Classical and quantum Nambu mechanics

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The classical and quantum features of Nambu mechanics are analyzed and fundamental issues are resolved. The classical theory is reviewed and developed utilizing varied examples. The quantum theory is discussed in a parallel presentation and illustrated with detailed specific cases. Quantization is carried out with standard Hilbert space methods. With the proper physical interpretation, obtained by allowing for different time scales on different invariant sectors of a theory, the resulting non-Abelian approach to quantum Nambu mechanics is shown to be fully consistent.

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I. INTRODUCTION

a. A brief historical overview. Nambu [1] introduced an elegant generalization of the classical Hamiltonian formalism by suggesting to supplant the Poisson brackets (PB) with 3- or *n*-linear, fully antisymmetric brackets, the classical Nambu brackets (CNB), a volume-element Jacobian determinant in a higher-dimensional space. These brackets, involving a dynamical quantity and two or more ‘‘Hamiltonians,’’ provide the time evolution of that quantity in a generalization of Hamilton's equations of motion for selected physical systems. It was gradually realized [2–4] that Nambu brackets in phase space describe the generic classical evolution of all systems with sufficiently many independent integrals of motion beyond those required for complete integrability of the systems. That is to say, all such ‘‘superintegrable systems’’ [5] are automatically described by Nambu's mechanics [6–8], whether or not one chooses to take cognizance of this alternate expression of their time development. This approach to time evolution for superintegrable systems is supplementary to the standard Hamiltonian dynamics evolution and provides additional tools for analyzing such systems. The power of Nambu's method is evident in manifesting and simultaneous accounting for a maximal number of the symmetries of these systems and in an efficient application of algebraic methods to yield results even without detailed knowledge of their specific dynamics.

As a bonus, the classical volume-preserving features of Nambu brackets suggest that they are useful for membrane theory [9]. There are in the literature several persuasive but inconclusive arguments that Nambu brackets are a natural language for describing extended objects, for example [10–20].

In his original paper [1], Nambu also introduced operator versions of his brackets as tools to implement the quantization of his approach to mechanics. He enumerated various logical possibilities involving them, arguing that some structures were either inconsistent or uninteresting, but he did not advocate the position that the remaining possibilities were untenable: Quantization was left as an open issue.

Unfortunately, subsequent unwarranted insistence on algebraic structures ill suited to the solution of the relevant physics problems resulted in a widely held belief that quantization of Nambu mechanics was problematic,[1] especially when that quantization was formulated as a one-parameter deformation of classical structures. In marked contrast to this prevailing pessimism, several illustrative superintegrable systems were quantized in [6] in a phase-space framework, both without and *with* the construction of quantum Nambu brackets (QNBs). However, the phase-space quantization utilized there, while most appropriate for comparing quantum expressions with their classical limits, is still unfamiliar to many readers and will not be used in this paper. Here, the quantization of all systems will be carried out in a conventional Hilbert space operator formalism.

It turns out [6] that all perceived difficulties in quantizing Nambu mechanics may be traced mathematically to the algebraic inconsistencies inherent in selecting constraints in a top-down approach, with little regard to the correct phase-space structure which already provides full and consistent answers, and with insufficient attention towards obtaining specific answers compatible with those produced in the quantized Hamiltonian description of these systems. Moreover, the physics underlying these perceived difficulties is simple and involves only basic principles in quantum mechanics.

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[1]A few representative statements from the literature are the following: ‘‘associated statistical mechanics and quantization are unlikely’’ [21]; ‘‘a quantum generalization of these algebras is shown to be impossible.’’ and ‘‘ ... the quantum analog of Nambu mechanics does not exist’’ [22,23]; ‘‘usual approaches to quantization have failed to give an appropriate solution ... ’’ [24]; ‘‘ ... direct application of deformation quantization to Nambu-Poisson structures is not possible’’ [14]; ‘‘the quantization of Nambu brackets turns out to be a quite non trivial problem’’ [25]; ‘‘this problem is still outstanding’’ [26].

b. Evolution scales in quantum physics. Some physicists might hold, without realizing it, the prejudice that continuous time evolution in quantum mechanics must always be formulated infinitesimally as a derivation. Accordingly, they implicitly *assume* the instantaneous temporal change in all dynamical variables is always given by nothing but a simple derivative, so that for all products of linear operators

$$\frac{d}{dt}(AB)=\left(\frac{d}{dt}A\right)B+A\left(\frac{d}{dt}B\right). \tag{1}$$

This assumption allows time development on physical Hilbert spaces to be expressed algebraically in terms of commutators with a Hamiltonian, since commutators are also derivations,[2]

$$[H,AB]=[H,A]B+A[H,B]. \tag{2}$$

Evidently, this approach leads to the simplest possible formalism. But is it really necessary to make this assumption and follow this approach?

It is not. Time evolution can also be expressed algebraically using quantum Nambu brackets. These quantum brackets are defined as totally antisymmetrized multilinear products of any number of linear operators acting on Hilbert space. When QNBs are used to implement time evolution in quantum mechanics, the result is usually *not* a derivation, but contains derivations entwined within more elaborate structures (although there are some interesting special exceptions that are described in the following).

This more general point of view towards time development can be arrived at just by realizing a physical idea. When a system has a number of conserved quantities, it is possible to partition the system's Hilbert space into invariant sectors. Time evolution on those various sectors may then be formulated using different time scales for the different sectors.[3] The resulting expression of instantaneous changes in time is then not a derivation, in general, when acting on the full Hilbert space and therefore is not given by a simple commutator. Remarkably, however, it often turns out to be given compactly in terms of QNBs. Conversely, if QNBs are used to describe time development, they usually impose different time scales on different invariant sectors of a system [6].

Nevertheless, so long as the different time scales are implemented in such a way as to produce evolving phase differences between nondegenerate energy eigenstates, there is no loss of information in this more general approach to time evolution. In the classical limit, this method is not really different from the usual Hamiltonian approach. A given classical trajectory has fixed values for all invariants and hence would have a fixed time scale in Nambu mechanics. Time development of any dynamical quantity along a single classical trajectory would therefore always be just a derivation, with no possibility of mixing time scales. Quantum mechanics, on the other hand, is more subtle, since the preparation of a state may yield a superposition of components from different invariant sectors. Such superpositions will, in general, involve multiple time scales in Nambu mechanics.

Technically, the various time scales arise in quantum Nambu mechanics as the entwined eigenvalues of generalized Jordan spectral problems, where selected invariants of the model in question appear as operators in the spectral equation. The resulting structure represents a new class of eigenvalue problems for mathematical physics. Fortunately, solutions of this new class can be found using traditional methods. (All this is explained explicitly in the context of the first example of Sec. III B.)

c. Related studies in mathematics. Algebras which involve multilinear products have also been considered at various times in the mathematical literature, partly as efforts to understand or generalize Jordan algebras [30–33] (cf. especially the ‘‘associator’’), but more generally following Higgins' study in the mid 1950s [34–37]. This eventually culminated in the investigations of certain cohomology questions, by Schlesinger and Stasheff [38], by Hanlon and Wachs [39,40], and by Azcárraga, Izquierdo, Perelomov, and Pérez Bueno [41,42,11], that led to results most relevant to Nambu's work.

d. Summary of material to follow. After a few motivational remarks on the geometry of Hamiltonian flows in phase space, Sec. II A, we describe the most important features of classical Nambu brackets, Sec. II B, with emphasis on practical, algebraic, evaluation methods. We delve into several examples, Sec. II C, to gain physical insight for the classical theory.

We then give a parallel discussion of the quantum theory, Sec. III A, so far as algebraic features and methods of evaluation are concerned. We define QNBs, as well as generalized Jordan products that naturally arise in conjunction with QNBs, when the latter are resolved into products of commutators. We define derivators as measures of the failure of the Leibniz rule for QNBs and discuss Jacobi and fundamental identities in a quantum setting. Then, we again turn to various examples, Sec. III B, to illustrate both the elegance and peculiarities of quantization. We deal with essentially the same examples in both classical and quantum frameworks, as a means of emphasizing the similarities and, more importantly, delineating the differences between CNBs and QNBs. The examples chosen are all models based on Lie symmetry algebras: $\mathrm{so}(3)=\mathrm{su}(2)$, $\mathrm{so}(4)=\mathrm{su}(2)\times\mathrm{su}(2)$, $\mathrm{so}(n)$, $\mathrm{u}(n)$, $\mathrm{u}(n)\times\mathrm{u}(m)$, and $g\times g$.

We conclude by summarizing our results and by suggesting some topics for further study. An Appendix discusses the formal solution of linear equations in Lie and Jordan algebras, with suggestions for techniques to bypass the effects of divisors of zero.

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[2]For simplicity we will assume, unless otherwise stated, that the operators have no *explicit* time dependence, although it is an elementary exercise to relax this assumption.

[3]In fact, the choice of time variables in the different invariant sectors of a quantum theory is very broad. They need not be just multiples of one another, but could have complicated functional dependencies, as discussed in [27] and [18]. The closest classical counterpart of this is found in the general method of *analytictime*, recently exploited so effectively in [28,29].

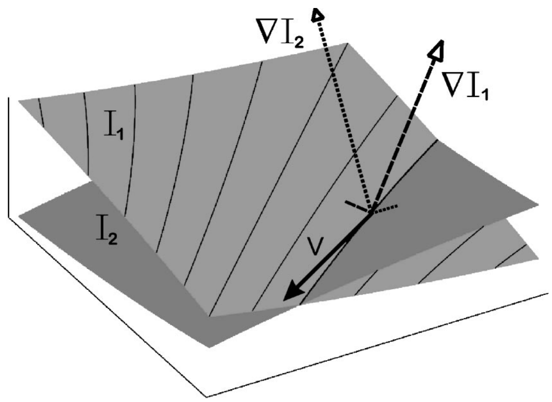

FIG. 1. Some classical phase-space geometry.

A hurried reader may wish to consider only Secs. II A and II B through Eq. (10), Sec. II C through Eq. (35), Sec. III A through Eq. (102), and Sec. III B through Eq. (153). This abridged material contains our main points.

II. CLASSICAL THEORY

We begin with a brief geometrical discussion of phase-space dynamics, to motivate the definition of classical Nambu brackets. We then describe properties of CNBs, with emphasis on practical evaluation methods, including various recursion relations among the brackets and simplifications that result from classical Lie symmetries being imposed on the entries in the brackets. We summarize the theory of the fundamental identity and explain its subsidiary role. We then go through several examples to gain physical insight for the classical formalism. All the examples are based on systems with Lie symmetries: $so(3)=su(2)$, $u(n)$, $so(4)=su(2)\times su(2)$, and $g\times g$.

A. Phase-space geometry

A Hamiltonian system with N degrees of freedom is *integrable* in the Liouville sense if it has N invariants in involution (globally defined and functionally independent) and *superintegrable* [5] if it has additional independent conservation laws up to a maximum total number of $2N-1$ invariants. For a maximally superintegrable system, the total multilinear cross product of the $2N-1$ local phase-space gradients of the invariants (each such gradient being perpendicular to its corresponding invariant isocline) is always locally tangent to the classical trajectory.

The illustrated surfaces (Fig. 1) are isoclines for two different invariants, respectively, I_1 and I_2. A particular trajectory lies along the intersection of these two surfaces. The local phase-space tangent v to this trajectory at the point depicted is given by the cross product of the local phase-space gradients of the invariants. (Other possible trajectories along the I_1 surface are also shown as contours representing other values for I_2, but the corresponding intersecting I_2 surfaces are not shown for other trajectories.)

Thus, in $2N$-dimensional phase space, for any phase-space function $A(\mathbf{x},\mathbf{p})$ with no explicit time dependence, the convective motion is fully specified by a phase-space Jacobian which amounts to the classical Nambu bracket,

$$\frac{dA}{dt}\equiv\mathbf{v}\cdot\boldsymbol{\nabla}A\propto\partial_{i_1}A\,\epsilon^{i_1i_2\cdots i_{2N}}\partial_{i_2}I_1\cdots\partial_{i_{2N}}I_{2N-1}$$
$$=\frac{\partial(A,I_1,\ldots,I_{2N-1})}{\partial(x_1,p_1,x_2,p_2,\ldots,x_N,p_N)},\qquad(3)$$

where $\mathbf{v}=(\dot{\mathbf{x}},\dot{\mathbf{p}})$ is the phase-space velocity, and the phase-space gradients are $\boldsymbol{\nabla}=(\partial_{\mathbf{x}},\partial_{\mathbf{p}})$. Evidently, the flow is solenoidal, $\boldsymbol{\nabla}\cdot\mathbf{v}=0$ (Liouville's theorem [1]). In short, a superintegrable system in phase space *can hardly avoid* having its classical evolution described by CNBs [6].

B. Properties of the classical brackets

a. Definitions. For a system with N degrees of freedom, and hence a $2N$-dimensional phase space, we define the maximal classical Nambu brackets (CNB) of rank $2N$ to be the determinant

$$\{A_1,A_2,\ldots,A_{2N}\}_{\mathrm{NB}}=\frac{\partial(A_1,A_2,\ldots,A_{2N})}{\partial(x_1,p_1,x_2,p_2,\ldots,x_N,p_N)}$$
$$=\epsilon^{i_1i_2\cdots i_{2N}}\partial_{i_1}A_1\cdots\partial_{i_{2N}}A_{2N}.\qquad(4)$$

These brackets are linear in their arguments, and completely antisymmetric in them. It may be thought of as the Jacobian induced by transforming to new phase-space variables A_i, the "elements" in the brackets. As expected for such a Jacobian, two functionally dependent elements cause the brackets to collapse to zero. So, in particular, adding to any element an arbitrary linear combination of the other elements will not change the value of the brackets.

Odd-dimensional brackets are also defined identically [1] in an odd-dimensional space.

b. Recursion relations. The simplest of these are immediate consequences of the properties of the totally antisymmetric Levi-Civita symbols

$$\frac{\partial(A_1,A_2,\ldots,A_k)}{\partial(z_1,z_2\ldots,z_k)}=\frac{\epsilon^{i_1\cdots i_k}}{(k-1)!}\left(\frac{\partial A_1}{\partial z_{i_1}}\right)\frac{\partial(A_2,\cdots,A_k)}{\partial(z_{i_2},\cdots,z_{i_k})}$$
$$=\frac{\epsilon^{j_1\cdots j_k}}{(k-1)!}\left(\frac{\partial A_{j_1}}{\partial z_1}\right)\frac{\partial(A_{j_2},\cdots,A_{j_k})}{\partial(z_2,\cdots,z_k)}.\qquad(5)$$

However, these $k=1+(k-1)$ resolutions are not especially germane to a phase-space discussion, since they reduce even brackets into products of odd brackets.

More usefully, any maximal even rank CNB can also be resolved into products of Poisson brackets. For example, for systems with two degrees of freedom, $\{A,B\}_{\mathrm{PB}}=\partial(A,B)/\partial(x_1,p_1)+\partial(A,B)/\partial(x_2,p_2)$, and the 4-bracket $\{A,B,C,D\}_{\mathrm{NB}}\equiv\partial(A,B,C,D)\partial(x_1,p_1,x_2,p_2)$ resolves as[4]

[4]These PB resolutions are somewhat simpler than their quantum counterparts, to be given below in Sec. III A, since ordering of products is not an issue here.

$$\{A,B,C,D\}_{\rm NB}=\{A,B\}_{\rm PB}\{C,D\}_{\rm PB}-\{A,C\}_{\rm PB}\{B,D\}_{\rm PB}-\{A,D\}_{\rm PB}\{C,B\}_{\rm PB} \quad (6)$$

in comportance with full antisymmetry under permutations of A,B,C, and D. The general result for maximal rank $2N$ brackets for systems with a $2N$-dimensional phase-space is[5]

$$\{A_1,A_2,\cdots,A_{2N-1},A_{2N}\}_{\rm NB} = \sum_{\substack{\text{all }(2N)!\text{ perms}\\ \{\sigma_1,\sigma_2,\ldots,\sigma_{2N}\}\\ \text{of the indices}\\ \{1,2,\ldots,2N\}}} \frac{\mathrm{sgn}(\sigma)}{2^N N!}\{A_{\sigma_1},A_{\sigma_2}\}_{\rm PB}\{A_{\sigma_3},A_{\sigma_4}\}_{\rm PB}\ldots \times\{A_{\sigma_{2N-1}},A_{\sigma_{2N}}\}_{\rm PB}, \quad (7)$$

where $\mathrm{sgn}(\sigma)=(-1)^{\pi(\sigma)}$ with $\pi(\sigma)$ the parity of the permutation $\{\sigma_1,\sigma_2,\ldots,\sigma_{2N}\}$. The sum only gives $(2N-1)!!=(2N)!/(2^N N!)$ distinct products of PBs on the right hand side (RHS), not $(2N)!$. Each such distinct product appears with net coefficient ± 1.

The proof of the relation (7) is elementary. Both right- and left-hand sides (LHS) of the expression are sums of $2N$th degree monomials linear in the $2N$ first-order partial derivatives of each of the As. Both sides are totally antisymmetric under permutations of the As. Hence, both sides are also totally antisymmetric under interchanges of partial derivatives. Thus, the two sides must be proportional. The only issue left is the constant of proportionality. This is easily determined to be 1, by comparing the coefficients of any given term appearing on both sides of the equation, e.g., $\partial_{x_1}A_1\partial_{p_1}A_2\cdots\partial_{x_N}A_{2N-1}\partial_{p_N}A_{2N}$.

For similar relations to hold for submaximal even rank Nambu brackets, these must first be defined. It is easiest to just *define* submaximal even rank CNBs by their Poisson brackets resolutions as in Eq. (7):[6]

$$\{A_1,A_2,\ldots,A_{2n-1},A_{2n}\}_{\rm NB} = \sum_{(2n)!\text{ perms }\sigma} \frac{\mathrm{sgn}(\sigma)}{2^n n!}\{A_{\sigma_1},A_{\sigma_2}\}_{\rm PB}\{A_{\sigma_3},A_{\sigma_4}\}_{\rm PB}\cdots \times\{A_{\sigma_{2n-1}},A_{\sigma n}\}_{\rm PB}, \quad (8)$$

only here we allow $n<N$. So defined, these submaximal CNBs enter in further recursive expressions. For example, for systems with three or more degrees of freedom, $\{A,B\}_{\rm PB}=\partial(A,B)/\partial(x_1,p_1)+\partial(A,B)/\partial(x_2,p_2)+\partial(A,B)/\partial(x_3,p_3)+\cdots$, and a general 6-bracket expression resolves as

$$\begin{aligned}\{A_1,A_2,A_3,A_4,A_5,A_6\}_{\rm NB} &=\{A_1,A_2\}_{\rm PB}\{A_3,A_4,A_5,A_6\}_{\rm NB}\\ &-\{A_1,A_3\}_{\rm PB}\{A_2,A_4,A_5,A_6\}_{\rm NB}\\ &+\{A_1,A_4\}_{\rm PB}\{A_2,A_3,A_5,A_6\}_{\rm NB}\\ &-\{A_1,A_5\}_{\rm PB}\{A_2,A_3,A_4,A_6\}_{\rm NB}\\ &+\{A_1,A_6\}_{\rm PB}\{A_2,A_3,A_4,A_5\}_{\rm NB}, \end{aligned} \quad (9)$$

with the 4-brackets resolvable into PBs as in Eq. (6). This permits the building-up of higher even rank brackets proceeding from initial PBs involving all degrees of freedom. The general recursion relation with this $2n=2+(2n-2)$ form is

$$\begin{aligned}\{A_1,A_2,\ldots,A_{2n-1},A_{2n}\}_{\rm NB} &=\{A_1,A_2\}_{\rm PB}\{A_3,\ldots,A_{2n}\}_{\rm NB}+\sum_{j=3}^{2n-1}(-1)^j\\ &\times\{A_1,A_j\}_{\rm PB}\{A_2,\ldots,A_{j-1},A_{j+1},\ldots,A_{2n}\}_{\rm NB}\\ &+\{A_1,A_{2n}\}_{\rm PB}\{A_2,\ldots,A_{2n-1}\}_{\rm NB}, \end{aligned} \quad (10)$$

and features $2n-1$ terms on the RHS. This recursive result is equivalent to taking Eq. (8) as a definition for $2n<2N$ elements, as can be seen by substituting the PB resolutions of the $(2n-2)$ brackets on the RHS of Eq. (10). Similar relations obtain when the $2n$ elements in the CNB are partitioned into sets of $(2n-2k)$ and $2k$ elements, with suitable antisymmetrization with respect to exchanges between the two sets.

