminate the longitudinal component of $\Pi^a_{{\rm GI}i}$ are possible. For the case that $\Pi^{c{\rm L}}_{{\rm GI}i}$ appears adjacent to and directly to the left of such a state $|\Psi\rangle$, we can make the replacement

$$\Pi^{c{\rm L}}_{{\rm GI}i}|\Psi\rangle=-\frac{\partial_i}{\partial^2}J^c_{0({\rm GI})}|\Psi\rangle \tag{49}$$

and, therefore, also

$$\begin{aligned}J^a_{0({\rm GI})}|\Psi\rangle&=gf^{abc}A^b_{{\rm GI}i}(\Pi^{c{\rm T}}_{{\rm GI}i}+\Pi^{c{\rm L}}_{{\rm GI}i})|\Psi\rangle\\&=\left\{J^{a{\rm T}}_{0({\rm GI})}-gf^{abc}A^b_{{\rm GI}i}\frac{\partial_i}{\partial^2}J^c_{0({\rm GI})}\right\}|\Psi\rangle,\end{aligned} \tag{50}$$

where $J^{a{\rm T}}_{0({\rm GI})}$ is defined as $J^{a{\rm T}}_{0({\rm GI})}\equiv gf^{abc}A^b_{{\rm GI}i}\Pi^{c{\rm T}}_{{\rm GI}i}$. Equation (50) can be iterated, leading to

$$J^b_{0({\rm GI})}\approx-\sum_{n=0}^{\infty}(-1)^n g^n f^{\vec{\alpha}bh}(\mathcal{T}^{\vec{\alpha}}_{(n)}J^{h{\rm T}}_{0({\rm GI})}) \tag{51}$$

where $\approx$ indicates that the replacement is valid only when the operators act on states $|\Psi\rangle$ that implement Gauss's law. When $J^{a\dagger}_{0({\rm GI})}$ stands directly to the right of $\langle\Psi|$ states, we can similarly make the replacement

$$J^{b\dagger}_{0({\rm GI})}\approx-\sum_{n=0}^{\infty}(-1)^n g^n f^{\vec{\alpha}bh}(\mathcal{T}^{\vec{\alpha}}_{(n)}J^{h{\rm T}}_{0({\rm GI})})^\dagger \tag{52}$$

where

$$\begin{aligned}&\{\mathcal{T}^{\vec{\alpha}}_{(n)}(\mathbf{r})J^{h{\rm T}}_{0({\rm GI})}(\mathbf{r})\}^\dagger\\&=\left\{\left[\left((J^{h{\rm T}\dagger}_{0({\rm GI})}(\mathbf{r}))\frac{\overleftarrow{\partial}_{j(n)}}{\partial^2}A^{\alpha(n)}_{{\rm GI}j(n)}(\mathbf{r})\right)\cdots\frac{\overleftarrow{\partial}_{j(2)}}{\partial^2}A^{\alpha(2)}_{{\rm GI}j(2)}(\mathbf{r})\right]\right.\\&\left.\quad\times\frac{\overleftarrow{\partial}_{j(1)}}{\partial^2}A^{\alpha(1)}_{{\rm GI}j(1)}(\mathbf{r})\right\}\end{aligned} \tag{53}$$

and where the arrows indicate that differentiation is applied to the left. Similarly, $\Pi^a_{{\rm GI}i}(\mathbf{r})$ and $\Pi^{a\dagger}_{{\rm GI}i}(\mathbf{r})$ can be expressed as

$$\Pi^b_{{\rm GI}j}\approx\Pi^{b{\rm T}}_{{\rm GI}j}+\frac{\partial_j}{\partial^2}\left(\sum_{n=0}^{\infty}(-1)^n g^n f^{\vec{\alpha}bh}(\mathcal{T}^{\vec{\alpha}}_{(n)}J^{h{\rm T}}_{0({\rm GI})})\right) \tag{54}$$

and

$$\Pi^{b\dagger}_{{\rm GI}j}\approx\Pi^{b{\rm T}\dagger}_{{\rm GI}j}+\frac{\partial_j}{\partial^2}\left(\sum_{n=0}^{\infty}(-1)^n g^n f^{\vec{\alpha}bh}(\mathcal{T}^{\vec{\alpha}}_{(n)}J^{h{\rm T}}_{0({\rm GI})})\right)^\dagger, \tag{55}$$

respectively. We can combine Eqs. (26) with Eqs. (51) and (52) to obtain

$$J^b_{0({\rm GI})}(\mathbf{y})\approx\partial^2\int d\mathbf{x}\mathcal{D}^{ba}(\mathbf{y},\mathbf{x})J^{a{\rm T}}_{0({\rm GI})}(\mathbf{x}) \tag{56}$$

and

$$J^{b\dagger}_{0({\rm GI})}(\mathbf{y})\approx\partial^2\int d\mathbf{x}J^{a{\rm T}\dagger}_{0({\rm GI})}(\mathbf{x})\mathcal{D}^{ab}(\mathbf{x},\mathbf{y}). \tag{57}$$

Equations (54) and (55) can be expressed as

$$\Pi^b_{{\rm GI}j}(\mathbf{y})\approx\Pi^{b{\rm T}}_{{\rm GI}j}(\mathbf{y})-\partial_j\int d\mathbf{x}\mathcal{D}^{ba}(\mathbf{y},\mathbf{x})J^{a{\rm T}}_{0({\rm GI})}(\mathbf{x}) \tag{58}$$

and

$$\Pi^{b\dagger}_{{\rm GI}j}(\mathbf{y})\approx\Pi^{b{\rm T}\dagger}_{{\rm GI}j}(\mathbf{y})-\partial_j\int d\mathbf{x}J^{a{\rm T}\dagger}_{0({\rm GI})}(\mathbf{x})\mathcal{D}^{ab}(\mathbf{x},\mathbf{y}), \tag{59}$$

respectively, where $J^{a{\rm T}\dagger}_{0({\rm GI})}(\mathbf{x})$ represents the Hermitian adjoint of $J^{a{\rm T}}_{0({\rm GI})}(\mathbf{x})$.

We can define an "effective" Hamiltonian $(\hat{H}_{\rm GI})_{\rm phys}$, which is obtained by making the replacements described by Eqs. (56)–(59) in $\hat{H}_{\rm GI}$ and excluding $H_{\mathcal{G}}$, since the latter will not contribute to any matrix elements in the physical space in which Gauss's law is implemented. With these replacements, we obtain

$$\begin{aligned}(\hat{H}_{\rm GI})_{\rm phys}=&\int d\mathbf{r}\left[\frac{1}{2}\Pi^{a{\rm T}\dagger}_{{\rm GI}i}(\mathbf{r})\Pi^{a{\rm T}}_{{\rm GI}i}(\mathbf{r})+\frac{1}{4}F^a_{{\rm GI}ij}(\mathbf{r})F^a_{{\rm GI}ij}(\mathbf{r})\right.\\&\left.+\psi^\dagger(\mathbf{r})(\beta m-i\alpha_i\partial_i)\psi(\mathbf{r})\right]-\int d\mathbf{r}j^a_i(\mathbf{r})A^a_{{\rm GI}i}(\mathbf{r})\\&-\frac{1}{2}\int d\mathbf{r}d\mathbf{x}d\mathbf{y}[j^b_0(\mathbf{x})+J^{b{\rm T}\dagger}_{0({\rm GI})}(\mathbf{x})]\\&\times\mathcal{D}^{ba}(\mathbf{x},\mathbf{r})\partial^2_{(\mathbf{r})}\mathcal{D}^{ac}(\mathbf{r},\mathbf{y})[j^c_0(\mathbf{y})+J^{c{\rm T}}_{0({\rm GI})}(\mathbf{y})].\end{aligned} \tag{60}$$

$(\hat{H}_{\rm GI})_{\rm phys}$ is not identical to $\hat{H}_{\rm GI}$. But $(\hat{H}_{\rm GI})_{\rm phys}$ can substitute for $\hat{H}_{\rm GI}$ as the generator of time evolution when we embed the theory within a space of states $|\Psi_\nu\rangle$ that satisfy the non-Abelian Gauss's law, $\mathcal{G}^a(\mathbf{x})|\Psi_\nu\rangle=0$. Because $\mathcal{G}^a(\mathbf{x})$ is Hermitian, the same state $|\Psi_\nu\rangle$ that obeys Eq. (48) also obeys $\langle\Psi_\nu|\mathcal{G}^a(\mathbf{x})=0$. Equation (20) demonstrates that when $\hat{H}_{\rm GI}$ appears in any "allowed" matrix element, $\Pi^a_{{\rm GI}i}$ and $J^a_{0({\rm GI})}$ always are situated where they abut a "ket" state vector $|\Psi_\alpha\rangle$ to their right; and $\Pi^{a\dagger}_{{\rm GI}i}$ and $J^{a\dagger}_{0({\rm GI})}$ always are situated where they abut a "bra" state vector $\langle\Psi_\beta|$ to their left. Since $\hat{H}_{\rm GI}$ will always be bracketed between two states $\langle\Psi_\beta|$ and $|\Psi_\alpha\rangle$ that implement Gauss's law, $\Pi^a_{{\rm GI}i}$ and $\Pi^{a\dagger}_{{\rm GI}i}$ can be replaced by their "soft" equivalents shown in Eqs. (58) and (59), respectively, and $J^b_{0({\rm GI})}$ and $J^{b\dagger}_{0({\rm GI})}$ can similarly be replaced as shown in Eqs. (56) and (57), respectively. $(\hat{H}_{\rm GI})_{\rm phys}$ can therefore always be substituted for $\hat{H}_{\rm GI}$ in matrix elements, as long as attention is paid to the need to restrict the space of state vectors to those that implement Gauss's law. For example, $\exp(-i\hat{H}_{\rm GI}t)|\Psi_\alpha\rangle$ can be replaced by $\exp(-i(\hat{H}_{\rm GI})_{\rm phys}t)|\Psi_\alpha\rangle$, since both will be required to project onto states that implement Gauss's law, as shown by

$$\exp[-i(\hat{H}_{\rm GI})t]|\Psi_\alpha\rangle=|\Psi_\nu\rangle\langle\Psi_\nu|\exp[-i(\hat{H}_{\rm GI})t]|\Psi_\alpha\rangle, \tag{61}$$

and

$$\begin{aligned}\langle\Psi_\nu|\exp[-i(\hat{H}_{\rm GI})t]|\Psi_\alpha\rangle &= \delta_{\nu\alpha}-it\langle\Psi_\nu|\hat{H}_{\rm GI})|\Psi_\alpha\rangle+\cdots+\frac{(-it)^n}{n!}\\ &\times\langle\Psi_\nu|\hat{H}_{\rm GI})|\Psi_{\mu_1}\rangle\langle\Psi_{\mu_1}|\hat{H}_{\rm GI})|\Psi_{\mu_2}\rangle\\ &\times\langle\Psi_{\mu_2}|\hat{H}_{\rm GI})|\Psi_{\mu_3}\rangle\cdots\langle\Psi_{\mu_{n-1}}|\hat{H}_{\rm GI})|\Psi_\alpha\rangle+\cdots.\end{aligned} \tag{62}$$

Each matrix element $\langle\Psi_{\mu_i}|\hat{H}_{\rm GI})|\Psi_{\mu_j}\rangle$ in Eq. (62) can be replaced by $\langle\Psi_{\mu_i}|(\hat{H}_{\rm GI})_{\rm phys}|\Psi_{\mu_j}\rangle$, so that $\exp[-i(\hat{H}_{\rm GI})t]|\Psi_\alpha\rangle$ can safely be replaced by $\exp(-i(\hat{H}_{\rm GI})_{\rm phys}t)|\Psi_\alpha\rangle$. The time evolution imposed by $\hat{H}_{\rm GI}$ on a state vector $|\Psi_\alpha\rangle$ for which $\mathcal{G}^c(\mathbf{x})|\Psi_\alpha\rangle=0$ takes place entirely within the space of states that implement Gauss's law. In the case of a state vector $|\chi\rangle$ for which $\mathcal{G}^c(\mathbf{x})|\chi\rangle=|\chi'\rangle$ where $|\chi'\rangle$ is nonvanishing,

$$\langle\chi|[\mathcal{G}^c(\mathbf{x}),\exp(-i\hat{H}_{\rm GI}t)]|\Psi_\alpha\rangle=\langle\chi'|\exp(-i\hat{H}_{\rm GI}t)|\Psi_\alpha\rangle=0 \tag{63}$$

because $\mathcal{G}^c(\mathbf{x})$ and $\hat{H}_{\rm GI}$ commute. This requires the part of χ that fails to implement Gauss's law to be orthogonal to $\exp(-i\hat{H}_{\rm GI}t)|\Psi_\alpha\rangle$. The only limitation on the validity of this argument is the non-normalizability of the states that implement Gauss's law, which complicates the algebraic properties of the $\{|\Psi_\alpha\rangle\}$ vector space. Nevertheless, Eqs. (61)–(63) show that we can restrict the space in which time evolution takes place to state vectors that implement Gauss's law without compromising the unitarity of the time-evolved $|\Psi_\alpha(t)\rangle$ or of the S matrix evaluated with such states. These considerations are also instrumental in allowing us to replace $\exp(-i\hat{H}_{\rm GI}t)$ with $\exp(-i(\hat{H}_{\rm GI})_{\rm phys}t)$. $\hat{H}_{\rm GI}$ and $(\hat{H}_{\rm GI})_{\rm phys}$ both commute with $\mathcal{G}^a(\mathbf{x})$ for all values of a, so that

$$\begin{aligned}\mathcal{G}^a(\mathbf{x})&\exp[-i(\hat{H}_{\rm GI})_{\rm phys}t]|\Psi_\alpha\rangle\\ &=\exp[-i(\hat{H}_{\rm GI})_{\rm phys}t]\mathcal{G}^a(\mathbf{x})|\Psi_\alpha\rangle=0\end{aligned} \tag{64}$$

as well as

$$\mathcal{G}^a(\mathbf{x})\exp[-i(\hat{H}_{\rm GI})t]|\Psi_\alpha\rangle=\exp[-i(\hat{H}_{\rm GI})t]\mathcal{G}^a(\mathbf{x})|\Psi_\alpha\rangle=0. \tag{65}$$

The state vectors $\exp[-i(\hat{H}_{\rm GI})t]|\Psi_\alpha\rangle$ and $\exp[-i(\hat{H}_{\rm GI})_{\rm phys}t]|\Psi_\alpha\rangle$ therefore are gauge-invariant and implement Gauss's law just as $|\Psi_\alpha\rangle$ does.

