Quantum corrections to the Schwarzschild and Kerr metrics

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We examine the corrections to the lowest order gravitational interactions of massive particles arising from gravitational radiative corrections. We show how the masslessness of the graviton and the gravitational self-interactions imply the presence of nonanalytic pieces $\sim\sqrt{-q^2},\sim q^2\ln - q^2$, etc., in the form factors of the energy-momentum tensor and that these correspond to long range modifications of the metric tensor $g_{\mu\nu}$ of the form $G^2m^2/r^2, G^2m\hbar/r^3$, etc. The former coincide with well known solutions from classical general relativity, while the latter represent new quantum mechanical effects, whose strength and form is necessitated by the low energy quantum nature of the general relativity. We use these results to define a running gravitational charge.

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I. INTRODUCTION

In this paper we will discuss the long distance classical and quantum corrections to the Schwarszchild and Kerr metrics using the techniques of effective field theory. We will show how the nonanalytic radiative corrections to the energy-momentum tensor can be used to obtain the classical nonlinear terms in these metrics at long distance, and calculate the analogous nonanalytic quantum corrections. For the Schwarzschild metric we consider the case of a massive scalar particle. Here we clear up some numerical disagreements in related calculations that have emerged in the literature. We then present the Kerr results, using a massive fermion as a source, and show that the spin-independent quantum corrections are the same as those of the scalar particle. We also elucidate various theoretical issues and compare with other results in the literature.

Effective field theory is ideally suited for discussing the quantum effects of general relativity at scales well below the Planck mass [1,2]. While it is expected that the degrees of freedom and the interactions of gravity will be modified beyond the Planck scale, at low energies these ingredients are accurately described by general relativity. Effective field theory separates the known quantum effects of the low energy particles from the unknown physics at high energy. The latter effects are represented by the most general series of effective Lagrangians consistent with the symmetry of general relativity. However the propagation of the low energy particles yields identifiable quantum effects that can be isolated by the techniques of effective field theory.

The present study builds on two sets of recent work. One of these is the use of effective field theory to study quantum corrections to the gravitational potential [1,3–9]. While the basic principles of these studies are the same, there are some differences and/or disagreements. Since there is not a universal definition of the meaning of a potential, different authors use different definitions of potential in terms of Feynman diagrams, and hence obtain different answers. Even in a case where the same definition is used, different results have been obtained. We will provide some clarification of these disagreements. A related paper [9] provides a full and detailed calculation of the scattering potential of scalar particles. In the present paper we note that a subset of diagrams is more readily interpreted as a change in the metric, and we calculate these effects.

The other precedent for the present paper is the calculation of the the leading quantum corrections to the Reissner-Nordström and Kerr-Newman metrics using effective field theory [10].[1] These metrics involve charged particles, so that the quantum corrections involved photon loops, not graviton loops. However, this provided a particularly clear laboratory for the study of metric corrections. Interestingly, we saw that the *classical* nonlinearities in the metric can be calculated straightforwardly using Feynman diagram techniques. At the same time we saw that there was a clear identification of certain nonanalytic terms with long-distance quantum effects in the metric. We use the insights of that study to investigate the present problems, which involve graviton loops. In the present case the interpretation is not as clear, although the calculations are well defined.

We will be using harmonic gauge throughout this paper. In this gauge, the Schwarzschild metric has the form [12,14],

$$g_{00}=\left(\frac{1-\dfrac{Gm}{r}}{1+\dfrac{Gm}{r}}\right)=1-2\frac{Gm}{r}+2\frac{G^2m^2}{r^2}+\ \ldots,$$

[1]These corrections have also been considered from the point of view of S-matrix theory in Ref. [11].

$$g_{0i}=0,$$

$$g_{ij}=-\delta_{ij}\left(1+\frac{Gm}{r}\right)^2-\frac{G^2m^2}{r^2}\left(\frac{1+\frac{Gm}{r}}{1-\frac{Gm}{r}}\right)\frac{r_ir_j}{r^2}$$

$$=-\delta_{ij}\left(1+2\frac{Gm}{r}+\frac{G^2m^2}{r^2}\right)-\frac{r_ir_j}{r^2}\frac{G^2m^2}{r^2}+\dots. \quad (1)$$

The Kerr metric [15] refers to a particle with spin and, keeping only terms up to first order in the angular momentum, has the harmonic gauge form

$$g_{00}=\left(\frac{1-\frac{Gm}{r}}{1+\frac{Gm}{r}}\right)+\dots=1-2\frac{Gm}{r}+2\frac{G^2m^2}{r^2}+\dots,$$

$$g_{0i}=\frac{2G}{r^2(r+mG)}(\vec{S}\times\vec{r})_i+\dots$$

$$=\left(\frac{2G}{r^3}-\frac{2G^2m}{r^4}\right)(\vec{S}\times\vec{r})_i+\dots,$$

$$g_{ij}=-\delta_{ij}\left(1+\frac{Gm}{r}\right)^2-\frac{G^2m^2}{r^2}\left(\frac{1+\frac{Gm}{r}}{1-\frac{Gm}{r}}\right)\frac{r_ir_j}{r^2}+\dots$$

$$=-\delta_{ij}\left(1+2\frac{Gm}{r}+\frac{G^2m^2}{r^2}\right)-\frac{r_ir_j}{r^2}\frac{G^2m^2}{r^2}+\dots. \quad (2)$$

We will show that using a particular set of Feynman diagrams we reproduce the former with the addition of a long distance quantum correction

$$g_{00}=1-2\frac{Gm}{r}+2\frac{G^2m^2}{r^2}+\frac{62G^2m\hbar}{15\pi r^3}+\dots,$$

$$g_{0i}=0,$$

$$g_{ij}=-\delta_{ij}\left(1+2\frac{Gm}{r}+\frac{G^2m^2}{r^2}+\frac{14G^2m\hbar}{15\pi r^3}\right)$$

$$-\frac{r_ir_j}{r^2}\left(\frac{G^2m^2}{r^2}+\frac{76G^2m\hbar}{15\pi r^3}\right)+\dots. \quad (3)$$

For the Kerr metric,

$$g_{00}=1-2\frac{Gm}{r}+2\frac{G^2m^2}{r^2}+\frac{62G^2m\hbar}{15\pi r^3}+\dots,$$

$$g_{0i}=\left(\frac{2G}{r^3}-\frac{2G^2m}{r^4}+\frac{36G^2\hbar}{15\pi r^5}\right)(\vec{S}\times\vec{r})_i+\dots,$$

$$g_{ij}=-\delta_{ij}\left(1+2\frac{Gm}{r}+\frac{G^2m^2}{r^2}+\frac{14G^2m\hbar}{15\pi r^3}\right)$$

$$-\frac{r_ir_j}{r^2}\left(\frac{G^2m^2}{r^2}+\frac{76G^2m\hbar}{15\pi r^3}\right)+\dots. \quad (4)$$

It is of course required that the classical spin-independent terms must be the same for a scalar particle and a fermion. We know of no firm requirement for the spin independence of the quantum corrections to g_{00} and g_{ij}, but from our calculation they are seen to be identical.

II. REVIEW

The metric is derived from the energy-momentum tensor of a source, using Einstein's equation as the equation of motion. At lowest order in the fields, the source particle is just a point particle in coordinate space. However, both classical fields and their quantum fluctuations modify the energy-momentum of a particle at long distance. These modifications can be found by the consideration of the radiative corrections to the energy-momentum tensor. When these are translated into a metric, they yield the classical nonlinearities and quantum modifications of the metric.

Let us review what was found in Ref. [10] for the conceptually simpler case of charged particles, as our calculation here will follow the same procedure. In that case the field around the particles was the electromagnetic field and the gravitational interaction was treated purely classically. The masslessness of the photon implies that there are long range fields around a charged particle and these carry energy and momentum. At the same time, in a Feynman diagram calculation of the renormalization of the energy momentum tensor of the charged particle, the masslessness of the photon leads to nonanalytic terms in the form factors having the structure $\sim\sqrt{-q^2},\sim q^2\ln-q^2$, where q is the momentum transfer, as well as analytic terms of order $q^2,q^4,\dots$. It was shown in detail how the $\sim\sqrt{-q^2}$ terms account for the classical energy-momentum of the electromagnetic field and how they exactly reproduce the classical nonlinearities in the metric that are present in the Reissner-Nordström and Kerr-Newman metrics. Nonanalytic terms of the form $q^2\ln-q^2$ also appear, and when they are included in the equations of motion they produce further corrections in these metrics. Explicit examination shows that these latter are linear in $\hbar$—i.e. they are quantum effects. Finally the analytic terms produce only delta functions (or derivatives of delta functions) in the metric, such that they vanish at long distance. Thus we saw that the long distance modifications of the metrics are obtained from the nonanlaytic terms in the formfactors of the energy momentum tensor.

The same effects are present in the purely gravitational case. If one expands the energy and momentum of the particle in powers of G, the lowest order result is that of a point particle. However, there is energy and momentum also carried by the gravitational field around the particle and this can be calculated via the one loop Feynman diagrams. Because the graviton is massless, there will also be nonanalytic terms of the forms $\sim\sqrt{-q^2},\sim q^2\ln-q^2$ in the form factors of the energy momentum tensor. Again these will produce long range modifications of the metric. If we include the relevant dimensionful couplings, this will have the schematic form

$$\text{metric}\sim Gm\int\frac{d^3q}{(2\pi)^3}e^{i\vec{q}\cdot\vec{r}}\frac{1}{\vec{q}^2}\left[1-aG\vec{q}^2\sqrt{\frac{m^2}{q}{}^2}\right.$$
$$-bG\vec{q}^2\log(\vec{q}^2)-cG\vec{q}^2+\ldots$$
$$\left.\sim Gm\left[\frac{1}{r}+\frac{aGm}{r^2}+\frac{bG\hbar}{r^3}+cG\delta^3(x)+\ldots\right].\quad (5)$$

Here a,b,c are dimensionless numbers and further numerical factors of order unity will be inserted later. We will examine the $\sqrt{-q^2}$ terms in detail and show how they correctly reproduce all the features of the classical metric. For the $q^2\ln -q^2$ terms we have included the factor of $\hbar$ that follows from dimensional analysis. The analytic correction to the energy momentum tensor yields the delta function term that is not relevant for the long distance behavior.

The nonanalytic terms come from the low energy propagation of gravitons, using the couplings of general relativity. Because these features are independent of the high energy behavior of gravity, they are unambiguous predictions of low energy general relativity. There is also no influence of other possible terms in the gravitational Lagrangian, such as R^2 or related corrections in the matter Lagrangian. These yield only analytic corrections to the form factor and hence do not provide long distance modifications of the metric. These are behaviors that are well known in the effective field theory of gravity [1].

Finally, we comment on some of the potential difficulties that are *not* present in our calculation. In general, quantum gravity calculations can present novel difficulties for field theory. In a general background geometry, the basis states for the "in" basis and the "out" basis may be significantly different as the geometry changes. It may be difficult to define the single particle states or to have well defined amplitudes. Fortunately, these complications are not present in our calculation. We have a perturbative treatment about flat space, and our external particles are well defined. Use of a plane wave basis allows one to extract information about momentum space transitions or, by constructing wave packets, localized states in coordinate space. The external gravitational coupling used for defining the metric is itself classical and the only quantum aspects to the calculation are the propagators internal to the loop diagrams. Thus our calculation is not beset with the subtleties that are peculiar to quantum gravity. The basic setup of the problem is analogous to the extraction of physical predictions in other field theories, with the modest change that we have gravitons within the loops. As described above, the low momentum component of such loops is well defined. Indeed, the closest model is the calculation of the quantum effects in the Reissner-Nordström metric, Ref. [10] which is a very similar calculation within QED.