These results may be extended beyond maximal CNBs to supermaximal brackets, in a useful way. All such supermaximal classical brackets vanish, for the simple reason that there are not enough independent partial derivatives to avoid repeating columns of the implicit matrix whose determinant is under consideration. Another way to say this is as it is impossible to antisymmetrize more than $2N$ coordinate and momentum indices in $2N$-dimensional phase space, so for any phase-space function V, we have $\epsilon^{[j_1 j_2\cdots j_{2N}}\partial^{i]}V\equiv 0$, with $\partial^i=\partial/\partial x^i, \partial^{1+i}=\partial/\partial p^i, 1\leqslant i$ (odd) $\leqslant 2N-1$. Consequently, $\partial^{j_1}A_1\cdots\partial^{j_{2N}}A_{2N}\,\epsilon^{[j_1 j_2\cdots j_{2N}}\partial^{i]}V=0$, for any $2N$ phase-space functions A_j, $j=1,\ldots,2N$, and any V, a result that may be thought of as the vanishing of the $(2N+1)$-th super-maximal CNB. As a further consequence, we have on $2N$-dimensional phase spaces other super-maximal identities

[5]This is essentially a special case of Laplace's theorem on the general minor expansions of determinants (cf. Chap. 4 in [43]), although it must be said that we have never seen it written, let alone used, in exactly this form, either in treatises on determinants or in textbooks on classical mechanics.

[6]This definition is consistent with the classical limits of quantum ⋆-brackets presented and discussed in [6], from which the same Poisson bracket resolutions follow as a consequence of taking the classical limit of ⋆-commutator resolutions of even ⋆ brackets. It is also consistent with taking symplectic traces of maximal CNBs, again as presented in [6] [see Eq. (14) to follow].

of the form

$$\{B_1,\ldots,B_k,V\}_{\rm NB}\{A_1,A_2,\ldots,A_{2N}\}_{\rm NB}$$
$$=\{B_1,\ldots,B_k,A_1\}_{\rm NB}\{V,A_2,\ldots,A_{2N}\}_{\rm NB}$$
$$+\{B_1,\ldots,B_k,A_2\}_{\rm NB}\{A_1,V,A_3,\ldots,A_{2N}\}_{\rm NB}+\cdots$$
$$+\{B_1,\ldots,B_k,A_{2N}\}_{\rm NB}\{A_1,A_2,\ldots,A_{2N-1},V\}_{\rm NB},\tag{11}$$

for any choice of V, k, As, and Bs. We have distinguished here a $(2N+1)$th phase-space function as V in anticipation of using the result later [cf. the discussion of the modified fundamental identity, (23) *et seq.*]. The expansions in Eqs. (8) and (10) also apply to the supermaximal case as well, where they provide vanishing theorems for the sums on the RHSs of those relations.

c. Reductions for classical Lie symmetries. When the phase-space functions involved in the classical brackets obey the Poisson brackets algebra (possibly even an infinite one), the NB reduces to become a sum of products, each product involving half as many phase-space functions (*reductio ad dimidium*). It follows as an elementary consequence of the PB resolution of even CNBs. For any PB Lie algebra given by

$$\{B_i,B_j\}_{\rm PB}=\sum_m c_{ij}^m B_m,\tag{12}$$

the PB resolution then gives (sum over all repeated ms is to be understood)

$$\{B_1,\ldots,B_{2k+1},A\}_{\rm NB}$$
$$=\sum_{(2k+1)!\ \text{perms}\ \sigma}\frac{\text{sgn}(\sigma)}{2^k k!}\{B_{\sigma_1},B_{\sigma_2}\}_{\rm PB}\{B_{\sigma_3},B_{\sigma_4}\}_{\rm PB}\cdots$$
$$\times\{B_{\sigma_{2k-1}},B_{\sigma_{2k}}\}_{\rm PB}\{B_{\sigma_{2k+1}},A\}_{\rm PB}$$
$$=\sum_{(2k+1)!\ \text{perms}\ \sigma}\frac{\text{sgn}(\sigma)}{2^k k!}c_{\sigma_1\sigma_2}^{m_1}c_{\sigma_3\sigma_4}^{m_2}\cdots c_{\sigma_{2k-1}\sigma_{2k}}^{m_k}$$
$$\times B_{m_1}B_{m_2}\cdots B_{m_k}\{B_{\sigma_{2k+1}},A\}_{\rm PB},\tag{13}$$

where A is arbitrary. Of course, if A is also an element of the Lie algebra then the last PB also reduces.

d. Traces. Define the *symplectic trace* of the classical brackets as

$$\sum_i\{x_i,p_i,A_1,\ldots,A_{2k}\}_{\rm NB}=(N-k)\{A_1,\ldots,A_{2k}\}_{\rm NB}.\tag{14}$$

A complete reduction of maximal CNBs to PBs follows by inserting $N-1$ conjugate pairs of phase-space coordinates and summing over them

$$\{A,B\}_{\rm PB}=\frac{1}{(N-1)!}\{A,B,x_{i_1},p_{i_1},\ldots,x_{i_{N-1}},p_{i_{N-1}}\}_{\rm NB},\tag{15}$$

where summation over all pairs of repeated indices is understood. Fewer traces lead to relations between CNBs of maximal rank, $2N$, and those of lesser rank, $2k$,

$$\{A_1,\ldots,A_{2k}\}_{\rm NB}$$
$$=\frac{1}{(N-k)!}\{A_1,\ldots,A_{2k},x_{i_1},p_{i_1},\ldots,x_{i_{N-k}},p_{i_{N-k}}\}_{\rm NB}.\tag{16}$$

This is consistent with the PB resolutions (8) used to define the lower rank CNBs previously, and provides another practical evaluation tool for these CNBs.

Through the use of such symplectic traces, Hamilton's equations for a general system—not necessarily superintegrable—admit an NB expression different from Nambu's original one, namely,

$$\frac{dA}{dt}=\{A,H\}_{\rm PB}=\frac{1}{(N-1)!}\{A,H,x_{i_1},p_{i_1},\ldots,x_{i_{N-1}},p_{i_{N-1}}\}_{\rm NB},\tag{17}$$

where H is the system Hamiltonian.

e. Derivations and the classical "Fundamental Identity." CNBs are all derivations with respect to each of their arguments [1]. For even brackets, this follows from Eq. (4) for maximal CNBs and from Eq. (8) [or Eq. (16)] for submaximal brackets,

$$\delta_{\mathbf{B}}A=\{A,B_1,B_2,\ldots,B_{2n-1}\}_{\rm NB},\tag{18}$$

where $\mathbf{B}$ is a shorthand for the string $B_1,B_2,\ldots,B_{2n-1}$. By derivation, we mean that Leibniz's elementary rule is satisfied,

$$\delta_{\mathbf{B}}(A\mathcal{A})=(\delta_{\mathbf{B}}A)\mathcal{A}+A(\delta_{\mathbf{B}}\mathcal{A})=\{A,B_1,\ldots,B_{2n-1}\}_{\rm NB}\mathcal{A}$$
$$+A\{\mathcal{A},B_1,\ldots,B_{2n-1}\}_{\rm NB}.\tag{19}$$

Moreover, when these derivations act on other *maximal* CNBs, they yield simple bracket identities [1,35,22],

$$\delta_{\mathbf{B}}\{C_1,\ldots,C_{2N}\}_{\rm NB}=\{\delta_{\mathbf{B}}C_1,\ldots,C_{2N}\}_{\rm NB}+\cdots$$
$$+\{C_1,\ldots,\delta_{\mathbf{B}}C_{2N}\}_{\rm NB},\tag{20}$$

alternatively

$$\{\{C_1,\ldots,C_{2N}\}_{\rm NB},B_1,\ldots,B_{2n-1}\}_{\rm NB}$$
$$=\{\{C_1,B_1,\ldots,B_{2n-1}\}_{\rm NB},\ldots,C_{2N}\}_{\rm NB}+\cdots$$
$$+\{C_1,\ldots,\{C_{2N},B_1,\ldots,B_{2n-1}\}_{\rm NB}\}_{\rm NB}.\tag{21}$$

In particular, any maximal CNB acting on any other maximal CNB always obeys the $(4N-1)$ element, $(2N+1)$ term identity [35,22]

$$0=\{\{A_1,A_2,\ldots,A_{2N}\}_{\rm NB},B_1,\ldots,B_{2N-1}\}_{\rm NB} - \sum_{j=1}^{2N}\{A_1,\ldots,\{A_j,B_1,\ldots,B_{2N-1}\}_{\rm NB},\ldots,A_{2N}\}_{\rm NB}. \tag{22}$$

This has been designated "the fundamental identity" (FI) [26], although its essentially subsidiary role should be apparent in this classical context.

f. Invariant coefficients. The fact that all CNBs are derivations, and that all supermaximal classical brackets vanish, leads to a slightly modified form of $4N$ element, $(2N+1)$-term fundamental identities, for a system in a $2N$-dimensional phase space [6]

$$\begin{aligned}\{B_1,\ldots,B_{2N-1},V\{A_1,A_2,\ldots,A_{2N}\}_{\rm NB}\}_{\rm NB} &= \{V\{B_1,\ldots,B_{2N-1},A_1\}_{\rm NB},A_2,\ldots,A_{2N}\}_{\rm NB}\\ &\quad+\{A_1,V\{B_1,\ldots,B_{2N-1},A_2\}_{\rm NB},A_3,\ldots,A_{4N-1}\}_{\rm NB}\\ &\quad+\cdots+\{A_1,\ldots,A_{2N-1},\\ &\quad\times V\{B_1,\ldots,B_{2N-1},A_{2N}\}_{\rm NB}\}_{\rm NB},\end{aligned} \tag{23}$$

for any choice of V, As, and Bs. This identity is just the sum of the supermaximal identity (11), for $k=2N-1$, plus V times the FI (22) for the derivation $\{B_1,\ldots,B_{2N-1},\{A_1,A_2,\ldots,A_{2N}\}_{\rm NB}\}_{\rm NB}$.

As a consequence of this modified FI, any proportionality constant V appearing in Eq. (3), i.e.,

$$\frac{dA}{dt}=V\{A,I_1,\ldots,I_{2N-1}\}_{\rm NB}, \tag{24}$$

has to be time invariant if it has no *explicit* time dependence [8]. As proof [6], since the time derivation satisfies the conditions for the above δ, we have

$$\begin{aligned}\frac{d}{dt}(V\{A_1,\ldots,A_{2N}\}_{\rm NB}) &= \dot V\{A_1,\ldots,A_{2N}\}_{\rm NB}\\ &\quad+V\{\dot A_1,\ldots,A_{2N}\}_{\rm NB}+\cdots\\ &\quad+V\{A_1,\ldots,\dot A_{2N}\}_{\rm NB}.\end{aligned} \tag{25}$$

Consistency with Eq. (24) requires this to be the same as

$$\begin{aligned}V\{V\{A_1,\ldots,A_{2N}\}_{\rm NB},I_1,\ldots,I_{2N-1}\}_{\rm NB} &= \dot V\{A_1,\ldots,A_{2N}\}_{\rm NB}\\ &\quad+V\{V\{A_1,I_1,\ldots,I_{2N-1}\}_{\rm NB},\ldots,A_{2N}\}_{\rm NB}+\cdots\\ &\quad+V\{A_1,\ldots,V\{A_{2N},I_1,\ldots,I_{2N-1}\}_{\rm NB}\}_{\rm NB}.\end{aligned} \tag{26}$$

By substitution of Eq. (23) with $B_j\equiv I_j$, $\dot V=0$ follows.

C. Illustrative classical examples

It is useful to consider explicit examples of classical dynamical systems described by Nambu brackets, to gain insight and develop intuition concerning CNBs. Previous classical examples were given by Nambu [1], and more recently, by Chatterjee [3], and by Gonera and Nutku [8,44]. We offer an eclectic selection based on those in [6].

a. SO(3) as a special case For example, consider a particle constrained to the surface of a unit radius 2-sphere S^2, but otherwise moving freely. Three independent invariants of this maximally superintegrable system are the angular momenta about the center of the sphere: L_x, L_y, L_z. Actually, no two of these are in involution, but this is quickly remedied, and moreover, it is not a hindrance since in the Nambu approach to mechanics all invariants are on a more equal footing.

To be more explicit, we may coordinate the upper and lower ($\pm$) hemispheres by projecting the particle's location onto the equatorial disk, $\{(x,y)|x^2+y^2\leqslant 1\}$. The invariants are then

$$L_z=xp_y-yp_x,\quad L_y=\pm\sqrt{1-x^2-y^2}\,p_x,\quad L_x=\mp\sqrt{1-x^2-y^2}\,p_y. \tag{27}$$

The last two are the de Sitter momenta, or nonlinearly realized axial charges corresponding to the "pions" x,y of this truncated σ model.

The Poisson brackets of these expressions close into the expected so(3) algebra,

$$\{L_x,L_y\}_{\rm PB}=L_z,\quad \{L_y,L_z\}_{\rm PB}=L_x,\quad \{L_z,L_x\}_{\rm PB}=L_y. \tag{28}$$

The usual Hamiltonian of the free particle system is the Casimir invariant [6]

$$\begin{aligned}H&=\tfrac{1}{2}(L_xL_x+L_yL_y+L_zL_z)\\ &=\tfrac{1}{2}(1-x^2)p_x^2+\tfrac{1}{2}(1-y^2)p_y^2-xyp_xp_y.\end{aligned} \tag{29}$$

Thus, it immediately follows algebraically that PBs of H with the $\mathbf{L}$ vanish, and their time-invariance holds,

$$\frac{d}{dt}\mathbf{L}=\{\mathbf{L},H\}_{\rm PB}=0. \tag{30}$$

So any one of the L's and this Casimir invariant constitute a pair of invariants in involution.

The corresponding so(3) CNB dynamical evolution, found in [6], is untypically concise:

$$\frac{dA}{dt}=\{A,H\}_{\rm PB}=\{A,L_x,L_y,L_z\}_{\rm NB}=\frac{\partial(A,L_x,L_y,L_z)}{\partial(x,p_x,y,p_y)}. \tag{31}$$

The simplicity of this result actually extends to more general contexts, upon use of suitable linear combinations. Special sums of such 4-brackets can be used to express time evolution for any classical system with a continuous symmetry algebra underlying the dynamics and whose Hamiltonian is just the quadratic Casimir invariant of that symmetry algebra. The system need not be superintegrable or even integrable in general.

Any simple Lie algebra allows a PB with a quadratic Casimir invariant to be rewritten as a *sum* of 4-brackets. Suppose

$$\{Q_a,Q_b\}_{\rm PB}=f_{abc}Q_c \tag{32}$$

in a basis where f_{abc} is totally antisymmetric. Then, for the following linear combination of Nambu 4-brackets weighted by the structure constants, use the PB resolution of the 4-brackets (6) to obtain (sum over repeated indices)

$$f_{abc}\{A,Q_a,Q_b,Q_c\}_{\rm NB}=3f_{abc}\{A,Q_a\}_{\rm PB}\{Q_b,Q_c\}_{\rm PB}$$
$$=3f_{abc}f_{bcd}\{A,Q_a\}_{\rm PB}Q_d. \tag{33}$$

Now, for simple Lie algebras (with appropriately normalized charges) one has

$$f_{abc}f_{bcd}=c_{\rm adjoint}\delta_{ad}, \tag{34}$$

where $c_{\rm adjoint}$ is a *number* [for example, $c_{\rm adjoint}=N$ for su(N)]. Thus, the classical 4-brackets reduce to a PB with the Casimir invariant Q_aQ_a,

$$f_{abc}\{A,Q_a,Q_b,Q_c\}_{\rm NB}=3c_{\rm adjoint}\{A,Q_a\}_{\rm PB}Q_a$$
$$=\frac{3}{2}c_{\rm adjoint}\{A,Q_aQ_a\}_{\rm PB}, \tag{35}$$

For su(2)=so(3), $c_{\rm adjoint}=2$, $f_{abc}\{A,Q_a,Q_b,Q_c\}_{\rm NB}=6\{A,L_x,L_y,L_z\}_{\rm NB}$, and we establish Eq. (31) above.

b. U(n) and isotropic oscillators. If we realize the u(n) algebra in the oscillator basis, where the phase-space "charges" $N_{jk}=(x_j-ip_j)(x_k+ip_k)/2$ obey the PB relations

$$\{N_{jk},N_{lm}\}_{\rm PB}=-i(N_{jm}\delta_{kl}-N_{lk}\delta_{jm}),\quad j,k,l,m=1,\ldots,n, \tag{36}$$

then the isotropic Hamiltonian is

$$H=\omega\sum_{i=1}^{n}N_i,\quad N_i\equiv N_{ii}. \tag{37}$$

This gives the n^2 conservation laws

$$\{H,N_{ij}\}_{\rm PB}=0. \tag{38}$$

However, only $2n-1$ of the N_{ij} are functionally independent for a classical system with a $2n$-dimensional phase space. This follows because all full phase-space Jacobians (i.e., maximal CNBs) involving $2n$ of the N_{ij} vanish. [For details, see the upcoming discussion surrounding Eq. (46).]

Following the logic that led to the previous *reductio ad dimidium* for general Lie symmetries, we obtain the main result for classical isotropic oscillator $2n$-brackets.

c. Classical isotropic oscillator brackets. (The $U(n)$ *reductio ad dimidium*): Let $N=N_1+N_2+\cdots+N_n$, and intercalate the $n-1$ nondiagonal charges N_{ii+1}, for $i=1,\ldots,n-1$, into classical Nambu $2n$-brackets with the n mutually involutive N_j, for $j=1,\ldots,n$, to find[7]

$$\{A,N_1,N_{12},N_2,N_{23},\ldots,N_{n-1},N_{n-1n},N_n\}_{\rm NB}$$
$$=(-i)^{n-1}\{A,N\}_{\rm PB}N_{12}N_{23}\cdots N_{n-1n}$$
$$=(-i)^{n-1}\{AN_{12}N_{23}\cdots N_{n-1n},N\}_{\rm PB}. \tag{39}$$

This result follows from the u(n) PB algebra of the charges (36). When the algebra is realized specifically by harmonic oscillators, the RHS factor may also be written as $N_{12}N_{23}\cdots N_{n-1n}=(N_2N_3\cdots N_{n-1})N_{1n}$.

Proof. Linearity in each argument and total antisymmetry of the CNB allows us to replace any one of the N_i by the sum N. Replace $N_n\to N$, to obtain

$$\{A,N_1,N_{12},N_2,\ldots,N_{n-1},N_{n-1n},N_n\}_{\rm NB}$$
$$=\{A,N_1,N_{12},N_2,\ldots,N_{n-1},N_{n-1n},N\}_{\rm NB}. \tag{40}$$

Now since $\{N,N_{ij}\}_{\rm PB}=0$, the PB resolution of the $2n$-brackets implies that N must appear "locked" in a PB with A, and therefore A cannot appear in any other PB. But then N_1 is in involution with all the remaining free N_{ij} except N_{12}. So N_1 must be locked in $\{N_1,N_{12}\}_{\rm PB}$. Continuing in this way, N_2 must be locked in $\{N_2,N_{23}\}_{\rm PB}$, etc., until, finally, N_{n-1} is locked in $\{N_{n-1},N_{n-1n}\}_{\rm PB}$. Thus, all $2n$ entries have been paired and locked in the indicated n PBs, i.e., they are all zipped-up. Consequently,

$$\{A,N_1,N_{12},N_2,\ldots,N_{n-1},N_{n-1n},N_n\}_{\rm NB}$$
$$=\{A,N\}_{\rm PB}\{N_1,N_{12}\}_{\rm PB}\cdots\{N_{n-1},N_{n-1n}\}_{\rm PB}. \tag{41}$$

All the paired N_{jk} Poisson brackets evaluate as $\{N_{j-1},N_{j-1j}\}_{\rm PB}=-iN_{j-1j}$, so

$$\{A,N_1,N_{12},N_2,\ldots,N_{n-1},N_{n-1n},N_n\}_{\rm NB}$$
$$=(-i)^{n-1}\{A,N\}_{\rm PB}N_{12}\cdots N_{n-1n}. \tag{42}$$

Finally, the PB with N may be performed either before or after the product of A with all the N_{j-1j}, since again $\{N,N_{ij}\}_{\rm PB}=0$, and the PB is a derivation. Hence,

$$\{A,N\}_{\rm PB}N_{12}\cdots N_{n-1n}=\{AN_{12}\cdots N_{n-1n},N\}_{\rm PB}.\ \text{QED}. \tag{43}$$

Remarkably, in Eq. (39), the invariants which are in involution [i.e., the Cartan subalgebra of $u(n)$] are separated out of the CNB into a single PB involving their sum (the Hamiltonian, $H=\omega N$), while the invariants which are not in involution [$n-1$ of them, corresponding in number to the rank of

[7]The nondiagonal charges are not real, but neither does this present a real problem. The proof leading to Eq. (39) also goes through if nondiagonal charges have their subscripts transposed. This allows replacing N_{ii+1} with real or purely imaginary combinations $N_{ii+1}\pm N_{i+1i}$ in the LHS $2n$-brackets, to obtain the alternative linear combinations $N_{ii+1}\mp N_{i+1i}$ in the product on the RHS.