In comparing $(\hat{H}_{\rm GI})_{\rm phys}$ with expressions for the Coulomb-gauge Hamiltonian in the literature, we note that the only significant difference between $(\hat{H}_{\rm GI})_{\rm phys}$ and the Coulomb-gauge Hamiltonian reported in Ref. [32] is that $\Pi^{b{\sf T}\dagger}_{{\rm GI}j}$, the Hermitian adjoint of the transverse gauge-invariant chromoelectric field, appears in Eq. (60) where the expression $\mathcal{J}^{-1}\Pi^{b{\sf T}}_{{\rm GI}j}\mathcal{J}$ appears in Ref. [32], where $\mathcal{J}=\det[\partial_i\cdot D_i]$. We will prove in Appendix B that

$$\Pi^{b{\sf T}\dagger}_{{\rm GI}j}=\mathcal{J}^{-1}\Pi^{b{\sf T}}_{{\rm GI}j}\mathcal{J}, \tag{66}$$

by using Eq. (11) and the identity

$$\frac{\delta}{\delta A^q_i(\mathbf{x})}\ln\mathcal{J}={\rm Tr}\left[(\partial\cdot D)^{-1}\frac{\delta}{\delta A^q_i(\mathbf{x})}\partial\cdot D\right] \tag{67}$$

where the trace in Eq. (67) extends to the coordinates and the color indices. With this demonstration, we see that Eq. (60) and the Coulomb-gauge Hamiltonian described in Eq. (4.65) in Ref. [32] are identical. It is also of interest to compare Eq. (60) with the Coulomb-gauge Hamiltonian in Ref. [11] as well as in the work of a number of other authors who used the same form of the Hamiltonian. The Hamiltonian in Ref. [11] differs from the Hamiltonian described by Eq. (4.65) in Ref. [32] in the fact that $\Pi^{b{\sf T}}_{{\rm GI}j}$ rather than $\Pi^{b{\sf T}\dagger}_{{\rm GI}j}$ appears in Ref. [11] in place of $\mathcal{J}^{-1}\Pi^{b{\sf T}}_{{\rm GI}j}\mathcal{J}$ in Ref. [32]; there is also the trivial difference that Ref. [11] deals with "pure glue" QCD so that the quark field is not included.

This discrepancy raises the question of the Hermiticity of the operator-valued transverse gauge-invariant chromoelectric field $\Pi^{b{\sf T}}_{{\rm GI}j}$, which is of considerable importance for determining the dynamical effects of $(\hat{H}_{\rm GI})_{\rm phys}$. One way of addressing this question is to use Eq. (11) and Eq. (65) in Ref. [5] to obtain

$$\Pi^{b\dagger}_{{\rm GI}j}(\mathbf{y})-\Pi^b_{{\rm GI}j}(\mathbf{y})=[\Pi^q_j(\mathbf{y}),R_{bq}(\mathbf{y})]=igf^{hcb}\frac{\partial}{\partial y_j}\mathcal{D}^{ch}(\mathbf{y},\mathbf{y}) \tag{68}$$

where the partial derivative acts on only the *first* $\mathbf{y}$ argument in $\mathcal{D}^{ch}(\mathbf{y},\mathbf{y})$. We might have expected that the transverse parts of $\Pi^{b\dagger}_{{\rm GI}j}(\mathbf{y})$ and $\Pi^b_{{\rm GI}j}(\mathbf{y})$ would be identical since any functionals of the form $[\delta_{i,j}-(\partial_i\partial_j/\partial^2)]\partial_j\xi(\mathbf{y})$ would necessarily vanish. Such a conclusion would not, however, be correct in this case, because in $(\partial/\partial y_j)\mathcal{D}^{qh}(\mathbf{y},\mathbf{y})$, the partial derivative differentiates only the *first* $\mathbf{y}$ in $\mathcal{D}^{qh}(\mathbf{y},\mathbf{y})$. We can make use of Eq. (38) and the fact that $f^{hcb}\delta_{hc}=0$ to express Eq. (68) as

$$\Pi^{b\dagger}_{\mathsf{GI}j}(\mathbf{y})-\Pi^{b}_{\mathsf{GI}j}(\mathbf{y})=ig^2f^{hcb}f^{\delta cs}\int d\mathbf{z}\frac{\partial}{\partial y_j}\left(\frac{1}{4\pi|\mathbf{y}-\mathbf{z}|}\right)\times A^{\delta}_{\mathsf{GI}k}(\mathbf{z})\frac{\partial}{\partial z_k}\mathcal{D}^{sh}(\mathbf{z},\mathbf{y}) \tag{69}$$

and we can extract the transverse parts to obtain

$$\Pi^{b\mathsf{T}\dagger}_{\mathsf{GI}j}(\mathbf{y})-\Pi^{b\mathsf{T}}_{\mathsf{GI}j}(\mathbf{y})=ig^2f^{hcb}f^{\delta cs}\left(\delta_{j,\ell}-\frac{\partial^{(\mathbf{y})}_j\partial^{(\mathbf{y})}_\ell}{\partial^2}\right)\times\int d\mathbf{z}\frac{\partial}{\partial y_\ell}\left(\frac{1}{4\pi|\mathbf{y}-\mathbf{z}|}\right)\times A^{\delta}_{\mathsf{GI}k}(\mathbf{z})\frac{\partial}{\partial z_k}\mathcal{D}^{sh}(\mathbf{z},\mathbf{y}). \tag{70}$$

Equation (70) makes it clear that $\Pi^{b\mathsf{T}\dagger}_{\mathsf{GI}j}(\mathbf{y})-\Pi^{b\mathsf{T}}_{\mathsf{GI}j}(\mathbf{y})$ is not the transverse projection of a gradient and therefore cannot be presumed to vanish.

Equally compelling evidence that $\Pi^{b\mathsf{T}}_{\mathsf{GI}j}$ is not identical to its Hermitian adjoint is provided by the observation that the commutators $[\Pi^{a\mathsf{T}}_{\mathsf{GI}i}(\mathbf{x}),\Pi^{b\mathsf{T}\dagger}_{\mathsf{GI}j}(\mathbf{y})]$ and $[\Pi^{a\mathsf{T}}_{\mathsf{GI}i}(\mathbf{x}),\Pi^{b\mathsf{T}}_{\mathsf{GI}j}(\mathbf{y})]$ differ. The latter vanishes, as is shown by Eqs. (40), (41). However, use of Eq. (11) and the commutation rules for the underlying Weyl-gauge fields lead to

$$[\Pi^{a\mathsf{T}}_{\mathsf{GI}i}(\mathbf{x}),\Pi^{b\mathsf{T}\dagger}_{\mathsf{GI}j}(\mathbf{y})]=g^2f^{hca}f^{pdb}\left(\delta_{ik}-\frac{\partial^{(\mathbf{x})}_i\partial^{(\mathbf{x})}_k}{\partial^2}\right)\times\left(\delta_{jl}-\frac{\partial^{(\mathbf{y})}_j\partial^{(\mathbf{y})}_l}{\partial^2}\right)\frac{\partial}{\partial y_l}\mathcal{D}^{dh}(\mathbf{y},\mathbf{x})\times\frac{\partial}{\partial x_k}\mathcal{D}^{cp}(\mathbf{x},\mathbf{y}), \tag{71}$$

and an alternate derivation based on Eqs. (42) and (68) confirms that result. Similarly to what we observed in connection with Eq. (68), the derivatives $\partial/\partial y_j$ and $\partial/\partial x_i$ each differentiate part, but not all of the $\mathbf{y}$ and $\mathbf{x}$ dependence, respectively, of the product $\mathcal{D}^{dh}(\mathbf{y},\mathbf{x})\mathcal{D}^{cp}(\mathbf{x},\mathbf{y})$ in Eq. (71). The transverse projections of

$$(\partial/\partial y_l)\mathcal{D}^{dh}(\mathbf{y},\mathbf{x})(\partial/\partial x_k)\mathcal{D}^{cp}(\mathbf{x},\mathbf{y})$$

therefore will not vanish, and $[\Pi^{a\mathsf{T}}_{\mathsf{GI}i}(\mathbf{x}),\Pi^{b\mathsf{T}\dagger}_{\mathsf{GI}j}(\mathbf{y})]\neq 0$. Since $[\Pi^{a\mathsf{T}}_{\mathsf{GI}i}(\mathbf{x}),\Pi^{b\mathsf{T}}_{\mathsf{GI}j}(\mathbf{y})]$ and $[\Pi^{a\mathsf{T}}_{\mathsf{GI}i}(\mathbf{x}),\Pi^{b\mathsf{T}\dagger}_{\mathsf{GI}j}(\mathbf{y})]$ differ $\Pi^{b\mathsf{T}}_{\mathsf{GI}j}(\mathbf{y})$ and $\Pi^{b\mathsf{T}\dagger}_{\mathsf{GI}j}(\mathbf{y})$ cannot be identical.

III. ISOMORPHISM AND ITS IMPLICATION FOR THE SCATTERING AMPLITUDE

In the preceding sections we have obtained a description of QCD that took the Weyl-gauge formulation as its point of departure, and arrived at a Hamiltonian in which all operator-valued fields—the gauge field, the chromoelectric field, as well as the quark field—are gauge invariant, and only the transverse components of the chromoelectric fields appear in the Hamiltonian $(\hat{H}_{\mathsf{GI}})_{\mathrm{phys}}$. It was necessary, however, to restrict use of this Hamiltonian to a space in which all state vectors implement the non-Abelian Gauss's law; and these state vectors are complicated constructions that are not easy to use. In this section we will show how isomorphisms can be established that enable us to identify $(\hat{H}_{\mathsf{GI}})_{\mathrm{phys}}$ with a Hamiltonian that can be used in a space of ordinary, conventional perturbative states.

To review the relation between gauge-invariant and perturbative states: In Ref. [1], a set of states was constructed in the form

$$|\Psi_i\rangle=\Psi|\phi_i\rangle \tag{72}$$

where the operator-valued Ψ was given as

$$\Psi=\sum_{n=0}^{\infty}\frac{i^n}{n!}\Psi_n \tag{73}$$

with

$$\Psi_n=\int d\mathbf{r}_1\cdots d\mathbf{r}_n\overline{\mathcal{A}^{q(1)}_{k(1)}}(\mathbf{r}_1)\cdots\overline{\mathcal{A}^{q(n)}_{k(n)}}(\mathbf{r}_n)\times\Pi^{q(1)}_{k(1)}(\mathbf{r}_1)\cdots\Pi^{q(n)}_{k(n)}(\mathbf{r}_n). \tag{74}$$

$|\phi_i\rangle$ designates one of a set of states that is annihilated by $\partial_j\Pi^b_j$. These $|\phi_i\rangle$ states—the so-called "Fermi" states—are related to "standard" perturbative states $|\mathsf{p}_i\rangle$ by

$$|\phi_i\rangle=\Xi|\mathsf{p}_i\rangle. \tag{75}$$

Ξ was given in Ref. [38], where it was also shown that $\partial_j\Pi^b_j(\mathbf{r})\Xi|\mathsf{p}_i\rangle=0$, where $|\mathsf{p}_i\rangle$ designates one of a set of "standard" perturbative states annihilated by all annihilation operators for fermion and transverse gauge field excitations. This set of perturbative states will be described more fully later in this section, and will turn out to be identical to perturbative states in QED, except for the fact that the gluon operators carry a Lie group index, while the photons do not. Since $\partial_j\Pi^b_j$ annihilates any $|\phi_i\rangle$ state, we can see that, in $|\Psi_i\rangle$ states, the negative chromoelectric field $\Pi^{q(\ell)}_{k(\ell)}(\mathbf{r}_\ell)$ in Ψ can be replaced by its transverse part $\Pi^{q(\ell)\mathsf{T}}_{k(\ell)}(\mathbf{r}_\ell)$, because the longitudinal parts vanish when acting on a $|\phi_i\rangle$ state. Furthermore, in Eq. (74), every transverse $\Pi^{q(\ell)\mathsf{T}}_{k(\ell)}(\mathbf{r}_\ell)$ is integrated with an $\overline{\mathcal{A}^{q(\ell)}_{k(\ell)}}(\mathbf{r}_\ell)$ in each variable, $\mathbf{r}_\ell$, and only the transverse components $\overline{\mathcal{A}^{q(\ell)\mathsf{T}}_{k(\ell)}}(\mathbf{r}_\ell)$ will survive this integration in the $|\Psi_i\rangle$ states, which become

$$|\Psi_i\rangle=\sum_{n=1}\frac{i^n}{n!}\int d\mathbf{r}_1\cdots d\mathbf{r}_n\overline{\mathcal{A}^{q(1)\mathsf{T}}_{k(1)}}(\mathbf{r}_1)\cdots\overline{\mathcal{A}^{q(n)\mathsf{T}}_{k(n)}}(\mathbf{r}_n)\times\Pi^{q(1)\mathsf{T}}_{k(1)}(\mathbf{r}_1)\cdots\Pi^{q(n)\mathsf{T}}_{k(n)}(\mathbf{r}_n)|\phi_i\rangle. \tag{76}$$

In Ref. [1], it was shown that

$$A^a_{\mathsf{GI}j}(\mathbf{r})\Psi|\phi_i\rangle=\Psi A^{a\mathsf{T}}_j(\mathbf{r})|\phi_i\rangle. \tag{77}$$

In Appendix D, we will use Eq. (76) to show that

$$\Pi^{c\mathsf{T}}_{\mathsf{GI}j}(\mathbf{r})\Psi|\phi_i\rangle=\Psi\Pi^{c\mathsf{T}}_j(\mathbf{r})|\phi_i\rangle. \tag{78}$$

Since the Hamiltonian $(\hat{H}_{\mathsf{GI}})_{\mathrm{phys}}$ consists of transverse fields only, Eqs. (77) and (78) afford us an opportunity to shift

$(\hat{H}_{\mathsf{GI}})_{\mathrm{phys}}$ from the left-hand side of Ψ to the right, with a concomitant substitution of transverse Weyl-gauge fields for the corresponding gauge-invariant fields. The one impediment to this process is that $\Pi^{b\mathsf{T}\dagger}_{\mathsf{GI}j}$, the Hermitian adjoint of $\Pi^{b\mathsf{T}}_{\mathsf{GI}j}$, also appears in $(\hat{H}_{\mathsf{GI}})_{\mathrm{phys}}$, and Eq. (78) only applies to $\Pi^{b\mathsf{T}}_{\mathsf{GI}j}$ and not to $\Pi^{b\mathsf{T}\dagger}_{\mathsf{GI}j}$. To remove that impediment, we use Eq. (66) to substitute $\mathcal{J}^{-1}\Pi^{b\mathsf{T}}_{\mathsf{GI}j}\mathcal{J}$ for $\Pi^{b\mathsf{T}\dagger}_{\mathsf{GI}j}$, and express $(\hat{H}_{\mathsf{GI}})_{\mathrm{phys}}$ as

$$(\hat{H}_{\mathsf{GI}})_{\mathrm{phys}}=\int d\mathbf{r}\Big\{\frac{1}{2}\mathcal{J}^{-1}\Pi^{b\mathsf{T}}_{\mathsf{GI}j}(\mathbf{r})\mathcal{J}\Pi^{a\mathsf{T}}_{\mathsf{GI}i}(\mathbf{r})+\frac{1}{4}F^{a}_{\mathsf{GI}ij}(\mathbf{r})F^{a}_{\mathsf{GI}ij}(\mathbf{r})+\psi^{\dagger}(\mathbf{r})(\beta m-i\alpha_i\partial_i)\psi(\mathbf{r})-j^{a}_{i}(\mathbf{r})A^{a}_{\mathsf{GI}i}(\mathbf{r})\Big\}-\frac{1}{2}\int d\mathbf{r}d\mathbf{x}d\mathbf{y}\times[j^{b}_{0}(\mathbf{x})+J^{b\mathsf{T}\dagger}_{0(\mathsf{GI})}(\mathbf{x})]\mathcal{D}^{ba}(\mathbf{x},\mathbf{r})\partial^{2}_{(\mathbf{r})}\mathcal{D}^{ac}(\mathbf{r},\mathbf{y})\times[j^{c}_{0}(\mathbf{y})+J^{c\mathsf{T}}_{0(\mathsf{GI})}(\mathbf{y})] \tag{79}$$

with

$$J^{a\mathsf{T}\dagger}_{0(\mathsf{GI})}=gf^{abc}\mathcal{J}^{-1}\Pi^{c\mathsf{T}}_{\mathsf{GI}i}\mathcal{J}A^{b}_{\mathsf{GI}i}.$$