III. LOWEST ORDER

Let us first consider the theory without loop corrections. The metric tensor is expanded as

$$g_{\mu\nu}\equiv\eta_{\mu\nu}+h^{(1)}_{\mu\nu}+\ldots,\quad (6)$$

where $\eta_{\mu\nu}=(1,-1,-1,-1)_{\rm diag}$ is the usual Minkowski metric and the superscript refers to the number of powers of the gravitational coupling which appear. The dynamical relation which connects $h^{(1)}_{\mu\nu}$ and the energy momentum tensor $T_{\mu\nu}$ is the Einstein equation, whose linearized form in harmonic gauge—$g^{\mu\nu}\Gamma^\lambda_{\mu\nu}=0$—is

$$\Box h^{(1)}_{\mu\nu}=-16\pi G\left(T_{\mu\nu}(x)-\frac{1}{2}\eta_{\mu\nu}T(x)\right),\quad (7)$$

where $T=\eta^{\mu\nu}T_{\mu\nu}$ represents the trace. The metric for a nearly static source is then recovered via the Green function in either coordinate or momentum space

$$h_{\mu\nu}(x)=-16\pi G\int d^3yD(x-y)\left(T_{\mu\nu}(y)-\frac{1}{2}\eta_{\mu\nu}T(y)\right)$$
$$=-16\pi G\int\frac{d^3q}{(2\pi)^3}e^{i\vec{q}\cdot\vec{r}}\frac{1}{\vec{q}^2}\left(T_{\mu\nu}(q)-\frac{1}{2}\eta_{\mu\nu}T(q)\right).\quad (8)$$

The matrix elements of $T_{\mu\nu}$ are described by

$$\langle p_2|T_{\mu\nu}(x)|p_1\rangle$$

and the conservation condition $\partial^\mu T_{\mu\nu}=0$ together with the requirement that $T_{\mu\nu}$ transform as a second rank Lorentz tensor demands the general (scalar field) form[2]

$$\langle p_2|T_{\mu\nu}(x)|p_1\rangle=\frac{e^{i(p_2-p_1)\cdot x}}{\sqrt{4E_2E_1}}[2P_\mu P_\nu F_1(q^2)$$
$$+(q_\mu q_\nu-\eta_{\mu\nu}q^2)F_2(q^2)],\quad (10)$$

where we have defined $P_\mu=\frac{1}{2}(p_1+p_2)_\mu$ and $q_\mu=(p_1-p_2)_\mu$. Conservation of energy and momentum requires $F_1(q^2=0)=1$ but there exists no constraint on $F_2(q^2)$.

The form factors F_1 and F_2 encode all the information about the distribution of energy and momentum for the heavy particle. Although these are defined in momentum space, coordinate space quantities can be studied by forming wave packets. For a very heavy particle, this involves momenta only very close to zero. At lowest order, in particular, we require only the form factors at zero momentum. However, the real quantum content of this paper concerns the quantum loops that modify the form factors. For the leading long distance corrections to this result, we will show by direct calculation how they are extracted by considering the nonanalytic terms in in expansion around zero momentum. The form factors also in principle contain, in their depen-

[2] Here we use the conventional normalization for the scalar field

$$\langle p_2|p_1\rangle=2E_1(2\pi)^3\delta^3(\vec{p}_2-\vec{p}_1)\quad (9)$$

dence on higher powers of q^2, more information about the short distance distribution of energy-momentum. However, in contrast with the nonanalytic terms, this is not universal but depends on the internal structure of the particles, and we do not consider them further.

For the case of a point mass m the lowest order form is [cf. Eq. (B5)]

$$\langle p_2|T^{(0)}_{\mu\nu}(0)|p_1\rangle = \frac{1}{\sqrt{4E_2E_1}}\left[2P_\mu P_\mu - \frac{1}{2}(q_\mu q_\nu - \eta_{\mu\nu}q^2)\right] \tag{11}$$

while in the case of the spin 1/2 system we have

$$\begin{aligned}\langle p_2|T^{(0)}_{\mu\nu}(0)|p_1\rangle &= \overline{u}(p_2)\frac{1}{2}(\gamma_\mu P_\nu + \gamma_\nu P_\mu)u(p_1)\\ &= \left[\frac{1}{m}P_\mu P_\nu - \frac{i}{4m}(\sigma_{\mu\lambda}q^\lambda P_\nu + \sigma_{\nu\lambda}q^\lambda P_\mu)\right]\\ &\quad\times u(p_1),\end{aligned} \tag{12}$$

where we use here the conventions of Bjorken and Drell and have employed the Gordon identity [13]. In either case, for a heavy point mass located at the origin we have the lowest order Breit frame result

$$\langle p_2|T^{(0)}_{\mu\nu}(0)|p_1\rangle \simeq m\,\delta_{\mu 0}\delta_{\nu 0}. \tag{13}$$

The Einstein equation then has the solution

$$h^{(1)}_{\mu\nu}(\vec{q}) = -\frac{8\pi Gm}{\vec{q}^2}\times\begin{cases}1, & \mu=\nu=0\\ 0, & \mu=0,\nu=i\\ \delta_{ij} & \mu=i,\nu=j\end{cases} + \dots, \tag{14}$$

which, using

$$\int\frac{d^3q}{(2\pi)^3}e^{i\vec{q}\cdot\vec{r}}\frac{1}{\vec{q}^2} = \frac{1}{4\pi r},\qquad \int\frac{d^3q}{(2\pi)^3}e^{i\vec{q}\cdot\vec{r}}\frac{q_j}{\vec{q}^2} = \frac{ir_j}{4\pi r^3}, \tag{15}$$

corresponds to the coordinate space result[3]

$$h^{(1)}_{\mu\nu}(\vec{r}) = f(r)\times\begin{cases}1, & \mu=\nu=0,\\ 0 & \mu=0,\nu=i\\ \delta_{ij}, & \mu=i,\nu=j,\end{cases} + \dots, \tag{16}$$

with

[3]Here the ellipses represent a very short range component associated with the q-dependent piece of $T_{\mu\nu}$.

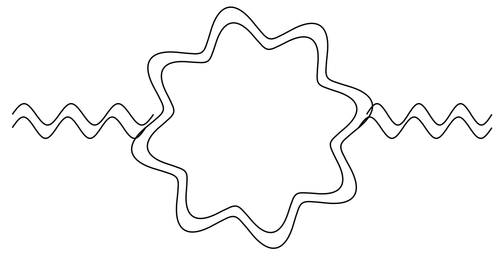

FIG. 1. The vacuum polarization diagram.

$$f(r) = -\frac{2Gm}{r}$$

and reproduces the well-known leading order piece of the Schwarzschild solution [14]. In the case of spin 1/2 there is an additional classical component which arises from the spin. Using

$$\langle p_2|T^{(0)}_{0i}(0)|p_1\rangle \simeq \chi_2^\dagger\vec{\sigma}\chi_1\times\vec{q} \tag{17}$$

we find the off-diagonal component of the metric

$$h^{(1)}_{0i}(\vec{q}) = -8\pi iG\frac{1}{\vec{q}^2}(\vec{S}\times\vec{q})_i \tag{18}$$

which corresponds to the coodinate space result

$$h^{(1)}_{0i}(\vec{r}) = \frac{2G}{r^3}(\vec{S}\times\vec{r})_i \tag{19}$$

and agrees to this order with the Kerr metric [15]. With this basic material in hand we now proceed to the inclusion of loop corrections.

IV. LOOP CORRECTIONS TO THE ENERGY MOMENTUM TENSOR—SPIN 0

Of course, the lowest order discussion given above is straightforward and familiar, while the purpose of the present paper is determine the nonanalytic corrections $\sim\sqrt{-q^2}$, $q^2\log -q^2$ to the form factors arising from the higher order gravitational self-interaction. The appearance of such terms was found in Ref. [10] (hereafter referred to as "I") to be associated with the feature that the graviton couples to the (massless) photon, and the same is expected to happen in the case of gravitational self-interaction since the graviton is itself massless. The relevant diagrams are shown in Fig. 1 and are similar to their electromagnetic analog considered in I, although the tensor nature of the graviton makes the calculation *considerably* more tedious. Details of the calculation are given in Appendix A and the results are [1,5]

$$F_1(q^2)=1+\frac{Gq^2}{\pi}\left(-\frac{3}{4}\log\frac{-q^2}{m^2}+\frac{1}{16}\frac{\pi^2 m}{\sqrt{-q^2}}\right)+\dots,$$

$$F_2(q^2)=-\frac{1}{2}+\frac{Gm^2}{\pi}\left(-2\log\frac{-q^2}{m^2}+\frac{7}{8}\frac{\pi^2 m}{\sqrt{-q^2}}\right)+\dots. \tag{20}$$

As found in the case of the electromagnetically corrected vertex studied in I, we observe that the $q^2=0$ value of the leading form factor $F_1(q^2)$ is unchanged from its lowest order value of unity, as required by energy-momentum conservation, while the form factor $F_2(q^2)$, which is not protected, *is* modified. Such higher order corrections are to be expected from the feature that gravity is nonlinear and must contain terms to all orders in the gravitational coupling.

The momentum space form factors imply a coordinate space structure of the energy momentum tensor which is modified at large distance. Using the integrals listed in Appendix A, we find the correction to the lowest order energy-momentum tensor to be

$$\begin{aligned}T_{00}(\vec r)&=\int\frac{d^3q}{(2\pi)^3}e^{i\vec q\cdot\vec r}\left(mF_1(q^2)+\frac{\vec q^2}{2m}F_2(q^2)\right)\\&=\int\frac{d^3q}{(2\pi)^3}e^{i\vec q\cdot\vec r}\Bigg[m+\pi Gm^2\left(-\frac{1}{16}+\frac{7}{16}\right)\\&\quad\times|\vec q|+\frac{Gm}{\pi}\vec q^2\log\vec q^2\left(\frac{3}{4}-1\right)\Bigg]\\&=m\delta^3(r)-\frac{3Gm^2}{8\pi r^4}-\frac{3Gm\hbar}{4\pi^2 r^5},\end{aligned}$$

$$T_{0i}(\vec r)=0,$$

$$\begin{aligned}T_{ij}(\vec r)&=\frac{1}{2m}\int\frac{d^3q}{(2\pi)^3}e^{i\vec q\cdot\vec r}(q_iq_j-\delta_{ij}\vec q^2)F_2(q^2)\\&=\int\frac{d^3q}{(2\pi)^3}e^{i\vec q\cdot\vec r}\Bigg[\frac{7\pi Gm^2}{16|\vec q|}(q_iq_j-\delta_{ij}\vec q^2)\\&\quad-(q_iq_j-\delta_{ij}\vec q^2)\frac{Gm}{\pi}\log\vec q^2\Bigg]\\&=-\frac{7Gm^2}{4\pi r^4}\left(\frac{r_ir_j}{r^2}-\frac{1}{2}\delta_{ij}\right)+\frac{2Gm\hbar}{\pi^2 r^5}\delta_{ij}.\end{aligned}\tag{21}$$

We have inserted factors of $\hbar$ where appropriate, although we continue to use $c=1$ units.