$SU(n)$] are effectively swept into a simple product. Time evolution for the isotropic oscillator is then given by [6]

$$(-i)^{n-1}N_{12}\cdots N_{n-1n}\frac{dA}{dt}=\omega\{A,N_1,N_{12},N_2,\ldots,N_{n-1},N_{n-1n},N_n\}_{\rm NB}. \tag{44}$$

This result reveals a possible degenerate situation for the Nambu approach.

When any two or more of the phase-space gradients entering into the brackets are parallel or when one or more of them vanish, the corresponding brackets also vanish, even if $dA/dt\neq 0$. Under these conditions, the brackets do not give any temporal change of A: Such changes are "lost" by the brackets. This can occur for the $u(n)$ brackets under consideration whenever $0=N_{12}\cdots N_{n-1n}$, i.e., whenever any $N_{i-1i}=0$ for some i. Initial classical configurations for which this is the case are not evolved by these particular brackets. This is not really a serious problem, since on the one hand, the configurations for which it happens are so easily cataloged and, on the other hand, there are other choices for the bracket entries which can be used to recover the lost temporal changes. It is just necessary to be aware of any such "kernel" when using any given brackets.

With that caveat in mind, there is another way to write Eq. (44) since the classical brackets are a derivation of each of their entries. Namely,

$$\frac{dA}{dt}=i^{n-1}\omega\{A,N_1,\ln(N_{12}),N_2,\ln(N_{23}),N_3,\ldots,N_{n-1},\ln(N_{n-1n}),N_n\}_{\rm NB}. \tag{45}$$

The logarithms intercalated between the diagonal N_j's on the RHS now have branch points corresponding to the classical bracket's kernel.

The selection of $2n-1$ invariants to be used in the maximal U(n) brackets is not unique, of course. In the list that we have selected, the indices, 1,2, . . . ,n, can be replaced by any permutation, $\sigma_1,\sigma_2,\ldots,\sigma_n$, so long as the correlations between indices for elements in the list are maintained. That is, we may replace the elements $N_1,N_{12},N_2,N_{23},\ldots,N_{n-1},N_{n-1n},N_n$ by $N_{\sigma_1},N_{\sigma_1\sigma_2}$, $N_{\sigma_2},N_{\sigma_2\sigma_3},\ldots,N_{\sigma_{n-1}},N_{\sigma_{n-1}\sigma_n},N_{\sigma_n}$, and the *reductio ad dimidium* still holds:

$$\{A,N_{\sigma_1},N_{\sigma_1\sigma_2},N_{\sigma_2},N_{\sigma_2\sigma_3},\ldots,N_{\sigma_{n-1}},N_{\sigma_{n-1}\sigma_n},N_{\sigma_n}\}_{\rm NB}=(-i)^{n-1}\{A,N\}_{\rm PB}N_{\sigma_1\sigma_2}N_{\sigma_2\sigma_3}\cdots N_{\sigma_{n-1}\sigma_n}. \tag{46}$$

Whatever list is selected, any invariant in that list is manifestly conserved by the $2n$-brackets. All other U(n) charges are also conserved by the brackets, even though they are not among the selected list of invariants. This last statement follows immediately from the $\{A,H\}_{\rm PB}$ factor on the RHS of Eq. (46).

d. SO(n+1) and free particles on n spheres. For a particle moving freely on the surface of an n sphere S^n, one now has a choice of $2n-1$ of the $n(n+1)/2$ invariant charges of so$(n+1)$, whose PB Lie algebra is conveniently written in terms of the $n(n-1)/2$ rotation generators, $L_{ab}=x^a p_b-x^b p_a$ for $a,b=1,\ldots,n$ and in terms of the de Sitter momenta, $P_a=\sqrt{1-q^2}p_a$ for $a=1,\ldots,n$, where $q^2=\Sigma_{a=1}^n(x^a)^2$. That PB algebra is

$$\{P_a,P_b\}_{\rm PB}=L_{ab},\quad \{L_{ab},P_c\}_{\rm PB}=\delta_{ac}P_b-\delta_{bc}P_a,$$

$$\{L_{ab},L_{cd}\}_{\rm PB}=L_{ac}\delta_{bd}-L_{ad}\delta_{bc}-L_{bc}\delta_{ad}+L_{bd}\delta_{ac}. \tag{47}$$

By direct calculation, one of several possible expressions for time evolution as a $2n$-brackets is [6]

$$(-1)^{n-1}P_2P_3\cdots P_{n-1}\frac{dA}{dt}=\frac{\partial(A,P_1,L_{12},P_2,L_{23},P_3,\ldots,P_{n-1},L_{n-1n},P_n)}{\partial(x_1,p_1,x_2,p_2,\ldots,x_n,p_n)}, \tag{48}$$

where $dA/dt=\{A,H\}_{\rm PB}$ and

$$H=\frac{1}{2}\sum_{a=1}^{n}P_aP_a+\frac{1}{4}\sum_{a,b=1}^{n}L_{ab}L_{ab}. \tag{49}$$

The CNB expressing classical time-evolution may also be written, more compactly, as a derivation

$$\frac{dA}{dt}=(-1)^{n-1}\{A,P_1,L_{12},\ln(P_2),L_{23},\ln(P_3),\ldots,\ln(P_{n-1}),L_{n-1n},P_n\}_{\rm NB}. \tag{50}$$

Once again, the branch points in the intercalated logarithms are indicators of this particular bracket's kernel.

e. SO(4)=SU(2)×SU(2) as another special case. The treatment of the 3-sphere S^3 also accords to the standard chiral model technology using left- and right-invariant Vielbeine [6]. Specifically, the two choices for such Dreibeine for the 3-sphere are [45]: $q^2=x^2+y^2+z^2$,

$${}^{(\pm)}V_a^i=\epsilon^{iab}x^b\pm\sqrt{1-q^2}g_{ai},\quad {}^{(\pm)}V^{ai}=\epsilon^{iab}x^b\pm\sqrt{1-q^2}\delta^{ai}. \tag{51}$$

The corresponding right and left conserved charges (left and right invariant, respectively) then are

$$\mathcal{R}^i={}^{(+)}V_a^i\frac{d}{dt}x^a={}^{(+)}V^{ai}p_a,$$

$$\mathcal{L}^i={}^{(-)}V_a^i\frac{d}{dt}x^a={}^{(-)}V^{ai}p_a. \tag{52}$$

Perhaps more intuitive are the linear combinations into axial and isospin charges (again linear in the momenta),

$$\tfrac{1}{2}(\mathcal{R}-\mathcal{L})=\sqrt{1-q^2}\mathbf{p}\equiv\mathbf{A},\quad \tfrac{1}{2}(\mathcal{R}+\mathcal{L})=\mathbf{x}\times\mathbf{p}\equiv\mathbf{I}. \tag{53}$$

It can easily be seen that the $\mathcal{L}$'s and the $\mathcal{R}$'s have PBs closing into the standard $\mathrm{su}(2)\times\mathrm{su}(2)$ algebra, i.e.,

$$\{\mathcal{L}_i,\mathcal{L}_j\}_{\mathrm{NB}}=-2\varepsilon_{ijk}\mathcal{L}_k,\ \ \{\mathcal{L}_i,\mathcal{R}_j\}_{\mathrm{NB}}=0,$$

$$\{\mathcal{R}_i,\mathcal{R}_j\}_{\mathrm{NB}}=-2\varepsilon_{ijk}\mathcal{R}_k. \tag{54}$$

Thus, they are seen to be constant, since the Hamiltonian (and also the Lagrangian) can, in fact, be written in terms of either quadratic Casimir invariant,

$$H=\tfrac{1}{2}\mathcal{L}\cdot\mathcal{L}=\tfrac{1}{2}\mathcal{R}\cdot\mathcal{R}. \tag{55}$$

The classical dynamics of this algebraic system is, like the single $SU(2)$ invariant dynamics that composes it, elegantly expressed on the six-dimensional phase space with maximal CNBs. We find various 6-bracket relations such as

$$\frac{\partial(A,H,\mathcal{R}_1,\mathcal{R}_2,\mathcal{L}_1,\mathcal{L}_2)}{\partial(x_1,p_1,x_2,p_2,x_3,p_3)}\equiv\{A,H,\mathcal{R}_1,\mathcal{R}_2,\mathcal{L}_1,\mathcal{L}_2\}_{\mathrm{NB}}$$
$$=-4\mathcal{L}_3\mathcal{R}_3\frac{dA}{dt}, \tag{56}$$

where $2H=\mathcal{R}_1^2+\mathcal{R}_2^2+\mathcal{R}_3^2=\mathcal{L}_1^2+\mathcal{L}_2^2+\mathcal{L}_3^2$ and A is an arbitrary function of the phase-space dynamical variables. Also,

$$\{A,\mathcal{R}_1,\mathcal{R}_2,\mathcal{L}_3,\mathcal{L}_1,\mathcal{L}_2\}_{\mathrm{NB}}=-4\mathcal{R}_3\frac{dA}{dt}, \tag{57}$$

and similarly $(\mathcal{R}\leftrightarrow\mathcal{L})$,

$$\{A,\mathcal{R}_1,\mathcal{R}_2,\mathcal{R}_3,\mathcal{L}_1,\mathcal{L}_2\}_{\mathrm{NB}}=-4\mathcal{L}_3\frac{dA}{dt}. \tag{58}$$

The kernels of these various brackets are evident from the factors multiplying dA/dt. None of these particular 6-bracket relations directly permits the $\mathcal{L}_3$ or $\mathcal{R}_3$ factors on their RHSs to be absorbed into logarithms, through use of the Leibniz rule. But, by subtracting the last two to obtain

$$\{A,\mathcal{R}_1,\mathcal{R}_2,\mathcal{L}_3-\mathcal{R}_3,\mathcal{L}_1,\mathcal{L}_2\}_{\mathrm{NB}}=4(\mathcal{L}_3-\mathcal{R}_3)\frac{dA}{dt}, \tag{59}$$

we can now introduce a logarithm to produce just a numerical factor multiplying the time derivative,

$$\{A,\mathcal{R}_1,\mathcal{R}_2,\ln(\mathcal{L}_3-\mathcal{R}_3)^2,\mathcal{L}_1,\mathcal{L}_2\}_{\mathrm{NB}}=8\frac{dA}{dt}. \tag{60}$$

Similarly, by adding Eqs. (57) and (58), we find

$$\{A,\mathcal{R}_1,\mathcal{R}_2,\ln(\mathcal{L}_3+\mathcal{R}_3)^2,\mathcal{L}_1,\mathcal{L}_2\}_{\mathrm{NB}}=-8\frac{dA}{dt}. \tag{61}$$

f. $G\times G$ chiral particles. In general, the preceding discussion also applies to all chiral models, with the algebra g for a chiral group G replacing $\mathrm{su}(2)$. The Vielbein-momenta combinations $V^{aj}p_a$ represent algebra generator invariants, whose quadratic Casimir group invariants yield the respective Hamiltonians.

That is to say, for [46] group matrices U generated by exponentiated constant group algebra matrices T, weighted by functions of the particle coordinates x, with $U^{-1}=U^{\dagger}$, we have

$$iU^{-1}\frac{d}{dt}U={}^{(+)}V_a^jT_j\frac{d}{dt}x^a={}^{(+)}V^{aj}p_aT_j,$$

$$iU\frac{d}{dt}U^{-1}={}^{(-)}V^{aj}p_aT_j. \tag{62}$$

It follows that PBs of left- and right-invariant charges (designated by $\mathcal{R}$'s and $\mathcal{L}$'s, respectively), as defined by the traces,

$$\mathcal{R}_j\equiv\frac{i}{2}\mathrm{tr}\left(T_jU^{-1}\frac{d}{dt}U\right)={}^{(+)}V^{aj}p_a,$$

$$\mathcal{L}_j\equiv\frac{i}{2}\mathrm{tr}\left(T_jU\frac{d}{dt}U^{-1}\right)={}^{(-)}V^{aj}p_a, \tag{63}$$

close to the identical PB Lie algebras,

$$\{\mathcal{R}_i,\mathcal{R}_j\}_{\mathrm{PB}}=-2f_{ijk}\mathcal{R}_k,\quad \{\mathcal{L}_i,\mathcal{L}_j\}_{\mathrm{PB}}=-2f_{ijk}\mathcal{L}_k, \tag{64}$$

and PB commute with each other,

$$\{\mathcal{R}_i,\mathcal{L}_j\}_{\mathrm{PB}}=0. \tag{65}$$

These two statements are implicit in [46] and throughout the literature, and are explicitly proven in [6].

The Hamiltonian for a particle moving freely on the $G\times G$ group manifold is the simple form

$$H=\tfrac{1}{2}(p_aV^{ai})(V^{bi}p_b), \tag{66}$$

with either choice, $V^{aj}={}^{(\pm)}V^{aj}$. That is,

$$H=\tfrac{1}{2}\mathcal{L}_j\mathcal{L}_j=\tfrac{1}{2}\mathcal{R}_j\mathcal{R}_j, \tag{67}$$

just as in the previous $\mathrm{SO}(4)=\mathrm{SU}(2)\times\mathrm{SU}(2)$ case. There are now several ways to present time evolution as CNBs for these models.

One way is as sums of 6-brackets. Making use of Eq. (34) and summing repeated indices:

$$f_{ijk}f_{imn}\{A,H,\mathcal{R}_j,\mathcal{R}_k,\mathcal{L}_m,\mathcal{L}_n\}_{\mathrm{NB}}$$
$$=f_{ijk}f_{imn}\{\mathcal{R}_j,\mathcal{R}_k\}_{\mathrm{PB}}\{\mathcal{L}_m,\mathcal{L}_n\}_{\mathrm{PB}}\{A,H\}_{\mathrm{PB}}$$
$$=4f_{ijk}f_{imn}f_{jkl}f_{mno}\mathcal{R}_l\mathcal{L}_o\{A,H\}_{\mathrm{PB}}$$
$$=4c^2_{\mathrm{adjoint}}\mathcal{R}_l\mathcal{L}_l\{A,H\}_{\mathrm{PB}}. \tag{68}$$

Thus, we have

$$\frac{dA}{dt}=\frac{1}{4c_{\text{adjoint}}^2\mathcal{R}_l\mathcal{L}_l}f_{ijk}f_{imn}\{A,H,\mathcal{R}_j,\mathcal{R}_k,\mathcal{L}_m,\mathcal{L}_n\}_{\text{NB}}. \tag{69}$$

The bracket kernel here is given by zeros of $(\mathcal{R}_l\pm\mathcal{L}_l)^2-4H=\pm 2\mathcal{R}_l\mathcal{L}_l$.

Another way to specify the time development for these chiral models is to use a maximal set of invariants in the CNB, selected from both left and right charges. Take n to be the dimension of the group G, then all charge indices range from 1 to n. For a point particle moving on the group manifold $G\times G$, the maximal brackets involve $2n$ elements. So, for example, we have (note the ranges of all the sums here are truncated to $n-1$, as are the indices on the Levi-Civita symbols)

$$\{A,H,\mathcal{L}_1,\ldots,\mathcal{L}_{n-1},\mathcal{R}_1,\ldots,\mathcal{R}_{n-1}\}_{\text{NB}}=\frac{1}{[(n-1)!]^2}\sum_{\text{all } i,j=1}^{n-1}\varepsilon_{i_1\cdots i_{n-1}}\varepsilon_{j_1\cdots j_{n-1}}\times\{A,H,\mathcal{L}_{i_1},\ldots,\mathcal{L}_{i_{n-1}},\mathcal{R}_{j_1},\ldots,\mathcal{R}_{j_{n-1}}\}_{\text{NB}}. \tag{70}$$

The RHS here vanishes for even n, so we take odd n, say $n=1+2s$. (To obtain a nontrivial result for even n, we may replace H by either $\mathcal{L}_n$ or $\mathcal{R}_n$. We leave this as an exercise in the classical case. The relevant combinatorics are discussed later, in the context of the quantized model.) So, since $\{H,\mathcal{L}_i\}_{\text{PB}}=0=\{H,\mathcal{R}_i\}_{\text{PB}}$, by the PB resolution we can write

$$\{A,H,\mathcal{L}_1,\ldots,\mathcal{L}_{n-1},\mathcal{R}_1,\ldots,\mathcal{R}_{n-1}\}_{\text{NB}}=K_n\sum_{\text{all } i,j=1}^{n-1}\varepsilon_{i_1\cdots i_{n-1}}\varepsilon_{j_1\cdots j_{n-1}}\{A,H\}_{\text{PB}}\{\mathcal{L}_{i_1},\mathcal{L}_{i_2}\}_{\text{PB}}\cdots\times\{\mathcal{L}_{i_{n-2}},\mathcal{L}_{i_{n-1}}\}_{\text{PB}}\{\mathcal{R}_{j_1},\mathcal{R}_{j_2}\}_{\text{PB}}\cdots\{\mathcal{R}_{j_{n-2}},\mathcal{R}_{j_{n-1}}\}_{\text{PB}}, \tag{71}$$

where[8]

$$K_{n=1+2s}=\frac{1}{4^s(s!)^2} \tag{72}$$

is a numerical combinatoric factor incorporating the number of equivalent ways to obtain the list of PBs in the product as written in Eq. (71).

Introducing a completely symmetric tensor $\sigma_{\{k_1\cdots k_s\}}$ defined by

$$\sigma_{\{k_1\cdots k_s\}}=\sum_{\text{all } i=1}^{n-1}\varepsilon_{i_1\cdots i_{n-1}}f_{i_1i_2k_1}\cdots f_{i_{n-2}i_{n-1}k_s}, \tag{73}$$

and using Eq. (64), we may rewrite Eq. (71) as (note the sums over ks and ms here are not truncated)

$$\{A,H,\mathcal{L}_1,\ldots,\mathcal{L}_{n-1},\mathcal{R}_1,\ldots,\mathcal{R}_{n-1}\}_{\text{NB}}=(-2)^{n-1}K_n\sum_{\text{all } k,m=1}^{n}\sigma_{\{k_1\cdots k_s\}}\sigma_{\{m_1\cdots m_s\}}\times\{A,H\}_{\text{PB}}\mathcal{L}_{k_1}\cdots\mathcal{L}_{k_s}\mathcal{R}_{m_1}\cdots\mathcal{R}_{m_s}. \tag{74}$$

Thus, we arrive at a maximal CNB expression of time evolution, for odd-dimensional G:

$$\frac{dA}{dt}=V\{A,H,\mathcal{L}_1,\ldots,\mathcal{L}_{n-1},\mathcal{R}_1,\ldots,\mathcal{R}_{n-1}\}_{\text{NB}}, \tag{75}$$

where the invariant factor V on the RHS is given by

$$\frac{1}{V}=\frac{1}{(s!)^2}\sum_{\text{all } k=1}^{n}\sigma_{\{k_1\cdots k_s\}}\mathcal{L}_{k_1}\cdots\mathcal{L}_{k_s}\sum_{\text{all } m=1}^{n}\sigma_{\{m_1\cdots m_s\}}\times\mathcal{R}_{m_1}\cdots\mathcal{R}_{m_s},$$

$$s\equiv\frac{n-1}{2}. \tag{76}$$

This factor determines the kernel of the brackets in question.

All this extends in a straightforward way to even-dimensional groups G and to the algebras of symmetry groups involving arbitrary numbers of factors, $G_1\times G_2\times\cdots$.

III. QUANTUM THEORY

We now consider the quantization of Nambu mechanics. Despite contrary claims in the literature, it turns out that the quantization is straightforward using the Hilbert space operator methods as originally proposed by Nambu. All that is needed is a properly consistent physical interpretation of the results, by allowing for dynamical time scales, as summarized in the Introduction. We provide a very detailed description of that interpretation in the following, but first we develop the techniques and machinery that are used to reach and implement it. Our presentation parallels the previous classical discussion as much as possible.

