

Comparison of cosmological Boltzmann codes: Are we ready for high precision cosmology?

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We compare three independent, cosmological linear perturbation theory codes to assess the level of agreement between them and to improve upon it by investigating the sources of discrepancy. By eliminating the major sources of numerical instability the final level of agreement between the codes was improved by an order of magnitude. The relative error is now below 10^{-3} for the dark matter power spectrum. For the cosmic microwave background anisotropies and using identical ionization histories the agreement is below the sampling variance up to $l=3000$, with close to 10^{-3} accuracy reached over most of this range of scales. The same level of agreement is also achieved for the polarization spectrum and the temperature-polarization cross-spectrum. Linear perturbation theory codes are thus well prepared for the present and upcoming high precision cosmological observations. The remaining systematic errors are likely to be dominated by the uncertainties in the recombination history.

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I. INTRODUCTION

Since the first detection of anisotropies in the cosmic microwave background (CMB) over a decade ago [1] progress has been steady and rapid. These measurements are in astonishing agreement with theoretical predictions of adiabatic cold dark matter models which have been refined over the last few years. The standard model emerging from these measurements suggests that we live in a spatially flat universe dominated by dark energy and dark matter, with a small amount of baryons and a spectrum of primordial fluctuations that is close to scale invariant. This picture has been given its most dramatic confirmation by the recent Wilkinson Microwave Anisotropy Probe (WMAP) results [2,3], which have reached percent level accuracy on degree scales, in combination with small-scale anisotropy measurements [4,5].

While the current observational situation is already impressive, future observations are even more promising. The *Planck* satellite and several ground based small scale CMB experiments (APEX [56], SPT [57], ACT [58]) will reach subpercent accuracy on scales above $10'$. A possible next generation CMB satellite dedicated to polarization could measure polarization to a comparable accuracy, as well as measure the projected dark matter potential using the lensing effect on CMB. High precision cosmology will also be achieved with other data sets, most notably large scale structure (LSS) and supernovae, both of which can supplement the information from the CMB to break the degeneracies [6–10]. Current constraints from galaxy clustering, weak lensing and the Ly- α forest are limited by either statistics or

systematic effects in the analysis and have not yet reached percent level precision. However, with better data and more work on systematics both of these aspects should improve dramatically. Comparison between the different probes will also provide additional cross-checks on the systematics.

High precision cosmological observations are of course pointless if they are not matched by theoretical predictions. The CMB and, to a lesser extent, LSS are unique in that they are sensitive to perturbations in linear regime. In this regime the evolution equations can in principle be solved to arbitrary precision and are thus limited only by the accuracy of the linear approximation itself. In practice the computational task is not quite so simple for various reasons: the evolution equations are complicated, the solutions are highly oscillatory and thus susceptible to numerical errors, the equations can be stiff and require different treatments in different regimes etc.

In this era of high precision cosmology it is worth revisiting the status of the theoretical calculations as well. The last of such comparison was performed almost a decade ago [11]. Informal comparisons between the different groups at the time led to a nominally stated accuracy of 1% for codes such as CMBFAST [12]. At the time both the CMB and LSS measurements were much less precise and in the case of CMB limited to large scales, where sampling variance limits the required accuracy. Thus 1% precision was more than sufficient for measurements then. Today, systematic errors at 1% level are already comparable to the statistical precision of the current observations such as WMAP and will certainly not suffice for the next generation of CMB experiments. Moreover, a decade ago the standard cosmological model

was a flat cold dark matter (CDM) model with $\Omega_m=1$ and there were only limited comparisons performed for the currently favored model with significant dark energy/cosmological constant or reionization optical depth. The goal of this paper is to revisit the accuracy of the current linear perturbation theory codes.