Note that the leading correction to $T_{\mu\nu}$ is classical in nature, since there are no factors of $\hbar$. We can show that this effect is generated by the energy and momentum that are carried by the gravitational field—Eq. (16)— surrounding the point mass. This field possesses an energy-momentum tensor [12]

$$\begin{aligned}8\pi G T^{\rm grav}_{\mu\nu}&=-\frac{1}{2}h^{(1)\lambda\kappa}[\partial_\mu\partial_\nu h^{(1)}_{\lambda\kappa}+\partial_\lambda\partial_\kappa h^{(1)}_{\mu\nu}-\partial_\kappa(\partial_\nu h^{(1)}_{\mu\lambda}\\&\quad+\partial_\mu h^{(1)}_{\nu\lambda})]-\frac{1}{2}\partial_\lambda h^{(1)}_{\sigma\nu}\partial^\lambda h^{(1)\sigma}_{\mu}+\frac{1}{2}\partial_\lambda h^{(1)}_{\sigma\nu}\partial^\sigma h^{(1)\lambda}_{\mu}\\&\quad-\frac{1}{4}\partial_\nu h^{(1)}_{\sigma\lambda}\partial_\mu h^{(1)\sigma\lambda}-\frac{1}{4}\eta_{\mu\nu}\Big(\partial_\lambda h^{(1)}_{\sigma\chi}\partial^\sigma h^{(1)\lambda\chi}\\&\quad-\frac{3}{2}\partial_\lambda h^{(1)}_{\sigma\chi}\partial^\lambda h^{(1)\sigma\chi}\Big)\\&\quad-\frac{1}{4}h^{(1)}_{\mu\nu}\Box h^{(1)}+\frac{1}{2}\eta_{\mu\nu}h^{(1)\alpha\beta}\Box h^{(1)}_{\alpha\beta}\end{aligned}\tag{22}$$

in terms of which the classical field correction to the point mass form of the energy-momentum tensor is determined to be

$$\begin{aligned}T^{\rm grav}_{00}(r)&=\frac{1}{8\pi G}\left(-\frac{3}{4}\vec\nabla f(r)\cdot\vec\nabla f(r)-3f(r)\vec\nabla^2 f(r)\right)+\dots\\&=-\frac{3Gm^2}{8\pi r^4}+\dots,\\T^{\rm grav}_{ij}(r)&=\frac{1}{8\pi G}\Big(-\frac{1}{2}\nabla_i f(r)\nabla_j f(r)+\frac{3}{4}\delta_{ij}\vec\nabla f(r)\cdot\vec\nabla f(r)\\&\quad-f(r)\nabla_i\nabla_j f(r)+\delta_{ij}f(r)\vec\nabla^2 f(r)\Big)+\dots\\&=-\frac{7Gm^2}{4\pi r^4}\left(\frac{r_ir_j}{r^2}-\frac{1}{2}\delta_{ij}\right)+\dots,\end{aligned}\tag{23}$$

where the ellipses indicate contributions localized about the origin. Obviously Eqs. (21) and (23) are identical, demonstrating the correspondence of the nonanalytic $\sqrt{-q^2}$ terms and the classical field energy, just as found in I for the electromagnetic case.

The remaining corrections to $T_{\mu\nu}$ contain an explicit factor of $\hbar$ and are thus intrinsically quantum mechanical in nature. The "physics" behind these modifications can be understood in terms of the position uncertainty associated with quantum mechanics, which implies the replacement of the distance r in the classical expression by the value $\sim r+\hbar/m$. Since for macroscopic distances $\hbar/m<<r$, expansion of the classical result in powers of $1/r$ leads qualitatively to the quantum modifications found in our loop calculation. We emphasize that both Eq. (21),(23) are long range effects which arise only because the graviton couples to a *massless* virtual particle—in this case the self-interaction. The explicit factor of $\hbar$ in the latter indicates clearly that these are quantum effects whose strength and form are necessitated by the quantum nature of the field theory.

V. CLASSICAL TERMS IN THE METRIC

Here we use this energy momentum tensor to calculate the associated metric. In I we were able to show that this proce-

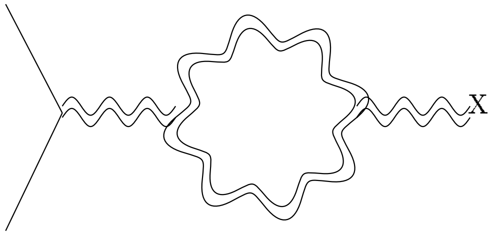

FIG. 2. Vacuum polarization modification of the energy-momentum tensor.

dure reproduced well-known results for classical metrics. We will demonstrate the same feature for the gravitational case. In this section we treat only the classical $\sqrt{-q^2}$ terms, which we denote by $\sqrt{q}$ superscripts to the form factors. The method here is made somewhat more complex by the necessity of dealing with the nonlinearity of the Einstein equation. Here we must consistently work to second order in G and to this order there is a nonlinear modification of the equations of motion relating the energy momentum tensor and the metric. This is worked out in Appendix A—Eq. (A16)—the result has the form to second order in G

$$\Box h^{(2)}_{\mu\nu} = -16\pi G\left(T^{\rm grav}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T^{\rm grav}\right) - \partial_\mu[f(r)\partial_\nu f(r)] - \partial_\nu[f(r)\partial_\mu f(r)]. \quad (24)$$

Noting that

$$\nabla_i[f(r)\nabla_j f(r)] + \nabla_j[f(r)\nabla_i f(r)] = 8G^2m^2\left(4\frac{r_ir_j}{r^6} - \frac{\delta_{ij}}{r^4}\right) = 4G^2m^2\nabla^2\left(\frac{\delta_{ij}}{r^2} - 2\frac{r_ir_j}{r^4}\right) \quad (25)$$

we find then that

$$h^{(2)}_{00}(r) = -16\pi G\int\frac{d^3q}{(2\pi)^3}e^{i\vec{q}\cdot\vec{r}}\frac{1}{\vec{q}^2}\left(\frac{m}{2}F_1^{\sqrt{q}}(-\vec{q}^2) - \frac{\vec{q}^2}{4m}F_2^{\sqrt{q}}(-\vec{q}^2)\right)$$
$$= -16\pi G\int\frac{d^3q}{(2\pi)^3}e^{i\vec{q}\cdot\vec{r}}\left(-\frac{G\pi m^2}{32|\vec{q}|} - \frac{7G\pi m^2}{32|\vec{q}|}\right) = \frac{2G^2m^2}{r^2},$$

$$h^{(2)}_{0i}(r) = 0,$$

$$h^{(2)}_{ij}(r) = -16\pi G\int\frac{d^3q}{(2\pi)^3}e^{i\vec{q}\cdot\vec{r}}\frac{1}{\vec{q}^2}\left[\frac{m}{2}F_1^{\sqrt{q}}(-\vec{q}^2)\delta_{ij} + \frac{1}{2m}\left(q_iq_j + \frac{1}{2}\delta_{ij}\vec{q}^2\right)F_2^{\sqrt{q}}(-\vec{q}^2)\right] + 4G^2m^2\left(\frac{\delta_{ij}}{r^2} - 2\frac{r_ir_j}{r^4}\right)$$
$$= -16\pi G\int\frac{d^3q}{(2\pi)^3}e^{i\vec{q}\cdot\vec{r}} \times\frac{1}{\vec{q}^2}\left[\delta_{ij}\left(-\frac{G\pi m^2|\vec{q}|}{32} + \frac{7G\pi m^2|\vec{q}|}{32}\right) + \frac{7G\pi m^2}{16}\frac{q_iq_j}{|\vec{q}|}\right] + 4G^2m^2\left(\frac{\delta_{ij}}{r^2} - 2\frac{r_ir_j}{r^4}\right)$$
$$= -\frac{G^2m^2}{r^2}\left(\delta_{ij} + \frac{r_ir_j}{r^2}\right). \quad (26)$$

Comparing with the Schwarzschild solution in harmonic coordinates— Eq. (1)—we find complete agreement.

VI. ADDITIONAL QUANTUM CORRECTIONS TO THE METRIC

Having identified the classical corrections, we could proceed in a similar fashion to calculate the quantum corrections using the $q^2\ln q^2$ nonanalytic terms. However there is one additional feature which needs to be included. There is a quantum modification of the equations of motion, which amounts to the addition of the vacuum polarization diagram of Fig. 2. In order to see that this is required, let us look at the quantum corrected effective action, which also has nonlocal long distance modifications. At one loop one finds the effective action

$$Z[h] = -\int d^4x d^4y \frac{1}{2}[h_{\mu\nu}(x)\Delta^{\mu\nu,\alpha\beta}(x-y)h_{\alpha\beta}(y) + O(h^3)] + Z_{\rm matter}[h,\phi]. \quad (27)$$

Here the renormalized action $\Delta^{\mu\nu,\alpha\beta}(x-y)$ contains

$$\Delta^{\mu\nu,\alpha\beta}(x-y) = \delta^4(x-y)D_2^{\mu\nu,\alpha\beta} + \hat{\Pi}^{\mu\nu,\alpha\beta}(x-y) + O(\partial^4), \quad (28)$$

where $D_2^{\mu\nu,\alpha\beta}$ is the differential operator following from the Einstein action and $\hat{\Pi}^{\mu\nu,\alpha\beta}(x-y)$ is the vacuum polarization function after renormalization, see Fig. 1. Following the steps in Appendix A we find that the vacuum polarization induces a change in the equations of motion

$$\Box h_{\mu\nu}(x)+P_{\mu\nu,\alpha\beta}\int d^4y\hat{\Pi}^{\alpha\beta,\gamma\delta}(x-y)h_{\gamma\delta}(y)$$
$$=-16\pi G\left(T^{\rm grav}_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}T^{\rm grav}\right)$$
$$-\partial_\mu[f(r)\partial_\nu f(r)]-\partial_\nu[f(r)\partial_\mu f(r)], \qquad (29)$$

where the projection operator $P_{\mu\nu,\alpha\beta}$ is given by

$$P_{\mu\nu,\alpha\beta}=I_{\mu\nu,\alpha\beta}-\frac{1}{2}\eta_{\mu\nu}\eta_{\alpha\beta},$$

$$I_{\mu\nu,\alpha\beta}=\frac{1}{2}(\eta_{\mu\alpha}\eta_{\nu\beta}+\eta_{\nu\alpha}\eta_{\mu\beta}). \qquad (30)$$

Equation (29) can be written, in harmonic gauge, as

$$\Box h_{\mu\nu}=-16\pi G\left(T^{\rm grav}_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}T^{\rm grav}\right)-\partial_\mu[f(r)\partial_\nu f(r)]$$
$$-\partial_\nu[f(r)\partial_\mu f(r)]$$
$$+16\pi G\int d^4yd^4zP_{\mu\nu,\alpha\beta}\hat{\Pi}^{\alpha\beta,\gamma\delta}(x-y)D(y-z)$$
$$\times\left(T^{\rm matt}_{\gamma\delta}(z)-\frac{1}{2}\eta_{\gamma\delta}T^{\rm matt}(z)\right), \qquad (31)$$

where the last term is just the vacuum polarization graph of Fig. 2.