A. Properties of the quantum brackets

a. Definition of QNBs. Define the quantum Nambu brackets, or QNBs [1], as fully antisymmetrized multilinear sums of operator products in an associative enveloping algebra

$$[A_1,A_2,\ldots,A_k]\equiv\sum_{\substack{\text{all } k! \text{ perms}\\ \{\sigma_1,\sigma_2,\ldots,\sigma_k\}\\ \text{of the indices}\\ \{1,2,\ldots,k\}}}\text{sgn}(\sigma)A_{\sigma_1}A_{\sigma_2}\cdots A_{\sigma_k}, \tag{77}$$

[8]The number of ways of picking the n PBs in the formula (71), taking into account both ε's, is $(n-2)(n-4)\cdots(1)\times(n-2)(n-4)\cdots(1)$, so $K_n=([(n-2)(n-4)\cdots(1)]/(n-1)!)^2$.

where $\mathrm{sgn}(\sigma)=(-1)^{\pi(\sigma)}$ with $\pi(\sigma)$ the parity of the permutation $\{\sigma_1,\sigma_2,\ldots,\sigma_k\}$. The brackets are unchanged by adding to any one element a linear combination of the others, in analogy with the usual row or column manipulations on determinants.

b. Recursion relations. There are various ways to obtain QNBs recursively, from products involving fewer operators. For example, a QNB involving k operators has both left- and right-sided resolutions of single operators multiplying QNBs of $k-1$ operators.

$$[A_1,A_2,\ldots,A_k]=\sum_{k!\ \text{perms}\ \sigma}\frac{\mathrm{sgn}(\sigma)}{(k-1)!}A_{\sigma_1}[A_{\sigma_2},\ldots,A_{\sigma_k}]$$
$$=\sum_{k!\ \text{perms}\ \sigma}\frac{\mathrm{sgn}(\sigma)}{(k-1)!}[A_{\sigma_1},\ldots,A_{\sigma_{k-1}}]A_{\sigma_k}. \tag{78}$$

On the RHS there are actually only k distinct products of single elements with $(k-1)$-brackets, each such product having a net coefficient ±1. The denominator compensates for replication of these products in the sum over permutations. (We leave it as an elementary exercise for the reader to prove this result.)

For example, the 2-brackets are obviously just the commutator $[A,B]=AB-BA$, while the 3-brackets may be written in either of two [1] or three convenient ways

$$[A,B,C]=A[B,C]+B[C,A]+C[A,B]$$
$$=[A,B]C+[B,C]A+[C,A]B$$
$$=\tfrac{3}{2}\{[A,B],C\}+\tfrac{1}{2}[\{A,B\},C]-[A,\{B,C\}]. \tag{79}$$

Summing the first two lines gives anticommutators containing commutators on the RHS

$$2[A,B,C]=\{A,[B,C]\}+\{B,[C,A]\}+\{C,[A,B]\}. \tag{80}$$

The last expression is to be contrasted to the Jacobi identity obtained by taking the difference of the first two RHS lines in Eq. (79):

$$0=[A,[B,C]]+[B,[C,A]]+[C,[A,B]]. \tag{81}$$

Similarly, for the 4-brackets,

$$[A,B,C,D]=A[B,C,D]-B[C,D,A]+C[D,A,B]$$
$$-D[A,B,C]$$
$$=-[B,C,D]A+[C,D,A]B-[D,A,B]C$$
$$+[A,B,C]D. \tag{82}$$

Summing these two lines gives

$$2[A,B,C,D]=[A,[B,C,D]]-[B,[C,D,A]]$$
$$+[C,[D,A,B]]-[D,[A,B,C]], \tag{83}$$

while taking the difference gives

$$0=\{A,[B,C,D]\}-\{B,[C,D,A]\}+\{C,[D,A,B]\}$$
$$-\{D,[A,B,C]\}. \tag{84}$$

There may be some temptation to think of the last of these as something like a generalization of the Jacobi identity, and, in principle, it is, but in a crucially limited way, so that temptation should be checked. The more appropriate and complete generalization of the Jacobi identity is given systematically below [cf. Eq. (119)].

c. Jordan products. Define a fully symmetrized, generalized Jordan operator product (GJP):

$$\{A_1,A_2,\ldots,A_k\}$$
$$\equiv\sum_{\substack{\text{all } k!\ \text{perms}\\ \{\sigma_1,\sigma_2,\ldots,\sigma_k\}\\ \text{of the indices}\\ \{1,2,\ldots,k\}}} A_{\sigma_1}A_{\sigma_2}\cdots A_{\sigma_k} \tag{85}$$

as introduced, in the bilinear form at least, by Jordan [31] to render non-Abelian algebras into Abelian algebras at the expense of nonassociativity. The generalization to multilinears was suggested by Kurosh [37], but the idea was not used in any previous physical application, as far as we know. A GJP also has left- and right-sided recursions,

$$\{A_1,A_2,\ldots,A_k\}$$
$$=\sum_{k!\ \text{perms}\ \sigma}\frac{1}{(k-1)!}A_{\sigma_1}\{A_{\sigma_2},A_{\sigma_3},\ldots,A_{\sigma_k}\}$$
$$=\sum_{k!\ \text{perms}\ \sigma}\frac{1}{(k-1)!}\{A_{\sigma_2},A_{\sigma_3},\ldots,A_{\sigma_{k-1}}\}A_{\sigma_k}. \tag{86}$$

On the RHS there are again only k distinct products of single elements with $(k-1)$ GJPs, each such product having a net coefficient $+1$. The denominator again compensates for replication of these products in the sum over permutations. (We leave it as another elementary exercise for the reader to prove this result.)

For example, a Jordan 2-product is obviously just an anticommutator $\{A,B\}=AB+BA$, while a 3-product is given by

$$\{A,B,C\}=\{A,B\}C+\{A,C\}B+\{B,C\}A$$
$$=A\{B,C\}+B\{A,C\}+C\{A,B\}$$
$$=\tfrac{3}{2}\{\{A,B\},C\}+\tfrac{1}{2}[[A,B],C]-[A,[B,C]]. \tag{87}$$

Equivalently, taking sums and differences, we obtain

$$2\,\{A,B,C\}=\{A,\{B,C\}\}+\{B,\{A,C\}\}+\{C,\{A,B\}\}, \tag{88}$$

as well as the companion of the Jacobi identity often encountered in superalgebras,

$$0=[A,\{B,C\}]+[B,\{A,C\}]+[C,\{A,B\}]. \tag{89}$$

Similarly for the 4-product,

$$\{A,B,C,D\}=A\{B,C,D\}+B\{C,D,A\}+C\{D,A,B\}+D\{A,B,C\}=\{A,B,C\}D+\{B,C,D\}A+\{C,D,A\}B+\{D,A,B\}C. \quad (90)$$

Summing gives

$$2\{A,B,C,D\}=\{A,\{B,C,D\}\}+\{B,\{C,D,A\}\}+\{C,\{D,A,B\}\}+\{D,\{A,B,C\}\}, \quad (91)$$

while subtracting gives

$$0=[A,\{B,C,D\}]+[B,\{C,D,A\}]+[C,\{D,A,B\}]+[D,\{A,B,C\}]. \quad (92)$$

Again the reader is warned off the temptation to think of the last of these as a bona fide generalization of the super-Jacobi identity. While it is a valid identity, of course, following from nothing but associativity, there is a superior and complete set of identities to be given later [cf. Eq. (119) to follow].

d. (Anti)Commutator resolutions. As in the classical case, Sec. II B, it is always possible to resolve even rank brackets into sums of commutator products, very usefully. For example,

$$[A,B,C,D]=[A,B][C,D]-[A,C][B,D]-[A,D][C,B]+[C,D][A,B]-[B,D][A,C]-[C,B][A,D]. \quad (93)$$

An arbitrary even bracket of rank $2n$ breaks up into $(2n)!/(2^n)=n!(2n-1)!!$ such products. Another way to say this is that even QNBs can be written in terms of GJPs of commutators. The general result is

$$[A_1,A_2,\ldots,A_{2n-1},A_{2n}]=\sum_{(2n)!\ \text{perms}\ \sigma}\frac{\text{sgn}(\sigma)}{2^n n!}\{[A_{\sigma_1},A_{\sigma_2}],[A_{\sigma_3},A_{\sigma_4}],\ldots,[A_{\sigma_{2n-1}},A_{\sigma_{2n}}]\}. \quad (94)$$

An even GJP also resolves into symmetrized products of anticommutators:

$$\{A_1,A_2,\ldots,A_{2n-1},A_{2n}\}=\sum_{(2n)!\ \text{perms}\ \sigma}\frac{1}{2^n n!}\{\{A_{\sigma_1},A_{\sigma_2}\},\{A_{\sigma_3},A_{\sigma_4}\},\ldots,\{A_{\sigma_{2n-1}},A_{\sigma_{2n}}\}\}. \quad (95)$$

The resolution (94) makes it transparent that all such even QNBs will vanish if one or more of the A_i are central (i.e., commute with all the other elements in the brackets). For instance, if any one A_i is a multiple of the unit operator, the $2n$-brackets will vanish. (This same statement does not apply to odd brackets, as Nambu realized originally for 3-brackets [1], and consequently, there are additional hurdles to be overcome when using odd QNBs.)

As in the classical bracket formalism, the proofs of the (anti)commutator resolution relations are elementary. Both left- and right-hand sides of the expressions are sums of $2n$th degree monomials linear in each of the As. Both sides are either totally antisymmetric, in the case of Eq. (94), or totally symmetric, in the case of Eq. (95), under permutations of the As. Thus, the two sides must be proportional. The only open issue is the constant of proportionality. This is easily determined to be 1, just by comparing the coefficients of any given term appearing on both sides of the equation, e.g., $A_1A_2\cdots A_{2N-1}A_{2N}$.

It is clear from the commutator resolution of even QNBs that totally symmetrized GJPs and totally antisymmetrized QNBs are not unrelated. In fact, the relationship is most pronounced in quantum mechanical applications where the operators form a Lie algebra.

e. Reductions for Lie algebras. In full analogy to the classical case above, when the operators involved in a QNB close into a Lie algebra, even if an infinite one, the Nambu brackets reduce in rank to become a sum of GJPs involving about half as many operators (*quantum reductio ad dimidium*). It follows as an elementary consequence of the commutator resolution of the Nambu brackets. First, consider even brackets, since the commutator reduction applies directly to that case. From the commutator resolution, it follows that for any Lie algebra given by

$$[B_i,B_j]=i\hbar\sum_m c_{ij}^m B_m, \quad (96)$$

we have for arbitrary A (sum over repeated ms)[9]

$$[B_1,\ldots,B_{2k+1},A]=\sum_{(2k+1)!\ \text{perms}\ \sigma}\frac{\text{sgn}(\sigma)}{2^k k!}\{[B_{\sigma_1},B_{\sigma_2}],[B_{\sigma_3},B_{\sigma_4}],\ldots,[B_{\sigma_{2k-1}},B_{\sigma_{2k}}],[B_{\sigma_{2k+1}},A]\}$$
$$=\sum_{(2k+1)!\ \text{perms}\ \sigma}\frac{\text{sgn}(\sigma)}{2^k k!}(i\hbar)^k c_{\sigma_1\sigma_2}^{m_1}c_{\sigma_3\sigma_4}^{m_2}\cdots c_{\sigma_{2k-1}\sigma_{2k}}^{m_k}\{B_{m_1},B_{m_2},\ldots,B_{m_k},[B_{\sigma_{2k+1}},A]\}. \quad (97)$$

For odd brackets, it is first necessary to resolve the QNB into products of single operators with even brackets, and then resolve the various even brackets into commutators. This gives a larger sum of terms for odd brackets, but again each term involves about half as many Jordan products compared

[9] After obtaining this result, and using it in [6], we learned that similar statements appeared previously in [42,11].

to the number of commutators resolving the original Nambu brackets. The mixture of algebraic structures in Eq. (97) suggests referring to this as a Nambu-Jordan-Lie (NJL) algebra.

f. The classical limit. Since Poisson brackets are straightforward classical limits of commutators

$$\lim_{\hbar\to 0}\left(\frac{1}{i\hbar}\right)[A,B]=\{A,B\}_{\mathrm{PB}},$$

it follows that the commutator resolution of all even QNBs directly specifies their classical limit. (For a detailed approach to the classical limit, including subdominant terms of higher order in $\hbar$, see, e.g., the Moyal bracket discussion in [6].)

For example, from

$$[A,B,C,D]=\{[A,B],[C,D]\}-\{[A,C],[B,D]\}-\{[A,D],[C,B]\}, \tag{98}$$

with due attention to a critical factor of 2 (i.e., the anticommutators on the RHS become just twice the ordinary products of their entries), the classical limit emerges as

$$\frac{1}{2}\lim_{\hbar\to 0}\left(\frac{1}{i\hbar}\right)^2[A,B,C,D] =\{A,B\}_{\mathrm{PB}}\{C,D\}_{\mathrm{PB}}-\{A,C\}_{\mathrm{PB}}\{B,D\}_{\mathrm{PB}} -\{A,D\}_{\mathrm{PB}}\{C,B\}_{\mathrm{PB}}=\{A,B,C,D\}_{\mathrm{NB}}. \tag{99}$$

And so it goes with all other even rank Nambu brackets. For the $2n$-brackets, one sees that

$$\frac{1}{n!}\lim_{\hbar\to 0}\left(\frac{1}{i\hbar}\right)^n[A_1,A_2,\ldots,A_{2n}] =\sum_{(2n)!\ \text{perms}\ \sigma}\frac{\mathrm{sgn}(\sigma)}{2^n n!}\{A_{\sigma_1},A_{\sigma_2}\}_{\mathrm{PB}}\{A_{\sigma_3},A_{\sigma_4}\}_{\mathrm{PB}}\cdots \times\{A_{\sigma_{2n-1}},A_{\sigma_{2n}}\}_{\mathrm{PB}} =\{A_1,A_2,\ldots,A_{2n}\}_{\mathrm{NB}}. \tag{100}$$

This is another way to establish that there are indeed $(2n-1)!!$ independent products of n Poisson brackets summing up to give the PB resolution of the classical Nambu $2n$-bracket. Once again due attention must be given to a critical additional factor of $n!$ [as in the denominator on the LHS of Eq. (100)] since the GJPs on the RHS of Eq. (94) will, in the classical limit, always replicate the same classical product $n!$ times.

g. The Leibniz rule failure and derivators. Define the *derivator* to measure the failure of the simplest Leibniz rule for QNBs,

$${}^{k+1}\boldsymbol{\Delta}_{\mathbf{B}}(A,\mathcal{A})\equiv(A,\mathcal{A}\,|\,B_1,\ldots,B_k) \equiv[A\mathcal{A},B_1,\ldots,B_k]-A[\mathcal{A},B_1,\ldots,B_k] -[A,B_1,\ldots,B_k]\mathcal{A}. \tag{101}$$

The first term on the RHS involves $(k+1)$-brackets acting on the product of A and $\mathcal{A}$, the order of the brackets being evident in the presuperscript of the $\boldsymbol{\Delta}_{\mathbf{B}}$ notation. This reads in an obvious way. For instance, ${}^4\boldsymbol{\Delta}_{\mathbf{B}}$ is a "4-delta of Bs." That notation also emphasizes that the Bs act *on* the pair of As. The second notation in Eq. (101) makes explicit all the Bs and is useful for computer code.

Any $\boldsymbol{\Delta}_{\mathbf{B}}$ acts on all pairs of elements in the enveloping algebra $\mathfrak{A}$ to produce another element in $\mathfrak{A}$,

$$\boldsymbol{\Delta}_{\mathbf{B}}:\mathfrak{A}\times\mathfrak{A}\mapsto\mathfrak{A}. \tag{102}$$

When $\boldsymbol{\Delta}_{\mathbf{B}}$ does not vanish the corresponding bracket with the Bs does not define a derivation on $\mathfrak{A}$. The derivator $\boldsymbol{\Delta}_{\mathbf{B}}(A,\mathcal{A})$ is linear in both A and $\mathcal{A}$, as well as linear in each of the Bs.

Less trivially, from explicit calculations, we find inhomogeneous recursion relations for these derivators:

$$(A,\mathcal{A}\,|\,B_1,\ldots,B_k) =\frac{1}{2}\sum_{k!\ \text{perms}\ \sigma}\frac{\mathrm{sgn}(\sigma)}{(k-1)!}[(A,\mathcal{A}\,|\,B_{\sigma_1},\ldots,B_{\sigma_{k-1}})B_{\sigma_k} +(-1)^kB_{\sigma_k}(A,\mathcal{A}\,|\,B_{\sigma_1},\ldots,B_{\sigma_{k-1}})] +\frac{1}{2}\sum_{k!\ \text{perms}\ \sigma}\frac{\mathrm{sgn}(\sigma)}{(k-1)!}([A,B_{\sigma_k}][B_{\sigma_1},\ldots,B_{\sigma_{k-1}},\mathcal{A}] -[A,B_{\sigma_1},\ldots,B_{\sigma_{k-1}}][B_{\sigma_k},\mathcal{A}]) +\frac{(-1)^{k+1}-1}{2}A[B_1,\ldots,B_k]\mathcal{A}. \tag{103}$$

Alternatively, we may write this so as to emphasize the number of distinct terms on the RHS and distinguish between the even and odd bracket cases. The first two terms under the sum on the RHS give a commutator/anticommutator for k odd/even, and the last term is absent for k odd.

For even (2n+2)-brackets, this becomes

$$2(A,\mathcal{A}|B_1,\ldots,B_{2n+1}) =[(A,\mathcal{A}|B_1,\ldots,B_{2n}),B_{2n+1}]+[A,B_{2n+1}] \times[B_1,\ldots,B_{2n},\mathcal{A}]-[A,B_1,\ldots,B_{2n}][B_{2n+1},\mathcal{A}] +(2\text{n signed permutations of the } B\text{s}), \tag{104}$$

where the first RHS line involves derivators of reduced rank, within commutators. For odd (2n+1)-brackets, it becomes

$$2(A,\mathcal{A}|B_1,\ldots,B_{2n}) =\{(A,\mathcal{A}|B_1,\ldots,B_{2n-1}),B_{2n}\}+[A,B_{2n}] \times[B_1,\ldots,B_{2n-1},\mathcal{A}]-[A,B_1,\ldots,B_{2n-1}][B_{2n},\mathcal{A}] +(2n-1\ \text{ signed permutations of the } B\text{s}) -2A[B_1,\ldots,B_{2n}]\mathcal{A}, \tag{105}$$

where the first RHS line involves derivators of reduced rank, within anticommutators. Note the additional inhomogeneity in the last RHS line of these results. It may be viewed as a type of quantum obstruction in the recursion relation for the odd (2n+1)-brackets.

The obstruction is clarified when we specialize to $n=1$, i.e., the 3-bracket case. Since commutators are always derivations, one has ${}^2\mathbf{\Delta}_B(A,\mathcal{A})=0$, and the first RHS line vanishes in Eq. (105) for the ${}^3\mathbf{\Delta}_\mathbf{B}(A,\mathcal{A})$ case. So we have just

$$(A,\mathcal{A}|B_1,B_2)=[A,B_2][B_1,\mathcal{A}]-[A,B_1][B_2,\mathcal{A}] -A[B_1,B_2]\mathcal{A}. \tag{106}$$

The first two terms on the RHS are $O(\hbar^2)$ while the last is $O(\hbar)$. It is precisely this last term which was responsible for some of Nambu's misgivings concerning his quantum 3-brackets. In particular, even in the extreme case when both A and $\mathcal{A}$ commute with the Bs, ${}^3\mathbf{\Delta}_\mathbf{B}(A,\mathcal{A})$ does not vanish:

$$(A,\mathcal{A}|B_1,B_2)|_{[A,B_i]=0=[\mathcal{A},B_i]}=-A\mathcal{A}[B_1,B_2]. \tag{107}$$

By contrast, for the even (2n+2)-brackets, all terms on the RHS of Eq. (104) are generically of the same order, $O(\hbar^{n+1})$, and all terms vanish if A and $\mathcal{A}$ commute with all the Bs. In terms of combinatorics, this seems to be the only feature for the simple, possibly failed, Leibniz rule that distinguishes between even and odd brackets. An even-odd QNB dichotomy has been previously noted [39] and stressed [11], for other reasons.

The size of the brackets involved in the derivators can be reduced when the operators obey a Lie algebra as in Eq. (96). The simplest situation occurs when the brackets are even. For this situation, we have

$$\begin{aligned}(A,\mathcal{A}|B_1,\ldots,B_{2k+1}) &= \sum_{(2k+1)!\ \text{perms}\ \sigma} \frac{\text{sgn}(\sigma)}{2^k k!}(i\hbar)^k c^{m_1}_{\sigma_1\sigma_2}c^{m_2}_{\sigma_3\sigma_4}\cdots c^{m_k}_{\sigma_{2k-1}\sigma_{2k}}\\ &\quad\times(\{B_{m_1},\ldots,B_{m_k},[B_{\sigma_{2k+1}},A\mathcal{A}]\}\\ &\quad -A\{B_{m_1},\ldots,B_{m_k},[B_{\sigma_{2k+1}},\mathcal{A}]\}\\ &\quad -\{B_{m_1},\ldots,B_{m_k},[B_{\sigma_{2k+1}},A]\}\mathcal{A}).\end{aligned} \tag{108}$$

h. Generalized Jacobi identities and quantum fundamental identities. We previously pointed out some elementary identities involving QNBs, e.g., Eqs. (84) and (92), which are suggestive of generalizations of the Jacobi identity for commutators. Those particular identities, while true, were not designated as "generalized Jacobi identities" (GJIs), for the simple fact that they do *not* involve the case where QNBs of a given rank act on QNBs of the same rank. Here, we explore QNB identities of the latter type. There are indeed acceptable generalizations of the usual commutators-acting-on-commutators Jacobi identity (i.e., quantum 2-brackets acting on quantum 2-brackets), and these generalizations are indeed valid for *all* higher rank QNBs (i.e., quantum n-brackets acting on quantum n-brackets). However, there is an essential distinction to be drawn between the even and odd quantum bracket cases [11,39].

