Kantowski-Sachs universe cannot be closed

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(Received 29 April 2003; published 30 October 2003)

In this paper, by analyzing the instability against spatially homogeneous and anisotropic perturbations of the Kantowski-Sachs type during different cosmological epochs, we show that it is a theoretical consequence of general relativity that the KS universe must be open or flat if it underwent a matter-dominated and/or radiation-dominated era in its past evolution, which theoretically confirms the flatness of our observable Universe.

DOI: 10.1103/PhysRevD.68.083512 PACS number(s): 98.80.−k, 95.30.Sf

The observations of the cosmic microwave background (CMB) anisotropy and of the spatial distribution of galaxies on a large scale are believed to have a cosmological origin and it would be very difficult to explain their existence and their isotropy if the hypothesis of the spatial homogeneity and isotropy of the Universe were not valid to a very good approximation on a large scale. There is a connection between isotropy and spatial homogeneity: unless a fundamental observer occupies a special position in the Universe, the isotropy implies spatial homogeneity. The assumption of spatial homogeneity and isotropy of space determines the metric completely except the sign of the curvature. Considerably more freedom is left if one assumes only spatial homogeneity of space [1,2]. All homogeneous cosmologies fall into two classes: the Bianchi models, for which the isometry group admits a three-dimensional simply transitive subgroup, and the Kantowski-Sachs models, for which the isometry subgroup is neither simply transitive nor admits a simply transitive subgroup [3]. There exist nine Bianchi types and, correspondingly, nine Bianchi cosmologies and each class has subclasses with extra symmetries. The Bianchi types are in general anisotropic so that they do not have all spatial directions at a point being equivalent. Many authors have addressed the evolution of spatially homogeneous cosmological models [4]. Spatially homogeneous cosmological models with a positive cosmological constant are investigated [5]. Exact string cosmological solutions have been found for the Kantowski-Sachs model by Barrow and Dabrowski [6]. Reula [7] showed that all small enough nonlinear perturbations decay exponentially during expanding phases of flat homogeneous cosmologies. Recently, Barrow *et al.* [8] showed that the Einstein static model is unstable to spatially inhomogeneous gravitational wave perturbations within the Bianchi type IX class of spatially homogeneous universes.

Recently, the Wilkinson Microwave Anisotropy Probe (WMAP) has provided high-resolution CMB data [9–11] for cosmology. Among the interesting conclusions that have been reached from these data are constraints on the present value Ω_0 of the total density parameter of the Universe. The new results indicate that while the Universe is close to being flat $\Omega_0 \simeq 1$, a closed universe is marginally preferred: $\Omega_0 > 1$ [9]. Especially, with a prior on the Hubble constant, one gets that $\Omega_0 = 1.03 \pm 0.05$ at 95% confidence level, while combining WMAP data with SNeIa leads to $\Omega_0 = 1.02 \pm 0.04$ or to $\Omega_0 = 1.02 \pm 0.02$, respectively, without and with a prior on the Hubble parameter. The latter may be regarded as the present best estimate of this parameter. In this paper, we study the instability against spatially homogeneous and anisotropic perturbations, which can be used as a probe of the curvature of space. We show that the anisotropy will not increase when the expansion rate is greater than certain values while it will increase when the expansion rate is less than that value or the universe is contracting. We find that the radiation-dominated and matter-dominated eras, which correspond to the scale factor $a(t) \sim t^{1/2}$ and $a(t) \sim t^{2/3}$, will not produce significant anisotropy when and only when the universe is spatially flat or open. In other words, we find a connection between isotropy and noncloseness: *the closed universe is unstable against spatially homogeneous and anisotropic perturbations of the Kantowski-Sachs type during the radiation-dominated and matter-dominated eras*.

In this paper, the cosmological anisotropy is the whole anisotropy of the scale factor and is different from CMB anisotropy. Why the cosmological anisotropy is still very tiny today begs for an explanation. Therefore, it is very interesting whether the degree of the cosmological anisotropy will increase during the evolution of the universe. We begin with the line element of anisotropic spacetime,

$$ds^2 = -dt^2 + a(t)^2[(1+\delta)^2 d\chi^2 + d\theta^2 + S^2(\theta) d\phi^2], \tag{1}$$

where

$$S(\theta) = \begin{cases} \sin\theta & \text{for} \quad k=1, \\ \theta & \text{for} \quad k=0, \\ \sinh\theta & \text{for} \quad k=-1, \end{cases} \tag{2}$$

and $k=0$ and -1 are just axisymmetric Bianchi type I and III universes while the $k=1$ model, or the closed anisotropic universe model, is referred to as the Kantowski-Sachs universe. Although only the closed models fall outside of the Bianchi classification, one can generally refer to them all as Kantowski-Sachs-like models for convenience [6]. In this paper, we analyze instability against spatially homogeneous and anisotropic perturbations within the Kantowski-Sachs-like models. The Einstein equations corresponding to the above setup are