We can define a "Hermitized" transverse gauge-invariant negative chromoelectric field $\mathcal{P}^{b\mathsf{T}}_{j}$

$$\mathcal{P}^{b\mathsf{T}}_{j}(\mathbf{r})=\mathcal{J}^{-1/2}\Pi^{b\mathsf{T}}_{\mathsf{GI}j}(\mathbf{r})\mathcal{J}^{1/2}. \tag{80}$$

As can be seen from Eq. (66), $\mathcal{P}^{b\mathsf{T}}_{j}$ is Hermitian, since

$$\mathcal{P}^{b\mathsf{T}\dagger}_{j}(\mathbf{r})=\mathcal{J}^{1/2}\Pi^{b\mathsf{T}\dagger}_{\mathsf{GI}j}(\mathbf{r})\mathcal{J}^{-1/2}=\mathcal{J}^{-1/2}[\mathcal{J}\,\Pi^{b\mathsf{T}\dagger}_{\mathsf{GI}j}(\mathbf{r})\mathcal{J}^{-1}]\mathcal{J}^{1/2}=\mathcal{P}^{b\mathsf{T}}_{j}(\mathbf{r}). \tag{81}$$

An important consideration for this argument is the fact that $\mathcal{J}^{1/2}$ is Hermitian, which is proven in Appendix C. In the same appendix, we also prove that the canonical commutation relations between $\Pi^{b\mathsf{T}}_{j}$'s and $A^{a}_{\mathsf{GI}j}$'s and that among $\Pi^{b\mathsf{T}}_{j}$'s remain unmodified with $\Pi^{b\mathsf{T}}_{j}$'s replaced by $\mathcal{P}^{b\mathsf{T}}_{\mathsf{GI}j}$'s. We then find that

$$\Pi^{b\mathsf{T}}_{\mathsf{GI}j}(\mathbf{r})=\mathcal{J}^{1/2}\mathcal{P}^{b\mathsf{T}}_{j}(\mathbf{r})\mathcal{J}^{-1/2}$$

and

$$\Pi^{b\mathsf{T}\dagger}_{\mathsf{GI}j}(\mathbf{r})=\mathcal{J}^{-1/2}\mathcal{P}^{b\mathsf{T}}_{j}(\mathbf{r})\mathcal{J}^{1/2}. \tag{82}$$

Equation (82) transforms from the non-Hermitian $\Pi^{b\mathsf{T}}_{\mathsf{GI}j}$ and $\Pi^{b\mathsf{T}\dagger}_{\mathsf{GI}j}$ to the Hermitian $\mathcal{P}^{b\mathsf{T}}_{j}$ (not, however, unitarily, since $\mathcal{J}^{1/2}$ is Hermitian and not the Hermitian adjoint of $\mathcal{J}^{-1/2}$). Transformations of this kind have previously been used by other workers [32,39]. It would be possible to make a compensating transformation on the states, but we prefer to leave the states untransformed and to extract

$$[\mathsf{H}]_0=\int d\mathbf{r}\Big[\frac{1}{2}\mathcal{P}^{b\mathsf{T}}_{j}(\mathbf{r})\mathcal{P}^{b\mathsf{T}}_{j}(\mathbf{r})+\frac{1}{4}\hat{F}^{a}_{\mathsf{GI}ij}(\mathbf{r})\hat{F}^{a}_{\mathsf{GI}ij}(\mathbf{r})+\psi^{\dagger}(\mathbf{r})\times(\beta m-i\alpha_i\partial_i)\psi(\mathbf{r})\Big] \tag{83}$$

from Eq. (79) in order to obtain a non-interacting part of $(\hat{H}_{\mathsf{GI}})_{\mathrm{phys}}$ that consists of Hermitian gauge-invariant fields and that can define interaction picture operators. As we will show in Appendix E, this process leads to the expression

$$(\hat{H}_{\mathsf{GI}})_{\mathrm{phys}}=[\mathsf{H}]_0+[\mathsf{H}]_1+[\mathsf{H}]_2 \tag{84}$$

where

$$[\mathsf{H}]_1=\int d\mathbf{r}\Big\{gf^{abc}\partial_iA^{a}_{\mathsf{GI}j}(\mathbf{r})A^{b}_{\mathsf{GI}i}(\mathbf{r})A^{c}_{\mathsf{GI}j}(\mathbf{r})+\frac{1}{4}g^2f^{abc}f^{ab'c'}A^{b}_{\mathsf{GI}i}(\mathbf{r})A^{c}_{\mathsf{GI}j}(\mathbf{r})A^{b'}_{\mathsf{GI}i}(\mathbf{r})A^{c'}_{\mathsf{GI}j}(\mathbf{r})-j^{a}_{i}(\mathbf{r})A^{a}_{\mathsf{GI}i}(\mathbf{r})\Big\}-\frac{1}{2}\int d\mathbf{r}d\mathbf{x}d\mathbf{y}\times[j^{b}_{0}(\mathbf{x})+\overline{\mathsf{J}}^{b\mathsf{T}}_{0(\mathsf{GI})}(\mathbf{x})]\mathcal{D}^{ba}(\mathbf{x},\mathbf{r})\partial^{2}_{(\mathbf{r})}\mathcal{D}^{ac}(\mathbf{r},\mathbf{y})\times[j^{c}_{0}(\mathbf{y})+\overline{\mathsf{J}}^{c\mathsf{T}}_{0(\mathsf{GI})}(\mathbf{y})] \tag{85}$$

and

$$[\mathsf{H}]_2=\mathcal{U}+\mathcal{V}+\frac{1}{2}\int d\mathbf{r}d\mathbf{x}d\mathbf{y}\{i\mathsf{k}^{b}_{0}(\mathbf{x})\mathcal{D}^{ba}(\mathbf{x},\mathbf{r})\partial^{2}_{(\mathbf{r})}\mathcal{D}^{ac}(\mathbf{r},\mathbf{y})\times[j^{c}_{0}(\mathbf{y})+\overline{\mathsf{J}}^{c\mathsf{T}}_{0(\mathsf{GI})}(\mathbf{y})]-[j^{b}_{0}(\mathbf{x})+\overline{\mathsf{J}}^{b\mathsf{T}}_{0(\mathsf{GI})}(\mathbf{x})]\times\mathcal{D}^{ba}(\mathbf{x},\mathbf{r})\partial^{2}_{(\mathbf{r})}\mathcal{D}^{ac}(\mathbf{r},\mathbf{y})i\mathsf{k}^{c}_{0}(\mathbf{y})+\mathsf{k}^{b}_{0}(\mathbf{x})\mathcal{D}^{ba}(\mathbf{x},\mathbf{r})\partial^{2}_{(\mathbf{r})}\mathcal{D}^{ac}(\mathbf{r},\mathbf{y})\mathsf{k}^{c}_{0}(\mathbf{y})\}, \tag{86}$$

where $\mathcal{U}$ and $\mathcal{V}$ as well as $\mathsf{k}^{b}_{0}(\mathbf{x})$ and $\overline{\mathsf{J}}^{b\mathsf{T}}_{0(\mathsf{GI})}$ are defined in Appendix E in Eqs. (E7), (E11), (E16), and (E15), respectively. $[\mathsf{H}]_0$, $[\mathsf{H}]_1$, and $[\mathsf{H}]_2$ are Hermitian, and all consist entirely of gauge-invariant, Hermitian, transverse gauge fields and gauge-invariant quark fields, which all obey "standard" commutation rules. Since $\mathcal{P}^{b\mathsf{T}}_{j}(\mathbf{y})$ and $\Pi^{b\mathsf{T}\dagger}_{\mathsf{GI}j}(\mathbf{y})$ have the same commutator with $A^{a}_{\mathsf{GI}i}(\mathbf{x})$, Eq. (42) also determines the commutation rule

$$[A^{b}_{\mathsf{GI}j}(\mathbf{y}),A^{a}_{\mathsf{GI}i}(\mathbf{x})]=[\mathcal{P}^{b\mathsf{T}}_{j}(\mathbf{y}),\mathcal{P}^{a\mathsf{T}}_{i}(\mathbf{x})]=0,$$

$$[\mathcal{P}_j^{b\mathsf{T}}(\mathbf{y}),A_{\mathsf{GI}i}^a(\mathbf{x})]=-i\,\delta_{ab}\left(\delta_{ij}-\frac{\partial_i\partial_j}{\partial^2}\right)\delta(\mathbf{x}-\mathbf{y}). \quad (87)$$

The sum $[\mathsf{H}]_0+[\mathsf{H}]_1$ is identical in form to the Coulomb-gauge QCD Hamiltonian used by Gribov [11,12], as well as by numerous other authors who have followed him in using this Hamiltonian. $[\mathsf{H}]_2$ consists of additional terms that are required because the transverse gauge-invariant negative chromoelectric field $\Pi_{\mathsf{GI}j}^{b\mathsf{T}}$ is not Hermitian. The elimination of $\Pi_{\mathsf{GI}j}^{b\mathsf{T}}$ and $\Pi_{\mathsf{GI}j}^{b\mathsf{T}\dagger}$ in favor of the Hermitian $\mathcal{P}_j^{b\mathsf{T}}$ is essential for the establishment of the isomorphism between $(\hat{H}_{\mathsf{GI}})_{\rm phys}$ and a Hamiltonian that can be used in a Fock space of perturbative states. We now proceed to the demonstration of this isomorphism.

Since both $A_{\mathsf{GI}i}^a(\mathbf{r})$ and $\mathcal{P}_j^{b\mathsf{T}}(\mathbf{r})$ are Hermitian and obey the commutation rule displayed in Eq. (87), we can represent them as

$$A_{\mathsf{GI}i}^c(\mathbf{r})=\sum_{\mathbf{k},n}\frac{\epsilon_i{}^n(\mathbf{k})}{\sqrt{2k}}[\alpha_n^c(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}}+\alpha_n^{c\dagger}(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{r}}] \quad (88)$$

and

$$\mathcal{P}_i^{c\mathsf{T}}(\mathbf{r})=-i\sum_{\mathbf{k},n}\epsilon_i{}^n(\mathbf{k})\sqrt{\frac{k}{2}}[\alpha_n^c(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}}-\alpha_n^{c\dagger}(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{r}}] \quad (89)$$

where n is summed over two transverse helicity modes and

$$[\alpha_n^a(\mathbf{k}),\alpha_\ell^{b\dagger}(\mathbf{q})]=\delta_{n,\ell}\delta_{a,b}\delta_{\mathbf{k},\mathbf{q}}$$

and

$$[\alpha_n^a(\mathbf{k}),\alpha_\ell^b(\mathbf{q})]=[\alpha_n^{a\dagger}(\mathbf{k}),\alpha_\ell^{b\dagger}(\mathbf{q})]=0. \quad (90)$$

Equations (88) and (89) can be inverted, leading to

$$\alpha_n^c(\mathbf{k})=\sqrt{\frac{k}{2}}\epsilon_i{}^n(\mathbf{k})\int d\mathbf{r}\left(A_{\mathsf{GI}i}^c(\mathbf{r})+\frac{i}{k}\mathcal{P}_i^{c\mathsf{T}}(\mathbf{r})\right)e^{-i\mathbf{k}\cdot\mathbf{r}} \quad (91)$$

and

$$\alpha_n^{c\dagger}(\mathbf{k})=\sqrt{\frac{k}{2}}\epsilon_i{}^n(\mathbf{k})\int d\mathbf{r}\left(A_{\mathsf{GI}i}^c(\mathbf{r})-\frac{i}{k}\mathcal{P}_i^{c\mathsf{T}}(\mathbf{r})\right)e^{i\mathbf{k}\cdot\mathbf{r}}. \quad (92)$$

Equations (91) and (92) show that $\alpha_n^c(\mathbf{k})$ and $\alpha_n^{c\dagger}(\mathbf{k})$ are gauge invariant and commute with the Gauss's law operator $\mathcal{G}^a(\mathbf{r})$. Equations (77), (78) and (80) demonstrate that any functional $\mathcal{F}(A_{\mathsf{GI}i}^a,\mathcal{P}_j^{b\mathsf{T}})$ will have the transformation property

$$\mathcal{F}(A_{\mathsf{GI}i}^a,\mathcal{P}_j^{b\mathsf{T}})\mathcal{J}^{-1/2}\Psi|\phi_\ell\rangle=\mathcal{J}^{-1/2}\Psi\mathcal{F}(A_i^{a\mathsf{T}},\Pi_j^{b\mathsf{T}})|\phi_\ell\rangle \quad (93)$$

leading to

$$\alpha_n^c(\mathbf{k})\mathcal{J}^{-1/2}\Psi|\phi_\ell\rangle=\mathcal{J}^{-1/2}\Psi\left\{\sqrt{\frac{k}{2}}\epsilon_i{}^n(\mathbf{k})\int d\mathbf{r}\times\left(A_i^{c\mathsf{T}}(\mathbf{r})+\frac{i}{k}\Pi_i^{c\mathsf{T}}(\mathbf{r})\right)e^{-i\mathbf{k}\cdot\mathbf{r}}\right\}|\phi_\ell\rangle, \quad (94)$$

and

$$\alpha_n^{c\dagger}(\mathbf{k})\mathcal{J}^{-1/2}\Psi|\phi_\ell\rangle=\mathcal{J}^{-1/2}\Psi\left\{\sqrt{\frac{k}{2}}\epsilon_i{}^n(\mathbf{k})\int d\mathbf{r}\times\left(A_i^{c\mathsf{T}}(\mathbf{r})-\frac{i}{k}\Pi_i^{c\mathsf{T}}(\mathbf{r})\right)e^{i\mathbf{k}\cdot\mathbf{r}}\right\}|\phi_\ell\rangle, \quad (95)$$

so that the isomorphism established in Eq. (93) between the gauge-invariant fields $A_{\mathsf{GI}i}^a$, $\mathcal{P}_j^{b\mathsf{T}}$ and the gauge-dependent Weyl-gauge fields $A_i^{a\mathsf{T}}$, $\Pi_j^{b\mathsf{T}}$, respectively, is transferred to a similar relation between the gauge-invariant creation and annihilation operators for transverse gluons, $\alpha_n^{c\dagger}(\mathbf{k})$ and $\alpha_n^c(\mathbf{k})$, and corresponding "standard" perturbative creation and annihilation operators $a_n^{c\dagger}(\mathbf{k})$ and $a_n^c(\mathbf{k})$. We can proceed by using the standard representation for the transverse part of the Weyl-gauge fields,

$$A_i^{c\mathsf{T}}(\mathbf{r})=\sum_{\mathbf{k},n}\frac{\epsilon_i{}^n(\mathbf{k})}{\sqrt{2k}}[a_n^c(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}}+a_n^{c\dagger}(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{r}}] \quad (96)$$

and

$$\Pi_i^{c\mathsf{T}}(\mathbf{r})=-i\sum_{\mathbf{k},n}\epsilon_i{}^n(\mathbf{k})\sqrt{\frac{k}{2}}[a_n^c(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}}-a_n^{c\dagger}(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{r}}], \quad (97)$$

which demonstrate that

$$\alpha_n^{c\dagger}(\mathbf{k})\mathcal{J}^{-1/2}\Psi|\phi_i\rangle=\mathcal{J}^{-1/2}\Psi a_n^{c\dagger}(\mathbf{k})|\phi_i\rangle,$$
$$\alpha_n^c(\mathbf{k})\mathcal{J}^{-1/2}\Psi|\phi_i\rangle=\mathcal{J}^{-1/2}\Psi a_n^c(\mathbf{k})|\phi_i\rangle. \quad (98)$$

Any $\alpha_n^c(\mathbf{k})$ will annihilate the gauge-invariant vacuum state $\mathcal{J}^{-1/2}\Psi\Xi|0\rangle$, because the transverse excitation operators $a_n^c(\mathbf{k})$ and $a_n^{c\dagger}(\mathbf{k})$ trivially commute with Ξ.