It is important to note that, historically, there have been some incorrect guesses and false starts in this direction that originated from the so-called fundamental identity obeyed by classical Nambu brackets (22). This simple identity apparently misled several investigators [22], most recently [26] and [24,14], to think of it as a "fundamental" generalization of the Jacobi identity, without taking care to preserve the Jacobi identity's traditional role of encoding nothing but associativity. These same investigators then insisted that a "correct quantization" of the classical Nambu brackets *must* satisfy an identity of the same form as Eq. (22).

Unfortunately for them, QNBs do *not* satisfy this particular identity, in general, and thereby pose a formidable problem to proponents of that identity's fundamental significance. This difficulty led [26,24,14], to seek alternative ways to quantize CNBs, ultimately culminating in the so-called Abelian deformation method [24,14]. This amounted to demanding that the quantized brackets satisfy the mathematical postulates of an "n-Lie algebra" as defined by Filippov [35] many years earlier. However, not only are those postulates *not* satisfied by generic QNBs, but more importantly, those postulates are *not* warranted by the physics of QNBs, as will be clear in the examples to follow.

The correct generalizations of the Jacobi identities which *do* encode associativity were found independently by groups of mathematicians [39] and physicists [41,11]. Interestingly, both groups were studying cohomology questions, so perhaps it is not surprising that they arrived at the same result. (Fortunately, for us the result is sufficiently simple in its combinatorics that we do not need to go through the cohomology issues.) The acceptable generalization of the Jacobi identity that was found is satisfied by all QNBs, although for odd QNBs there is a significant difference in the form of the final result: It contains an "inhomogeneity." The correct generalization is obtained just by totally antisymmetrizing the action of n-brackets on other n-brackets. Effectively, this amounts to antisymmetrizing the form of the RHS of Eq. (22) over all permutations of the As and Bs including all exchanges of As with Bs.

We illustrate the correct quantum identity for the case of three-brackets acting on 3-brackets, where the classical result is

$$\begin{aligned}0&=\{\{A,B,C\}_{NB},D,E\}_{NB}-\{\{A,D,E\}_{NB},B,C\}_{NB}\\ &\quad-\{A,\{B,D,E\}_{NB},C\}_{NB}-\{A,B,\{C,D,E\}_{NB}\}_{NB},\end{aligned} \tag{109}$$

i.e., Eq. (22) for $n=3$. For ease in writing, we let $A_1\equiv A$, $A_2\equiv B$, $A_3\equiv C$, $B_1\equiv D$, and $B_2\equiv E$. Consider $[[A,B,C],D,E]$. This QNB corresponds to the first term on the RHS of Eq. (109). If we antisymmetrize $[[A,B,C],D,E]$ over all $5!$ permutations of A,B,C,D, and E, we obtain, with a common overall coefficient of $12=2!3!$, a total of $10=5!/(2!3!)$ distinct terms as follows:

$$[[A,B,C],D,E]+[[A,D,E],B,C]+[[D,B,E],A,C] +[[D,E,C],A,B]-[[D,B,C],A,E]-[[E,B,C],D,A] -[[A,D,C],B,E]-[[A,E,C],D,B]-[[A,B,D],C,E] -[[A,B,E],D,C]. \quad (110)$$

Now we determine the coefficient of any given monomial produced by this sum.[10] Since the expression is totally antisymmetrized in all the five elements, the result must be proportional, to $[A,B,C,D,E]$. To determine the constant of proportionality it suffices to consider the monomial $ABCDE$. This particular monomial can be found in only three terms out of the ten in Eq. (110), namely, in

$$[[A,B,C],D,E],$$
$$[A,[B,C,D],E]=-[[D,B,C],A,E], \quad \text{and}$$
$$[A,B,[C,D,E]]=[[D,E,C],A,B]. \quad (111)$$

The various terms are obtained just by ''shifting'' the interior brackets from left to right within the exterior brackets, while keeping all the bracket entries in a fixed left-to-right order, and keeping track of the sgn(σ) factors. (Call this the ''shifting bracket argument.'') The monomial $ABCDE$ appears in each of these terms with coefficient $+1$, for a total of $+3 \times ABCDE$. Thus, we conclude with a five-element, 11-term identity

$$[[A,B,C],D,E]+[[A,D,E],B,C]+[[D,B,E],A,C] +[[D,E,C],A,B]-[[D,B,C],A,E] -[[E,B,C],D,A]-[[A,D,C],B,E] -[[A,E,C],D,B]-[[A,B,D],C,E] -[[A,B,E],D,C]=3[A,B,C,D,E]. \quad (112)$$

This is the prototypical generalization of the Jacobi identity for odd QNBs, and like the Jacobi identity, it is antisymmetric in all of its elements. The RHS here is the previously designated inhomogeneity.

The totally antisymmetrized action of odd n QNBs on other odd n QNBs results in $(2n-1)$*-brackets*.

We recognize in the first four terms of Eq. (112) those QNB combinations which correspond to the individual terms on the RHS of Eq. (109). However, *the signs are changed* for three of the four QNB terms relative to those in Eq. (109). One might hope that changing these signs in the QNB combinations will lead to some simplification, and indeed it does, but it does not cause the resulting expression to vanish, as it did in Eq. (109). To see this, consider in the same way the effects of antisymmetrizing the QNBs corresponding to each of the other three terms on the RHS of Eq. (109). The second RHS term would have as correspondent $-[[A,D,E],B,C]$, which, when totally antisymmetrized, gives an overall common coefficient of $2!3!$ multiplying

$$-[[A,D,E],B,C]-[[A,B,C],D,E]-[[B,D,C],A,E] -[[B,C,E],A,D]+[[B,D,E],A,C] +[[C,D,E],B,A]+[[A,B,E],D,C] +[[A,C,E],B,D]+[[A,D,B],E,C] +[[A,D,C],B,E]=-3[A,B,C,D,E]. \quad (113)$$

The third RHS term of Eq. (109) would have as correspondent $-[A,[B,D,E],C]=[[B,D,E],A,C]$, which, when totally antisymmetrized, gives an overall common coefficient of $2!3!$ multiplying

$$[[B,D,E],A,C]+[[B,A,C],D,E]+[[A,D,C],B,E] +[[A,C,E],B,D]-[[A,D,E],B,C] -[[C,D,E],A,B]-[[B,A,E],D,C] -[[B,C,E],A,D]-[[B,D,A],E,C] -[[B,D,C],A,E]=-3[A,B,C,D,E]. \quad (114)$$

The fourth and final RHS term of Eq. (109) would have as correspondent $-[A,B,[C,D,E]]=[[C,D,E],B,A]$, which, when totally antisymmetrized, gives an overall common coefficient of $2!3!$ multiplying

$$[[C,D,E],B,A]+[[C,B,A],D,E]+[[B,D,A],C,E] +[[B,A,E],C,D]-[[B,D,E],C,A] -[[A,D,E],B,C]-[[C,B,E],D,A] -[[C,A,E],B,D]-[[C,D,B],E,A] -[[C,D,A],B,E]=-3[A,B,C,D,E]. \quad (115)$$

Adding Eqs. (112), (113), (114), and (115) leads to the sum of QNB combinations that corresponds to the antisymmetrized form of the RHS of Eq. (109); namely,

$$([[A,B,C],D,E]-[[A,D,E],B,C]-[A,[B,D,E],C] -[A,B,[C,D,E]]) \pm(\text{nine distinct permutations of all four terms}) =-6[A,B,C,D,E]. \quad (116)$$

This result shows that the simple combination of QNB terms that corresponds to Eq. (109) (without full antisymmetrization) cannot possibly vanish unless the 5-brackets $[A,B,C,D,E]$ vanish.

A similar consideration of the action of a 4-bracket on a 4-bracket illustrates the general form of the GJI for even brackets and shows the essential differences between the even and odd bracket cases. We proceed as above by starting with the combination $[[A,B,C,D],E,F,G]$, and then totally antisymmetrizing with respect to *A*, *B*, *C*, *D*, *E*, *F*, and *G*. We

[10] This line of argument is an adaptation of that in [39]. Equivalent methods are used in [41,11].

find $35=7!/(3!4!)$ distinct terms in the resulting sum. Now we determine the coefficient of any given monomial that would appear in this sum. Since the expression is again totally antisymmetrized in all the seven elements, the result must be proportional to $[A,B,C,D,E,F,G]$. To determine the constant of proportionality, it suffices to consider the monomial $ABCDEFG$ and use the shifting bracket argument, which shows that this particular monomial can be found in only four terms out of the 35 in the sum, namely, in

$$\begin{aligned}&[[A,B,C,D],E,F,G],\\ &-[A,[B,C,D,E],F,G], \quad [A,B,[C,D,E,F],G], \quad \text{and}\\ &-[A,B,C,[D,E,F,G]].\end{aligned} \tag{117}$$

The monomial $ABCDEFG$ appears in these four terms with coefficients $+1,-1,+1$, and -1, for a total of $0\times ABCDEFG$. Thus, we conclude that ([39,41], and also [11], especially Eq. (32))

$$[[A,B,C,D],E,F,G]\pm(34 \text{ distinct permutations})=0. \tag{118}$$

This is the prototypical generalization of the Jacobi identity for even QNBs and constitutes the full antisymmetrization of all arguments of the analogous FI. There is no RHS inhomogeneity in this case.

The totally antisymmetrized action of even n QNBs on other even n QNBs results in zero.

The generalized Jacobi identity for arbitrary n-brackets follows from the same simple analysis of coefficients of any given monomial, as in Eqs. (111) and (117). The shifting bracket argument actually leads to a larger set of results, whenever the actions of any brackets are totally antisymmetrized. We present that larger generalization here, calling it the *quantum Jacobi identity* or QJI. The GJI is the QJI for $k=n-1$.

i. QJIs for QNBs. These are given as

$$\begin{aligned}\sum_{(n+k)!\ \text{perms}\ \sigma} &\operatorname{sgn}(\sigma)[[A_{\sigma_1},\dots,A_{\sigma_n}],A_{\sigma_{n+1}},\dots,A_{\sigma_{n+k}}]\\ &=[A_1,\dots,A_{n+k}]\times n!k!\\ &\quad\times\begin{cases}(k+1) & \text{if } n \text{ is odd}\\ \frac{1}{2}[1+(-1)^k] & \text{if } n \text{ is even.}\end{cases}\end{aligned} \tag{119}$$

This result is proven just by computing the coefficient of the $A_1\cdots A_{n+k}$ monomial using the shifting bracket argument as given previously to establish Eqs. (111) and (117). Other arguments leading to the same result may be found in [41,11].

This is the quantum identity that most closely corresponds to the general classical result [see the second talk under [6], Eq. (28)] for any even n and any odd k (only $n=2N$, $k=2N-1$ is the FI),

$$\begin{aligned}&\{\{A_1,A_2,\dots,A_n\}_{NB},B_1,\dots,B_k\}_{NB}\\ &\quad-\sum_{j=1}^{n}\{A_1,\dots,\{A_j,B_1,\dots,B_k\}_{NB},\dots,A_n\}_{NB}\\ &=\{B_1,\{B_2,\dots,B_k\}_{NB},A_1,\dots,A_n\}_{NB}-\cdots\\ &\quad+\{B_k,\{B_1,\dots,B_{k-1}\}_{NB},A_1,\dots,A_n\}_{NB}.\end{aligned} \tag{120}$$

While this classical identity holds without requiring full antisymmetrization over all exchanges of As and Bs, in contrast the quantum identity *must* be totally antisymmetrized if it is to be a consequence of only the associativity of the underlying algebra of Hilbert space operators. Note that the $n!k!$ on the RHS of Eq. (119) may be replaced by just 1 if we sum *only* over permutations in which the $A_{i\leq n}$ are interchanged with the $A_{i>n}$ in $[[A_{\sigma_1},\dots,A_{\sigma_n}],A_{\sigma_{n+1}},\dots,A_{\sigma_{n+k}}]$ and ignore all permutations of the $A_1,A_2,\dots,A_n$ among themselves and of the $A_{n+1},\dots,A_{n+k}$ among themselves.

There is an important specialization of the QJI result [39,41]: For any even n and any odd k

$$\begin{aligned}\sum_{(n+k)!\ \text{perms}\ \sigma} &\operatorname{sgn}(\sigma)[[A_{\sigma_1},\dots,A_{\sigma_n}],A_{\sigma_{n+1}},\dots,A_{\sigma_{n+k}}]\\ &=0.\end{aligned} \tag{121}$$

In particular, when $k=n-1$, for n even, the vanishing RHS obtains. All other n-not-even and/or k-not-odd cases of the QJI have the $[A_1,\dots,A_{n+k}]$ inhomogeneity on the RHS.

The QJI also permits us to give the correct form of the so-called fundamental identities valid for all QNBs. We accordingly call these *quantum fundamental identities* (QFIs) and present them in their general form.

j. QFIs for QNBs. These are given as

$$\begin{aligned}\sum_{(n+k)!\ \text{perms}\ \sigma} &\operatorname{sgn}(\sigma)\Bigg([[A_{\sigma_1},\dots,A_{\sigma_n}],A_{\sigma_{n+1}},\dots,A_{\sigma_{n+k}}]\\ &\quad-\sum_{j=1}^{n}[A_{\sigma_1},\dots,[A_{\sigma_j},A_{\sigma_{n+1}},\dots,A_{\sigma_{n+k}}],\dots,A_{\sigma_n}]\Bigg)\\ &=[A_1,\dots,A_{n+k}]\times n!k!\\ &\quad\times\begin{cases}0 & \text{if } k \text{ is odd}\\ (1-n)(k+1) & \text{if } k \text{ is even and } n \text{ is odd}\\ [1-n(k+1)] & \text{if } k \text{ is even and } n \text{ is even.}\end{cases}\end{aligned} \tag{122}$$

Aside from the trivial case of $n=1$, the only way the RHS vanishes without conditions on the full $(n+k)$-brackets is when k is odd. All $n>1$, even k result in the $[A_1,\dots,A_{n+k}]$ inhomogeneity on the RHS.

Partial antisymmetrizations of the individual terms in the general QFI may also be entertained. The result is to find more complicated inhomogeneities, and does not seem to be very informative. At best these partial antisymmetrizations show in a supplemental way how the fully antisymmetrized results are obtained.

In certain isolated, special cases [cf. the su(2) example of the next section, for which $k=3$], the bracket effects of select Bs can act as a derivation (essentially because the k brackets are equivalent, in their effects, to commutators). If that is the case, then the quantum version of the simple identity in Eq. (22) holds trivially. It is also possible, in principle, for that simple identity to hold, again in very special situations, if the quantum brackets are not a derivation, through various cancellations among terms. As an aid to finding such peculiar situations, it is useful to resolve the quantum correspondents of the terms in the classical FI into the derivators introduced previously (101). From the definition of $[A_1, \ldots ,A_n]$ in Eq. (77) and some straightforward manipulations, we find

$$[[A_1, \ldots ,A_n],\mathbf{B}]-\sum_{j=1}^{n}[A_1, \ldots ,[A_j,\mathbf{B}], \ldots ,A_n]$$

$$=\sum_{n!\ \text{perms}\ \sigma}\text{sgn}(\sigma)\Bigg[\frac{1}{(n-1)!}(A_{\sigma_1},[A_{\sigma_2}, \ldots ,A_{\sigma_n}]|\mathbf{B}) + \frac{1}{(n-2)!}A_{\sigma_1}(A_{\sigma_2},[A_{\sigma_3}, \ldots ,A_{\sigma_n}]|\mathbf{B}) + \frac{1}{2!(n-3)!}[A_{\sigma_1},A_{\sigma_2}](A_{\sigma_3},[A_{\sigma_4}, \ldots ,A_{\sigma_n}]|\mathbf{B})+\cdots + \frac{1}{(n-2)!}[A_{\sigma_1},A_{\sigma_2}, \ldots ,A_{\sigma_{n-2}}](A_{\sigma_{n-1}},A_{\sigma_n}|\mathbf{B})\Bigg], \tag{123}$$

with the abbreviation $\mathbf{B}\equiv B_1, \ldots ,B_k$. The terms on the RHS are a sum over $j=1, \ldots ,n-1$ of derivators between solitary As (i.e. 1-brackets) and various $(n-j)$-brackets, left multiplied by complementary rank $(j-1)$-brackets. (There is a similar identity that involves right multiplication by the complementary brackets.)

For example, suppose $n=2$. Then we have, for any number of Bs

$$[[A_1,A_2],\mathbf{B}]-[[A_1,\mathbf{B}],A_2]-[A_1,[A_2,\mathbf{B}]] =(A_1,A_2|\mathbf{B})-(A_2,A_1|\mathbf{B}). \tag{124}$$

In principle, this can vanish, even when the action of the Bs is not a derivation, if the k derivator is symmetric in the first two arguments. That is, if $(A_1,A_2|B_1, \ldots ,B_k) =\frac{1}{2}(A_1,A_2|B_1, \ldots ,B_k)+\frac{1}{2}(A_2,A_1|B_1, \ldots ,B_k)$. However, we have not found a compelling (nontrivial) physical example where this is the case.

B. Illustrative quantum examples

As in the classical situation, it is useful to consider explicit examples of quantized dynamical systems described by quantum Nambu brackets, to gain insight and develop intuition. However, for quantum systems, it is more than useful; it is crucial to examine detailed cases to appreciate how quandaries that have been hinted at in the past are actually resolved, especially since the exact classical phase-space geometry in Sec. II A is no longer applicable. Similar studies have been attempted before, but have reached conclusions sharply opposed to ours.[11] Here, we demonstrate how the simplest Nambu mechanical systems are quantized consistently and elegantly by conventional operator methods.

a. SU(2) as a special case. The commutator algebra of the charges $(L_0\equiv L_z,L_\pm\equiv L_x\pm iL_y)$ is

$$[L_+,L_-]=2\hbar L_0,\quad [L_0,L_-]=-\hbar L_-,\quad [L_0,L_+]=\hbar L_+, \tag{125}$$

giving rise to $[L_-,L_0^2]=\{[L_-,L_0],L_0\}=\hbar\{L_-,L_0\}$, etc. The invariant quadratic Casimir is

$$I=L_+L_-+L_0(L_0-\hbar)=L_-L_++(L_0+\hbar)L_0. \tag{126}$$

We use the algebra and the commutator resolution of the 4-brackets

$$[A,B,C,D]=\{[A,B],[C,D]\}-\{[A,C],[B,D]\} -\{[A,D],[C,B]\} \tag{127}$$

to obtain [6] the quantization of Eq. (31):

$$[A,L_0,L_+,L_-]=2\hbar[A,I], \tag{128}$$

and the more elaborate

$$[A,I,L_+,L_-]=2\hbar\{[A,I],L_0\}=2\hbar[\{A,L_0\},I]. \tag{129}$$

Since I and L_0 commute, the nested commutator-anticommutator can also be written using the 3-brackets: $[\{A,L_0\},I]=[A,I,L_0]-[L_0I,A]$.

So for SU(2) invariant systems with $H=I/2$, Eq. (128) gives rise to the complete analog of classical time development as a derivation (31), namely,

$$i\hbar^2\frac{dA}{dt}=\hbar[A,H]=\frac{1}{4}[A,L_0,L_+,L_-], \tag{130}$$

where the QNB in question happens to be a derivation too [6], and thus satisfies an effective FI [see Eq. (152)]. By contrast, Eq. (129) gives rise to

$$i\hbar^2\left\{\frac{dA}{dt},L_0\right\}=\hbar\{[A,H],L_0\}=\frac{1}{4}[A,I,L_+,L_-]. \tag{131}$$

Since the latter of these is manifestly not a derivation, one should not expect, as we have stressed, Leibniz rule and classical-like fundamental identities to hold. Of course, since a derivation is entwined in the structure, substitution $A\to A\mathcal{A}$ and application of Leibniz's rule to just the time derivation alone will necessarily yield correct but complicated

[11]"The quantization of Nambu structures turns out to be a nontrivial problem even (or especially) in the simplest cases" [24].

expressions (but not particularly informative results, given the consistency of the structure already established). However, in more general contexts and in fanciful situations where the operators playing the role of L_0 were invertible, one might envision trying to solve for the time derivative through formal resolvent methods (see the Appendix), e.g.,

$$4i\hbar^2\frac{dA}{dt}=\sum_{n=0}^{\infty}(-L_0)^n[A,I,L_+,L_-](L_0)^{-n-1}, \tag{132}$$

where the new brackets implicitly defined by the RHS would now be a derivation (here, just the commutator with I). It is clear from our discussion, however, that the QNBs and the entwinings they imply, are still the preferred presentation of the quantized Nambu mechanics.