The vacuum polarization has been calculated by ’tHooft and Veltman [17], and in momentum space it contains a factor of $q^4\log(-q^2)$ which is the source of the nonlocality. The specific form is

$$\hat{\Pi}_{\alpha\beta,\gamma\delta}=-\frac{2G}{\pi}\log(-q^2)\left[\frac{21}{120}q^4I_{\alpha\beta,\gamma\delta}+\frac{23}{120}q^4\eta_{\alpha\beta}\eta_{\gamma\delta}\right.$$
$$-\frac{23}{120}q^2(\eta_{\alpha\beta}q_\gamma q_\delta+\eta_{\gamma\delta}q_\alpha q_\beta)$$
$$-\frac{21}{240}q^2(q_\alpha q_\delta\eta_{\beta\gamma}+q_\beta q_\delta\eta_{\alpha\gamma}$$
$$\left.+q_\alpha q_\gamma\eta_{\beta\delta}+q_\beta q_\gamma\eta_{\alpha\delta})+\frac{11}{30}q_\alpha q_\beta q_\gamma q_\delta\right]. \qquad (32)$$

When we employ this form along with the graviton propagator, we find for that the vacuum polarization contributes a shift in the metric

$$\delta h^{(2)\text{vac pol}}_{\mu\nu}(x)=32G^2\int\frac{d^3q}{(2\pi)^3}e^{i\vec{q}\cdot\vec{r}}\log(\vec{q}^2)\left[\frac{21}{120}T^{\rm matt}_{\mu\nu}(q)\right.$$
$$\left.+\left(\frac{1}{240}\eta_{\mu\nu}-\frac{11}{60}\frac{q^\mu q^\nu}{\vec{q}^2}\right)T^{\rm matt}(q)\right]. \qquad (33)$$

In terms of components, we find,

$$\delta h^{(2)\text{vac pol}}_{00}=-\frac{43G^2m\hbar}{15\pi r^3},$$

$$\delta h^{(2)\text{vac pol}}_{ij}=\frac{G^2m\hbar}{15\pi r^3}\left(\delta_{ij}+44\frac{r_ir_j}{r^2}\right). \qquad (34)$$

The remaining corrections come from the logarithms in the vertex correction. Using the energy momentum tensor shown above plus the integrals listed Appendix A we find

$$\delta h^{(2)\text{vertex}}_{00}(r)=-16\pi G\int\frac{d^3q}{(2\pi)^3}e^{i\vec{q}\cdot\vec{r}}\frac{1}{\vec{q}^2}\left(\frac{m}{2}F_1(-\vec{q}^2)\right.$$
$$\left.-\frac{\vec{q}^2}{4m}F_2(-\vec{q}^2)\right)$$
$$=-16\pi G\int\frac{d^3q}{(2\pi)^3}e^{i\vec{q}\cdot\vec{r}}\frac{Gm}{\pi}\left(\frac{3}{8}+\frac{1}{2}\right)\log\vec{q}^2$$
$$=\frac{7G^2m\hbar}{\pi r^3},$$

$$\delta h^{(2)\text{vertex}}_{0i}(r)=0,$$

$$\delta h^{(2)\text{vertex}}_{ij}(r)=-16\pi G\int\frac{d^3q}{(2\pi)^3}e^{i\vec{q}\cdot\vec{r}}\frac{1}{\vec{q}^2}\log\vec{q}^2$$
$$\times\left[\frac{m}{2}F_1(-\vec{q}^2)\right.$$
$$\left.\times\delta_{ij}+\frac{1}{2m}\left(q_iq_j+\frac{1}{2}\delta_{ij}\vec{q}^2\right)F_2(-\vec{q}^2)\right]$$
$$=-16\pi G\int\frac{d^3q}{(2\pi)^3}e^{i\vec{q}\cdot\vec{r}}\frac{Gm}{\pi}\left[\delta_{ij}\left(\frac{3}{8}-\frac{1}{2}\right)\right.$$
$$\left.-\frac{q_iq_j}{\vec{q}^2}\right]$$
$$=-\frac{G^2m\hbar}{\pi r^3}\left(\delta_{ij}+8\frac{r_ir_j}{r^2}\right), \qquad (35)$$

where we have shown only the effects of the quantum logarithms. Adding these corrections to the vacuum polarization and classical terms reproduces the metric displayed in Eq. (3).[4]

[4]Logarithmic corrections in many gauge theories can be analyzed using the renormalization group and the results are described by anomalous dimensions. In general relativity the situation is different because of the dimensionful nature of the gravitational coupling. The one loop correction is of order $Gq^2\ln q^2$, and higher orders will involve higher powers of q^2 and logarithms—hence higher orders will be power suppressed at low energy.

VII. FERMIONS AND SPIN

Having understood the spinless sector, we now turn our attention to the case of a particle with spin, in particular spin one-half. The general form for the spin 1/2 matrix element of the energy-momentum tensor can be written as [18]

$$\langle p_2|T_{\mu\nu}|p_1\rangle=\overline{u}(p_2)\bigg[F_1(q^2)P_\mu P_\nu\frac{1}{m}-F_2(q^2)\times\left(\frac{i}{4m}\sigma_{\mu\lambda}q^\lambda P_\nu+\frac{i}{4m}\sigma_{\nu\lambda}q^\lambda P_\mu\right)+F_3(q^2)(q_\mu q_\nu-\eta_{\mu\nu}q^2)\frac{1}{m}\bigg]u(p_1). \quad (36)$$

The normalization condition $F_1(q^2=0)=1$ corresponds to energy-momentum conservation as found before, while the second normalization condition $F_2(q^2=0)=1$ is required by the constraint of angular momentum conservation. This can be seen by defining

$$\hat{M}_{12}=\int d^3x(T_{01}x_2-T_{02}x_1)$$

$$\overset{q\to 0}{\to}-i(\nabla_q)_2\int d^3xe^{i\vec{q}\cdot\vec{r}}T_{01}(\vec{r})+i(\nabla_q)_1\int d^3xe^{i\vec{q}\cdot\vec{r}}T_{02}(\vec{r}), \quad (37)$$

whereby

$$\lim_{q\to 0}\langle p_2|\hat{M}_{12}|p_1\rangle=\frac{1}{2}=\frac{1}{2}\overline{u}_\uparrow(p)\sigma_3u_\uparrow(p)F_2(q^2), \quad (38)$$

i.e., $F_2(q^2=0)=1$, as found explicitly in our calculation.

The Feynman diagrams for fermions are shown in Fig. 2. We find, as shown in the Appendix A

$$F_1(q^2)=1+\frac{Gq^2}{\pi}\left(\frac{\pi^2m}{16\sqrt{-q^2}}-\frac{3}{4}\log\frac{-q^2}{m^2}\right)+\dots,$$

$$F_2(q^2)=1+\frac{Gq^2}{\pi}\left(\frac{\pi^2m}{4\sqrt{-q^2}}+\frac{1}{4}\log\frac{-q^2}{m^2}\right)+\dots,$$

$$F_3(q^2)=\frac{Gm^2}{\pi}\left(\frac{7\pi^2m}{16\sqrt{-q^2}}-\log\frac{-q^2}{m^2}\right)+\dots. \quad (39)$$

We convert this into an energy-momentum tensor. Writing $\vec{S}=\vec{\sigma}/2$ for the spin, the general relation to the fermion form factors is

$$T_{00}(\vec{r})=\int\frac{d^3q}{(2\pi)^3}e^{i\vec{q}\cdot\vec{r}}\left(mF_1(-\vec{q}^2)+\frac{\vec{q}^2}{m}F_3(-\vec{q}^2)\right),$$

$$T_{0i}(\vec{r})=i\int\frac{d^3q}{(2\pi)^3}e^{i\vec{q}\cdot\vec{r}}\frac{1}{2}(\vec{S}\times\vec{q})_iF_2(-\vec{q}^2),$$

$$T_{ij}(\vec{r})=\frac{1}{m}\int\frac{d^3q}{(2\pi)^3}e^{i\vec{q}\cdot\vec{r}}(q_iq_j-\delta_{ij}\vec{q}^2)F_3(-\vec{q}^2). \quad (40)$$

Using our results (39) for the form factors this becomes

$$T_{00}(\vec{r})=\int\frac{d^3q}{(2\pi)^3}e^{i\vec{q}\cdot\vec{r}}\left(m+\frac{3Gm^2\pi}{8}\times|\vec{q}|-\frac{Gm}{4\pi}\vec{q}^2\log\vec{q}^2\right)+\dots$$
$$=m\delta^3(\vec{r})-\frac{3Gm^2}{8\pi r^4}-\frac{3Gm\hbar}{4\pi r^5}+\dots,$$

$$T_{0i}(\vec{r})=\frac{i}{2}\int\frac{d^3q}{(2\pi)^3}e^{i\vec{q}\cdot\vec{r}}(\vec{S}\times\vec{q})_i\times\left(1-\frac{Gm\pi}{4}|\vec{q}|-\frac{G}{4\pi}\vec{q}^2\log\vec{q}^2\right)+\dots$$
$$=\frac{1}{2}(\vec{S}\times\vec{\nabla})_i\delta^3(\vec{r})+\left(-\frac{Gm}{2\pi r^6}+\frac{15G\hbar}{4\pi^2r^7}\right)(\vec{S}\times\vec{r})_i+\dots,$$

$$T_{ij}(\vec{r})=\int\frac{d^3q}{(2\pi)^3}e^{i\vec{q}\cdot\vec{r}}\left(\frac{7Gm^2\pi}{16|\vec{q}|}-\frac{Gm}{\pi}\log\vec{q}^2\right)\times(q_iq_j-\delta_{ij}\vec{q}^2)+\dots$$
$$=-\frac{7Gm^2}{4\pi r^4}\left(\frac{r_ir_j}{r^2}-\frac{1}{2}\delta_{ij}\right)+\frac{2Gm\hbar}{\pi^2r^5}\delta_{ij}+\dots. \quad (41)$$

We can again check the classical piece of this result against our expectations of the energy-momentum carried by the gravitational field. The spin-independent pieces are identical to that found for the spinless case. In the case of the off-diagonal component of the energy-momentum tensor, Eq. (A16) yields

$$T_{0i}^{\rm grav}=\frac{1}{8\pi G}\left(-\frac{1}{2}h_{0j}^{(1)}\nabla_i\nabla_jh_{00}^{(1)}+\frac{1}{2}\nabla_jh_{ki}^{(1)}\nabla_kh_{j0}^{(1)}\right)$$
$$=\frac{1}{16\pi Gm}\{-[(\vec{S}\times\vec{\nabla})_jf(r)]\nabla_i\nabla_jf(r)+[\nabla_jf(r)]\nabla_i(\vec{S}\times\vec{\nabla})_jf(r)\}$$
$$=-\frac{Gm}{2\pi r^6}(\vec{S}\times\vec{r})_i \quad (42)$$

in agreement with the result obtained from Eq. (41).