At this point, we can establish an isomorphism between two Fock spaces: The "standard" Weyl-gauge Fock space consists of

$$|\mathbf{k}\rangle=a_n^{c\dagger}(\mathbf{k})|0\rangle \quad (99)$$

$$\cdots$$

$$|\mathbf{k}_1\cdots\mathbf{k}_i\cdots\mathbf{k}_N\rangle=K[a_{n_1}^{c_1\dagger}(\mathbf{k}_1)\cdots a_{n_i}^{c_i\dagger}(\mathbf{k}_i)\cdots a_{n_N}^{c_N\dagger}(\mathbf{k}_N)]|0\rangle, \quad (100)$$

with K the normalization constant and the gauge-invariant states that implement the non-Abelian Gauss's law can be represented as

$$|\overline{\mathbf{k}}\rangle=\frac{1}{C}\alpha_n^{c\dagger}(\mathbf{k})\mathcal{J}^{-1/2}\Psi\Xi|0\rangle \tag{101}$$

$$\cdots$$

$$|\overline{\mathbf{k}}_1\cdots\overline{\mathbf{k}}_i\cdots\overline{\mathbf{k}}_N\rangle=\frac{K}{C}[\alpha_{n_1}^{c_1\dagger}(\mathbf{k}_1)\cdots\alpha_{n_i}^{c_i\dagger}(\mathbf{k}_i)\cdots\alpha_{n_N}^{c_N\dagger}(\mathbf{k}_N)]\times\mathcal{J}^{-1/2}\Psi\Xi|0\rangle \tag{102}$$

where $|0\rangle$ designates the perturbative vacuum annihilated by $a_n^c(\mathbf{k})$ as well as by the annihilation operators for quarks and antiquarks, $q_{\mathbf{p},s}$ and $\overline{q}_{\mathbf{p},s}$ respectively. The additional normalization constant C^{-1} must be introduced to compensate for the fact that $|C|^2=|\mathcal{J}^{-1/2}\Psi\Xi|0\rangle|^2=\langle 0|\Xi^\star\Psi^\star\mathcal{J}^{-1}\Psi\Xi|0\rangle$, which formally is a universal positive constant, is not finite; and the state $\mathcal{J}^{-1/2}\Psi\Xi|0\rangle$ is not normalizable. However, once C is introduced, the $|\overline{\mathbf{k}}_1\cdots\overline{\mathbf{k}}_i\cdots\overline{\mathbf{k}}_N\rangle$ states form a satisfactory Fock space that is gauge invariant as well as isomorphic to the space of $|\mathbf{k}_1\cdots\mathbf{k}_i\cdots\mathbf{k}_N\rangle$ states. We can now use Eqs. (88), and (89) to express $[\mathsf{H}]_0$ as

$$[\mathsf{H}]_0=\sum_{\mathbf{k},c} k\alpha_n^{c\dagger}(\mathbf{k})\alpha_n^c(\mathbf{k})+\sum_{\mathbf{p},s}\mathcal{E}_{\mathbf{p}}(q_{\mathbf{p},s}^\dagger q_{\mathbf{p},s}+\overline{q}_{\mathbf{p},s}^\dagger\overline{q}_{\mathbf{p},s}) \tag{103}$$

with the subscript s labeling the color, flavor and helicity of the quarks. In this form, $[\mathsf{H}]_0$ can be seen to describe the energy of non-interacting gauge-invariant transverse gluons of energy k and quarks and anti-quarks, respectively of energy $\mathcal{E}_{\mathbf{p}}=\sqrt{m^2+|\mathbf{p}|^2}$. We can also define another Hamiltonian, $\mathcal{H}=\mathcal{H}_0+\mathcal{H}_1+\mathcal{H}_2$, in which each component part is identical in form to $[\mathsf{H}]_0+[\mathsf{H}]_1+[\mathsf{H}]_2$, respectively, but with the substitutions

$$\mathcal{P}_j^{b\mathsf{T}}\to\Pi_j^{b\mathsf{T}} \text{ and } A_{\mathsf{GI}i}^a\to A_i^{a\mathsf{T}}$$

everywhere—including the replacement of $A_{\mathsf{GI}i}^a$ by $A_i^{a\mathsf{T}}$ in the inverse Faddeev-Popov operator $\mathcal{D}^{ab}(\mathbf{x},\mathbf{y})$—so that $\mathcal{H}$ is characteristic of the Coulomb gauge, but nevertheless is a functional of transverse Weyl-gauge unconstrained fields. For example, $\mathcal{H}_0$ is

$$\mathcal{H}_0=\int d\mathbf{r}\left[\frac{1}{2}\Pi_i^{a\mathsf{T}}(\mathbf{r})\Pi_i^{a\mathsf{T}}(\mathbf{r})+\frac{1}{4}\hat{F}_{ij}^a(\mathbf{r})\hat{F}_{ij}^a(\mathbf{r})+\psi^\dagger(\mathbf{r})(\beta m-i\alpha_i\partial_i)\psi(\mathbf{r})\right] \tag{104}$$

where $\hat{F}_{ij}^a=\partial_jA_i^a-\partial_iA_j^a$. Using Eqs. (96) and (97), we can express $\mathcal{H}_0$ in the form

$$\mathcal{H}_0=\sum_{\mathbf{k},c} ka_n^{c\dagger}(\mathbf{k})a_n^c(\mathbf{k})+\sum_{\mathbf{p},s}\mathcal{E}_{\mathbf{p}}(q_{\mathbf{p},s}^\dagger q_{\mathbf{p},s}+\overline{q}_{\mathbf{p},s}^\dagger\overline{q}_{\mathbf{p},s}). \tag{105}$$

We can then use Eq. (93) to establish that

$$[\mathsf{H}]_0\mathcal{J}^{-1/2}\Psi\Xi|n\rangle=\mathcal{J}^{-1/2}\Psi\Xi\mathcal{H}_0|n\rangle \tag{106}$$

as well as

$$[\mathsf{H}]_1\mathcal{J}^{-1/2}\Psi\Xi|n\rangle=\mathcal{J}^{-1/2}\Psi\Xi\mathcal{H}_1|n\rangle \tag{107}$$

and

$$[\mathsf{H}]_2\mathcal{J}^{-1/2}\Psi\Xi|n\rangle=\mathcal{J}^{-1/2}\Psi\Xi\mathcal{H}_2|n\rangle. \tag{108}$$

The state vector $|n\rangle$ represents one of the $|\mathbf{k}_1\cdots\mathbf{k}_i\cdots\mathbf{k}_N\rangle$, the "standard" perturbative eigenstates of $\mathcal{H}_0$.

We can use the relations between the Weyl-gauge and the gauge-invariant states that we established in the preceding discussion to extend the isomorphism we have demonstrated to include scattering transition amplitudes. For this purpose, we define

$$\mathcal{H}_{\rm int}=\mathcal{H}_1+\mathcal{H}_2 \text{ and } [\mathsf{H}]_{\rm int}=[\mathsf{H}]_1+[\mathsf{H}]_2. \tag{109}$$

The transition amplitude between gauge-invariant states is given by

$$\overline{\mathsf{T}}_{f,i}=\frac{1}{C^2}\langle f|\Xi^\star\Psi^\star\mathcal{J}^{-1/2}\left\{[\mathsf{H}]_{\rm int}+[\mathsf{H}]_{\rm int}\times\frac{1}{E_i-(\hat{H}_{\mathsf{GI}})_{\rm phys}+i\epsilon}[\mathsf{H}]_{\rm int}\right\}\mathcal{J}^{-1/2}\Psi\Xi|i\rangle, \tag{110}$$

where $|i\rangle$ and $|f\rangle$ each designate one of the $|n\rangle$ states; $|i\rangle$ represents an incident and $|f\rangle$ a final state in a scattering process. With the results of the preceding discussion, we can express this as

$$\begin{aligned}\overline{\mathsf{T}}_{f,i}&=\frac{1}{C^2}\langle f|\Xi^\star\Psi^\star\mathcal{J}^{-1}\Psi\Xi|n\rangle\langle n|\times\left\{\mathcal{H}_{\rm int}+\mathcal{H}_{\rm int}\frac{1}{E_i-\mathcal{H}_0-\mathcal{H}_{\rm int}+i\epsilon}\mathcal{H}_{\rm int}\right\}|i\rangle\\&=\frac{1}{C^2}\langle 0|\Xi^\star\Psi^\star\mathcal{J}^{-1}\Psi\Xi|0\rangle\langle f|\times\left\{\mathcal{H}_{\rm int}+\mathcal{H}_{\rm int}\frac{1}{E_i-\mathcal{H}_0-\mathcal{H}_{\rm int}+i\epsilon}\mathcal{H}_{\rm int}\right\}|i\rangle\end{aligned} \tag{111}$$

where we sum over the complete set of perturbative states $|n\rangle\langle n|$. The third line of Eq. (111) follows from

$$\frac{1}{C^2}\langle f|\Xi^\star\Psi^\star\mathcal{J}^{-1}\Psi\Xi|n\rangle=\frac{1}{C^2}\langle 0|a_f(\mathbf{k}_f)\Xi^\star\Psi^\star\mathcal{J}^{-1}\Psi\Xi a_n^\dagger(\mathbf{k}_n)|0\rangle$$
$$=\frac{1}{C^2}\langle 0|\Xi^\star\Psi^\star\mathcal{J}^{-1/2}\alpha_f(\mathbf{k}_f)\alpha_n^\dagger(\mathbf{k}_n)\mathcal{J}^{-1/2}\Psi\Xi|0\rangle$$
$$=\delta_{f,n}\delta(\mathbf{k}_f-\mathbf{k}_n)\frac{1}{C^2}\langle 0|\Xi^\star\Psi^\star\mathcal{J}^{-1}\Psi\Xi|0\rangle-\frac{1}{C^2}\langle 0|\Xi^\star\Psi^\star\mathcal{J}^{-1/2}\alpha_n^\dagger(\mathbf{k}_n)\alpha_f(\mathbf{k}_f)\mathcal{J}^{-1/2}\Psi\Xi|0\rangle \tag{112}$$

and the observation that the last term on the last line of Eq. (112) vanishes trivially. With the isomorphism of the states $|\mathbf{k}_1\cdots\mathbf{k}_i\cdots\mathbf{k}_N\rangle$ and $|\overline{\mathbf{k}_1}\cdots\overline{\mathbf{k}_i}\cdots\overline{\mathbf{k}_N}\rangle$ that we have established,

$$\overline{\mathsf{T}}_{f,i}=T_{f,i} \tag{113}$$

where

$$T_{f,i}=\langle f|\left\{\mathcal{H}_{\rm int}+\mathcal{H}_{\rm int}\frac{1}{E_i-\mathcal{H}_0-\mathcal{H}_{\rm int}+i\epsilon}\mathcal{H}_{\rm int}\right\}|i\rangle \tag{114}$$

is a transition amplitude that can be evaluated with Feynman graphs and rules, because it is based on "standard" perturbative states that are not required to implement Gauss's law and need not be gauge invariant.

In the remainder of this section, we will discuss the relation of our formulation of the scattering transition amplitude to approaches to this problem in Coulomb-gauge formulations of QCD. As was pointed out in Sec. II B, the effective Hamiltonian $(\hat{H}_{\sf GI})_{\rm phys}$ described in Eq. (60) is identical to one obtained by Christ and Lee [32], who treated gauge fields as coordinates and applied the apparatus of transformations from Cartesian to curvilinear coordinates to the problem of formulating Coulomb-gauge QCD. Here, we will show that $\hat{H}_{\sf GI}$—the precursor of $(\hat{H}_{\sf GI})_{\rm phys}$, described in Eq. (20)—is identical in form to the Hamiltonian given in Eq. (6.15) in Ref. [32], which leads to the Coulomb-gauge perturbative rules formulated by Christ and Lee. For this purpose, $\hat{H}_{\sf GI}$ will be expressed in terms of $\mathcal{P}_j^{b\sf T}$ and $A_{{\sf GI}i}^a$, and then Weyl ordered. The equivalence of Christ and Lee's results with Schwinger's [36] was already confirmed in Ref. [32].