It is useful to provide here some history on the development of relativistic perturbation theory and Boltzmann codes. Initial work on perturbation theory, including the classification of perturbations into scalar, vector and tensor, was done by Lifshitz [13]. Later papers clarified the gauge issues for scalar modes [14,15]. The main ingredients for computing the CMB spectrum were put in place already by the early 1970's [16–21], in those days still without cold dark matter (CDM). Work in the eighties introduced CDM and several computational advancements, such as the use of the multipole moment hierarchy to solve the equations for photon distribution function and the introduction of polarization [22–25]. Code development became an active area of research in the early nineties and there were several codes in addition to the ones mentioned above developed around that time [26–28]. As new cosmological models or parameters were introduced the corresponding CMB spectrum was calculated, such as the case for open models [29,30], tensor modes [31] and massive neutrinos [32,33]. A new method to compute the anisotropies based on line of sight integration was introduced in 1996 [12]. The resulting public domain code named CMBFAST was roughly two orders of magnitude faster than the traditional Boltzmann codes. The main subsequent developments were the improved treatment of polarization including E and B modes [34–36], inclusion of lensing effect on the CMB [37–40], spatially closed models [41–43], improvements in the recombination calculation [44,45] and introduction of additional cosmological parameters, such as dark energy/quintessence [46,47].

The principal guidelines in deciding which codes to include in the current comparison were independence and accuracy. While there was a lot of code development activity after COBE, most of the codes were not being updated after CMBFAST was made public. Two exceptions to this are the codes developed by Sugiyama [48,49], hereafter NS, and White [36,41,50], hereafter MW, both of which are included in the current comparison. These two codes are completely independent of CMBFAST and are traditional Boltzmann codes without the line of sight integration. NS code is based on gauge invariant formalism, while MW code and CMBFAST use synchronous gauge formalism. Other, more recent codes, such as CAMB [42] and CMBEASY [51], originally started as translations of CMBFAST into f90 and C++, respectively, and are thus not independent. There was a lot of subsequent work put into these codes later, so the extent to which the possible numerical errors in CMBFAST are also present in these codes is unclear and we do not explore it in this paper. We also do not use the COSMICS package [52] in this comparison. CMBFAST Boltzmann evolution equations originate from COSMICS and are thus not independent. At the time of CMBFAST development the two codes were extensively compared, but many of subsequent developments (polarization, non-flat geom-

etries, lensing, dark energy) were not included in the COSMICS package.

The goal of this paper is to test the numerical accuracy of the linear perturbation theory codes, which are used extensively in the parameter determinations. When we started the project the initial agreement was no worse than 1–2%, consistent with the stated accuracy. We will show below that the final agreement is much more impressive than that and is essentially sampling variance limited up to the highest multipole moment we used in comparison ($l=3000$). This is not to say that the theoretical predictions are this accurate, since the physics used in the codes is the same and there could be additional effects not included in any of the current versions. However, the numerical approximations, which are present in all of the codes, appear to be under control and do not lead to systematic errors of significance for the current and next generation of experiments. One of such effects emphasized in this paper is the physics of recombination, which should be revisited with an independent analysis to confirm the accuracy of the recombination code currently used [44].

II. COMPARISON OF RESULTS

In the current code comparison we limit ourselves to the simplest model with a cosmological constant. Even though the model was chosen prior to recent WMAP results it is in fact very close to their best fit model. Our standard model has $\Omega_{CDM}=0.3$, $\Omega_b=0.04$, $\Omega_\Lambda=0.66$ and $H_0=70$ km s Mpc. We assume a scale invariant $n=1$ primordial power spectrum with and without reionization (since the results are for the most part unchanged in the two cases we will only show those without reionization in the following). We do not include gravitational lensing effect in the current comparison, since it is not implemented in all of the codes (efforts to verify the lensing code accuracy in CMBFAST are currently underway). For the same reason we also do not include the tensors in our comparison. Accuracy of the tensor calculation is unlikely to be critical for the current or future generation of experiments, since tensors are already known to be subdominant and only contribute on large scales, where the sampling variance errors are large. Similar arguments also apply to the massive neutrinos or more general forms of dark energy, which are thus not explored in more detail here. In all of the comparisons we used the same recombination outputs. We have found some small differences between the different implementations of RECFAST [44], but these appear not to be important at the current level of precision.