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$$3H^2+\frac{2H\dot{\delta}}{1+\delta}+\frac{k}{a^2}=\kappa\rho, \tag{3}$$

$$2\frac{\ddot{a}}{a}+H^2+\frac{k}{a^2}=-\kappa p, \tag{4}$$

$$2\frac{\ddot{a}}{a}+H^2+\frac{3H\dot{\delta}}{1+\delta}+\frac{\ddot{\delta}}{1+\delta}=-\kappa p, \tag{5}$$

where $\kappa=8\pi G$ and p and ρ are the pressure and energy density of the perfect fluid, respectively. The energy conservation of the perfect fluid is expressed as

$$\frac{d\rho}{dt}=-\left(3H+\frac{\dot{\delta}}{1+\delta}\right)(\rho+p). \tag{6}$$

Substituting Eq. (5) into Eq. (4), we have

$$\ddot{\delta}+3\frac{\dot{a}}{a}\dot{\delta}-\frac{k}{a^2}(1+\delta)=0. \tag{7}$$

It does not lose generality that we assume $\delta(t_0)\geqslant 0$ for definiteness. In the following, we investigate the generic evolution of the anisotropy in the power law expansion and exponential expansion. Fortunately, Eq. (7) can be solved exactly in the above cases. For the power law expansion $a(t)=a(t/t_0)^q$, the solutions for Eq. (7) are the following.

Case (i): $q=0$,

$$\delta(t)=\begin{cases}\dfrac{1+\delta_0+a_0\dot{\delta}_0}{2}\exp\left(\dfrac{t-t_0}{a_0}\right)+\dfrac{1+\delta_0-a_0\dot{\delta}_0}{2}\exp\left(\dfrac{t_0-t}{a_0}\right)-1 & \text{for}\quad k=+1,\\ \dot{\delta}_0 t+\delta_0-t_0\dot{\delta}_0 & \text{for}\quad k=0,\\ \left[(1+\delta_0)\cos\left(\dfrac{t_0}{a_0}\right)-a_0\dot{\delta}_0\sin\left(\dfrac{t_0}{a_0}\right)\right]\cos\left(\dfrac{t}{a_0}\right)+\left[(1+\delta_0)\sin\left(\dfrac{t_0}{a_0}\right)+a_0\dot{\delta}_0\cos\left(\dfrac{t_0}{a_0}\right)\right]\sin\left(\dfrac{t}{a_0}\right)-1 & \text{for}\quad k=-1,\end{cases} \tag{8}$$

where $\delta_0\equiv\delta(t_0)$ and $\dot{\delta}_0\equiv\dot{\delta}(t_0)$.

Case (ii): $k=0$ and $q\neq 0$,

$$\delta(t)=\begin{cases}\dfrac{t_0^{3q}\dot{\delta}_0}{t^{3q-1}}+\delta_0-\dfrac{\dot{\delta}_0 t_0}{1-3q} & \text{for}\quad q\neq 1/3,\\ t_0\dot{\delta}_0\ln\left(\dfrac{t}{t_0}\right)+\delta_0 & \text{for}\quad q=1/3.\end{cases} \tag{9}$$

Case (iii): $k\neq 0$ and $q=1$,

$$\delta(t)=C_1 t^{-1+\sqrt{1+kt_0^2/a_0^2}}+C_2 t^{-1-\sqrt{1+kt_0^2/a_0^2}}-1, \tag{10}$$

where integral constants C_1 and C_2 can be fixed by initial values δ_0 and $\dot{\delta}_0$.

Case (iv): $k\neq 0$ and $q\neq 0,1$,

$$\delta(t)=\left(\frac{t}{t_0}\right)^{(1-3q)/2}\left\{C_1 Z_\nu^{(1)}\left[\frac{\sqrt{-k}t_0}{a_0(1-q)}\left(\frac{t}{t_0}\right)^{1-q}\right]+C_2 Z_\nu^{(2)}\left[\frac{\sqrt{-k}t_0}{a_0(1-q)}\left(\frac{t}{t_0}\right)^{1-q}\right]\right\}-1, \tag{11}$$

where $\nu=(1-3q)/2(1-q)$, C_1 and C_2 are integral constants, and $Z_\nu^{(1)}$ and $Z_\nu^{(2)}$ are Bessel functions for $k=-1$ and the modified Bessel functions for $k=+1$, respectively, which can be fixed by the initial values. For example, if $k=-1$ and $\nu\neq$ integer, we have

$$C_1=\frac{t_0^{3\nu/(3-2\nu)}[(3-2\nu)^2(1+\delta_0)Z_{1-\nu}(x_0)+4a_0\dot{\delta}_0 Z_{-\nu}(x_0)]}{(3-2\nu)^2[Z_{\nu-1}(x_0)Z_{-\nu}(x_0)+Z_{1-\nu}(x_0)Z_\nu(x_0)]}, \tag{12}$$

$$C_2=\frac{t_0^{3\nu/(3-2\nu)}[(3-2\nu)^2(1+\delta_0)Z_{\nu-1}(x_0)+4a_0\dot{\delta}_0 Z_\nu(x_0)]}{(3-2\nu)^2[Z_{\nu-1}(x_0)Z_{-\nu}(x_0)+Z_{1-\nu}(x_0)Z_\nu(x_0)]}, \tag{13}$$

where $x_0=\sqrt{-k}t_0/(1-q)a_0$.