It might be useful to view solving for dA/dt as a problem of implementing scale transformations in the generalized Jordan algebra context. Evidently, there is little if any discussion of that mathematical problem in the literature [47].

The physics described by the first QNB (130), is standard time evolution, just encoded in an unusual way as quantum 4-brackets. However, the other QNB, in Eq. (131), illustrates the idea discussed in the Introduction. Physically, these Nambu brackets are an entwined form of time evolution, where the Jordan algebraic eigenvalues σ of L_0, defined by $\{A,L_0\}=\sigma A$, set the time scales for the various sectors of the theory: i.e., the formalism gives dynamical time scales. To see this, resolve the identity, $1=\Sigma_\lambda\mathbb{P}_\lambda$, in terms of L_0 projections, $\mathbb{P}_\lambda^2=\mathbb{P}_\lambda$, and use this in turn to resolve any operator A as a sum of left and right eigenoperators of L_0,

$$A=\sum_{\lambda,\rho}\mathbb{P}_\lambda A_{\lambda\rho}\mathbb{P}_\rho,\quad A_{\lambda\rho}=\mathbb{P}_\lambda A\mathbb{P}_\rho. \tag{133}$$

These eigenoperators obey

$$L_0A_{\lambda\rho}=\lambda A_{\lambda\rho},\quad A_{\lambda\rho}L_0=\rho A_{\lambda\rho}. \tag{134}$$

Following such a decomposition, since $[\mathbb{P}_\lambda,H]=0=[\mathbb{P}_\lambda,L_0]$, Eq. (131) can be written as a sum of terms

$$\begin{aligned}i\hbar^2\left\{\frac{dA_{\lambda\rho}}{dt},L_0\right\}&=\hbar\{[A_{\lambda\rho},H],L_0\}\\&=\hbar[\{A_{\lambda\rho},L_0\},H]\\&=i\hbar^2(\lambda+\rho)\frac{dA_{\lambda\rho}}{dt}.\end{aligned} \tag{135}$$

That is to say, the sum of the left and right eigenvalues of the operator $A_{\lambda\rho}$ gives a Jordan eigenvalue $\sigma=\lambda+\rho$,

$$\{A_{\lambda\rho},L_0\}=(\lambda+\rho)A_{\lambda\rho}, \tag{136}$$

and this Jordan eigenvalue sets the time scale for the instantaneous evolution of the eigenoperator. Since a general operator is a sum of eigenoperators, this construction will, in general, give a mixture of time scales. Stated precisely

$$i\hbar^2\left\{\frac{dA}{dt},L_0\right\}=i\hbar^2\sum_{\lambda,\rho}(\lambda+\rho)\mathbb{P}_\lambda\frac{dA_{\lambda\rho}}{dt}\mathbb{P}_\rho=\frac{1}{4}[A,I,L_+,L_-]. \tag{137}$$

This highlights the differences between Eq. (131) and conventional time evolution for operators in the Heisenberg picture, as in Eq. (130), since the time scales in Eq. (137) depend on the angular momentum eigenvalues.

This example illustrates our introductory remarks about different time scales for the different invariant sectors of a system. It also shows why the action of the 4-brackets in question is not a derivation. The simple Leibniz rule for generic A and $\mathcal{A}$, that would equate $[A\mathcal{A},I,L_+,L_-]$ with $A[\mathcal{A},I,L_+,L_-]+[A,I,L_+,L_-]\mathcal{A}$, will fail for (nonvanishing) products $A_{\lambda_1\mu}\mathcal{A}_{\mu\rho_2}$, unless $\lambda_1+\mu=\mu+\rho_2=\lambda_1+\rho_2$, i.e., unless $\lambda_1=\mu=\rho_2$. If restricted and applied to a single angular momentum sector, the 4-brackets under consideration does indeed give just the time derivative of all diagonal (i.e., those with $\lambda=\rho$) angular momentum eigenoperators on that sector. When projected onto any such sector, the action of these 4-brackets is therefore a derivation, since the time scale will be fixed. But without such a projection, when acting on the full Hilbert space of the system, where more than one value of λ is encountered, these 4-brackets are *not* simple derivations, not even if they act on only diagonal L eigenoperators, if two or more L_0 Jordan eigenvalues are involved. At best, we may think of it as some sort of dynamically scaled derivation, since it gives time derivatives scaled by angular momentum eigenvalues.

The quantum brackets in Eq. (131) have a kernel, just as their classical limits do, but the quantum case evinces the linear superpositions inherent in quantum mechanics. This is evident in Eq. (137), where a given eigenoperator is left unchanged by the brackets if $\lambda=-\rho$, rather than simply $\lambda=0=\rho$. This quantum effect is linked to the fact that Jordan algebras are not division rings, as discussed in the Appendix. For this and other reasons, having to do with the fact that the 4-brackets in Eq. (131) are not a derivation, it is not possible to simply divide the LHS of Eq. (131) by L_0 and then absorb the $1/L_0$ on the RHS directly into the brackets, as we did in previous classical cases, such as Eq. (45). While the result in Eq. (130) does indeed have the expected form produced by such naive manipulations (such manipulations being valid for the classical limits of the expressions), this result cannot be derived in this way. It is legitimately obtained only through the commutator resolution, as above.

Other choices for the invariants in the 4-brackets lead to some even more surprising results and offer additional insight into the quantum tricks that the QNBs are capable of playing. For example,

$$\begin{aligned}[A,L_0^2,L_+,L_-]&=2\hbar\{[A,L_0^2],L_0\}+\hbar\{[A,L_+],\{L_-,L_0\}\}\\&\quad+\hbar\{[A,L_-],\{L_+,L_0\}\}\\&=2\hbar\{[A,I],L_0\}+2\hbar^3[L_0,A]\\&\quad+\hbar[L_-,[L_+,[A,L_0]]]\\&\quad+\hbar[L_+,[L_-,[A,L_0]]].\end{aligned} \tag{138}$$

This shows an interesting effect in addition to the dynamical time scales evident in the first term of the last equality, namely; *quantum rotation terms*, as given by the last three $O(\hbar^4)$ terms in Eq. (138). If A is not invariant under rotations about the polar axis, so that $[L_0,A]\neq 0$, the last three terms in Eq. (138) may generate changes in A, even though A is time invariant. The effect is a purely quantum one; it disappears completely in the classical limit. The QNB in question is algebraically covariant, but not algebraically invariant (as opposed to *time* invariant). Therefore, it may and does lead to nontrivial tensor products when it acts on other algebraically covariant As. (As a general rule of thumb, if the QNB is allowed to do something, then it will.)

Mathematically, it is sometimes useful to think of nested multicommutators, such as those in Eq. (138), as higher partial derivatives. This manifestation provides another reason why general QNBs are not derivations (i.e., first derivatives only) and do not obey the simple Leibniz rule.

Combining Eq. (138) with Eq. (129) also yields

$$[A,(L_+L_-),L_+,L_-]=2\hbar^2[A,I]-2\hbar^3[L_0,A]-\hbar[L_-,[L_+,[A,L_0]]]-\hbar[L_+,[L_-,[A,L_0]]]. \tag{139}$$

Since every commutator is inherently $O(\hbar)$, the first term on the RHS of this last result is $O(\hbar^3)$, while the last three are $O(\hbar^4)$. All vanish in the classical limit

$$\lim_{\hbar\to 0}\left(\frac{1}{i\hbar}\right)^2[A,(L_+L_-),L_+,L_-]=0. \tag{140}$$

This illustrates how nontrivial QNBs can collapse to nothing as CNBs. The $O(\hbar^4)$ terms are quantum rotations, as in Eq. (138), but with changed signs.

The two results (138) and (139) are simple illustrations of the failure of QNBs to obey the elementary Leibniz rule. As derivators

$$(L_0,L_0|L_+,A,L_-)=2\hbar^3[L_0,A]+\hbar[L_-,[L_+,[A,L_0]]]+\hbar[L_+,[L_-,[A,L_0]]], \tag{141}$$

whose RHS is $O(\hbar^4)$ and thus, vanishes in the classical limit as $O(\hbar^2)$, and

$$(L_+,L_-|L_+,A,L_-)=2\hbar^2[A,I]-2\hbar^3[L_0,A]-\hbar[L_-,[L_+,[A,L_0]]]-\hbar[L_+,[L_-,[A,L_0]]], \tag{142}$$

whose RHS is inherently $O(\hbar^3)$, due to the $[A,I]$ term, and also vanishes in the classical limit.[12] The LHS of either of these expressions would vanish identically were the 4-brackets involved actually derivations, but they are not. There are some special situations, such as when A is in the enveloping algebra and is invariant under rotations about the polar axis, i.e., $A=A(L_0,(L_+L_-))$, for which the derivators do vanish. However, for general A, including most of those in the enveloping algebra which are not polar invariants, these derivators do not vanish, and so, in general, the QNBs in Eqs. (138) and (139) are not derivations.

[12] If f is a function in the $su(2)$ enveloping algebra, then $[f,I]=0$, and $(J_+,J_-|J_+,f,J_-)$ is again inherently $O(\hbar^4)$.

Another simple example of a nontrivial QNB, with a trivial classical limit, is

$$[L_0L_+,L_0^2,L_+,L_-]=-4\hbar^3(2L_0-\hbar)L_+. \tag{143}$$

This would again vanish were the generic 4-brackets derivations, but as already stressed, they are not. The corresponding $^4\mathbf{\Delta}$ here gives

$$2(L_0,L_+|L_0^2,L_+,L_-)=[\{L_0,L_+\},L_0^2,L_+,L_-]=-8\hbar^3\{L_0,L_+\}, \tag{144}$$

a purely quantum effect for 4-brackets. Its classical limit is

$$\lim_{\hbar\to 0}\frac{1}{\hbar^2}[\{L_0,L_+\},L_0^2,L_+,L_-]=-8\lim_{\hbar\to 0}\hbar\{L_0,L_+\}=0. \tag{145}$$

The RHS has one too many powers of $\hbar$ to contribute classically.

A class of such results, evocative of those found in deformed Lie algebras, is given by

$$(g(L_0),L_+|L_0^2,L_+,L_-)=2\hbar L_+(I+\hbar L_0-L_0^2)[g(L_0+2\hbar)-2g(L_0+\hbar)+g(L_0)]-4\hbar^2L_+(\hbar+2L_0)[g(L_0+2\hbar)-g(L_0+\hbar)]. \tag{146}$$

The choice $g(L_0)=L_0$ reduces to the particular case in Eq. (144).

b. 4-brackets sums for any Lie algebra. How does the quantum 4-bracket method extend to other examples, perhaps even to models that are not integrable? In complete parallel with the classical example, any Lie algebra will allow a commutator with a quadratic Casimir invariant to be rewritten as a sum of 4-brackets. Suppose

$$[Q_a,Q_b]=i\hbar f_{abc}Q_c \tag{147}$$

in a basis where f_{abc} is totally antisymmetric. Then, through the use of the commutator resolution (93), for a structure-constant-weighted sum of quantum 4-brackets, we find

$$f_{abc}[A,Q_a,Q_b,Q_c]=3f_{abc}\{[A,Q_a],[Q_b,Q_c]\}=3i\hbar f_{abc}f_{bcd}\{[A,Q_a],Q_d\}. \tag{148}$$

Again, for simple Lie algebras, use Eq. (34) to obtain a commutator with the Casimir invariant Q_aQ_a,

$$3i\hbar\, c_{\mathrm{adjoint}}\{[A,Q_a],Q_a\}=3i\hbar c_{\mathrm{adjoint}}[A,Q_aQ_a]. \tag{149}$$

Thus, we obtain the quantization of the classical result in Eq. (35),

$$f_{abc}[A,Q_a,Q_b,Q_c]=3i\hbar c_{\text{adjoint}}[A,Q_aQ_a]. \quad (150)$$

This becomes Eq. (35) in the classical limit, with an appropriate factor of 2 included, as in Eq. (99). A slightly different route to this result is to use the left- and right-sided resolutions of the 4-brackets into 3-brackets (82), and then note that the trilinear invariant reduces to the quadratic Casimir invariant:

$$f_{abc}[Q_a,Q_b,Q_c]=3i\hbar c_{\text{adjoint}}Q_aQ_a. \quad (151)$$

Thus, as in Eq. (130), this particular linear combination of quantum 4-brackets acts as a derivation. As a corollary, we have the 4-bracket effective fundamental identity (EFI) [6], i.e., one with three of the entries being related by a Lie algebra:

$$\begin{aligned} f_{abc}[Q_a,Q_b,Q_c,[A,B,\ldots,D]] &= f_{abc}[[Q_a,Q_b,Q_c,A],B,\ldots,D] \\ &\quad + f_{abc}[A,[Q_a,Q_b,Q_c,B],\ldots,D]+\cdots \\ &\quad + f_{abc}[A,B,\ldots,[Q_a,Q_b,Q_c,D]]. \end{aligned} \quad (152)$$

By using Eq. (150), all models with dynamics based on simple Lie algebras with $H=\frac{1}{2}Q_aQ_a$ can be quantized through the use of summed quantum 4-brackets to describe their time evolution as a derivation:

$$i\hbar^2\frac{dA}{dt}=\hbar[A,H]=\frac{1}{6ic_{\text{adjoint}}}f_{abc}[A,Q_a,Q_b,Q_c]. \quad (153)$$

This special combination of sums of 4-brackets leads to an exception to the generic QNB feature of dynamically scaled time derivatives. It shows that QNBs can be used to describe conventional time evolution for many systems, not only those that are superintegrable or integrable.

c. U(n) and isotropic quantum oscillators. The previous results on Nambu-Jordan-Lie algebras can be applied to harmonic oscillators. For the isotropic oscillator, the NJL approach quickly leads to a compact result. A set of operators can be chosen that produces only one term in the sum of Jordan-Kurosh products.

Consider the n-dimensional oscillator using the standard raising/lowering operator basis, but normalized in a way ($\sqrt{2}a\equiv x+ip$, $\sqrt{2}b\equiv x-ip$) that makes the classical limit more transparent:

$$[a_i,b_j]=\hbar\delta_{ij}, \quad [a_i,a_j]=0=[b_i,b_j]. \quad (154)$$

Construct the usual bilinear charges that realize the $u(n)$ algebra:

$$N_{ij}\equiv b_ia_j, \quad [N_{ij},N_{kl}]=\hbar(N_{il}\delta_{jk}-N_{kj}\delta_{il}). \quad (155)$$

Then, the isotropic Hamiltonian is

$$H=\omega\sum_{i=1}^{n}(N_i+\tfrac{1}{2}\hbar), \quad N_i\equiv N_{ii}, \quad (156)$$

which gives the n^2 conservation laws

$$[H,N_{ij}]=0. \quad (157)$$

Consideration of the isotropic oscillator dynamics using QNBs yields the main result for oscillator $2n$-brackets.

d. Isotropic oscillator quantum brackets. [The U(n) quantum *reductio ad dimidium*.] Let $N=N_1+N_2+\ldots+N_n$ and intercalate the $n-1$ nondiagonal operators N_{ii+1}, for $i=1,\ldots,n-1$, into $2n$-brackets along with the n mutually commuting N_j, for $j=1,\ldots,n$, and along with an arbitrary A to find[13]

$$\begin{aligned} &[A,N_1,N_{12},N_2,N_{23},\ldots,N_{n-1},N_{n-1n},N_n] \\ &\quad=\hbar^{n-1}\{[A,N],N_{12},N_{23},\ldots,N_{n-1n}\} \\ &\quad=\hbar^{n-1}[\{A,N_{12},N_{23},\ldots,N_{n-1n}\},N]. \end{aligned} \quad (158)$$

This result shows that the QNB on the LHS will indeed vanish, not just when A is one of the $2n-1$ charges listed along with A in the brackets, but also if A is one of the remaining U(n) charges, by virture of the explicit commutator with $N=H/\omega$ appearing on the RHS of Eqs. (158) and (157). The classical limit of Eq. (158) is Eq. (39), of course.

The nondiagonal operators N_{ii+1} do not all commute among themselves nor with all the N_j, but their non-Abelian properties are encountered in the above Jordan and Nambu products in a minimal way. Only adjacent entries in the list $N_{12},N_{23},N_{34},\ldots,N_{n-1n}$ fail to commute. Also in the list of $2n-1$ generators within the original QNB, each N_j fails to commute only with the adjacent N_{j-1j} and N_{jj+1}. Such a list of invariants constitutes a "Hamiltonian path" through the algebra.[14]

Proof. Linearity in each argument and total antisymmetry of the Nambu brackets allow us to replace any one of the N_i by the sum N. Replace $N_n\rightarrow N$, hence obtain

$$\begin{aligned} &[A,N_1,N_{12},N_2,\ldots,N_{n-1},N_{n-1n},N_n] \\ &\quad=[A,N_1,N_{12},N_2,\ldots,N_{n-1},N_{n-1n},N]. \end{aligned} \quad (159)$$

[13]Analogously to the classical case, the nondiagonal charges are not hermitian. But the proof leading to Eq. (158) also goes through if nondiagonal charges have their subscripts transposed. This allows replacing N_{ii+1} with Hermitian or anti-Hermitian combinations $N_{ii+1}\pm N_{i+1i}$ in the LHS $2n$-brackets, to obtain the alternative linear combinations $N_{ii+1}\mp N_{i+1i}$ in the GJP on the RHS of Eq. (158).

[14]There are other Hamiltonian paths through the algebra. As previously mentioned in the case of CNBs, a different set of $2n-1$ invariants which leads to an equivalent *reductio ad dimidium* can be obtained just by taking an arbitrarily ordered list of the mutually commuting N_i and then intercalating nondiagonal generators to match adjacent indices on the N_i. That is, for any permutation of the indices $\{\sigma_1,\ldots,\sigma_n\}$, we have: $[A,N_{\sigma_1},N_{\sigma_1\sigma_2},N_{\sigma_2},N_{\sigma_2\sigma_3},N_{\sigma_3},\ldots,N_{\sigma_{n-1}},N_{\sigma_{n-1}\sigma_n},N_{\sigma_n}]=\hbar^{n-1}\{[A,N],N_{\sigma_1\sigma_2},N_{\sigma_2\sigma_3},\ldots,N_{\sigma_{n-1}\sigma_n}\}=\hbar^{n-1}[\{f,N_{\sigma_1\sigma_2},N_{\sigma_2\sigma_3},\ldots,N_{\sigma_{n-1}\sigma_n}\},N]$.

Now, since $[N,N_{ij}]=0$, the commutator resolution of the $2n$-brackets implies that N must appear locked in a commutator with A and therefore A cannot appear in any other commutator. But then N_1 commutes with all the remaining free N_{ij} except N_{12}. So N_1 must be locked in $[N_1,N_{12}]$. Continuing in this way, N_2 must be locked in $[N_2,N_{23}]$, etc., until finally N_{n-1} is locked in $[N_{n-1},N_{n-1n}]$. Thus, all $2n$ entries have been paired and locked in the indicated n commutators, i.e., they are all zipped-up. Moreover, these n commutators can and will appear as products ordered in all $n!$ possible ways with coefficients $+1$ since interchanging a pair of commutators requires interchanging two pairs of the original entries in the brackets. We conclude that

$$[A,N_1,N_{12},N_2,\ldots,N_{n-1},N_{n-1n},N_n] =\{[A,N],[N_1,N_{12},],\ldots,[N_{n-1},N_{n-1n}]\}. \quad (160)$$

Now all the paired N_{ij} commutators evaluate as $[N_{i-1},N_{i-1i}]=\hbar N_{i-1i}$, so we have

$$[A,N_1,N_{12},N_2,\ldots,N_{n-1},N_{n-1n},N_n] =\hbar^{n-1}\{[A,N],N_{12},\ldots,N_{n-1n}\}. \quad (161)$$

Finally the commutator with N may be performed either before or after the Jordan product of A with all the N_{i-1i}, since again $[N,N_{ij}]=0$. Hence,

$$\{[A,N],N_{12},\ldots,N_{n-1n}\} =[\{A,N_{12},\ldots,N_{n-1n}\},N]. \quad \text{QED} \quad (162)$$

In analogy with the classical situation, the quantum invariants which are in involution [i.e., the Cartan subalgebra of $\mathrm{u}(n)$] are separated out of the QNB into a single commutator involving their sum, the oscillator Hamiltonian, while the invariants which do not commute [$n-1$ of them, corresponding in number to the rank of $\mathrm{SU}(n)$] are swept into a generalized Jordan product. Thus, we have been led to a more complicated Jordan-Kurosh eigenvalue problem for $\mathrm{u}(n)$ invariant dynamics, as the entwined effect of several mutually noncommuting N_{ij}s. The individual N_{ij} may not be diagonalized simultaneously, so it may not be obvious what the general formalism of projection operators will lead to in this case, as compared to Eq. (133) *et seq*., but in fact it can be carried through by rearranging the terms in the Jordan product, as we shall explain.