The required accuracy depends on the scale one is probing and the information one is extracting from the power spectra. For a given C_l there are $2l+1$ independent multipole moments in the sky and the relative error on it will be roughly given by $1/\sqrt{l}$. However, the ultimate goal is not the spectrum C_l , but a small set of cosmological parameters, so one must combine C_l 's together. If the errors in the calculation of C_l are correlated then one needs a more stringent accuracy criterion. In the limit of only one parameter being determined from the data (for example, the overall amplitude of the spectrum assuming its shape is known) the number of modes up to l is l^2 and the required theoretical precision is

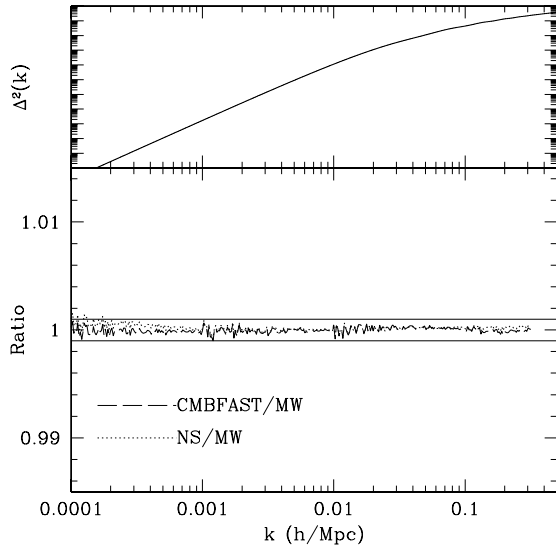


FIG. 1. Dark matter power spectrum for the 3 codes (top) and ratios between them (bottom). Also shown are $1 \pm 0.1\%$ horizontal lines. The relative errors are below 0.1%.

$\sqrt{2}/l$. This theoretical limit is not reached in practice, since there is always more than one parameter determined from the CMB data and since the sky coverage is always less than unity (due to the finite sky coverage or galactic contamination). To account for this we will roughly double this limit, so that we assume the required precision at a given l is determined by

$$\frac{\delta C_l}{C_l} = \frac{3}{l}. \quad (1)$$

This corresponds to 0.1% accuracy at $l=3000$, the maximum l used in comparison here. Note that for $l < 30$ the required accuracy is only 10% and there is thus little point in attempting to achieve very high accuracy on large scales.

While for the CMB the sampling variance always limits the required theoretical precision, this is less of an issue for the 3D matter power spectrum. Fortunately, the matter power spectrum is also much easier to compute with high accuracy. Figure 1 shows the comparison between the matter power spectra among the three codes. We have assumed the same initial conditions in all the codes, so the comparison of the transfer functions at the end tests the accuracy of relating the primordial spectrum of fluctuations to the final matter power spectrum, both in normalization and shape. We see that the agreement is remarkable, at least at the level of 10^{-3} . This is comparable or better than the accuracy in the CMB, so any matter power spectrum normalization from the CMB (such as σ_8) is limited by the accuracy in the CMB spectra. Computing the dark matter (as well as baryon or massive neutrino) power spectra is thus essentially exact for the current purposes. It is easy to understand why the dark matter transfer function can be computed so accurately. The evolution equation for the dark matter is a simple second order differential equation, its solutions are smooth and have a simple power law (or logarithmic) growth both in radiation and mat-

ter domination epochs. Dark matter exhibits no oscillatory behavior and only couples to gravity. As a result its evolution can be computed numerically to an exquisite precision.

We turn next to the CMB comparison. In general, high accuracy of CMB anisotropies is much more difficult to achieve than that of the dark matter power spectrum. There are several reasons for this.

(1) Before recombination, the evolution of baryons and photons is tightly coupled due to the high probability of a photon scattering off an electron. This leads to a stiff system of differential equations and a special treatment must be used before recombination, switching to the regular one at later times when the mean free path to Thomson scattering increases.

(2) As is well known, the CMB spectra have a lot of structure with prominent acoustic peaks, unlike the dark matter where the spectrum is a slowly varying function of scale. The structure is even sharper for polarization spectrum and polarization-temperature cross-correlation (where the spectrum can be positive or negative). The phases of the acoustic peaks depend sensitively on the numerical accuracy. They also depend sensitively on the recombination history, which must be computed very accurately.