Now let us calculate the metric components. In this case we find the relation of the metric to the fermion form factors is given by

$$h_{00}(\vec{r}) = -16\pi G \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \frac{1}{\vec{q}^2}\left(\frac{m}{2}F_1(-\vec{q}^2) - \frac{\vec{q}^2}{2m}F_3(-\vec{q}^2)\right),$$

$$h_{0i}(\vec{r}) = -16\pi G \frac{i}{2}\int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \frac{1}{\vec{q}^2} F_2(-\vec{q}^2)(\vec{S}\times\vec{q})_i,$$

$$h_{ij}(\vec{r}) = -16\pi G \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \frac{1}{\vec{q}^2}\left[\frac{m}{2}F_1(-\vec{q}^2)\delta_{ij} + \frac{1}{m}\left(q_iq_j + \frac{1}{2}\delta_{ij}\vec{q}^2\right)F_3(-\vec{q}^2)\right] + \frac{4G^2m^2}{r^2}\left(\delta_{ij} - 2\frac{r_ir_j}{r^2}\right). \tag{43}$$

With the form factors calculated above this yields

$$\begin{aligned} h_{00}^{\text{vertex}}(\vec{r}) &= -16\pi G \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \frac{1}{\vec{q}^2}\left(\frac{m}{2} - \frac{Gm^2\pi|\vec{q}|}{4} + \frac{7Gm\vec{q}^2}{8}\log\vec{q}^2\right) + \dots \\ &= -\frac{2Gm}{r} + \frac{2G^2m^2}{r^2} + \frac{7G^2m\hbar}{\pi r^3} + \dots, \end{aligned}$$

$$\begin{aligned} h_{0i}^{\text{vertex}}(\vec{r}) &= -16\pi G \frac{i}{2}\int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \frac{1}{\vec{q}^2}\left(1 - \frac{Gm\pi|\vec{q}|}{4} - \frac{G\vec{q}^2}{4\pi}\log\vec{q}^2\right)(\vec{S}\times\vec{q})_i + \dots \\ &= \left(\frac{2G}{r^3} - \frac{2G^2m}{r^4} + \frac{3G^2\hbar}{\pi r^5}\right)(\vec{S}\times\vec{r})_i + \dots, \end{aligned}$$

$$\begin{aligned} h_{ij}^{\text{vertex}}(\vec{r}) &= -16\pi G \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \frac{1}{\vec{q}^2}\left(\frac{m}{2}\delta_{ij} - \left(\frac{Gm^2\pi|\vec{q}|}{32} - \frac{3Gm\vec{q}^2}{8\pi}\log\vec{q}^2\right)\delta_{ij} + \left(q_iq_j + \frac{1}{2}\vec{q}^2\delta_{ij}\right)\left(\frac{7Gm^2\pi}{16|\vec{q}|} - \frac{Gm}{\pi}\log\frac{\vec{q}^2}{m^2}\right)\right) \\ &\quad + \frac{4G^2m^2}{r^2}\left(\delta_{ij} - 2\frac{r_ir_j}{r^2}\right) + \dots \\ &= -\delta_{ij}\frac{2Gm}{r} - \frac{G^2m^2}{r^2}\left(\delta_{ij} + \frac{r_ir_j}{r^2}\right) - \frac{G^2m\hbar}{\pi r^3}\left(\delta_{ij} + 8\frac{r_ir_j}{r^2}\right) + \dots. \end{aligned} \tag{44}$$

We observe that the diagonal components of the vertex correction are identical to those found for the spinless case, as expected, and that there exists a nonvanishing nondiagonal term associated with the spin. The diagonal components of the vacuum polarization are also clearly identical to the bosonic case, but there is a new off-diagonal component associated with the spin

$$\begin{aligned} h_{0i}^{(2)\text{vac pol}} &= 32G^2 \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \log\vec{q}^2 \ \frac{21}{240} iF_2(q^2)(\vec{S}\times\vec{q})_i \\ &= \frac{21G^2\hbar}{5\pi r^5}(\vec{S}\times\vec{r})_i \end{aligned} \tag{45}$$

Again these are added together in order to yield the result quoted in the introduction. We have thus reproduced the Kerr metric—Eq. (2)—in harmonic gauge together with the associated quantum corrections.

VIII. DISCUSSION OF THE METRIC AND GRAVITATIONAL POTENTIAL

The quantum correction to the Schwarzschild metric has previously been discussed by Duff [19]. While that discussion properly identifies $\ln -q^2$ terms as the source of the quantum effects, the calculation is incomplete because it only includes the effect of the vacuum polarization diagram. This can be traced to the assumption of a ‘‘classical source,’’ which meant that the vertex diagrams were not included. However, any source has a gravitational field surrounding it and that field has a quantum component. The effective field theory treatment demonstrates the existence of quantum corrections due to the vertex diagrams—they are of the same order as those due to vacuum polarization and they must be included. In this sense, there is no fully classical source in gravity. If one takes the mass of a particle to infinity, the gravitational coupling also grows and the quantum effects do not decouple. Rather for a heavy mass it is long distances which determines the classical limit, as the quantum effects become smaller than the classical effects in the limit of large distance. However, the vertex corrections are as important as the vacuum polarization for the quantum correction to the metric and they must be included.

The bosonic diagrams that we have considered have also been parts of the calculations of the quantum corrections to the Newtonian potential. We have shown them in detail because there has been numerical disagreements in the literature. We believe that our results are the correct ones. There appears to have been a numerical error in the original result of Ref. [1]. We have identified the location of that error and carefully reconsidered that value. The identity of Eq. (B8) makes it easy to repeat this part of the calculation. The authors of Ref. [4] also appear to be in error. Their calculation would lead to the wrong classical terms, which certainly indicates an error and implies that the quoted quantum portion is also not trustworthy. In addition, our fermionic calculation serves as an independent confirmation of the bosonic result, as the calculational details are quite different even though the result is the same.

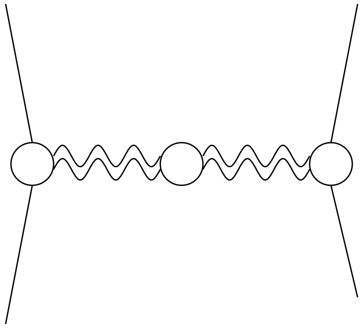

FIG. 3. Diagrams contributing to the one-particle-reducible potential.

If we use our present result to define the one-particle-reducible potential, including the diagrams in Fig. 3, we obtain the result

$$V(r)=-\frac{Gm_1m_2}{r}\left(1-\frac{G(m_1+m_2)}{r}-\frac{167}{30\pi}\frac{G\hbar}{r^2}+\ldots\right). \tag{46}$$

This potential is not itself the scattering potential. In a separate work [9] we calculate the other diagrams which are required to fully define the scattering amplitude. These include box diagrams and several triangle diagrams. However, the 1PR potential represents the sets of diagrams that are used to define the running charge in QED and QCD and these diagrams can be used for a similar definition here. We propose that the quantum correction from these diagrams be used to define a running gravitational coupling appropriate for harmonic gauge. This results in

$$G(r)=G\left(1-\frac{167}{30\pi}\frac{G\hbar}{r^2}\right). \tag{47}$$

The fact that this definition is independent of the masses of the objects involved suggests that it has a universal character appropriate for the running charge. Our work shows that this form is independent of spin. Note also that the charge becomes weaker at shorter distances. This is in accord with a heuristic expectation that the gravitational interaction at large distances feels the total mass of the object, but when probed at small distances gravity will see a smaller effect because the quantum fluctuations spread out the energy contained in the fields. That the running gravitational coupling varies with a power of r rather than the logarithm is required by the dimensional gravitational coupling constant.

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APPENDIX A: THE EQUATIONS OF MOTION

The method for describing the quantum gravitational field has been understood since the classic works of 't Hooft and Veltman [17]. In particular, the gauge fixing procedure and the introduction of ghost fields are fully described there. We use their calculation of the vacuum polarization diagram, where the gauge treatment is particularly crucial. In this appendix we do not repeat the standard features that can be found in their work. However, we do make use of a somewhat specific treatment of the equations of motion, so that we describe in this appendix the features that are needed for our calculation. Further relevant details concerning the coupling of the gravitons to the quantum fields are presented in Appendix B.

The full gravitational action is given by

$$S_g=\int d^4x\sqrt{-g}\left(\frac{1}{16\pi G}R+\mathcal{L}_m\right), \tag{A1}$$

where $\mathcal{L}_m$ is the Lagrange density for matter. Variation of Eq. (A1) yields the Einstein equation

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=-8\pi GT_{\mu\nu}, \tag{A2}$$

where the energy-momentum tensor $T_{\mu\nu}$ is given by

$$T_{\mu\nu}=\frac{2}{\sqrt{-g}}\frac{\partial}{\partial g^{\mu\nu}}(\sqrt{-g}\mathcal{L}_m). \tag{A3}$$

We work in the weak field limit, with an expansion in powers of the gravitational coupling G

$$\begin{aligned} g_{\mu\nu}&\equiv\eta_{\mu\nu}+h^{(1)}_{\mu\nu}+h^{(2)}_{\mu\nu}+\ldots,\\ g^{\mu\nu}&=\eta^{\mu\nu}-h^{(1)\mu\nu}-h^{(2)\mu\nu}+h^{(1)\mu\lambda}h^{(1)\nu}_{\lambda}\\ &\quad+\ldots, \end{aligned} \tag{A4}$$

where here the superscript indicates the number of powers of G which appear and indices are understood to be raised or lowered by $\eta_{\mu\nu}$. We shall also need the determinant which is given by

$$\sqrt{-g}=\exp\frac{1}{2}\operatorname{tr}\log g=1+\frac{1}{2}(h^{(1)}+h^{(2)})-\frac{1}{4}h^{(1)}_{\alpha\beta}h^{(1)\alpha\beta}+\frac{1}{8}h^{(1)2}+\ldots. \tag{A5}$$

The corresponding curvatures are given by

$$R^{(1)}_{\mu\nu}=\frac{1}{2}[\partial_\mu\partial_\nu h^{(1)}+\partial_\lambda\partial^\lambda h^{(1)}_{\mu\nu}-\partial_\mu\partial_\lambda h^{(1)\lambda}{}_\nu-\partial_\nu\partial_\lambda h^{(1)\lambda}{}_\mu],$$

$$R^{(1)}=\Box h^{(1)}-\partial_\mu\partial_\nu h^{(1)\mu\nu},$$

$$R^{(2)}_{\mu\nu}=\frac{1}{2}[\partial_\mu\partial_\nu h^{(2)}+\partial_\lambda\partial^\lambda h^{(2)}_{\mu\nu}-\partial_\mu\partial^\lambda h^{(2)}_{\lambda\nu}-\partial_\nu\partial^\lambda h^{(2)}_{\lambda\mu}]$$
$$-\frac{1}{4}\partial_\mu h^{(1)}_{\alpha\beta}\partial_\nu h^{(1)\alpha\beta}-\frac{1}{2}\partial_\alpha h^{(1)}_{\mu\lambda}\partial^\alpha h^{(1)}_{\lambda\nu}$$
$$+\frac{1}{2}\partial_\alpha h^{(1)}_{\mu\lambda}\partial^\lambda h^{(1)\alpha}_{\nu}+\frac{1}{2}h^{(1)\lambda\alpha}[\partial_\lambda\partial_\nu h^{(1)}_{\mu\alpha}+\partial_\lambda\partial_\mu h^{(1)}_{\nu\alpha}$$
$$-\partial_\mu\partial_\nu h^{(1)}_{\lambda\alpha}-\partial_\lambda\partial_\alpha h^{(1)}_{\mu\nu}]+\frac{1}{2}\left(\partial_\beta h^{(1)\beta\alpha}-\frac{1}{2}\partial^\alpha h^{(1)}\right)$$
$$\times(\partial_\mu h^{(1)}_{\nu\alpha}+\partial_\nu h^{(1)}_{\mu\alpha}-\partial_\alpha h^{(1)}_{\mu\nu}),$$

$$R^{(2)}=\Box h^{(2)}-\partial^\mu\partial^\nu h^{(2)}_{\mu\nu}-\frac{3}{4}\partial_\mu h^{(1)}_{\alpha\beta}\partial^\mu h^{(1)\alpha\beta}$$
$$+\frac{1}{2}\partial_\alpha h^{(1)}_{\mu\lambda}\partial^\lambda h^{(1)\mu\alpha}+\frac{1}{2}h^{(1)\lambda\alpha}(2\partial_\lambda\partial^\beta h^{(1)}_{\alpha\beta}-\Box h^{(1)}_{\lambda\alpha}$$
$$-\partial_\lambda\partial_\alpha h^{(1)})+\left(\partial^\beta h^{(1)\alpha}_{\beta}-\frac{1}{2}\partial^\alpha h^{(1)}\right)$$
$$\times\left(\partial^\sigma h^{(1)}_{\sigma\alpha}-\frac{1}{2}\partial_\alpha h^{(1)}\right). \qquad \text{(A6)}$$