Our QNB description of time evolution for isotropic quantum oscillators therefore becomes

$$\omega[A,N_1,N_{12},N_2,\ldots,N_{n-1},N_{n-1n},N_n] =i\hbar^n\left\{\frac{dA}{dt},N_{12},\ldots,N_{n-1n}\right\}, \quad (163)$$

whose classical limit is precisely Eq. (44). This result encodes both dynamical time scales and, in higher orders of $\hbar$, group rotation terms, as a consequence of the generalized Jordan eigenvalue problem involving noncommuting elements, $N_{12},\ldots,N_{n-1n}$.

The specific oscillator realization of U(3) explicates this last point. The RHS of Eq. (163) becomes

$$\left\{\frac{dA}{dt},N_{12},N_{23}\right\}=3\left\{(N_2+\tfrac{1}{2}\hbar)N_{13},\frac{dA}{dt}\right\}+\tfrac{1}{2}\hbar\left[N_{13},\frac{dA}{dt}\right] +\left[\left[N_{23},\frac{dA}{dt}\right],N_{12}\right]. \quad (164)$$

We have rearranged terms so as to produce just a simple Jordan product (anticommutator), not a generalized one, and rotations of the time derivative. This leads to a Jordan spectral problem involving only a commutative product, $N_2N_{13}=N_{13}N_2$, to set the dynamical time scales for the various invariant sectors of the theory. The additional rotation terms in Eq. (164) are similar to those encountered previously in the SU(2) examples, (138) and (139), with a notable difference: The rotation is performed on dA/dt, not or A. As in those previous SU(2) cases, however, the rotations are higher order in $\hbar$ than the anticommutator term, and so they drop out of the classical limit. Decompositions similar to Eq. (164) apply to all the other $U(n)$ cases described by Eq. (163), as can be seen from the list of operators in the GJP of that equation, by noting that only adjacent N_{ii+1}elements in the list fail to commute.

It should also be apparent from the form of Eq. (164) that one cannot simply divide the LHS by $N_{12}N_{23}$ and then naively sweep the $(N_{12}N_{23})^{-1}$ factor on the RHS into a logarithm. This is permitted in the classical limit, as in Eq. (45), but operators are not as easily manipulated on Hilbert space. Perhaps it is useful to think of this as a problem of implementing scale transformations in the generalized Jordan-Kurosh algebra context, but here the rotation terms complicate the problem. These terms also complicate the issue of the quantum bracket's kernel, although that issue for just the first RHS term in Eq. (164) is the familiar one for Jordan algebras.

The result (163) helps to clarify why the Leibniz rules fail when time evolution is expressed using QNBs for the isotropic oscillator, for here this failure has been linked to the intervention of a Jordan product involving noncommuting invariants. The Leibniz failure can be summarized in derivators. For the U(n) case,

$$\begin{aligned}(A,\mathcal{A}|N_1,N_{12},N_2,N_{23},\ldots,N_{n-1},N_{n-1n},N_n) &=\hbar^{n-1}[\{A\mathcal{A},N_{12},N_{23},\ldots,N_{n-1n}\},N]\\ &-\hbar^{n-1}A[\{\mathcal{A},N_{12},N_{23},\ldots,N_{n-1n}\},N]\\ &-\hbar^{n-1}[\{A,N_{12},N_{23},\ldots,N_{n-1n}\},N]\mathcal{A}. \quad (165)\end{aligned}$$

For U(2), this reduces to

$$(A,\mathcal{A}|N_1,N_{12},N_2)=\hbar[N,A][N_{12},\mathcal{A}]-\hbar[N_{12},A][N,\mathcal{A}]. \quad (166)$$

In the classical limit, the derivator vanishes, as expected.

e. $SO(n+1)$ and quantum particles on n-spheres. For a quantum particle moving freely on the surface of an n-sphere, it is a delicate matter to express the quantum so$(n+1)$ charges in terms of the canonical coordinates and momenta, but it can be done (cf. the discussion in [6], and references cited therein). The quantum version of the classical PB algebra is then obtained without any modifications.

$$[P_a,P_b]=i\hbar L_{ab},$$

$$[L_{ab},P_c]=i\hbar(\delta_{ac}P_b-\delta_{bc}P_a),$$

$$[L_{ab},L_{cd}]=i\hbar(L_{ac}\delta_{bd}-L_{ad}\delta_{bc}-L_{bc}\delta_{ad}+L_{bd}\delta_{ac}). \tag{167}$$

Hence, the symmetric Hamiltonian has the same quadratic form as the classical expression (49). So a natural choice for the QNB involves the same set of $2n-1$ invariants as selected in the classical case: $[A,P_1,L_{12},P_2,L_{23},P_3,\ldots,P_{n-1},L_{n-1n},P_n]$.

The QNB results in an entwined time derivative, with attendant dynamical time scales and quantum rotation terms,

$$\frac{1}{(i\hbar)^n n}[A,P_1,L_{12},P_2,L_{23},P_3,\ldots,P_{n-1},L_{n-1n},P_n]$$
$$=(-1)^{n-1}\left\{P_2,P_3,\ldots,P_{n-1},\frac{dA}{dt}\right\}$$
$$+\text{quantum rotation terms}, \tag{168}$$

where $i\hbar dA/dt=[A,H]$. As in the previous SU(2) and U(n) quantum examples, the operator entwinement on the RHS is not trivially eliminated through simple logarithms, as it is in the classical situation in going from Eqs. (48) to (50), but leads to Jordan-Kurosh spectral problems on the Hilbert space of the system. The kernel of the quantum bracket is similarly impacted.

If one of the P's or L's in the $2n$-brackets of Eq. (168) were replaced by H, the occurrence of an entwined time derivative would be manifest [see Eq. (177) below for the 3-sphere case]. Otherwise, with the invariants as chosen, it is laborious to obtain $[A,H]$ by direct calculation. Likewise, the explicit form of the quantum rotation terms in Eq. (168), for general n, are laboriously obtained by direct calculation and will be given elsewhere. They may be constructed through an embedding of the orthogonal group into the unitary group treated previously. Suffice it here to say that they are higher order in $\hbar$, as expected, and to consider the case of the 3-sphere, for comparison to the chiral charge methods given below.

For the 3-sphere, it is convenient to define the usual duals (sum repeated indices),

$$L_i=\tfrac{1}{2}\varepsilon_{ijk}L_{jk}. \tag{169}$$

Then, the algebra becomes the well known [compare to Eqs. (53) and (54)]

$$[L_i,L_j]=i\hbar\varepsilon_{ijk}L_k,\quad [L_i,P_j]=i\hbar\varepsilon_{ijk}P_k,$$
$$[P_i,P_j]=i\hbar\varepsilon_{ijk}L_k, \tag{170}$$

and the group invariant Hamiltonian is $H=\frac{1}{2}(P_iP_i+L_iL_i)$. By direct calculation, we then obtain [cf. third RHS line in Eq. (87), and also recall for a particle on the 3-sphere, $L_iP_i=0$]

$$[A,L_1,L_2,P_1,P_2,P_3]=3i\hbar^3\left\{\frac{dA}{dt},P_3\right\}-3\hbar^2\{[A,L_iP_i],L_3\}$$
$$+\tfrac{1}{4}\hbar^2\sum_i\,([[[A,L_i-P_i],L_i-P_i],L_3+P_3]$$
$$-[[[A,L_i+P_i],L_i+P_i],L_3-P_3]). \tag{171}$$

Once again, the quantum rotation terms represent higher-order corrections, in powers of $\hbar$, corresponding to group rotations of A. For example, if A is the 3-sphere bilinear

$$A_{ab}\equiv(L_a+P_a)(L_b-P_b), \tag{172}$$

then $dA_{ab}/dt=0$ for a particle moving freely on the surface of the sphere, but the corresponding quantum group rotation induced by the 3-sphere 6-brackets is *not* zero. The 6 brackets reduce entirely to those quantum rotation terms. Explicitly, we find

$$[A_{ab},P_1,L_3,P_2,L_1,P_3]=4i\hbar^5\sum_c\,(\varepsilon_{b2c}A_{ac}-\varepsilon_{a2c}A_{cb}). \tag{173}$$

As in all previous cases, the quantum rotations disappear in the classical limit

$$\lim_{\hbar\to 0}[A_{ab},P_1,L_3,P_2,L_1,P_3]/\hbar^3=0. \tag{174}$$

f. $SO(4)=SU(2)\times SU(2)$ as another special case. We consider this particular example in more detail, as a bridge to general chiral models, choosing bracket elements that exhibit dynamical time scales both with and without group rotations. Use L_i and R_i for the mutually commuting su(2) charges:

$$[L_i,L_j]=i\hbar\varepsilon_{ijk}L_k,\quad [R_i,R_j]=i\hbar\varepsilon_{ijk}R_k,\quad [L_i,R_j]=0. \tag{175}$$

[Again, compare to Eqs. (53) and (54) and note the normalization here differs from that used earlier: $\mathcal{L}_i=-2L_i$, $\mathcal{R}_i=-2R_i$.] Define the usual quadratic Casimir invariants for the left and right algebras,

$$I_L=\sum_i L_i^2,\quad I_R=\sum_i R_i^2. \tag{176}$$

Then, for a Hamiltonian of the form $H\equiv F(I_L,I_R)$, where F is any function of the left and right Casimir invariants, we find

$$[A,H,R_1,R_2,L_1,L_2]=(i\hbar)^2\{[A,H],L_3,R_3\}$$
$$=(i\hbar)^2[\{A,L_3,R_3\},H]. \tag{177}$$

This is the quantum analog of the classical result in Eq. (56). Aside from trivial normalization factors, the difference lies in the particular ordering of operators in the quantum expression. Physically, Eq. (177) is simply time evolution generated by the Hamiltonian, with the Jordan-Kurosh, simultaneous eigenvalues of L_3 and R_3 setting the time scales for the various sectors of the theory. In particular, this is true for $H \propto I_L$ or $H \propto I_R$, as is relevant for a quantum particle moving freely on the 3-sphere.

The dynamical time scale structure produced by these quantum 6-brackets are a simple extension of the structure found previously for the SU(2) example. The Jordan-Kurosh eigenvalues σ are now defined by

$$\{A_\sigma, L_3, R_3\} = \sigma A_\sigma . \tag{178}$$

A complete set of operators consists of all doubly projected eigenoperators $A_{\lambda_1\rho_1,\lambda_2\rho_2}$, where

$$L_3 A_{\lambda_1\rho_1,\lambda_2\rho_2} = \lambda_1 A_{\lambda_1\rho_1,\lambda_2\rho_2}, \quad A_{\lambda_1\rho_1,\lambda_2\rho_2} L_3 = \lambda_2 A_{\lambda_1\rho_1,\lambda_2\rho_2},$$
$$R_3 A_{\lambda_1\rho_1,\lambda_2\rho_2} = \rho_1 A_{\lambda_1\rho_1,\lambda_2\rho_2}, \quad A_{\lambda_1\rho_1,\lambda_2\rho_2} R_3 = \rho_2 A_{\lambda_1\rho_1,\lambda_2\rho_2}. \tag{179}$$

Hence, $\{A, L_3, R_3\} = \sigma_{12} A$, with

$$\sigma_{12} = 2\lambda_1\rho_1 + \lambda_1\rho_2 + \rho_1\lambda_2 + 2\lambda_2\rho_2, \tag{180}$$

since

$$\{A, L_3, R_3\} = \{L_3, R_3\} A + L_3 A R_3 + R_3 A L_3 + A\{L_3, R_3\}. \tag{181}$$

So the time scales for the various sectors of the theory are set jointly by the eigenvalues of L_3 and R_3.

The simple Leibniz rule for generic A and $\mathcal{A}$, that would equate $[A\mathcal{A}, H, R_1, R_2, L_1, L_2]$ with $A[\mathcal{A}, H, R_1, R_2, L_1, L_2] + [A, H, R_1, R_2, L_1, L_2]\mathcal{A}$, will fail for products $A_{\lambda_1\rho_1,\lambda_2\rho_2}\mathcal{A}_{\lambda_2\rho_2,\lambda_3\rho_3}$, unless

$$\sigma_{12} = \sigma_{23} = \sigma_{13} . \tag{182}$$

There are no higher-order quantum group rotation terms in this particular case, due to our choice for the invariants in the brackets $[A, H, R_1, R_2, L_1, L_2]$. The more general situation is revealed by a different choice, as follows.

g. 3-sphere chiral 6-brackets. We take all five of the fixed elements in the 6-brackets to be charges in the su(2) $\times$ su(2) algebra, and not Casimir invariants, to find

$$[A, L_1, L_2, L_3, R_1, R_2] = \frac{3}{2}(i\hbar)^2 [\{A, R_3\}, I_L] + \frac{1}{2}(i\hbar)^2 \sum_i [[[A, L_i], L_i] R_3], \tag{183}$$

or, equivalently,

$$[A, R_1, R_2, R_3, L_1, L_2] = \frac{3}{2}(i\hbar)^2 [\{A, L_3\}, I_R] + \frac{1}{2}(i\hbar)^2 \sum_i [[[A, R_i], R_i] L_3]. \tag{184}$$

The first terms (single commutators) on the RHSs of Eqs. (183) and (184) are inherently $O(\hbar^3)$, and give scaled time derivatives, while the second terms (triple commutators) are $O(\hbar^5)$, and give additional group rotations. Perhaps these results may be interpreted as group *covariant* Hamiltonian flows, with "quantum connections" given as the triple commutator, higher-order effects in $\hbar$. Note the LHS of Eq. (183) manifestly vanishes when A is one of L_1, L_2, L_3, R_1, or R_2, while the RHS manifestly vanishes for the remaining choice $A = R_3$, as well as R_1 and R_2. Similarly, the RHS of Eq. (184) manifestly vanishes for $A = L_j$, $j = 1, 2, 3$, including the one case excepted by the LHS of that equation.

We may add or subtract Eqs. (183) and (184) to gain $L \leftrightarrow R$ symmetry between left- and right-hand sides, but the resulting quantum expressions do not permit easy conversions into logarithms, as in the classical case [cf. Eq. (59)]. Nonetheless, for the free particle on the 3-sphere, with $H = 2I_L = 2I_R$, we may write the sum and difference as

$$\begin{aligned}[A, L_1, L_2, R_3 \pm L_3, R_1, R_2] &= \frac{-3i\hbar^3}{4}\left\{\frac{dA}{dt}, L_3 \pm R_3\right\} \\ &\quad + \frac{1}{2}(i\hbar)^2 \sum_i ([[[A, R_i], R_i], L_3] \\ &\quad \pm [[[A, L_i], L_i], R_3]).\end{aligned} \tag{185}$$

This is the by-now-familiar form, consisting of an entwined time derivative and group rotations.

As a simple example to isolate and accentuate the group rotation effects, take A to be any bilinear $A_{ab} \equiv L_a R_b$ of specific left and right charges. Since commutators are indeed derivations, all functions of the six possible L_a and R_b charges commute with the Casimir invariants, so the first terms on the RHSs of Eqs. (183) and (184) vanish for $A = A_{ab}$ (i.e., A_{ab} for a particle moving freely on the surface of a 3-sphere has no time derivatives). The second terms on the RHSs of Eqs. (183) and (184) do *not* vanish for $A = A_{ab}$ but are just rotations of the L_a and R_b charges, respectively, about the z axis:

$$\sum_i [[[A_{ab}, L_i], L_i], R_3] = 2i\hbar^3 \sum_c \varepsilon_{b3c} A_{ac},$$
$$\sum_i [[[A_{ab}, R_i], R_i], L_3] = 2i\hbar^3 \sum_{c=1,2,3} \varepsilon_{a3c} A_{cb}. \tag{186}$$

So, for this particular example,

$$[A_{ab},L_1,L_2,L_3,R_1,R_2]=-i\hbar^5\sum_c \varepsilon_{b3c}A_{ac},$$

$$[A_{ab},R_1,R_2,R_3,L_1,L_2]=-i\hbar^5\sum_c \varepsilon_{a3c}A_{cb}. \quad (187)$$

Let us establish this result in detail by proceeding from the SU(2)×SU(2) chiral form of the *reductio ad dimidium*, namely,

$$[A,L_1,L_2,L_3,R_1,R_2]=(i\hbar)^2\sum_j \{[A,L_j],L_j,R_3\}. \quad (188)$$

This enables us to compute how the bilinear is transformed by the 6-brackets. First

$$[L_aR_b,L_1,L_2,L_3,R_1,R_2]=(i\hbar)^2\sum_j \{[L_aR_b,L_j],L_j,R_3\}$$
$$=(i\hbar)^3\sum_{j,k}\varepsilon_{ajk}\{L_kR_b,L_j,R_3\}. \quad (189)$$

But then

$$\{L_kR_b,L_j,R_3\}=L_kR_b\{L_j,R_3\}+L_j\{L_kR_b,R_3\}+R_3\{L_kR_b,L_j\}$$
$$=2L_kL_jR_bR_3+L_jL_k\{R_b,R_3\}+\{L_k,L_j\}R_3R_b, \quad (190)$$

so, summing repeated indices,

$$\varepsilon_{ajk}\{L_kR_b,L_j,R_3\}=i\hbar\,\varepsilon_{ajk}\varepsilon_{kjm}L_m\left(R_bR_3-\frac{1}{2}\{R_b,R_3\}\right)$$
$$=\frac{1}{2}(i\hbar)^2\varepsilon_{ajk}\varepsilon_{kjm}L_m\varepsilon_{b3c}R_c$$
$$=\hbar^2L_aR_c\varepsilon_{b3c}. \quad (191)$$

This confirms by direct calculation that the chosen brackets do not just produce entwined time derivatives, but more elaborately, the brackets combine entwined dA/dt with infinitesimal group rotations of A. Since group rotations are symmetries of the system's dynamics, this is not an inconsistent combination (cf. covariant derivatives in Yang-Mills theory).

h. Quantum $G\times G$ chiral particles. Consider next models whose dynamics are invariant under chiral groups, $G\times G$. For example, a particle moving freely on the group manifold is of this type. Let n be the dimension of the group G, and write the charge algebra underlying the group $G\times G$ as

$$[L_i,L_j]=i\hbar f_{ijk}L_k,\quad [R_i,R_j]=i\hbar f_{ijk}R_k,\quad [L_i,R_j]=0. \quad (192)$$

Then for odd $n\equiv 1+2s$, with sums over repeated indices understood to run from 1 to n,

$$[A,L_1,\ldots,L_n,R_1,\ldots,R_{n-1}]$$
$$=\frac{1}{n!(n-1)!}\varepsilon_{i_1\ldots i_n}\varepsilon_{j_i\ldots j_{n-1}n}\times[A,L_{i_1},\ldots,L_{i_n},R_{j_1},\ldots,R_{j_{n-1}}]$$
$$=K_n\varepsilon_{i_1\ldots i_n}\varepsilon_{j_i\ldots j_{n-1}n}\{[A,L_{i_1}],[L_{i_2},L_{i_3}],\ldots,\times[L_{i_{n-1}},L_{i_n}],[R_{j_1},R_{j_2}],\ldots,[R_{j_{n-2}},R_{j_{n-1}}]\}, \quad (193)$$

where $K_n=[4^s(s!)^2]^{-1}$ [the same numerical combinatoric factor introduced earlier in the classical example (72)] incorporates the number of equivalent ways to obtain the list of commutators in the generalized Jordan products as written.[15] So

$$[A,L_1,\ldots,L_n,R_1,\ldots,R_{n-1}]$$
$$=K_n(i\hbar)^{n-1}\varepsilon_{i_1\ldots i_n}\varepsilon_{j_i\ldots j_{n-1}n}(f_{i_2i_3k_1}\cdots f_{i_{n-1}i_nk_s})\times(f_{j_1j_2m_1}\cdots f_{j_{n-2}j_{n-1}m_s})\times\{[A,L_{i_1}],L_{k_1},\ldots,L_{k_s},R_{m_1},\ldots,R_{m_s}\}. \quad (194)$$

This leads to some mixed symmetry tensors that are familiar from classical invariant theory for Lie groups,

$$\tau_{n\{m_1\ldots m_s\}}\equiv\varepsilon_{j_1\ldots j_{n-1}n}f_{j_1j_2m_1}\cdots f_{j_{n-2}j_{n-1}m_s}. \quad (195)$$

Need has not dictated obtaining elegant expressions for these tensors, except in special cases, but undoubtedly they exist.[16]

In terms of these, the reduction becomes

$$[A,L_1,\ldots,L_n,R_1,\ldots,R_{n-1}]$$
$$=K_n(i\hbar)^{n-1}\tau_{i_1\{k_1\ldots k_s\}}\tau_{n\{m_1\ldots m_s\}}\times\{[A,L_{i_1}],L_{k_1},\ldots,L_{k_s},R_{m_1},\ldots,R_{m_s}\}. \quad (196)$$

Results for even n are similar, only in that case the arbitrary A must be locked in a commutator with an R.