Equation (82) demonstrates that the functional dependence of $\hat{H}_{\sf GI}$ on $\mathcal{P}_j^{b\sf T}$ and $A_{{\sf GI}i}^a$ is the same as the functional dependence of

$$\overline{H}\equiv\mathcal{J}^{1/2}\hat{H}_{\sf GI}\mathcal{J}^{-1/2} \tag{115}$$

on $\Pi_j^{b\sf T}$ and $A_{{\sf GI}i}^a$. $\overline{H}$ was used by Christ and Lee to generate the path integral representation of the Coulomb gauge [32], and they showed that

$$\overline{H}=\hat{H}_{\sf GI}^W+\mathcal{V}_1+\mathcal{V}_2 \tag{116}$$

where the superscript W designates Weyl-ordering with respect to $\Pi_{{\sf GI}j}^{b\sf T}$ and $A_{{\sf GI}i}^a$. The additional terms $\mathcal{V}_1$ and $\mathcal{V}_2$ are given by

$$\mathcal{V}_1=\frac{1}{8}g^2f^{lbc}f^{lad}\int d\mathbf{r}\partial_j\mathcal{D}^{ab}(\mathbf{r},\mathbf{r})\partial_j\mathcal{D}^{cd}(\mathbf{r},\mathbf{r}) \tag{117}$$

and

$$\mathcal{V}_2=\frac{1}{8}g^2f^{lna}f^{kbm}\int d\mathbf{x}d\mathbf{y}d\mathbf{z}[\delta_{kn}\delta_{ji}\delta(\mathbf{y}-\mathbf{x})+D_j\mathcal{D}^{kn}(\mathbf{y},\mathbf{x})\overleftarrow{\partial}_i]\mathcal{D}^{ac}(\mathbf{x},\mathbf{z})\partial_{(\mathbf{z})}^2\mathcal{D}^{cb}(\mathbf{z},\mathbf{y})[\delta_{lm}\delta_{ij}\delta(\mathbf{x}-\mathbf{y})+D_i\mathcal{D}^{lm}(\mathbf{x},\mathbf{y})\overleftarrow{\partial}_j] \tag{118}$$

where the partial derivative ∂_j to the left of $\mathcal{D}^{ab}(\mathbf{r},\mathbf{r})$ acts only on its first argument. When a partial derivative with a left arrow on top appears to the right of $\mathcal{D}^{ab}$ with two identical arguments, it acts only on its second argument. The case of two identical arguments of $\mathcal{D}$ is understood as $\partial_j\mathcal{D}^{ab}(\mathbf{x},\mathbf{x})\equiv\lim_{\mathbf{y}\to\mathbf{x}}(\partial/\partial x_j)\mathcal{D}^{ab}(\mathbf{x},\mathbf{y})$ and $\mathcal{D}^{ab}(\mathbf{x},\mathbf{x})\overleftarrow{\partial}_j\equiv\lim_{\mathbf{y}\to\mathbf{x}}(\partial/\partial x_j)\mathcal{D}^{ab}(\mathbf{y},\mathbf{x})$, where the limit is taken *after* the partial derivative has been evaluated. This convention will be followed consistently in the following discussions. Since the commutator of $\mathcal{P}_j^{b\sf T}$ and $A_{{\sf GI}i}^a$ is identical to that of $\Pi_{{\sf GI}j}^{b\sf T}$ and $A_{{\sf GI}i}^a$, an equation parallel to Eq. (116),

$$\hat{H}_{\sf GI}=\hat{H}_{\sf GI}^W+\mathcal{V}_1+\mathcal{V}_2 \tag{119}$$

will be proven below. The superscript W designates Weyl ordering, but in this case with respect to $\mathcal{P}_j^{b\sf T}$ and $A_{{\sf GI}i}^a$. The parallel structure refers to the fact that, as was pointed out above, $\hat{H}_{\sf GI}$ has the same functional dependence on $\mathcal{P}_j^{b\sf T}$ and $A_{{\sf GI}i}^a$ as $\overline{H}$ has on $\Pi_{{\sf GI}j}^{b\sf T}$ and $A_{{\sf GI}i}^a$. Since the fermion variables commute with $\mathcal{P}_j^{b\sf T}$ and $A_{{\sf GI}i}^a$, we may drop them for the proof of Eq. (119); we will also drop $H_{\mathcal{G}}$, since it makes no contributions in the space of gauge-invariant states. It follows from Eq. (58) that for a physical state $|\Psi\rangle$,

$$\Pi_{{\sf GI}j}^b(\mathbf{r})|\Psi\rangle=-\mathcal{E}_j^b(\mathbf{r})|\Psi\rangle \tag{120}$$

with

$$\mathcal{E}_j^b(\mathbf{r})=\mathcal{J}^{1/2}\bigg[-\mathcal{P}_j^{b\mathsf{T}}(\mathbf{r})+\int d\mathbf{x}\partial_j\mathcal{D}^{bq}(\mathbf{r},\mathbf{x})\times D_i(\mathbf{x})\mathcal{P}_i^{q\mathsf{T}}(\mathbf{x})\bigg]\mathcal{J}^{-1/2}. \tag{121}$$

In terms of the Weyl-ordered chromoelectric field operator of Schwinger [9],

$$E_j^b(\mathbf{r})=-\mathcal{P}_j^{b\mathsf{T}}(\mathbf{r})+\frac{1}{2}\int d\mathbf{x}[\partial_j\mathcal{D}^{bq}(\mathbf{r},\mathbf{x})D_i(\mathbf{x})\mathcal{P}_i^{q\mathsf{T}}(\mathbf{x})+D_i(\mathbf{x})\mathcal{P}_i^{q\mathsf{T}}(\mathbf{x})\partial_j\mathcal{D}^{bq}(\mathbf{r},\mathbf{x})], \tag{122}$$

we have

$$\mathcal{E}_j^b(\mathbf{r})=E_j^b(\mathbf{r})+\Delta_j^b(\mathbf{r})$$
$$\mathcal{E}_j^b(\mathbf{r})^\dagger=E_j^b(\mathbf{r})-\Delta_j^b(\mathbf{r}). \tag{123}$$

The Hamiltonian $\hat{H}_{\mathsf{GI}}$, in the absence of the fermion field and without $H_{\mathcal{G}}$ can be written as

$$\hat{H}_{\mathsf{GI}}=K+\frac{1}{4}\int d\mathbf{r}F_{\mathsf{GI}ij}^a(\mathbf{r})F_{\mathsf{GI}ij}^a(\mathbf{r}) \tag{124}$$

where the kinetic energy

$$K=\frac{1}{2}\int d\mathbf{r}\mathcal{E}_j^b(\mathbf{r})^\dagger\mathcal{E}_j^b(\mathbf{r})=\frac{1}{2}\int d\mathbf{r}[E_j^b(\mathbf{r})E_j^b(\mathbf{r})+v(\mathbf{r})] \tag{125}$$

with

$$v(\mathbf{r})=-\Delta_j^b(\mathbf{r})\Delta_j^b(\mathbf{r})+[E_j^b(\mathbf{r}),\Delta_j^b(\mathbf{r})]. \tag{126}$$

To evaluate $\Delta_j^b(\mathbf{r})$, we observe that

$$\mathcal{E}_j^b(\mathbf{r})=-\mathcal{P}_j^{b\mathsf{T}}(\mathbf{r})+\frac{1}{2}[\mathcal{P}_j^{b\mathsf{T}}(\mathbf{r}),\ln(\mathcal{J})]+\int d\mathbf{x}\partial_j\mathcal{D}^{bq}(\mathbf{r},\mathbf{x})D_i(\mathbf{x})\mathcal{P}_i^{q\mathsf{T}}(\mathbf{x})-\frac{1}{2}\int d\mathbf{x}\partial_j\mathcal{D}^{bq}(\mathbf{r},\mathbf{x})D_i[\mathcal{P}_i^{q\mathsf{T}}(\mathbf{x}),\ln(\mathcal{J})] \tag{127}$$

and

$$\mathcal{E}_j^b(\mathbf{r})^\dagger=-\mathcal{P}_j^{b\mathsf{T}}(\mathbf{r})-\frac{1}{2}[\mathcal{P}_j^{b\mathsf{T}}(\mathbf{r}),\ln(\mathcal{J})]+\int d\mathbf{x}D_i(\mathbf{x})\mathcal{P}_i^{q\mathsf{T}}(\mathbf{x})\partial_j\mathcal{D}^{bq}(\mathbf{r},\mathbf{x})+\frac{1}{2}\int d\mathbf{x}\partial_j\mathcal{D}^{bq}(\mathbf{r},\mathbf{x})D_i[\mathcal{P}_i^{q\mathsf{T}}(\mathbf{x}),\ln(\mathcal{J})] \tag{128}$$

so that

$$\frac{1}{2}[\mathcal{E}_j^b(\mathbf{r})+\mathcal{E}_j^b(\mathbf{r})^\dagger]=E_j^b(\mathbf{r}) \tag{129}$$

and, therefore, that

$$\frac{1}{2}[\mathcal{E}_j^b(\mathbf{r})-\mathcal{E}_j^b(\mathbf{r})^\dagger]=\Delta_j^b(\mathbf{r}). \tag{130}$$

With Eqs. (120) and (68), this leads to

$$\Delta_j^b(\mathbf{r})=-\frac{i}{2}gf^{bch}\frac{\partial}{\partial r_j}\mathcal{D}^{ch}(\mathbf{r},\mathbf{r}). \tag{131}$$

In Appendix F, we shall prove that

$$\frac{1}{2}\int d\mathbf{r}v(\mathbf{r})=\mathcal{V}_1. \tag{132}$$

In the form given in Eq. (124) with K as described in Eq. (125), the effective Hamiltonian ($\hat{H}_{\mathsf{GI}}$) is identified with that of Schwinger [36]. The next step towards the proof of Eq. (119) follows from the operator identity given in Ref. [32]

$$\frac{1}{2}\int d\mathbf{r}E_j^b(\mathbf{r})E_j^b(\mathbf{r})=\frac{1}{2}\int d\mathbf{r}[E_j^b(\mathbf{r})E_j^b(\mathbf{r})]^W-\frac{1}{8}\int d\mathbf{x}d\mathbf{y}d\mathbf{z}[\mathcal{P}_i^{a\mathsf{T}}(\mathbf{x}),D_k\mathcal{D}^{bc}(\mathbf{x},\mathbf{z})\overleftarrow{\partial}_j]\times[\mathcal{P}_k^{b\mathsf{T}}(\mathbf{y}),D_i\mathcal{D}^{ac}(\mathbf{y},\mathbf{z})\overleftarrow{\partial}_j]. \tag{133}$$

Using the commutation relation (E10), we can show that the second term on the right-hand side of Eq. (133) is the same as $\mathcal{V}_2$ (the same proof is also given in Ref. [32]) and Eq. (119) is established.

IV. DISCUSSION

In this work we have used earlier results [1,5,6] to express the Weyl-gauge Hamiltonian entirely in terms of operator-valued fields that are gauge invariant as well as path independent. These gauge-invariant fields have many features in common with Coulomb-gauge fields: Their commutation rules agree with those given by Schwinger in his Coulomb-gauge formulation of QCD [9,36], except for differences in operator order; these differences can be ascribed to the fact that Schwinger imposed Weyl order in his work while we do not make any *ad hoc* changes in operator order. The gauge-invariant gauge field is transverse and Hermitian, but the gauge-invariant chromoelectric field is neither transverse nor Hermitian. Even the transverse part of the gauge-invariant chromoelectric field is not Hermitian. That fact is important for relating the Hamiltonian we obtained in Eq. (84) with those given by Gribov [11], Schwinger [36], and Christ and Lee [32].

The relation between the Coulomb-gauge Hamiltonian for QCD and the Weyl-gauge Hamiltonian expressed in terms of gauge-invariant fields closely parallels the relation between the two corresponding QED Hamiltonians. The Weyl-gauge Hamiltonian for QCD is represented entirely in terms of gauge-invariant fields in Eqs. (20) and (24). When formulated in terms of gauge-invariant fields, QCD must be embedded in a space of gauge-invariant states that obey the non-Abelian Gauss's law. Within such a space of gauge-invariant states, further transformation of the QCD Hamiltonian we have constructed can be effected. Thus transformed, the Hamiltonian consists of two parts. One part, $(\hat{H}_{\mathrm{GI}})_{\mathrm{phys}}$—displayed in Eq. (60)—is identical to the Coulomb-gauge Hamiltonian. It is a functional of transverse gauge-invariant chromoelectric fields, gauge-invariant gauge fields (which are inherently transverse), as well as gauge-invariant quark fields. The other part, $H_{\mathcal{G}}$—displayed in Eq. (24)—makes only vanishing contributions to matrix elements within the space of gauge-invariant states that are required for the Hamiltonian to act consistently as the time-evolution operator. $H_{\mathcal{G}}$ does affect the field equations and "remembers" that the formulation is for the Weyl, and not the Coulomb gauge. This situation is precisely the same as in QED, in which the Weyl-gauge Hamiltonian, expressed in terms of the gauge-invariant field (in that case, simply the transverse part of the gauge field), is the sum of two terms, given in Eqs. (45) and (46); the former is the Coulomb-gauge Hamiltonian, and the latter makes only vanishing contributions to matrix elements within the space of gauge-invariant states, but is necessary for reproducing the Euler-Lagrange equations for Weyl-gauge QED.

In spite of the similarity between QCD and QED in the relation between the Weyl and Coulomb gauges summarized in the preceding paragraph, there is an important difference between the gauge-invariant states for the two theories: gauge-invariant and perturbative states in QED are unitarily equivalent, and in a Hamiltonian formulation, this unitary equivalence permits us to use perturbative states in evaluating scattering amplitudes in QED in algebraic and covariant gauges without compromising the implementation of Gauss's law [22,23]. But there can be no unitary equivalence between gauge-invariant states and perturbative states in QCD. And the gauge-invariant states in QCD are complicated, not normalizable, and very cumbersome to use. In order to make effective use of the Weyl-gauge QCD Hamiltonian represented in terms of gauge-invariant fields, some relation is required that allows us to circumvent the absence of the unitary equivalence between gauge-invariant and perturbative states that afflicts non-Abelian gauge theories. In Sec. III we establish such a relation in the form of an isomorphism that enables us to consistently carry out calculations in QCD with an equivalent Hamiltonian that is a functional of the original gauge-dependent Weyl-gauge fields and that is used with standard perturbative states. In the case of QCD, this isomorphism has been demonstrated for the Weyl gauge only. An extension to a somewhat larger class of algebraic gauges defined by $A_0+\gamma A_3=0$ with $\gamma\geqslant 0$ should not be difficult [40] but, in contrast to QED, there is no indication that further extensions—to covariant gauges, for example—are possible. Finally, in Sec. III, we show that the effective Hamiltonian $(\hat{H}_{\mathrm{GI}})_{\mathrm{phys}}$—and therefore also $\mathcal{H}=\mathcal{H}_0+\mathcal{H}_1+\mathcal{H}_2$—can be expressed in appropriately Weyl-ordered forms and shown to be equivalent to results obtained by Schwinger [36] and by Christ and Lee [32]. The Hamiltonian used by Gribov in Ref. [11] is equivalent to only $\mathcal{H}=\mathcal{H}_0+\mathcal{H}_1$. $\mathcal{H}_2$ does not appear in that work, because the non-Hermiticity of the transverse chromoelectric field was not taken into account.