In order to define the propagator, we must make a gauge choice and we shall work in harmonic gauge—$g^{\mu\nu}\Gamma^\lambda_{\mu\nu}=0$—which reads, to second order in the field expansion

$$0=\partial^\beta h^{(1)}_{\beta\alpha}-\frac{1}{2}\partial_\alpha h^{(1)}$$
$$=\left(\partial^\beta h^{(2)}_{\beta\alpha}-\frac{1}{2}\partial_\alpha h^{(2)}-\frac{1}{2}h^{(1)\lambda\sigma}(\partial_\lambda h^{(1)}_{\sigma\alpha}+\partial_\sigma h^{(1)}_{\lambda\alpha}-\partial_\alpha h^{(1)}_{\sigma\lambda})\right) \qquad \text{(A7)}$$

Using these results, the Einstein equation reads, in lowest order,

$$\Box h^{(1)}_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}\Box h^{(1)}-\partial_\mu\left(\partial^\beta h^{(1)}_{\beta\nu}-\frac{1}{2}\partial_\nu h^{(1)}\right)$$
$$-\partial_\nu\left(\partial^\beta h^{(1)}_{\beta\mu}-\frac{1}{2}\partial_\mu h^{(1)}\right)$$
$$=-16\pi G T^{\rm matt}_{\mu\nu} \qquad \text{(A8)}$$

which, using the gauge condition Eq. (A7), can be written as

$$\Box\left(h^{(1)}_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}h^{(1)}\right)=-16\pi G T^{\rm matt}_{\mu\nu} \qquad \text{(A9)}$$

or in the equivalent form

$$\Box h^{(1)}_{\mu\nu}=-16\pi G\left(T^{\rm matt}_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}T^{\rm matt}\right). \qquad \text{(A10)}$$

As shown in Sec. II, this equation has the familiar solution for a static point mass

$$h^{(1)}_{\mu\nu}=\delta_{\mu\nu}f(r), \qquad \text{(A11)}$$

where

$$f(r)=-\frac{2Gm}{r}.$$

In second order the validity of the Einstein equation requires that

$$R^{(2)}_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}R^{(2)}-\frac{1}{2}h^{(1)}_{\mu\nu}R^{(1)}=0. \qquad \text{(A12)}$$

It is useful to write this equation in the form

$$\Box h^{(2)}_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}\Box h^{(2)}-\partial_\mu\left(\partial^\beta h^{(2)}_{\beta\nu}-\frac{1}{2}\partial_\nu h^{(2)}\right)$$
$$-\partial_\nu\left(\partial^\beta h^{(2)}_{\beta\mu}-\frac{1}{2}\partial_\mu h^{(2)}\right)$$
$$\equiv-16\pi G T^{\rm grav}_{\mu\nu}, \qquad \text{(A13)}$$

where $T^{\rm grav}_{\mu\nu}$ can be identified as the energy-momentum carried by the gravitational field and can be read off as [16]

$$8\pi G T^{\rm grav}_{\mu\nu}=-\frac{1}{2}h^{(1)\lambda\kappa}[\partial_\mu\partial_\nu h^{(1)}_{\lambda\kappa}+\partial_\lambda\partial_\kappa h^{(1)}_{\mu\nu}-\partial_\kappa(\partial_\nu h^{(1)}_{\mu\lambda}$$
$$+\partial_\mu h^{(1)}_{\nu\lambda})]-\frac{1}{2}\partial_\lambda h^{(1)}_{\sigma\nu}\partial^\lambda h^{(1)\sigma}_{\mu}$$
$$+\frac{1}{2}\partial_\lambda h^{(1)}_{\sigma\nu}\partial^\sigma h^{(1)\lambda}_{\mu}-\frac{1}{4}\partial_\nu h^{(1)}_{\sigma\lambda}\partial_\mu h^{(1)\sigma\lambda}$$
$$-\frac{1}{4}\eta_{\mu\nu}\left(\partial_\lambda h^{(1)}_{\sigma\chi}\partial^\sigma h^{(1)\lambda\chi}-\frac{3}{2}\partial_\lambda h^{(1)}_{\sigma\chi}\partial^\lambda h^{(1)\sigma\chi}\right)$$
$$-\frac{1}{4}h^{(1)}_{\mu\nu}\Box h^{(1)}+\frac{1}{2}\eta_{\mu\nu}h^{(1)\alpha\beta}\Box h^{(1)}_{\alpha\beta}. \qquad \text{(A14)}$$

Using the gauge condition Eq. (A7), Eq. (A13) becomes

$$\Box\left(h^{(2)}_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}h^{(2)}\right)=-16\pi G T^{\rm grav}_{\mu\nu}$$
$$+\partial_\mu\left[h^{(1)\lambda\sigma}\left(\partial_\lambda h^{(1)}_{\sigma\nu}-\frac{1}{2}\partial_\nu h^{(1)}_{\lambda\sigma}\right)\right]$$
$$+\partial_\nu\left[h^{(1)\lambda\sigma}\left(\partial_\lambda h^{(1)}_{\sigma\mu}-\frac{1}{2}\partial_\mu h^{(1)}_{\lambda\sigma}\right)\right]$$
$$-\eta_{\mu\nu}\partial^\alpha\left[h^{(1)\lambda\sigma}\left(\partial_\lambda h^{(1)}_{\alpha\sigma}-\frac{1}{2}\partial_\alpha h^{(1)}_{\lambda\sigma}\right)\right] \qquad \text{(A15)}$$

and, using the lowest order solution Eq. (A11) we find the form

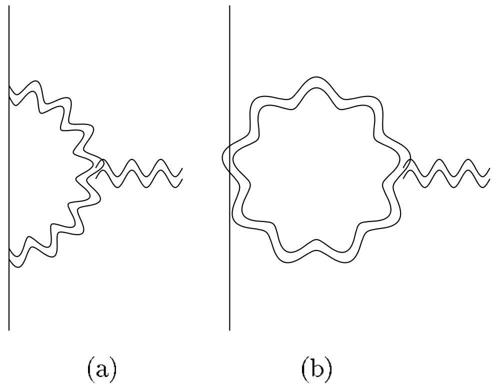

FIG. 4. Gravitational radiative correction diagrams leading to nonanalytic components of form factors.

$$\Box h^{(2)}_{\mu\nu} = -16\pi G\left(T^{\rm grav}_{\mu\nu} - \frac{1}{2}\,\eta_{\mu\nu}T^{\rm grav}\right) - \partial_\mu[f(r)\partial_\nu f(r)] - \partial_\nu[f(r)\partial_\mu f(r)]. \tag{A16}$$

For use in the spin 1/2 case we note that the corresponding off-diagonal equation reads

$$\Box h^{(2)}_{0i} = -16\pi G T^{\rm grav}_{0i} - \nabla_i(h^{(1)}_{0j}\nabla_j h^{(1)}_{00}) + \nabla_i(h^{(1)}_{jk}\nabla_j h^{(1)}_{0k}). \tag{A17}$$

However, using the lowest order solutions found above we easily verify that

$$\nabla_i(h^{(1)}_{0j}\nabla_j h^{(1)}_{00}) = \nabla_i(h^{(1)}_{jk}\nabla_j h^{(1)}_{0k}) = 0. \tag{A18}$$

Thus the off-diagonal Einstein equation in second order has the simple form

$$\Box h^{(2)}_{0i} = -16\pi G T^{\rm grav}_{0i} \tag{A19}$$

while the general form in second order is seen to be given by Eq. (A16).

APPENDIX B: DETAILS OF THE BOSONIC AND FERMIONIC VERTEX CORRECTIONS

1. Spin zero

Here we show the calculation of the nonanalytic terms in vertex correction, following the method of Ref. [1]. Such pieces arise from the diagrams in Fig. 4, wherein the external graviton couples to the massless graviton fields in the loop. We have found that a symmetric ordering of the momentum is useful, using the following integrals:

$$I = \int \frac{d^dk}{(2\pi)^d}\,\frac{1}{\left(k-\frac{q}{2}\right)^2\left(k+\frac{q}{2}\right)^2\left[\left(p-k+\frac{q}{2}\right)^2 - m^2\right]} = \frac{i}{32\pi^2 m^2}[-L-S] + \dots,$$

$$I_\mu = \int \frac{d^dk}{(2\pi)^d}\,\frac{k_\mu}{\left(k-\frac{q}{2}\right)^2\left(k+\frac{q}{2}\right)^2\left[\left(p-k+\frac{q}{2}\right)^2 - m^2\right]} = \frac{i}{32\pi^2 m^2}\left[P_\mu\left(1+\frac{q^2}{2m}\right)L + \frac{q^2}{4m^2}S\right] + \dots,$$

$$I_{\mu\nu} = \int \frac{d^dk}{(2\pi)^d}\,\frac{k_\mu k_\nu}{\left(k-\frac{q}{2}\right)^2\left(k+\frac{q}{2}\right)^2\left[\left(p-k+\frac{q}{2}\right)^2 - m^2\right]} = \frac{i}{32\pi^2 m^2}\left[-P_\mu P_\nu\frac{q^2}{2m^2}\left(L+\frac{1}{4}S\right) - (q_\mu q_\nu - \eta_{\mu\nu}q^2)\times\left(\frac{1}{4}L + \frac{1}{8}S\right)\right] + \dots,$$

$$I_{\mu\nu\alpha} = \int \frac{d^dk}{(2\pi)^d}\,\frac{k_\mu k_\nu k_\alpha}{\left(k-\frac{q}{2}\right)^2\left(k+\frac{q}{2}\right)^2\left(\left(p-k+\frac{q}{2}\right)^2 - m^2\right)} = \frac{i}{32\pi^2 m^2}\left[P_\mu P_\nu P_\alpha\left(\frac{q^2}{6m^2}\right)L + [(q_\mu q_\nu - \eta_{\mu\nu}q^2)P_\alpha + (q_\mu q_\alpha - \eta_{\mu\alpha}q^2)P_\nu + (q_\nu q_\alpha - \eta_{\nu\alpha}q^2)P_\mu]\frac{L}{12}\right] + \dots, \tag{B1}$$

where $S = \pi^2 m/\sqrt{-q^2}$, $L = \log(-q^2/m^2)$. From Fig. 4(a), we then have

$$A^{\mu\nu}_{(a)} = iP^{\alpha,\lambda\kappa}iP^{\gamma\delta,\rho\sigma}i \times \int \frac{d^4\ell}{(2\pi)^4}\,\frac{\tau_{\alpha\beta}(p,p'-\ell)\,\tau_{\gamma\delta}(p'-\ell,p')\,\tau^{\mu\nu}_{\rho\sigma,\lambda\kappa}(\ell,q)}{\ell^2(\ell-q)^2[(\ell-p')^2-m^2]} \tag{B2}$$

while from Fig. 4(b),

$$A^{\mu\nu}_{(b)} = \frac{i}{2}P^{\alpha\beta,\lambda\kappa}iP^{\gamma\delta,\rho\sigma}\tau_{\alpha\beta,\gamma\delta}(p,p')\int \frac{d^4\ell}{(2\pi)^4}\,\frac{\tau^{\mu\nu}_{\lambda\kappa,\rho\sigma}(\ell,q)}{\ell^2(\ell-q)^2}. \tag{B3}$$