As in the classical case (70), a somewhat simpler choice for the invariants in the maximal brackets requires us to compute (note the range of the sums)

[15] The number of ways of choosing the n commutators in Eq. (193) is $n(n-2)(n-4)\ldots(1)\times(n-2)(n-4)\ldots(1)$, so $K_n=[n(n-2)(n-4)\ldots(1)\times(n-2)(n-4)\ldots(1)]/n!(n-1)!$.

[16] S. Meshkov has suggested that similar tensors and invariants constructed from them appear in nuclear shell theory.

$$[A,F(I_L,I_R),L_1,\ldots,L_{n-1},R_1,\ldots,R_{n-1}]$$
$$=\frac{1}{(n-1)!(n-1)!}\sum_{\text{all } i,j=1}^{n-1}\varepsilon_{i_1\ldots i_{n-1}}\varepsilon_{j_i\ldots j_{n-1}n}\times[A,F(I_L,I_R),L_{i_1},\ldots,L_{i_{n-1}},R_{j_1},\ldots,R_{j_{n-1}}],\tag{197}$$

where $F(I_L,I_R)$ is any function of the left and right Casimir invariants. The RHS here vanishes for even n, so again we take odd n, say $n=1+2s$. Then by the commutator resolution, since $[F(I_L,I_R),L_i]=0=[F(I_L,I_R),R_i]$, we can write

$$[A,F(I_L,I_R),L_1,\ldots,L_{n-1},R_1,\ldots,R_{n-1}]$$
$$=K_n\sum_{\text{all } i,j=1}^{n-1}\varepsilon_{i_1\ldots i_{n-1}}\varepsilon_{j_i\ldots j_{n-1}n}\{[A,F(I_L,I_R)],\times[L_{i_1},L_{i_2}],\ldots,[L_{i_{n-2}},L_{i_{n-1}}],[R_{j_1},R_{j_2}],\ldots,\times[R_{j_{n-2}},R_{j_{n-1}}]\},\tag{198}$$

where again $1/K_n=4^s(s!)^2$. So, for a Hamiltonian of the form $H=F(I_L,I_R)$, we have

$$[A,H,L_1,\ldots,L_{n-1},R_1,\ldots,R_{n-1}]$$
$$=\frac{1}{(s!)^2}(-4\hbar^2)^s\sum_{\text{all } k,m=1}^{n}\sigma_{\{k_1\ldots k_s\}}\sigma_{\{m_1\ldots m_s\}}\times\{[A,H],L_{k_1},\ldots,L_{k_s},R_{m_1},\ldots,R_{m_s}\},\tag{199}$$

with $2s\equiv n-1$ and the completely symmetric tensor $\sigma_{\{k_1\ldots k_s\}}$ defined as in the classical situation Eq. (73). Note the range of the sum in Eq. (73) is truncated from that in Eq. (199), although the sum may be trivially extended just by adding a fixed extra index to the Levi-Civita symbol:

$$\sigma_{\{k_1\ldots k_s\}}=\sum_{\text{all } i=1}^{n}\varepsilon_{ni_1\ldots i_{n-1}}f_{i_2i_3k_1}\ldots f_{i_{n-2}i_{n-1}k_s}.\tag{200}$$

The commutator of A with the function of Casimir invariants can be computed after the generalized Jordan product (GJP), again since $[H,L_i]=0=[H,R_i]$. So, with the sums over repeated ks and ms understood,

$$[A,H,L_1,\ldots,L_{n-1},R_1,\ldots,R_{n-1}]$$
$$=K_n(i\hbar)^{n-1}\sigma_{\{k_1\ldots k_s\}}\sigma_{\{m_1\ldots m_s\}}[\{A,L_{k_1},\ldots,L_{k_s},\times R_{m_1},\ldots,R_{m_s}\},H].\tag{201}$$

The GJP spectral equation,

$$\lambda A_\lambda+\text{higher-order quantum rotation terms}$$
$$=K_n(i\hbar)^{n-1}\sigma_{\{k_1\ldots k_s\}}\sigma_{\{m_1\ldots m_s\}}\times\{A_\lambda,L_{k_1},\ldots,L_{k_s},R_{m_1},\ldots,R_{m_s}\},\tag{202}$$

must now be solved to find the time scales λ that govern the QNB generated time evolution,

$$i\hbar\lambda\frac{d}{dt}A_\lambda+\text{higher-order quantum rotation terms}=\lambda[A_\lambda,H]$$
$$=[A_\lambda,H,L_1,\ldots,L_{n-1},R_1,\ldots,R_{n-1}].\tag{203}$$

All this extends in a straightforward way to the algebras of symmetry groups involving arbitrary numbers of factors. Rather than pursue that generalization, however, we focus instead on unitary factors, where the τ and σ tensors simplify. For a touch of variety, we take the left and right group factors to be different unitary groups.

i. $U(n)\times U(m)$ models. For systems with $\mathrm{U}(n)\times\mathrm{U}(m)$ group invariant dynamics, with the proper choice of charge basis, the structure-constant-weighted sums of the previous formulas can be made to reduce to single terms as in the case of the previous $\mathrm{U}(n)$ example. We take the oscillator basis for each of the algebras, so that the charges obey the commutators

$$[N_{ij},N_{kl}]=\hbar(N_{il}\delta_{jk}-N_{kj}\delta_{il}),$$
$$[M_{ab},M_{cd}]=\hbar(M_{ad}\delta_{bc}-M_{cb}\delta_{ad}),\quad [N_{ij},M_{ab}]=0,\tag{204}$$

for $i,j,k,l=1,\ldots,n$, and $a,b,c,d=1,\ldots,m$. As before, we denote the mutually commuting diagonal charges as $N_{jj}=N_j$ and $M_{aa}=M_a$, with (central charge) sums $N=\Sigma_{j=1}^{n}N_{jj}$, $M=\Sigma_{a=1}^{m}M_{aa}$. Then, as for the single $\mathrm{U}(n)$ results, we have either

$$[A,N_1,N_{12},N_2,\ldots,N_{n-1},N_{n-1n},N_n,M_1,M_{12},M_2,\ldots,M_{m-1},M_{m-1m}]$$
$$=\hbar^{n+m-2}\{[A,N],N_{12},\ldots,N_{n-1n},M_{12},\ldots,M_{m-1m}\}=\hbar^{n+m-2}[\{A,N_{12},\ldots,N_{n-1n},M_{12},\ldots,M_{m-1m}\},N],\tag{205}$$

or similarly with $M\leftrightarrow N$, as well as other such relations that follow from choosing different Hamiltonian paths through the algebras.

Replacement of one of the diagonal charges with an arbitrary function of the left and right Casimir invariants, as well as the two central sums, leads to similar results. These may now be used to discuss time development for systems whose Hamiltonians are of the form

$$H=F(N,M,I_N,I_M), \tag{206}$$

for which

$$[A,H,N_1,N_{12},N_2,\ldots,N_{n-1},N_{n-1n},M_1,M_{12},M_2,\ldots,M_{m-1},M_{m-1m}]$$
$$=\hbar^{n+m-2}[\{A,N_{12},\ldots,N_{n-1n},M_{12},\ldots,M_{m-1m}\},H]=\hbar^{n+m-2}\{[A,H],N_{12},\ldots,N_{n-1n},M_{12},\ldots,M_{m-1m}\}$$
$$=i\hbar^{n+m-1}\left\{\frac{dA}{dt},N_{12},\ldots,N_{n-1n},M_{12},\ldots,M_{m-1m}\right\}. \tag{207}$$

The effect of the remaining, noncommuting charges in the generalized Jordan product is once again to set time scales for the various invariant sectors of the theory. So, if

$$\{A_\sigma,N_{12},\ldots,N_{n-1n},M_{12},\ldots,M_{m-1m}\}$$
$$=\sigma A_\sigma+\text{higher-order quantum rotation terms}, \tag{208}$$

then

$$i\hbar^{n+m-1}\sigma\frac{dA_\sigma}{dt}$$
$$=[A_\sigma,H,N_1,N_{12},N_2,\ldots,N_{n-1},N_{n-1n},M_1,M_{12},$$
$$\times M_2,\ldots,M_{m-1},M_{m-1m}]$$
$$+\text{higher-order quantum rotation terms}, \tag{209}$$

with quite elaborate sums of such terms describing the time evolution of general operators.

All this extends to the algebras of symmetry groups involving arbitrary numbers of unitary group factors.

C. Summary table

For convenience, we summarize the results of all the previous sections as a table of the key formulas (see Table I).

An empirical methodology suggested by the above examples argues for the following check list in quantizing a general classical system of the type (24). If V is trivial (i.e., numerical), the QNB corresponding to the CNB involved is a prime candidate for an ‘‘exceptional’’ derivation quantization, provided the derivation property checks (and thus the EFI). In the generic case, if V is a function of the invariants, manipulation of the classical expression may be useful, to result in a simpler V and in new CNB entries which would still combine into the Hamiltonian in the PB resolution. The corresponding QNB would then be expected to yield the entwined structures to be studied as above, with the Hamiltonian appearing as an entwined commutator (time derivative), and with the respective time scale eigenvalue problems to be solved.

TABLE I. Key formulas.

Model symmetry	Classical dynamics	Quantum dynamics
SO(3)	(31)	(130) (131)
Any Lie (4-brackets sum)	(35)	(150)
U(N) (oscillators)	(44)	(163)
SO($N+1$)	(48)	(168)
$SO(4)=SU(2)\otimes SU(2)$	(56) (57)	(177) (183)
	(58) (59)	(184) (185)
$G\otimes G$	(71) (75)	(201) (207)

IV. CONCLUSIONS

In this paper, we have demonstrated and illustrated through simple, explicit examples, how Nambu brackets provide a consistent, elegant description, both classically and quantum mechanically, especially of superintegrable systems using even QNBs. This description can be equivalent to classical and quantum Hamiltonian mechanics, but it is broader in its conceptualizations and may have more possible uses. In particular, we have explained in detail how QNBs are consistent, after all, given due consideration to multiple time scales set by invariants entwining the time derivatives, and how reputed inconsistencies have instead involved unsuitable and untenable conditions. We have also emphasized additional complications that distinguish odd QNBs.

We believe the physical interpretations of entwined time derivatives, with their dynamical time scales, and group rotations, in the general situation, explain the perceived failure of the classical Leibniz rules and the classical FI in a transparent way, and are the only ingredients required for a successful non-Abelian quantum implementation of the most general Nambu brackets as descriptions of dynamics. Perhaps this approach is equivalent to the Abelian deformation approach [14], but that has not been shown. However, ultimately it should not be necessary to argue, physically, that if the Abelian deformation approach to quantization of Nambu brackets is indeed logically complete and consistent, then it must give specific results equivalent to the more traditional noncommutative operator approach given here. There is, after all, not very much freedom in the quantization of free particles and simple harmonic oscillators.

Moreover, Hanlon and Wachs [39] announced the result that even QNB algebras (designated by them as ‘‘Lie k algebras’’) are ‘‘Koszul’’ (also see [41]), and therefore have duals which are commutative and totally associative. Is it possible that the Abelian deformation quantization of Nambu bracketss is precisely this dual, and is in that sense equivalent mathematically to the non-Abelian structures we have discussed?

Other such mathematical issues and areas for further study are raised by our analysis, such as the following: A complete mathematical classification of Jordan-Kurosh eigenvalue problems; a corresponding treatment of quantum rotation terms; a study of both classical and quantum topological effects in terms of Nambu brackets; and the behavior of the brackets in the large N limit (as one way to obtain a field theory).

There are also several open avenues for physical applications, the most promising involving membranes and other extended objects. In that regard, given the quantum dichotomy of even and odd brackets, it would appear that extended objects with alternate-dimensional world volumes are more amenable to QNBs. While volume preserving diffeomorphisms are based on classical geometrical concepts, perhaps relying too strongly on associativity, their ultimate generalization to noncommutative geometries, and their uses in field, string, and membrane theories, should be possible. We hope the developments in this paper contribute towards completion of such enterprises.

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APPENDIX: FORMAL DIVISION

We are often interested in solving nonlinear algebraic equations in both Lie and special Jordan algebras. This is hampered by the fact that these are not division rings.

Nevertheless, there is a *formal* series solution to construct inverses in both special Jordan and Lie algebras as contained in an associative embedding algebra $\mathfrak{A}$. For the former, consider[17]

$$a\circ b = b\circ a \equiv ab + ba = c. \tag{A1}$$

Suppose b and c are given. We wish to solve for a. We assume the inverse b^{-1} exists in the enveloping algebra. So we seek to construct either right b_R^{-1} or left b_L^{-1} inverses under Jordan multiplication $\circ$, so that $a = c\circ b_R^{-1} = b_L^{-1}\circ c$. A formal series solution for b_R^{-1} is obtained from the inverse b^{-1} in the enveloping algebra by writing $a = cb^{-1} - bab^{-1}$ and iterating. Thus,

$$a = cb^{-1} - bcb^{-2} + b^2ab^{-2} = \left(\sum_{n=0}^{\infty} (-1)^n b^n c b^{-n}\right) b^{-1} \equiv c\circ b_R^{-1}. \tag{A2}$$

Similarly, for the left inverse $a = b^{-1}c - b^{-1}ab$, so

$$a = b^{-1}c - b^{-2}cb + b^{-2}ab^2 = b^{-1}\left(\sum_{n=0}^{\infty} (-1)^n b^{-n} c b^{n}\right) \equiv b_L^{-1}\circ c. \tag{A3}$$

Requiring formally that these two inverses give the same a leads to an expression that involves only Jordan products of elements from the enveloping algebra,

$$a = \frac{1}{2}\sum_{n=0}^{\infty} (-1)^n (b^n c b^{-n-1} + b^{-n-1} c b^n) = \frac{1}{2}\sum_{n=0}^{\infty} (-1)^n (b^n c b^n)\circ b^{-2n-1}. \tag{A4}$$

However, it involves an infinite number of such products. This raises convergence issues, even when b^{-1} exists in the enveloping algebra.

As the simplest possible illustration of the convergence issues, suppose b and c commute, $bc = cb$. Then, either of the above series for left or right inverses gives an ill-defined result, $a = b^{-1}c\Sigma_{n=0}^{\infty}(-1)^n$. Evidently, the proper way to interpret the (Borel summable) series in this case is $\Sigma_{n=0}^{\infty}(-1)^n = 1/[1-(-1)] = \frac{1}{2}$, to produce the solution to the original equation when $ab = ba$ as well as $bc = cb$. Namely, $2ab = c$ and $a = \frac{1}{2}b^{-1}c$. Nevertheless, convergence is a problem.

For the Lie case, the same formal approach may be considered. Let

$$a \diamondsuit b = -b \diamondsuit a \equiv ab - ba = c. \tag{A5}$$

Suppose b and c are given and solve for a. That is, construct either right b_R^{-1} or left b_L^{-1} inverses under Lie multiplication $\diamondsuit$ so that $a = c \diamondsuit b_R^{-1} = -b_L^{-1} \diamondsuit c$. Again, these are given by formal series solutions obtained by writing $a = cb^{-1} + bab^{-1}$ and iterating. Thus,

$$a = cb^{-1} + bcb^{-2} + b^2ab^{-2} = \left(\sum_{n=0}^{\infty} b^n c b^{-n}\right) b^{-1} \equiv c \diamondsuit b_R^{-1}. \tag{A6}$$

Similarly, for the left inverse $a = -b^{-1}c + b^{-1}ab$, so

$$a = -b^{-1}c - b^{-2}cb + b^{-2}ab^2 = -b^{-1}\left(\sum_{n=0}^{\infty} b^{-n} c b^{n}\right) \equiv -b_L^{-1} \diamondsuit c. \tag{A7}$$

[17]Jordan would include a factor of 1/2 in the definition of $\circ$.

Requiring that these two inverses give the same a leads to an expression that involves only Lie products of elements from the enveloping algebra,

$$a=\frac{1}{2}\sum_{n=0}^{\infty}(b^n c b^{-n-1}-b^{-n-1}cb^n)$$
$$\equiv\frac{1}{2}\sum_{n=0}^{\infty}(b^n c b^n)\diamondsuit b^{-2n-1}. \tag{A8}$$

But, once more, it involves an infinite number of such products. Again this raises convergence issues, even when b^{-1} exists in the enveloping algebra.

As an illustration of convergence issues in this case, follow Wigner's counsel and take 2×2 matrices, $b=\sigma_y=b^{-1}$ and $c=2i\sigma_z$. Then, for even n, $(b^n c b^n)\diamondsuit b^{-2n-1}=[c,b]=4\sigma_x$; while for odd n, $(b^n c b^n)\diamondsuit b^{-2n-1}=[bcb,b]=-[c,b]=-4\sigma_x$. Again, the series gives an ill-defined result, $a=\frac{1}{2}4\sigma_x\Sigma_{n=0}^{\infty}(-1)^n$. This shows clearly that convergence is again a problem. As before, the proper way to interpret the sum in this particular example is $\Sigma_{n=0}^{\infty}(-1)^n=\frac{1}{2}$, to produce the obvious solution to the original equation, $a=\sigma_x$.

The failed convergence for these series is accompanied by a basic problem: divisors of zero. Even when b is invertible in the enveloping algebra, so that the only solution of $ab=0$ is $a=0$, this does *not* hold for the Jordan or Lie products. The Lie case is most familiar and easily seen. $a\diamondsuit b=0$ always has an infinite number of nonvanishing solutions $a\neq0$. Namely, $a=\kappa b$ for any parameter $\kappa\neq0$. Moreover, there can and will be other independent solutions for higher-dimensional enveloping algebras. That is to say, Lie algebras are not division rings, even when they only involve commutators of invertible elements from $\mathfrak{A}$. The same is true for the Jordan case, in general. For instance, using the 2×2 matrices as an example, again with $b=\sigma_y$, we have $a\circ b=0$ for $a=\kappa\sigma_z+\lambda\sigma_x$ for any parameters κ and λ. So, as stressed already, special Jordan algebras are not division rings—not even when they involve only anticommutators of invertible elements from $\mathfrak{A}$.

Perhaps one way to avoid these difficulties and place the formal series constructions on a firmer footing would be through regularizing *deformations* of the algebras. This works for the specific Jordan or Lie examples above, as illustrations of the method. Rather than considering the Jordan or Lie products, we take $ab+\lambda ba=c$. (This deformation was actually analyzed in Jordan's original paper [31], before he settled on the $\lambda=1$ case.) This yields a convergent series for the right inverse b_R^{-1} when $|\lambda|<1$ and a convergent series for the left inverse b_L^{-1} when $|\lambda|>1$. For the right inverse, write $a=cb^{-1}-\lambda bab^{-1}$ and iterate. Thus

$$a=cb^{-1}-\lambda bcb^{-2}+\lambda^2 b^2 ab^{-2}$$
$$=\left(\sum_{n=0}^{\infty}(-\lambda)^n b^n c b^{-n}\right)b^{-1}. \tag{A9}$$

For the simplest situation where $bc=cb$, this gives

$$a=b^{-1}c\left(\sum_{n=0}^{\infty}(-\lambda)^n\right)=b^{-1}c\frac{1}{1+\lambda}. \tag{A10}$$

Now the correct result emerges in the limit $\lambda\rightarrow1$, but strictly speaking this is *not* within the radius of convergence of the series. The series must first be summed to obtain a meromorphic function, by analytic continuation, and the limit applied to that function.

The same method works for the simple Lie example given above. Again, suppose $b=\sigma_y=b^{-1}$ and let $c=2i\sigma_z$. The series for the right inverse now gives $a=\frac{1}{2}4\sigma_x\Sigma_{n=0}^{\infty}\lambda^n=[2/(1-\lambda)]\sigma_x$. The limit $\lambda\rightarrow-1$ converts both this solution and the original equation $ab+\lambda ba=c$ into the Lie problem of interest $a\diamondsuit b=c$.

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