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APPENDIX A

In this section we will prove Eqs. (31) and (30). We use Eqs. (34)–(36) and expand the product $\mathcal{D}^{ah}(\mathbf{y},\mathbf{x})\overleftarrow{\overleftarrow{\partial\cdot D^{hb}}}_{(\mathbf{x})}$ in Eq. (31) as a series in powers of g, and observe that $\mathrm{O}(n)$ terms originate from $\mathcal{D}^{ah}_{(n)}(\mathbf{y},\mathbf{x})(\delta_{hb}\overleftarrow{\partial^2}_{(\mathbf{x})})$ and from $\mathcal{D}^{ah}_{(n-1)}(\mathbf{y},\mathbf{x})(-gf^{hqb}\overleftarrow{\partial_i}A^q_{\mathrm{GI}i})$. For example, the first part of the $n=0$ term of Eq. (31) originates from the $\delta_{hb}\overleftarrow{\partial^2}$ part of $\overleftarrow{\partial\cdot D^{hb}}(\mathbf{x})$ and is

$$\frac{-\delta_{ah}}{4\pi|\mathbf{y}-\mathbf{x}|}(\delta_{hb}\overleftarrow{\partial^2}_{(\mathbf{x})})=\partial^2_{(\mathbf{x})}\frac{-\delta_{ab}}{4\pi|\mathbf{x}-\mathbf{y}|}=\delta_{ab}\delta(\mathbf{x}-\mathbf{y}) \tag{A1}$$

and the second part of the $n=0$ term of Eq. (31), which stems from the $-gf^{hqb}\overleftarrow{\partial_i}A^q_{\mathrm{GI}i}$ in $\overleftarrow{\overleftarrow{\partial\cdot D^{hb}}}(\mathbf{x})$, is

$$\frac{-\delta_{ah}}{4\pi|\mathbf{y}-\mathbf{x}|}(-gf^{hqb}\overleftarrow{\partial}_i A^q_{\mathsf{Gl}i}(\mathbf{x}))$$
$$=-gf^{aqb}\partial_{i(\mathbf{y})}\frac{1}{4\pi|\mathbf{y}-\mathbf{x}|}A^q_{\mathsf{Gl}i}(\mathbf{x}). \tag{A2}$$

The first part of the $n=1$ term of Eq. (31)

$$\left\{gf^{qah}\int\frac{d\mathbf{z}}{4\pi|\mathbf{y}-\mathbf{z}|}A^q_{\mathsf{Gl}j}(\mathbf{z})\frac{\partial}{\partial z_j}\left(\frac{1}{4\pi|\mathbf{z}-\mathbf{x}|}\right)\right\}\delta_{hb}\overleftarrow{\partial^2}_{(\mathbf{x})}$$
$$=gf^{aqb}\partial_{i(\mathbf{y})}\frac{1}{4\pi|\mathbf{y}-\mathbf{x}|}A^q_{\mathsf{Gl}i}(\mathbf{x}) \tag{A3}$$

exactly cancels the second part of the $n=0$ term, and this pattern of cancellation can easily be seen to hold in general—the first part of the $n+1$ term cancelling the second part of the nth term. For the general term,

$$\mathcal{D}^{ah}_{(n)}(\mathbf{y},\mathbf{x})\delta_{hb}\overleftarrow{\partial^2}_{(\mathbf{x})}=-f^{\alpha_1 as_1}f^{s_1\alpha_2 s_2}\cdots f^{s_{n-1}\alpha_n h}g^n\int\frac{d\mathbf{z}(1)}{4\pi|\mathbf{y}-\mathbf{z}(1)|}A^{\alpha_1}_{\mathsf{Gl}l_1}(\mathbf{z}(1))\frac{\partial}{\partial z(1)_{l_1}}$$
$$\times\int\frac{d\mathbf{z}(2)}{4\pi|\mathbf{z}(1)-\mathbf{z}(2)|}\cdots A^{\alpha_{(n-1)}}_{\mathsf{Gl}l}[\mathbf{z}(n-1)]\frac{\partial}{\partial z(n-1)_l}\frac{\partial}{\partial z(n-1)_k}\frac{1}{4\pi|\mathbf{z}(n-1)-\mathbf{x}|}A^{\alpha_n}_{\mathsf{Gl}k}(\mathbf{x}) \tag{A4}$$

and $\{\mathcal{D}^{ah}_{(n-1)}(\mathbf{y},\mathbf{x})\}\{-gf^{hqb}\overleftarrow{\partial}_i A^b_{\mathsf{Gl}i}(\mathbf{x})\}$ is

$$-\mathcal{D}^{ah}_{(n-1)}(\mathbf{y},\mathbf{x})gf^{hqb}\overleftarrow{\partial}_i A^b_{\mathsf{Gl}i}(\mathbf{x})=f^{\alpha_1 as_1}f^{s_1\alpha_2 s_2}\cdots f^{s_{n-2}\alpha_{n-1}h}f^{hqb}g^n\int\frac{d\mathbf{z}(1)}{4\pi|\mathbf{y}-\mathbf{z}(1)|}A^{\alpha_1}_{\mathsf{Gl}l_1}[\mathbf{z}(1)]\frac{\partial}{\partial z(1)_{l_1}}$$
$$\times\int\frac{d\mathbf{z}(2)}{4\pi|\mathbf{z}(1)-\mathbf{z}(2)|}A^{\alpha_2}_{\mathsf{Gl}l_2}[\mathbf{z}(2)]\frac{\partial}{\partial z(2)_{l_2}}\cdots A^{\alpha_n}_{\mathsf{Gl}i}[\mathbf{z}(n-1)]\frac{\partial}{\partial z(n-1)_l}$$
$$\times\frac{\partial}{\partial z(n-1)_i}\frac{1}{4\pi|\mathbf{z}(n-1)-\mathbf{x}|}A^q_{\mathsf{Gl}i}(\mathbf{x}) \tag{A5}$$

so that, relabeling dummy indices $h\to s_{n-1}$ and $q\to\alpha_n$, we obtain

$$\mathcal{D}^{ah}_{(n)}(\mathbf{y},\mathbf{x})\overleftarrow{\partial^2_{(\mathbf{x})}}-\mathcal{D}^{ah}_{(n-1)}(\mathbf{y},\mathbf{x})gf^{hqb}\overleftarrow{\partial}_i A^b_{\mathsf{Gl}i}(\mathbf{x})=0$$

and a consistent pattern of cancellations is established with $\delta_{ab}\delta(\mathbf{y}-\mathbf{x})$ remaining as the only surviving term in $\mathcal{D}^{ah}(\mathbf{y},\mathbf{x})\partial\cdot\overleftarrow{D^{hb}}(\mathbf{x})$. A similar argument can be used to demonstrate Eq. (30).

APPENDIX B

In this section we shall prove the identity given in Eq. (66). By definition—Eq. (11)—we have

$$\mathcal{J}^{-1}\Pi^b_{\mathsf{Gl}j}(\mathbf{x})\mathcal{J}=\Pi^b_{\mathsf{Gl}j}(\mathbf{x})+I^b_j(\mathbf{x}) \tag{B1}$$

with

$$I^b_j(\mathbf{x})=\mathcal{J}^{-1}[\Pi^b_{\mathsf{Gl}j}(\mathbf{x}),\mathcal{J}]=R_{bc}(\mathbf{x})\mathcal{J}^{-1}[\Pi^c_j(\mathbf{x}),\mathcal{J}] \tag{B2}$$

so that

$$I_j^b(\mathbf{x}) = -iR_{bc}(\mathbf{x})\frac{\delta \ln \mathcal{J}}{\delta A_j^c(\mathbf{x})} = -iR_{bc}(\mathbf{x})\int d\mathbf{y}\int d\mathbf{z}\mathcal{D}^{mn}(\mathbf{y},\mathbf{z})\frac{\delta}{\delta A_j^c(\mathbf{x})}\partial\cdot D^{nm}(\mathbf{z})\,\delta(\mathbf{z}-\mathbf{y})$$

$$= igf^{lnm}R_{bc}(\mathbf{x})\int d\mathbf{y}\int d\mathbf{z}\mathcal{D}^{mn}(\mathbf{y},\mathbf{z})\frac{\delta A_{\mathrm{GI}i}^l(\mathbf{z})}{\delta A_j^c(\mathbf{x})}\frac{\partial}{\partial z_i}\delta(\mathbf{z}-\mathbf{y})$$

$$= -gf^{lnm}R_{bc}(\mathbf{x})\int d\mathbf{y}\int d\mathbf{z}\mathcal{D}^{mn}(\mathbf{y},\mathbf{z})[\Pi_j^c(\mathbf{x}),A_{\mathrm{GI}i}^l(\mathbf{z})]\frac{\partial}{\partial z_i}\delta(\mathbf{z}-\mathbf{y}) \tag{B3}$$

where we have used Eq. (67). Substituting Eq. (39) and using

$$\Pi_j^c(\mathbf{x}) = R_{bc}(\mathbf{x})\Pi_{\mathrm{GI}j}^b(\mathbf{x}), \tag{B4}$$

we obtain that

$$I_j^b(\mathbf{x}) = igf^{bca}\frac{\partial}{\partial x_j}\mathcal{D}^{ac}(\mathbf{x},\mathbf{x}) + \frac{\partial}{\partial x_j}\phi^b(\mathbf{x}) \tag{B5}$$

where the gradient acts only on the first argument of $\mathcal{D}^{ac}$ and the longitudinal term comes from the second term of Eq. (39),

$$\phi^b(\mathbf{x}) = igf^{bca}\int d\mathbf{y}\frac{\partial}{\partial y_j}\mathcal{D}^{ac}(\mathbf{y},\mathbf{y})\mathcal{D}^{mn}(\mathbf{x},\mathbf{y})\overleftarrow{D_j^{nm}}(\mathbf{y}) \tag{B6}$$

with $\partial/\partial y_i$ acting on the first argument of $\mathcal{D}^{ac}(\mathbf{y},\mathbf{y})$. Comparing the transverse part of Eq. (B5) with that of Eq. (68), Eq. (66) is proved.

APPENDIX C

To prove the Hermiticity of the Faddeev-Popov determinant $\mathcal{J}$ as an operator in the Hilbert space of states, we recall the criterion that an operator is Hermitian if its expectation values with respect to all states are real. In the coordinate representation of states for which A_{GI} is diagonalized and corresponds to a c-number field configuration, the expectation value of an operator which is a functional of the operator A_{GI} is equal to the same functional of the c-number field configuration A_{GI}. For each c-number field configuration, the Faddeev-Popov operator, $\partial_j D_j$, with D_j denoting the covariant derivative, $D_j^{ab} = \delta^{ab}\partial_j - gf^{abc}A_{\mathrm{GI}j}^c$, becomes an operator with respect to space coordinates and group indices. We have

$$\partial_j^\dagger = -\partial_j \tag{C1}$$

and

$$D_j^\dagger = -D_j, \tag{C2}$$

with the dagger referring to space coordinates and group indices. Therefore

$$(\partial_j D_j)^\dagger = D_j\partial_j = \partial_j D_j, \tag{C3}$$

where the last step follows from the transversality of A_{GI}. Therefore the Faddeev-Popov operator is Hermitian with space coordinates and group indices for any field configuration. Its determinant, $\mathcal{J}$, must be real. The Hermiticity of $\mathcal{J}$ in the Hilbert space of states is established according to our criterion, and the Hermiticity of $\mathcal{J}^{1/2}$ is an obvious corollary.

To derive the commutation relations among $\mathcal{P}_j^l(\mathbf{x})$'s and $A_{\mathrm{GI}j}^l(\mathbf{x})$'s, we notice that

$$\mathcal{P}_j^l(\mathbf{x}) = \Pi_{\mathrm{GI}j}^{l\mathsf{T}}(\mathbf{x}) + \frac{1}{2}[\Pi_{\mathrm{GI}j}^{l\mathsf{T}}(\mathbf{x}),\ln\mathcal{J}] \tag{C4}$$

with the second term a functional of $A_{\mathrm{GI}j}^l(\mathbf{x})$ only. Then we have

$$[\mathcal{P}_i^a(\mathbf{x}),A_{\mathrm{GI}j}^b(\mathbf{y})] = [\Pi_{\mathrm{GI}i}^{a\mathsf{T}}(\mathbf{x}),A_{\mathrm{GI}j}^b(\mathbf{y})]. \tag{C5}$$

Furthermore,

$$\begin{aligned}[\mathcal{P}_i^a(\mathbf{x}),\mathcal{P}_j^b(\mathbf{y})] &= [\Pi_{\mathrm{GI}i}^{a\mathsf{T}}(\mathbf{x}),\Pi_{\mathrm{GI}j}^{b\mathsf{T}}(\mathbf{y})] \\ &\quad + \frac{1}{2}[\Pi_{\mathrm{GI}i}^{a\mathsf{T}}(\mathbf{x}),[\Pi_{\mathrm{GI}j}^{b\mathsf{T}}(\mathbf{y}),\ln\mathcal{J}]] \\ &\quad + \frac{1}{2}[[\Pi_{\mathrm{GI}i}^{a\mathsf{T}}(\mathbf{x}),\ln\mathcal{J}],\Pi_{\mathrm{GI}j}^{b\mathsf{T}}(\mathbf{y})] \\ &= [\Pi_{\mathrm{GI}i}^{a\mathsf{T}}(\mathbf{x}),\Pi_{\mathrm{GI}j}^{b\mathsf{T}}(\mathbf{y})] = 0,\end{aligned} \tag{C6}$$

where the Jacobian identity

$$\begin{aligned}&[\Pi_{\mathrm{GI}i}^{a\mathsf{T}}(\mathbf{x}),[\Pi_{\mathrm{GI}j}^{b\mathsf{T}}(\mathbf{y}),\ln\mathcal{J}]] + [[\Pi_{\mathrm{GI}i}^{a\mathsf{T}}(\mathbf{x}),\ln\mathcal{J}],\Pi_{\mathrm{GI}j}^{b\mathsf{T}}(\mathbf{y})] \\ &\quad = -[\ln\mathcal{J},[\Pi_{\mathrm{GI}i}^{a\mathsf{T}}(\mathbf{x}),\Pi_{\mathrm{GI}j}^{b\mathsf{T}}(\mathbf{y})]] = 0\end{aligned} \tag{C7}$$

has been employed. Therefore the commutation relations among $\mathcal{P}_j^l(\mathbf{x})$'s and $A_{\mathrm{GI}j}^l(\mathbf{x})$'s remain canonical.