Here the coupling to matter via one-graviton and two-graviton vertices can be found by expanding the spin zero matter Lagrangian

$$\sqrt{-g}\mathcal{L}_m = \sqrt{-g}\left(\frac{1}{2}D_\mu\phi g^{\mu\nu}D_\nu\phi - \frac{1}{2}m^2\phi^2\right) \tag{B4}$$

via

$$\sqrt{-g}\mathcal{L}^{(0)}_m = \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2),$$

$$\sqrt{-g}\mathcal{L}_m^{(1)} = -\frac{1}{2}h^{(1)\mu\nu}\bigg(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}(\partial_\alpha\phi\partial^\alpha\phi - m^2\phi^2)\bigg),$$

$$\sqrt{-g}\mathcal{L}_m^{(2)} = -\frac{1}{2}h^{(2)\mu\nu}\bigg(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}(\partial_\alpha\phi\partial^\alpha\phi - m^2\phi^2)\bigg) + \frac{1}{2}\bigg(h^{(1)\mu\lambda}h^{(1)\nu}{}_\lambda - \frac{1}{2}h^{(1)}h^{(1)\mu\nu}\bigg)\partial_\mu\phi\partial_\nu\phi - \frac{1}{8}\bigg(h^{(1)\alpha\beta}h^{(1)}_{\alpha\beta} - \frac{1}{2}h^{(1)2}\bigg)(\partial^\alpha\phi\partial_\alpha\phi - m^2\phi^2). \tag{B5}$$

The one- and two-graviton vertices are then, respectively,

$$\tau_{\alpha\beta}(p,p') = \frac{-i\kappa}{2}[p_\alpha p'_\beta + p'_\alpha p_\beta - \eta_{\alpha\beta}(p\cdot p' - m^2)],$$

$$\tau_{\alpha\beta,\gamma\delta}(p,p') = i\kappa^2\bigg[I_{\alpha\beta,\rho\xi}I^{\xi}_{\sigma,\gamma\delta}(p^\rho p'^\sigma + p'^\rho p^\sigma) - \frac{1}{2}(\eta_{\alpha\beta}I_{\rho\sigma,\gamma\delta} + \eta_{\gamma\delta}I_{\rho\sigma,\alpha\beta})p'^\rho p^\sigma - \frac{1}{2}\bigg(I_{\alpha\beta,\gamma\delta} - \frac{1}{2}\eta_{\alpha\beta}\eta_{\gamma\delta}\bigg)(p\cdot p' - m^2)\bigg], \tag{B6}$$

where we have defined $\kappa^2 = 32\pi G$. We also require the triple graviton vertex $\tau^{\mu\nu}_{\alpha\beta,\gamma\delta}(k,q)$ whose form is

$$\begin{aligned}\tau^{\mu\nu}_{\alpha\beta,\gamma\delta}(k,q) = \frac{i\kappa}{2}\bigg\{ & P_{\alpha\beta,\gamma\delta}\bigg[k^\mu k^\nu + (k-q)^\mu(k-q)^\nu + q^\mu q^\nu - \frac{3}{2}\eta^{\mu\nu}q^2\bigg] + 2q_\lambda q_\sigma[I^{\lambda\sigma,}{}_{\alpha\beta}I^{\mu\nu,}{}_{\gamma\delta} + I^{\lambda\sigma,}{}_{\gamma\delta}I^{\mu\nu,}{}_{\alpha\beta} - I^{\lambda\mu,}{}_{\alpha\beta}I^{\sigma\nu,}{}_{\gamma\delta}\\ & - I^{\sigma\nu,}{}_{\alpha\beta}I^{\lambda\mu,}{}_{\gamma\delta}] + [q_\lambda q^\mu(\eta_{\alpha\beta}I^{\lambda\nu,}{}_{\gamma\delta} + \eta_{\gamma\delta}I^{\lambda\nu,}{}_{\alpha\beta}) + q_\lambda q^\nu(\eta_{\alpha\beta}I^{\lambda\mu,}{}_{\gamma\delta} + \eta_{\gamma\delta}I^{\lambda\mu,}{}_{\alpha\beta}) - q^2(\eta_{\alpha\beta}I^{\mu\nu,}{}_{\gamma\delta}\\ & + \eta_{\gamma\delta}I^{\mu\nu,}{}_{\alpha\beta}) - \eta^{\mu\nu}q^\lambda q^\sigma(\eta_{\alpha\beta}I_{\gamma\delta,\lambda\sigma} + \eta_{\gamma\delta}I_{\alpha\beta,\lambda\sigma})] + [2q^\lambda(I^{\sigma\nu,}{}_{\alpha\beta}I_{\gamma\delta,\lambda\sigma}(k-q)^\mu + I^{\sigma\mu,}{}_{\alpha\beta}I_{\gamma\delta,\lambda\sigma}(k-q)^\nu\\ & - I^{\sigma\nu,}{}_{\gamma\delta}I_{\alpha\beta,\lambda\sigma}k^\mu - I^{\sigma\mu,}{}_{\gamma\delta}I_{\alpha\beta,\lambda\sigma}k^\nu) + q^2(I^{\sigma\mu,}{}_{\alpha\beta}I^{\nu}{}_{\gamma\delta,\sigma} + I^{\nu}{}_{\alpha\beta,\sigma}I^{\sigma\mu,}{}_{\gamma\delta}) + \eta^{\mu\nu}q^\lambda q_\sigma(I_{\alpha\beta,\lambda\rho}I^{\rho\sigma,}{}_{\gamma\delta}\\ & + I_{\gamma\delta,\lambda\rho}I^{\rho\sigma,}{}_{\alpha\beta})] + \bigg[(k^2 + (k-q)^2)\bigg(I^{\sigma\mu,}{}_{\alpha\beta}I^{\nu}{}_{\gamma\delta,\sigma} + I^{\sigma\nu,}{}_{\alpha\beta}I^{\mu}{}_{\gamma\delta,\sigma} - \frac{1}{2}\eta^{\mu\nu}P_{\alpha\beta,\gamma\delta}\bigg)\\ & - (k^2\eta_{\gamma\delta}I^{\mu\nu,}{}_{\alpha\beta} + (k-q)^2\eta_{\alpha\beta}I^{\mu\nu,}{}_{\gamma\delta})\bigg]\bigg\}.\end{aligned} \tag{B7}$$

Before presenting our results, we note a simplification—it can be easily seen that the terms in the 3-graviton vertex function proportional to k^2 or $(k-q)^2$ do *not* produce nonanalytic pieces when inserted into either Eq. (B2) or Eq. (B3) and can be dropped.

A further enormous simplification of indices results from the identity [5]

$$P^{\xi\zeta,\alpha\beta}\tau^{\mu\nu}{}_{\alpha\beta,\gamma\delta}(k,q)P^{\gamma\delta,\kappa\rho} = \tau^{\mu\nu,\xi\zeta,\kappa\rho}(k,q) \tag{B8}$$

for all the terms which lead to nonanalytic corrections. This can be verified straightforwardly. The resulting integrals are still tedious, but can be done directly.

Decomposing the remaining piece of this vertex into the four bracketed terms, we list our results in terms of the contributions from each bracket separately: Fig. 4(a)

$$\begin{aligned}F_1(q^2) &= \frac{Gq^2}{\pi}\bigg(\bigg[\frac{1}{4} - 2 + 1 + 0\bigg]\log(-q^2) + \bigg[\frac{1}{16} - 1 + 1 + 0\bigg]\frac{\pi^2 m}{\sqrt{-q^2}}\bigg)\\ &= \frac{Gq^2}{\pi}\bigg(-\frac{3}{4}\log(-q^2) + \frac{1}{16}\frac{\pi^2 m}{\sqrt{-q^2}}\bigg),\end{aligned}$$

$$\begin{aligned}F_2(q^2) &= \frac{Gm^2}{\pi}\bigg(\bigg[\frac{13}{3} - 1 + 0 - 1\bigg]\log(-q^2) + \bigg[\frac{7}{8} - 1 + 2 - 1\bigg]\frac{\pi^2 m}{\sqrt{-q^2}}\bigg)\\ &= \frac{Gm^2}{\pi}\bigg(\frac{7}{3}\log(-q^2) + \frac{7}{8}\frac{\pi^2 m^2}{\sqrt{-q^2}}\bigg),\end{aligned}$$

Fig. 4(b)

$$F_1(q^2)=\frac{Gq^2}{\pi}\{([0+2+0-2]\log(-q^2)\}=0,$$

$$F_2(q^2)=\frac{Gm^2}{\pi}\left(\left[-\frac{25}{3}+0+2+2\right]\log(-q^2)\right)=\frac{Gm^2}{\pi}\left(-\frac{13}{3}\log(-q^2)\right). \quad \text{(B9)}$$

2. Spin 1/2

For the case of spin 1/2 we require some additional formalism in order to extract the gravitational couplings. In this case the matter Lagrangian reads

$$\sqrt{e}\mathcal{L}_m=\sqrt{e}\bar{\psi}(i\gamma^a e_a^\mu D_\mu-m)\psi \quad \text{(B10)}$$

and involves the vierbein e_a^μ which links global coordinates with those in a locally flat space. The vierbein is in some sense the "square root" of the metric tensor $g_{\mu\nu}$ and satisfies the relations

$$e^a{}_\mu e^b{}_\nu\eta_{ab}=g_{\mu\nu}, \quad e^a{}_\mu e_{a\nu}=g_{\mu\nu},$$
$$e^{a\mu}e_{b\mu}=\delta_b^a, \quad e^{a\mu}e_a^\nu=g^{\mu\nu}. \quad \text{(B11)}$$

The covariant derivative is defined via

$$D_\mu\psi=\partial_\mu\psi+\frac{i}{4}\sigma^{ab}\omega_{\mu ab}, \quad \text{(B12)}$$

where

$$\omega_{\mu ab}=\frac{1}{2}e_a{}^\nu(\partial_\mu e_{b\nu}-\partial_\nu e_{b\mu})-\frac{1}{2}e_b^\nu(\partial_\mu e_{a\nu}-\partial_\nu e_{a\mu})$$
$$+\frac{1}{2}e_a{}^\rho e_b{}^\sigma(\partial_\sigma e_{c\rho}-\partial_\rho e_{c\sigma})e_\mu{}^c. \quad \text{(B13)}$$