For the exponentially expansion $a(t)=a_0\exp[H(t-t_0)]$, the solutions of Eq. (7) are as follows:

$$\delta(t)=C_1\exp\left[-\frac{3}{2}H(t-t_0)\right]Z_{3/2}\left(\frac{\sqrt{-k}}{Ha_0}\exp[H(t-t_0)]\right)+C_2\exp\left[-\frac{3}{2}H(t-t_0)\right]Z_{-3/2}\left(\frac{\sqrt{-k}}{Ha_0}\exp[H(t-t_0)]\right)-1, \quad (14)$$

where the integral constants C_1 and C_2 can also be fixed by initial values δ_0 and $\dot{\delta}_0$. Using the asymptotic expressions of Bessel functions and modified Bessel functions at $x\gg 1$, we find that the anisotropy will increase rapidly if the expansion rate $a(t)$ is slower than $a(t)\sim t^{1/3}$ for $k=-1,0$ or $a(t)\sim t$ for $k=+1$. Therefore, during the matter-dominated era and radiation-dominated era, the open and flat universes are stable against the spatially homogeneous anisotropic perturbation during the matter- and radiation-dominated eras, while the closed universe is not and the anisotropy increase exponentially.

The recent data from WMAP indicate that the universe is almost flat [10,12]. In the following, we will rigorously prove the conclusion in the flat universe case. Equation (7) can be rewritten as

$$\frac{\dot{\delta}(t)}{\dot{\delta}_0}=\left[\frac{a_0}{a(t)}\right]^3. \quad (15)$$

The scale factor $a(t)$ is determined by the pressure and energy density of the perfect fluid in the Einstein equations (3)–(5). For example, we can apply these equations specifically to the initial stage of cosmological evolution which is assumed to be governed by the ordinary scalar field. In this paper, these equations are applicable to matter with the energy-momentum tensor of arbitrary form so that we should discuss the varied form of $a(t)$. From Eq. (15), we have

$$\delta_0+|\dot{\delta}_0|\cdot\left|\int_{t_0}^{t}\left[\frac{a_0}{a(t)}\right]^3 dt\right|\leqslant\delta(t)\leqslant\delta_0+|\dot{\delta}_0|\cdot\left|\int_{t_0}^{t}\left[\frac{a_0}{a(t)}\right]^3 dt\right|. \quad (16)$$

Next, we discuss the convergence of definite integral in Eq. (16). At $t\gg t_0$, we rewrite $a(t)$ as $f(t)/t^q$, if $q>\frac{1}{3}$ and $f(t)\leqslant\text{const}\leqslant+\infty$, then the integral is convergent. If $q\leqslant\frac{1}{3}$ and $f(t)>\text{const}\geqslant 0$, then the integral is divergent. This argument can be extended to the following form by using the consensus Cauchy criterion: if $[a(t_0)/a(t)]^3=f(t)\cdot g(t)$ for $t\gg t_0$, $\int_{t_0}^{\infty}g(t)dt$ is a convergent integral and $f(t)$ is a monotonic and boundary function, then $I\equiv\int[a(t_0)/a(t)]^3dt$ is a convergent integral. Therefore, we have rigorously proved that a flat universe is stable against a spatially homogeneous and anisotropic perturbation of Kantowski-Sachs type during the radiation- and matter-dominated eras.

Now, let summarize the key points of the above discussions. In this paper, we focus on the Kantowski-Sach-like model of cosmology and discuss the evolution of the cosmological anisotropy. We showed that for open and flat universes, the expansion rate of the scale factor must be greater than $a(t)\sim t^{1/3}$ so that the cosmological anisotropy does not increase while this constraint becomes $a(t)\sim t$ for the closed universe. The implication of these constraints are that (i) the isotropic universe must also be open or flat if it underwent the radiation- and matter-dominated eras in its past evolution, (ii) the oscillation universe and static universe, corresponding to $q=0$, are generally unstable against the spatially homogeneous anisotropic perturbation unless an unnatural fine-tuning is introduced (for example, fine-tuning $1+\delta_0+a_0\dot{\delta}_0=0$ is needed for the static universe model), and (iii) together with the recent observation, which favors a closed and flat universe, it will be better to say that the Universe is flat.

As is known to all, when constructing spatially homogeneous cosmological models, we simply choose a three-dimensional Lie group G, choose a basis of left invariant dual vector fields on G, and choose a time-dependent left invariant metric $h_{ab}(t)$ on G. Then we can express the space-time metric $g_{\mu\nu}$ in term of h_{ab} [13]. The Kantowski-Sachs-like model considered here can be treated by a similar technique [14]. Although there are many other spatially homogeneous cosmological models that do not isotropize and their cosmological anisotropy evolution may also be interesting to study, they are beyond the scope of this paper.

The authors thank P. Steinhardt and J. Barrow for helpful comments. This work was partially supported by the National Nature Science Foundation of China under Grant No. 19875016 and the Foundation of Shanghai Development for Science and Technology under Grant No. JC 14035.

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