APPENDIX D

In this section we shall prove Eq. (78). Using Eq. (76), we define

$$\Psi_n^{\mathsf{T}} = \int d\mathbf{r_1}\ldots d\mathbf{r_n}\overline{\mathcal{A}_{j_1}^{b_1\mathsf{T}}}(\mathbf{r}_1)\ldots\overline{\mathcal{A}_{j_n}^{b_n\mathsf{T}}}(\mathbf{r}_n)\Pi_{j_1}^{b_1\mathsf{T}}(\mathbf{r}_1)\ldots \times\Pi_{j_n}^{b_n\mathsf{T}}(\mathbf{r}_n), \tag{D1}$$

from which it follows that

$$[\Pi_i^a(\mathbf{x}),\Psi_n^{\mathsf{T}}]=n\int d\mathbf{y}[\Pi_i^a(\mathbf{x}),\overline{\mathcal{A}_j^{b\mathsf{T}}}(\mathbf{y})]\Psi_{n-1}^{\mathsf{T}}\Pi_j^b(\mathbf{y}). \tag{D2}$$

This leads to

$$\Pi_i^a(\mathbf{x})\Psi^{\mathsf{T}}=\Psi^{\mathsf{T}}\Pi_i^a(\mathbf{x})+i\int d\mathbf{y}[\Pi_i^a(\mathbf{x}),\overline{\mathcal{A}_j^{b\mathsf{T}}}(\mathbf{y})]\Psi^{\mathsf{T}}\Pi_j^b(\mathbf{y}). \tag{D3}$$

The commutator involved can be calculated from the relation

$$\overline{\mathcal{A}_j^{b\mathsf{T}}}(\mathbf{x})=A_{\mathsf{GI}j}^b(\mathbf{x})-A_j^{b\mathsf{T}}(\mathbf{x}), \tag{D4}$$

which implies that

$$[\Pi_i^a(\mathbf{x}),\overline{\mathcal{A}_j^{b\mathsf{T}}}(\mathbf{y})]=-i\frac{\delta A_{\mathsf{GI}j}^b(\mathbf{y})}{\delta A_i^a(\mathbf{x})}+i\,\delta_{ab}\delta_{ij}^{\mathsf{T}}(\mathbf{x}-\mathbf{y}). \tag{D5}$$

The functional derivative was calculated in Ref. [32] and the commutator in Ref. [5], and the result can also be deduced from Eq. (39) with the aid of Eq. (11), which gives rise to

$$[\Pi_i^a(\mathbf{x}),A_{\mathsf{GI}j}^b(\mathbf{y})]=-i\bigg(R_{ba}(\mathbf{x})\,\delta_{ij}\delta(\mathbf{x}-\mathbf{y}) +R_{la}(\mathbf{x})\frac{\partial}{\partial x_i}\mathcal{D}^{lk}(\mathbf{x},\mathbf{y})\overleftarrow{D_j^{kb}}(\mathbf{y})\bigg). \tag{D6}$$

Substituting this into Eq. (D3) and using Eq. (11), we find that

$$\Pi_{\mathsf{GI}i}^a(\mathbf{x})\Psi|\phi\rangle=\Psi\Pi_i^{a\mathsf{T}}(\mathbf{x})|\phi\rangle -\frac{\partial}{\partial x_i}\int d\mathbf{y}\mathcal{D}^{al}(\mathbf{x},\mathbf{y})\overleftarrow{D_j^{lb}}(\mathbf{y})\Psi\Pi_j^{b\mathsf{T}}(\mathbf{y})|\phi\rangle. \tag{D7}$$

Taking the transverse part of both sides, we end up with

$$\Pi_{\mathsf{GI}i}^{a\mathsf{T}}(\mathbf{x})\Psi|\phi\rangle=\Psi\Pi_i^{a\mathsf{T}}(\mathbf{x})|\phi\rangle. \tag{D8}$$

The identity Eq. (78) is proved.

APPENDIX E

In this appendix we will show how to obtain Eq. (84) from Eq. (79). In order to obtain the bilinear product $\mathcal{P}_j^{b\mathsf{T}}(\mathbf{r})\mathcal{P}_j^{b\mathsf{T}}(\mathbf{r})$ for inclusion in a non-interacting part of $(\hat{H}_{\mathsf{GI}})_{\mathrm{phys}}$ that can define interaction picture operators, we now express $\mathcal{J}^{-1/2}\mathcal{P}_j^{b\mathsf{T}}(\mathbf{r})\mathcal{J}\mathcal{P}_j^{b\mathsf{T}}(\mathbf{r})\mathcal{J}^{-1/2}$ as

$$\begin{aligned}\mathcal{J}^{-1/2}&\mathcal{P}_j^{b\mathsf{T}}(\mathbf{r})\mathcal{J}\mathcal{P}_j^{b\mathsf{T}}(\mathbf{r})\mathcal{J}^{-1/2}\\ &=\mathcal{P}_j^{b\mathsf{T}}(\mathbf{r})\mathcal{P}_j^{b\mathsf{T}}(\mathbf{r})+\mathcal{P}_j^{b\mathsf{T}}(\mathbf{r})\mathcal{J}^{1/2}[\mathcal{P}_j^{b\mathsf{T}}(\mathbf{r}),\mathcal{J}^{-1/2}]\\ &\quad-[\mathcal{P}_j^{b\mathsf{T}}(\mathbf{r}),\mathcal{J}^{-1/2}]\mathcal{J}^{1/2}\mathcal{P}_j^{b\mathsf{T}}(\mathbf{r})-[\mathcal{P}_j^{b\mathsf{T}}(\mathbf{r}),\mathcal{J}^{-1/2}]\\ &\quad\times\mathcal{J}[\mathcal{P}_j^{b\mathsf{T}}(\mathbf{r}),\mathcal{J}^{-1/2}]\end{aligned} \tag{E1}$$

and make use of the formula

$$\begin{aligned}[\mathcal{P}_j^{b\mathsf{T}}(\mathbf{r}),\mathcal{J}^{-1/2}]&=-\frac{1}{2}\mathcal{J}^{-3/2}[\mathcal{P}_j^{b\mathsf{T}}(\mathbf{r}),\mathcal{J}]\\ &=-\frac{1}{2}\mathcal{J}^{-1/2}[\mathcal{P}_j^{b\mathsf{T}}(\mathbf{r}),\ln(\mathcal{J})].\end{aligned} \tag{E2}$$

We also observe that the commutator of $\mathcal{P}_j^{b\mathsf{T}}(\mathbf{r})$ and any functional of $A_{\mathsf{GI}i}^a(\mathbf{r}')$ commutes with any other functional of $A_{\mathsf{GI}i}^a(\mathbf{r}')$, and that, in fact,

$$\begin{aligned}[\mathcal{P}_j^{b\mathsf{T}}(\mathbf{y}),A_{\mathsf{GI}i}^a(\mathbf{x})]&=[\Pi_{\mathsf{GI}j}^{b\mathsf{T}}(\mathbf{y}),A_{\mathsf{GI}i}^a(\mathbf{x})]\\ &=-i\,\delta_{ab}\bigg(\delta_{ij}-\frac{\partial_i\partial_j}{\partial^2}\bigg)\delta(\mathbf{x}-\mathbf{y}).\end{aligned} \tag{E3}$$

With these observations, we obtain

$$\begin{aligned}\mathcal{J}^{-1/2}&\mathcal{P}_j^{b\mathsf{T}}(\mathbf{r})\mathcal{J}\mathcal{P}_j^{b\mathsf{T}}(\mathbf{r})\mathcal{J}^{-1/2}\\ &=\mathcal{P}_j^{b\mathsf{T}}(\mathbf{r})\mathcal{P}_j^{b\mathsf{T}}(\mathbf{r})-\frac{1}{2}[\mathcal{P}_j^{b\mathsf{T}}(\mathbf{r}),[\mathcal{P}_j^{b\mathsf{T}}(\mathbf{r}),\ln(\mathcal{J})]]\\ &\quad-\frac{1}{4}[\mathcal{P}_j^{b\mathsf{T}}(\mathbf{r}),\ln(\mathcal{J})][\mathcal{P}_j^{b\mathsf{T}}(\mathbf{r}),\ln(\mathcal{J})].\end{aligned} \tag{E4}$$

Equations (66) and (68) show that

$$\begin{aligned}[\mathcal{P}_j^{b\mathsf{T}}(\mathbf{y}),\ln(\mathcal{J})]&=[\Pi_{\mathsf{GI}j}^{b\mathsf{T}}(\mathbf{y}),\ln(\mathcal{J})]\\ &=igf^{hcb}\delta_{j,k}^{\mathsf{T}}(\mathbf{y})\lim_{\mathbf{x}\to\mathbf{y}}\frac{\partial}{\partial y_k}\mathcal{D}^{ch}(\mathbf{y},\mathbf{x}).\end{aligned} \tag{E5}$$

With Eq. (38) this can be rewritten in the form

$$[\Pi_{\mathsf{GI}j}^{b\mathsf{T}}(\mathbf{y}),\ln(\mathcal{J})]=ig^2f^{bdh}f^{\delta ds}\delta_{j,k}^{\mathsf{T}}(\mathbf{y})\int d\mathbf{z}\frac{\partial}{\partial y_k}\bigg(\frac{1}{4\pi|\mathbf{y}-\mathbf{z}|}\bigg) \times A_{\mathsf{GI}l}^{\delta}(\mathbf{z})\frac{\partial}{\partial z_l}\mathcal{D}^{sh}(\mathbf{z},\mathbf{y}) \tag{E6}$$

where $\delta_{j,k}^{\mathsf{T}}(\mathbf{y})=[\delta_{jk}-(\partial_j^{(\mathbf{y})}\partial_k^{(\mathbf{y})}/\partial^2)]\delta(\mathbf{y})$. In this form, it is clear that, to leading order, $[\Pi_{\mathsf{GI}j}^{b\mathsf{T}}(\mathbf{y}),\ln(\mathcal{J})]$ is a g^2 term and that the limit $\mathbf{x}\to\mathbf{y}$ has already been carried out. We can use Eqs. (E4) and (E6) to obtain an expression for $\mathcal{U}\equiv -\frac{1}{4}\int d\mathbf{r}[\mathcal{P}_j^{b\mathsf{T}}(\mathbf{r}),\ln(\mathcal{J})][\mathcal{P}_j^{b\mathsf{T}}(\mathbf{r}),\ln(\mathcal{J})]$, which becomes an interaction term in $(\hat{H}_{\mathsf{GI}})_{\mathrm{phys}}$, given by

$$\begin{aligned}\mathcal{U}=&\frac{1}{8}g^4f^{bdh}f^{\delta ds}f^{bd'h'}f^{\delta'd's'}\int d\mathbf{y}d\mathbf{z}d\mathbf{z}'\\ &\times\bigg\{\delta_{j,k}^{\mathsf{T}}(\mathbf{y})\delta_{j,k'}^{\mathsf{T}}(\mathbf{y})\frac{\partial}{\partial y_k}\bigg(\frac{1}{4\pi|\mathbf{y}-\mathbf{z}|}\bigg)A_{\mathsf{GI}l}^{\delta}(\mathbf{z})\frac{\partial}{\partial z_l}\mathcal{D}^{sh}(\mathbf{z},\mathbf{y})\\ &\times\frac{\partial}{\partial y_{k'}}\bigg(\frac{1}{4\pi|\mathbf{y}-\mathbf{z}'|}\bigg)A_{\mathsf{GI}l'}^{\delta'}(\mathbf{z}')\frac{\partial}{\partial z'_{l'}}\mathcal{D}^{s'h'}(\mathbf{z}',\mathbf{y})\bigg\}.\end{aligned} \tag{E7}$$

Similarly, from Eq. (E5), we see that

$$[\mathcal{P}_j^{b\mathsf{T}}(\mathbf{y}),[\mathcal{P}_j^{b\mathsf{T}}(\mathbf{y}),\ln(\mathcal{J})]] = igf^{hdb}\left[\mathcal{P}_j^{b\mathsf{T}}(\mathbf{y}),\delta_{j,k}^{\mathsf{T}}(\mathbf{y})\lim_{\mathbf{x}\to\mathbf{y}}\frac{\partial}{\partial y_k}\mathcal{D}^{dh}(\mathbf{y},\mathbf{x})\right] \tag{E8}$$

in which we represent $\mathcal{P}_j^{b\mathsf{T}}(\mathbf{y})$ as $\lim_{\mathbf{r}\to\mathbf{y}}\mathcal{P}_j^{b\mathsf{T}}(\mathbf{r})$ so that

$$[\mathcal{P}_j^{b\mathsf{T}}(\mathbf{y}),[\mathcal{P}_j^{b\mathsf{T}}(\mathbf{y}),\ln(\mathcal{J})]] = igf^{hdb}\lim_{\mathbf{r}\to\mathbf{y}}\delta_{j,k}^{\mathsf{T}}(\mathbf{y})\lim_{\mathbf{x}\to\mathbf{y}}\frac{\partial}{\partial y_k}[\mathcal{P}_j^{b\mathsf{T}}(\mathbf{r}),\mathcal{D}^{dh}(\mathbf{y},\mathbf{x})]. \tag{E9}$$

Using Eq. (34), we obtain

$$\begin{aligned}[\mathcal{P}_j^{b\mathsf{T}}(\mathbf{r}),\mathcal{D}^{dh}(\mathbf{y},\mathbf{x})] &= -gf^{s\alpha t}\int d\mathbf{z}\mathcal{D}^{ds}(\mathbf{y},\mathbf{z})[\mathcal{P}_j^{b\mathsf{T}}(\mathbf{r}),A_{\mathsf{GI}\ell}^{\alpha}(\mathbf{z})]\partial_\ell^{(\mathbf{z})}\mathcal{D}^{th}(\mathbf{z},\mathbf{x})\\ &= -igf^{sbt}\int d\mathbf{z}\mathcal{D}^{ds}(\mathbf{y},\mathbf{z})\left(\delta_{j,\ell}-\frac{\partial_j^{(\mathbf{z})}\partial_\ell^{(\mathbf{z})}}{\partial^2}\right)\\ &\quad\times\delta(\mathbf{z}-\mathbf{r})\partial_\ell^{(\mathbf{z})}\mathcal{D}^{th}(\mathbf{z},\mathbf{x}).\end{aligned} \tag{E10}$$

After integration over all of space,

$$\int d\mathbf{y}[\mathcal{P}_j^{b\mathsf{T}}(\mathbf{y}),[\mathcal{P}_j^{b\mathsf{T}}(\mathbf{y}),\ln(\mathcal{J})]]$$

becomes another interaction term in $(\hat{H}_{\mathsf{GI}})_{\mathrm{phys}}$, given by

$$\mathcal{V}=\frac{1}{4}g^2f^{hdb}f^{sbt}\int d\mathbf{y}\lim_{\mathbf{r}\to\mathbf{y}}\{\delta_{j,k}^{\mathsf{T}}(\mathbf{y})\delta_{j,\ell}^{\mathsf{T}}(\mathbf{r})\times[\partial_k^{(\mathbf{y})}\mathcal{D}^{ds}(\mathbf{y},\mathbf{r})\partial_\ell^{(\mathbf{r})}\mathcal{D}^{th}(\mathbf{r},\mathbf{y})]\}. \tag{E11}$$