The connection with the metric tensor can be made via the expansion

$$e^a{}_\mu=\delta_\mu^a+c_\mu^{(1)a}+c_\mu^{(2)a}+\dots, \quad \text{(B14)}$$

where, as before, the superscript indicates the number of powers of the gravitational coupling G which are present. The inverse of this matrix is

$$e_a^\mu=\delta_a^\mu-c_a^{(1)\mu}-c_a^{(2)\mu}+c_b^{(1)\mu}c_a^{(1)b}+\dots \quad \text{(B15)}$$

and we find

$$g_{\mu\nu}=\eta_{\mu\nu}+c_{\mu\nu}^{(1)}+c_{\nu\mu}^{(1)}+c_{\mu\nu}^{(2)}+c_{\nu\mu}^{(2)}+c^{(1)a}{}_\mu c_{a\nu}^{(1)}+\dots,$$
$$g^{\mu\nu}=\eta^{\mu\nu}-c^{(1)\mu\nu}-c^{(1)\nu\mu}-c^{(2)\mu\nu}-c^{(2)\nu\mu}+c^{(1)a\mu}c_a^{(1)\nu}$$
$$+c^{(1)\mu a}c_a^{(1)\nu}+c^{(1)\mu a}c_a^{(1)\nu}+\dots. \quad \text{(B16)}$$

For our purposes we shall use only the symmetric component of the c-matrices, since these are physical and can be connected to the metric tensor, while their antisymmetric components are associated with freedom of homogeneous transformations of the local Lorentz frames and do not contribute to nonanalyticity. We then find

$$c_{\mu\nu}^{(1)}\to\frac{1}{2}(c_{\mu\nu}^{(1)}+c_{\nu\mu}^{(1)})=\frac{1}{2}h_{\mu\nu}^{(1)}$$

We then have

$$\det e=1+c+\frac{1}{2}c^2-\frac{1}{2}c_a^b c_b^a+\dots$$
$$=1+\frac{1}{2}h+\frac{1}{8}h^2-\frac{1}{8}h_a^b h_b^a+\dots. \quad \text{(B17)}$$

Using these forms the matter Lagrangian has the expansion

$$\sqrt{e}\mathcal{L}_m^{(0)}=\bar{\psi}\left(\frac{i}{2}\gamma^\alpha\delta_\alpha^\mu\partial_\mu^{LR}-m\right)\psi,$$

$$\sqrt{e}\mathcal{L}_m^{(1)}=-\frac{1}{2}h^{(1)\alpha\beta}\bar{\psi}i\gamma_\alpha\partial_\beta^{LR}\psi-\frac{1}{2}h^{(1)}\bar{\psi}\left(\frac{i}{2}\not\partial^{LR}-m\right)\psi,$$

$$\sqrt{e}\mathcal{L}_m^{(2)}=-\frac{1}{2}h^{(2)\alpha\beta}\bar{\psi}i\gamma_\alpha\partial_\beta^{LR}\psi-\frac{1}{2}h^{(2)}\bar{\psi}\left(\frac{i}{2}\not\partial^{LR}-m\right)\psi$$
$$-\frac{1}{8}h_{\alpha\beta}^{(1)}h^{(1)\alpha\beta}\bar{\psi}i\gamma^\gamma\partial_\lambda^{LR}\psi+\frac{1}{16}(h^{(1)})^2\bar{\psi}i\gamma^\gamma\partial_\gamma^{LR}\psi$$
$$-\frac{1}{8}h^{(1)}\bar{\psi}i\gamma^\alpha h_\alpha^\lambda\partial_\lambda^{LR}\psi+\frac{3}{16}h_{\delta\alpha}^{(1)}h^{(1)\alpha\mu}\bar{\psi}i\gamma^\delta\partial_\mu^{LR}\psi$$
$$+\frac{1}{4}h_{\alpha\beta}^{(1)}h^{(1)\alpha\beta}\bar{\psi}m\psi-\frac{1}{8}(h^{(1)})^2\bar{\psi}m\psi$$
$$+\frac{i}{16}h_{\delta\nu}^{(1)}(\partial_\beta h_\alpha^{(1)\nu}-\partial_\alpha h_\beta^{(1)\nu})\epsilon^{\alpha\beta\delta\epsilon}\bar{\psi}\gamma_\epsilon\gamma_5\psi, \quad \text{(B18)}$$

where

$$\bar{\psi}\partial_\alpha^{LR}\psi\equiv\bar{\psi}\partial_\alpha\psi-(\partial_\alpha\bar{\psi})\psi.$$

The corresponding one- and two-graviton vertices are found then to be

$$\tau_{\alpha\beta}(p,p')=\frac{-i\kappa}{2}\left[\frac{1}{4}[\gamma_\alpha(p+p')_\beta+\gamma_\beta(p+p')_\alpha]-\frac{1}{2}\eta_{\alpha\beta}\left(\frac{1}{2}(\not p+\not p')-m\right)\right],$$

$$\tau_{\alpha\beta,\gamma\delta}(p,p')=i\kappa^2\Bigg\{-\frac{1}{2}\left(\frac{1}{2}(\not p+\not p')-m\right)P_{\alpha\beta,\gamma\delta}-\frac{1}{16}\{\eta_{\alpha\beta}[\gamma_\gamma(p+p')_\delta+\gamma_\delta(p+p')_\gamma]+\eta_{\gamma\delta}[\gamma_\alpha(p+p')_\beta+\gamma_\beta(p+p')_\alpha]\}+\frac{3}{16}(p+p')^\epsilon\gamma^\xi(I_{\xi\phi,\alpha\beta}I^\phi{}_{\epsilon,\gamma\delta}+I_{\xi\phi,\gamma\delta}I^\phi{}_{\epsilon,\alpha\beta})+\frac{i}{8}\epsilon^{\rho\sigma\eta\lambda}\gamma_\lambda\gamma_5(I^\nu{}_{\alpha\beta,\eta}I_{\gamma\delta,\sigma\nu}k'_\rho-I^\nu{}_{\gamma\delta,\eta}I_{\alpha\beta,\sigma\nu}k_\rho)\Bigg\}. \quad \text{(B19)}$$

With these results in hand the loop integrations can now be performed, as before, yielding, for spin 1/2:

Fig. 4(a)

$$F_1(q^2)=\frac{Gq^2}{\pi}\left(\left[\frac{1}{4}-\frac{3}{4}-\frac{1}{2}+\frac{1}{4}\right]\log(-q^2)+\left[\frac{1}{16}+0-1+1\right]\frac{\pi^2 m}{\sqrt{-q^2}}\right)=\frac{Gq^2}{\pi}\left(-\frac{3}{4}\log(-q^2)+\frac{1}{16}\frac{\pi^2 m}{\sqrt{-q^2}}\right),$$

$$F_2(q^2)=\frac{Gq^2}{\pi}\left(\left[\frac{1}{6}-\frac{7}{4}+\frac{3}{4}+\frac{13}{12}\right]\log(-q^2)+\left[0+0-\frac{1}{2}+\frac{3}{4}\right]\frac{\pi^2 m}{\sqrt{-q^2}}\right)=\frac{Gq^2}{\pi}\left(\frac{1}{4}\log(-q^2)+\frac{1}{4}\frac{\pi^2 m}{\sqrt{-q^2}}\right),$$

$$F_3(q^2)=\frac{Gm^2}{\pi}\left(\left[\frac{4}{3}+0-1+\frac{1}{4}\right]\log(-q^2)+\left[\frac{7}{16}+0+0+0\right]\frac{\pi^2 m}{\sqrt{-q^2}}\right)=\frac{Gm^2}{\pi}\left(\frac{7}{12}\log(-q^2)+\frac{7}{16}\frac{\pi^2 m^2}{\sqrt{-q^2}}\right),$$

Fig. 4(b)

$$F_1(q^2)=\frac{Gq^2}{\pi}\left(\left[0+\frac{11}{4}-\frac{1}{2}-\frac{9}{4}\right]\log(-q^2)\right)=0,$$

$$F_2(q^2)=\frac{Gq^2}{\pi}\left(\left[0+\frac{7}{4}-\frac{1}{2}-\frac{5}{4}\right]\log(-q^2)\right)=0,$$

$$F_3(q^2)y=\frac{Gm^2}{\pi}\left(\left[-\frac{10}{3}+0+0+\frac{7}{4}\right]\log(-q^2)\right)=-\frac{Gm^2}{\pi}\frac{19}{12}\log(-q^2). \quad \text{(B20)}$$

All calculations of the form factors where done by hand and by computer. To do the various contractions of indices and integrations by computer, a computer algorithm for MAPLE 7[5] (TM) were developed and used to do the calculations.

APPENDIX C: USEFUL INTEGRALS

Here we collect the integrals used to calculate the long range corrections to the energy momentum tensor and the metric. For the classical correction to the energy momentum tensor we use

$$\int\frac{d^3q}{(2\pi)^3}e^{i\vec q\cdot\vec r}|\vec q|=-\frac{1}{\pi^2 r^4},$$

$$\int\frac{d^3q}{(2\pi)^3}e^{i\vec q\cdot\vec r}q_j|\vec q|=\frac{-4ir_j}{\pi^2 r^6},$$

$$\int\frac{d^3q}{(2\pi)^3}e^{i\vec q\cdot\vec r}\frac{q_iq_j}{|\vec q|}=\frac{1}{\pi^2 r^4}\left(\delta_{ij}-4\frac{r_ir_j}{r^2}\right), \quad \text{(C1)}$$

and the quantum effect use

$$\int\frac{d^3q}{(2\pi)^3}e^{i\vec q\cdot\vec r}\vec q^2\log\vec q^2=\frac{3}{\pi r^5},$$

$$\int\frac{d^3q}{(2\pi)^3}e^{i\vec q\cdot\vec r}q_j\vec q^2\log\vec q^2=\frac{i15r_j}{\pi r^7},$$

$$\int\frac{d^3q}{(2\pi)^3}e^{i\vec q\cdot\vec r}q_iq_j\log\vec q^2=\frac{1}{\pi r^5}\delta_{ij}. \quad \text{(C2)}$$

For the metric we require

$$\int\frac{d^3q}{(2\pi)^3}e^{i\vec q\cdot\vec r}\frac{1}{|\vec q|}=\frac{1}{2\pi^2 r^2},$$

$$\int\frac{d^3q}{(2\pi)^3}e^{i\vec q\cdot\vec r}\frac{q_j}{|\vec q|}=\frac{ir_j}{\pi^2 r^4},$$

[5]MAPLE and MAPLE V are registered trademarks of Waterloo Maple, Inc.

$$\int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \frac{q_i q_j}{|\vec{q}|^3} = \frac{1}{2\pi^2 r^2}\left(\delta_{ij} - 2\frac{r_i r_j}{r^2}\right), \tag{C3}$$

and

$$\int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \log \vec{q}^2 = -\frac{1}{2\pi r^3},$$

$$\int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} q_j \log \vec{q}^2 = \frac{-i3r_j}{2\pi r^5},$$

$$\int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \left(\frac{q_i q_j}{\vec{q}^2}\right) \log \vec{q}^2 = -\frac{r_i r_j}{2\pi r^5}. \tag{C4}$$

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