$\mathcal{V}$ is singular since the leading terms in $\mathcal{D}^{ds}(\mathbf{y},\mathbf{r})$ and $\mathcal{D}^{th}(\mathbf{r},\mathbf{y})$ $[-\delta_{ds}(4\pi|\mathbf{y}-\mathbf{r}|)^{-1}$ and $-\delta_{th}(4\pi|\mathbf{y}-\mathbf{r}|)^{-1}$, respectively], are not eliminated by the structure constants in $\mathcal{V}$. Christ and Lee called attention to such singularities in their work [32], and conjectured that they might be useful in cancelling unresolved divergences in Coulomb-gauge QCD. The same remark applies to $\mathcal{V}$. We continue by eliminating the non-Hermitian chromoelectric fields from $J_{0(\mathsf{GI})}^{a\mathsf{T}}$ and $J_{0(\mathsf{GI})}^{a\mathsf{T}\dagger}$, obtaining

$$J_{0(\mathsf{GI})}^{c\mathsf{T}}(\mathbf{y})=gf^{cqp}A_{\mathsf{GI}j}^{q}(\mathbf{y})\Pi_{\mathsf{GI}j}^{p\mathsf{T}}(\mathbf{y}) =gf^{cqp}A_{\mathsf{GI}j}^{q}(\mathbf{y})\left\{\mathcal{P}_j^{p\mathsf{T}}(\mathbf{y})-\frac{1}{2}[\mathcal{P}_j^{p\mathsf{T}}(\mathbf{y}),\ln(\mathcal{J})]\right\} \tag{E12}$$

and

$$J_{0(\mathsf{GI})}^{c\mathsf{T}\dagger}(\mathbf{y})=gf^{cqp}A_{\mathsf{GI}j}^{q}(\mathbf{y})\Pi_{\mathsf{GI}j}^{p\mathsf{T}\dagger}(\mathbf{y}) =gf^{cqp}A_{\mathsf{GI}j}^{q}(\mathbf{y})\left\{\mathcal{P}_j^{p\mathsf{T}}(\mathbf{y})+\frac{1}{2}[\mathcal{P}_j^{p\mathsf{T}}(\mathbf{y}),\ln(\mathcal{J})]\right\}, \tag{E13}$$

and

$$J_{0(\mathsf{GI})}^{c\mathsf{T}}(\mathbf{y})=\bar{\mathsf{J}}_{0(\mathsf{GI})}^{c\mathsf{T}}(\mathbf{y})+i\mathsf{k}_0^c(\mathbf{y})$$

and

$$J_{0(\mathsf{GI})}^{c\mathsf{T}\dagger}(\mathbf{y}) =\bar{\mathsf{J}}_{0(\mathsf{GI})}^{c\mathsf{T}}(\mathbf{y})-i\mathsf{k}_0^c(\mathbf{y}) \tag{E14}$$

where

$$\bar{\mathsf{J}}_{0(\mathsf{GI})}^{c\mathsf{T}}=gf^{cqp}A_{\mathsf{GI}j}^{q}\mathcal{P}_j^{p\mathsf{T}} \tag{E15}$$

and, using Eq. (38), $i\mathsf{k}_0^c$ can be identified as an additional, auxiliary gluon color-charge density in which

$$\mathsf{k}_0^c(\mathbf{y})=-\frac{1}{2}g^3f^{cqp}f^{hdp}f^{\gamma ds}A_{\mathsf{GI}i}^{q}(\mathbf{y})\delta_{i,j}^{\mathsf{T}}(\mathbf{y})\times\int d\mathbf{z}\frac{\partial}{\partial y_j}\left(\frac{1}{4\pi|\mathbf{y}-\mathbf{z}|}\right)A_{\mathsf{GI}k}^{\gamma}(\mathbf{z})\frac{\partial}{\partial z_k}\mathcal{D}^{sh}(\mathbf{z},\mathbf{y}). \tag{E16}$$

This representation enables us to express H_{C}—the nonlocal interaction involving quark and gluon color-charge densities in Eq. (79)—in the manifestly Hermitian form

$$\begin{aligned}H_{\mathsf{C}}=-\frac{1}{2}\int d\mathbf{r}d\mathbf{x}d\mathbf{y}[j_0^b(\mathbf{x})+\bar{\mathsf{J}}_{0(\mathsf{GI})}^{b\mathsf{T}}(\mathbf{x})\\ -i\mathsf{k}_0^b(\mathbf{x})]\mathcal{D}^{ab}(\mathbf{x},\mathbf{r})\partial_{(\mathbf{r})}^2\mathcal{D}^{ac}(\mathbf{r},\mathbf{y})\\ \times[j_0^c(\mathbf{y})+\bar{\mathsf{J}}_{0(\mathsf{GI})}^{c\mathsf{T}}(\mathbf{y})+i\mathsf{k}_0^c(\mathbf{y})]\end{aligned} \tag{E17}$$

in which all operator-valued fields are Hermitian as well as gauge invariant. When we have eliminated all the $\Pi_{\mathsf{GI}j}^{p\mathsf{T}}$ and $\Pi_{\mathsf{GI}j}^{p\mathsf{T}\dagger}$ from $(\hat{H}_{\mathsf{GI}})_{\mathrm{phys}}$ and replaced them with $\mathcal{P}_j^{p\mathsf{T}}$ and the other expressions obtained in this process, we obtain Eq. (84).

APPENDIX F

To establish Eq. (132), we quote an identity in [32],

$$f^{abc}\int d\mathbf{r}[D_j^{am}X^m(\mathbf{r})Y^b(\mathbf{r})Z^c(\mathbf{r})+X^a(\mathbf{r})D_j^{bm}Y^m(\mathbf{r})Z^c(\mathbf{r}) +X^a(\mathbf{r})Y^b(\mathbf{r})D_j^{cm}Z^m(\mathbf{r})]=0. \tag{F1}$$

The proof follows from the observation that the ordinary derivative terms of the covariant derivatives, D_j's, in

Eq. (F1) add up to a total derivative and the structure constant terms of D_j's add up to zero on account of the Jacobian identity

$$f^{lab}f^{lmc}+f^{lbc}f^{lma}+f^{lca}f^{lmb}=0. \tag{F2}$$

Notice that the functions X, Y and Z may carry other color or vector indices and the dependence on other coordinates.

According to Eq. (E10),

$$[\Pi_j^c(\mathbf{r}),\mathcal{D}^{ab}(\mathbf{x},\mathbf{y})]=-\int d\mathbf{z}d\mathbf{z}'\,\mathcal{D}^{am}(\mathbf{x},\mathbf{z})\times[\Pi_j^c(\mathbf{x}),(\partial\cdot D)^{mn}]\delta(\mathbf{z}-\mathbf{z}')\times\mathcal{D}^{nb}(\mathbf{z}',\mathbf{y}) \tag{F3}$$

and, with Eq. (D6), we have

$$\begin{aligned}[E_j^b(\mathbf{r}),\Delta_j^b(\mathbf{r})]&=-[\Pi_{\mathrm{GI},j}^b(\mathbf{r}),\Delta_j^b(\mathbf{r})]\\&=-R_{ba}(\mathbf{r})[\Pi_j^a(\mathbf{r}),\Delta_j^b(\mathbf{r})]\\&=\frac{1}{2}g^2f^{lab}f^{lmn}\partial_j\mathcal{D}^{am}(\mathbf{r},\mathbf{r})\partial_j\mathcal{D}^{nb}(\mathbf{r},\mathbf{r})\\&\quad+\frac{g^2}{2}f^{lab}f^{cmn}\int d\mathbf{x}D_i^{ck}\mathcal{D}^{kl}(\mathbf{x},\mathbf{r})\overleftarrow{\partial}_j\\&\quad\times\mathcal{D}^{ma}(\mathbf{x},\mathbf{r})\overleftarrow{\partial}_j\partial_i\mathcal{D}^{nb}(\mathbf{x},\mathbf{r}),\end{aligned} \tag{F4}$$

where the symmetry property Eq. (37) is employed to obtain the second term on the right-hand side. Upon relabeling the dummy color indices, we convert the second term of the right-hand side of Eq. (F4) to

$$\begin{aligned}&\frac{g^2}{4}f^{lab}f^{cmn}\int d\mathbf{x}D_i^{ck}\mathcal{D}^{kl}(\mathbf{x},\mathbf{r})\overleftarrow{\partial}_j\mathcal{D}^{ma}(\mathbf{x},\mathbf{r})\overleftarrow{\partial}_j\partial_i\mathcal{D}^{nb}(\mathbf{x},\mathbf{r})\\&\quad+\frac{g^2}{4}f^{lab}f^{cmn}\int d\mathbf{x}\mathcal{D}^{cl}(\mathbf{x},\mathbf{r})\overleftarrow{\partial}_jD_i^{mk}\\&\quad\times\mathcal{D}^{ka}(\mathbf{x},\mathbf{r})\overleftarrow{\partial}_j\partial_i\mathcal{D}^{nb}(\mathbf{x},\mathbf{r})\\&=-\frac{g^2}{4}f^{lab}f^{cmb}\partial_j\mathcal{D}^{lc}(\mathbf{r},\mathbf{r})\partial_j\mathcal{D}^{am}(\mathbf{r},\mathbf{r}),\end{aligned} \tag{F5}$$

where the last step follows from the identities Eqs. (F1) and (30). We have then

$$[E_j^b(\mathbf{r}),\Delta_j^b(\mathbf{r})]=\frac{1}{4}g^2(2f^{lbm}f^{lan}-f^{lam}f^{lbn})\times\partial_j\mathcal{D}^{ab}(\mathbf{r},\mathbf{r})\partial_j\mathcal{D}^{mn}(\mathbf{r},\mathbf{r}). \tag{F6}$$

Combining it with

$$-\Delta_j^b(\mathbf{r})\Delta_j^b(\mathbf{r})=\frac{1}{4}g^2f^{lab}f^{lmn}\partial_j\mathcal{D}^{ab}(\mathbf{r},\mathbf{r})\partial_j\mathcal{D}^{mn}(\mathbf{r},\mathbf{r}) \tag{F7}$$

according to Eq. (126) and using the Jacobian identity Eq. (F2), we end up with Eq. (132).

[1] L. Chen, M. Belloni, and K. Haller, Phys. Rev. D **55**, 2347 (1997).

[2] In this work we use nonrelativistic notation, in which all space-time indices are subscripted and designate contravariant components of contravariant quantities such as A_i^a or j_i^a, and covariant components of covariant quantities such as ∂_i.

[3] K. Haller, L. Chen, and Y.S. Choi, Phys. Rev. D **60**, 125010 (1999).

[4] P.A.M. Dirac, Can. J. Phys. **33**, 650 (1955).

[5] K. Haller, Int. J. Mod. Phys. A **16**, 2789 (2001).

[6] M. Belloni, L. Chen, and K. Haller, Phys. Lett. B **403**, 316 (1997).

[7] The transformation of the quark field from the $\mathcal{C}$ to the $\mathcal{N}$ representation, $\mathcal{U}_{\mathcal{C}}^{-1}\psi\mathcal{U}_{\mathcal{C}}=V_{\mathcal{C}}{}^{-1}\psi$, and the transformation that leads to the gauge-invariant quark field in the $\mathcal{C}$ representation, $\psi_{\mathrm{GI}}=V_{\mathcal{C}}\psi$, transform in opposite directions. In the former case we transform the quark field that is represented by ψ in the $\mathcal{C}$ representation to the $\mathcal{N}$ representation; in the latter case, we transform the quark field that is gauge-invariant in the $\mathcal{N}$ representation—namely ψ—to its corresponding $\mathcal{C}$ representation form. The transformations therefore proceed in opposite directions in the two cases.

[8] A similar equation appears in L. D. Faddeev and A. A. Slavnov, *Gauge Fields: An Introduction to Quantum Theory*, second edition (Addison-Wesley, Redwood City, CA, 1991).

[9] J. Schwinger, Phys. Rev. **125**, 1043 (1962).

[10] Superscripts T and L designate transverse and longitudinal components, respectively.

[11] V. N. Gribov, "Instability of non-Abelian Gauge Theories and Impossibility of Choice of Coulomb Gauge," Lecture at the 12th Winter School of the Leningrad Nuclear Physics Institute.

[12] V.N. Gribov, Nucl. Phys. **B139**, 1 (1978).

[13] M. Lavelle and D. McMullan, Phys. Rep. **279**, 1 (1997).

[14] K. Cahill and D.R. Stump, Phys. Rev. D **20**, 540 (1979).

[15] A. Cucchieri and D. Zwanziger, Nucl. Phys. B (Proc. Suppl.) **106**, 694 (2002).

[16] D. Zwanziger, Suppl. Prog. Theor. Phys. **131**, 233 (1998).

[17] R.E. Cutkosky, Phys. Rev. D **30**, 447 (1984); Phys. Rev. Lett. **51**, 538 (1983); **51**, 1603E (1983).

[18] R. Jackiw, I. Muzinich, and C. Rebbi, Phys. Rev. D **17**, 1576 (1978). For a review and further references, see R. Jackiw, in *Current Algebra and Anomalies*, edited by S. B. Treiman *et al.* (World Scientific, Singapore, 1985); R. Rajaraman, *Solitons and Instantons* (North-Holland, Amsterdam, 1982).

[19] For example, S. Weinberg, *The Quantum Theory of Fields* (Cambridge Univ. Press, Cambridge, UK, 1995), Vol. II, Sec. 15.4.

[20] A. Bassetto, G. Nardelli, and R. Soldati, *Yang-Mills Theories*

in Algebraic Non-covariant Gauges (World Scientific, Singapore, 1991).

[21] I.M. Singer, Commun. Math. Phys. **60**, 7 (1978). The author points out that the proof that the Gribov ambiguity will occur in all gauges does not apply to algebraic gauges for which no conditions at infinity are imposed.

[22] K. Haller, Phys. Rev. D **36**, 1830 (1987).

[23] K. Haller and E. Lim-Lombridas, Found. Phys. **24**, 217 (1994).

[24] S.N. Gupta, Proc. Phys. Soc., London, Sect. A **63**, 681 (1950); K. Bleuler, Helv. Phys. Acta **23**, 567 (1950).

[25] See Ref. [23] and earlier references cited therein.

[26] E. Fermi, Rev. Mod. Phys. **4**, 131 (1932); for a review, see K. Nishijima, *Fields and Particles* (Benjamin Books, New York, 1969), Sec. 2-6.

[27] K. Haller, Phys. Rev. D **36**, 1839 (1987).

[28] Non-normalizable states in QED can be entirely avoided by the substitution of $\mathcal{G}^{(+)}(\mathbf{r})|\Psi\rangle=0$ for $\mathcal{G}(\mathbf{r})|\Psi(\mathbf{r})\rangle=0$ [24]. In the case of QCD, however, the imposition of $\mathcal{G}^{a(+)}(\mathbf{r})|\Psi\rangle=0$ for all values of the Lie index a has, as one of its consequences, that $\mathcal{G}^{a}(\mathbf{r})|\Psi(\mathbf{r})\rangle=0$ also obtains [27]. Avoidance of non-normalizable states in QCD by a method similar to the one provided for QED by Ref. [24] is therefore impossible.

[29] J. Goldstone and R. Jackiw, Phys. Lett. **74B**, 81 (1978).

[30] R. Jackiw, Rev. Mod. Phys. **52**, 661 (1980).

[31] P. A. M. Dirac, *Lectures on Quantum Mechanics* (Yeshiva University Press, New York, 1964); also see Can. J. Math. **2**, 129 (1950); P.G. Bergmann and I. Goldberg, Phys. Rev. **98**, 531 (1955). For a different approach to this problem, see L. Faddeev and R. Jackiw, Phys. Rev. Lett. **60**, 1692 (1988); R. Jackiw, *Diverse Topics in Theoretical and Mathematical Physics* (World Scientific, Singapore, 1995).

[32] N.H. Christ and T.D. Lee, Phys. Rev. D **22**, 939 (1980).

[33] T. D. Lee, *Particle Physics and Introduction to Field Theory* (Harwood, New York, 1981), Chap. 18.

[34] M. Creutz, I.J. Muzinich, and T.N. Tudron, Phys. Rev. D **19**, 531 (1979).

[35] J.L. Gervais and B. Sakita, Phys. Rev. D **18**, 453 (1978).

[36] J. Schwinger, Phys. Rev. **127**, 324 (1962); **130**, 402 (1963).

[37] G.C. Rossi and M. Testa, Nucl. Phys. **B163**, 109 (1980); **B176**, 477 (1980); **B237**, 442 (1984).

[38] M. Belloni, L. Chen, and K. Haller, Phys. Lett. B **373**, 185 (1996).

[39] K. Gawedzki, Phys. Rev. **26**, 3593 (1982).

[40] K. Haller, Phys. Lett. B **251**, 575 (1990).