(3) The time dependence of the multipole moments is highly oscillatory and requires fine time sampling. To obtain a C_l at a given l one must integrate over all the Fourier modes k . This k -mode dependence is also highly oscillatory and again requires fine sampling to achieve a sufficient accuracy. For traditional Boltzmann codes this can be computationally expensive, so approximations have been developed to reduce the number of evaluations. This is in principle avoided in the line-of-sight integration approach used in CMBFAST, which however, introduces its own approximations. Among these are the time sampling of the sources, treatment of reionization, l sampling, cutoff in the photon and neutrino hierarchy etc. At the time of CMBFAST first release the main goal was to reduce the computational time while still maintaining 1% accuracy. The approximation criteria were often chosen aggressively to reduce the run time. We have found that many of these approximations can be significantly improved in accuracy if original criteria are made slightly more conservative, without a significant increase in the run time. A few examples of these in CMBFAST are an increase in the Fourier mode k sampling, increase in the time sampling, decreased tolerance parameter of the differential equation integrator for low k modes and higher multipole moment in the cutoff for the neutrino and photon hierarchies. We also changed the time of switch from the tight coupling approximation to the full integration, which is now earlier than before (we refer the reader to the technical documentation of CMBFAST for more details).

Figure 2 shows the ratios between the codes for the temperature spectrum C_l^{TT} . We also show the cosmic variance error [Eq. (1)] and $\pm 0.1\%$ error lines. We see that the agreement is well within the cosmic variance limits and close to 0.1% for almost all l . The only exception is around $l \sim 10$, where there is a somewhat larger error of up to 0.5%. This is caused by the line of sight integration method as implemented in CMBFAST, where one uses integration by parts to

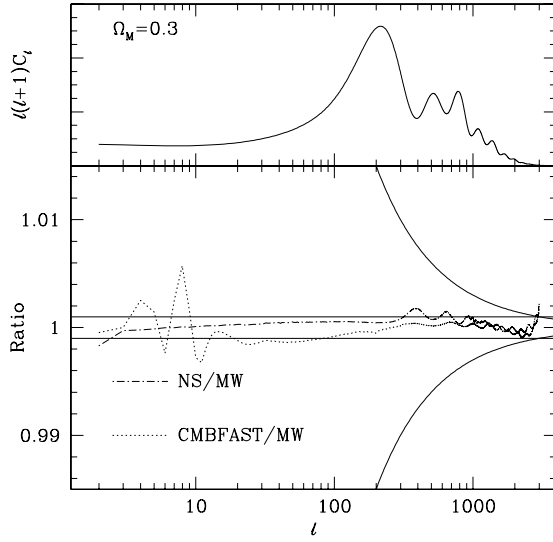


FIG. 2. C_l^{TT} for the 3 codes (top) and ratios between them (bottom). Also shown is the sampling variance limit $1 \pm 3/l$ and $1 \pm 0.1\%$ horizontal lines.

rewrite the sources into a single term that multiplies the spherical Bessel functions. This form requires very precise cancellations in the integrals over the visibility function on large scales. The error is, however, harmless, since it is two orders of magnitude smaller than the sampling variance. The agreement between NS and MW is equally remarkable and even better than MW/CMBFAST on large scales.

Figure 3 shows the same comparison for C_l^{EE} , the E-type polarization power spectrum. The agreement is roughly at the same level as for C_l^{TT} , close to 0.1% across all the scales. The exception again is CMBFAST at $l < 20$, where the error can be up to 1%, caused by imperfect cancellations in the line of sight integration over the recombination epoch at $z \sim 1100$. This discrepancy is not a real problem, since the comparisons here are for no reionization model and even a small amount of reionization increases the polarization

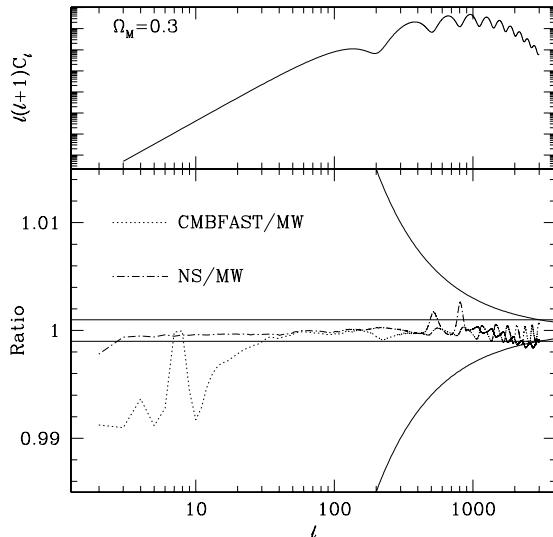


FIG. 3. Same as Fig. 2 for C_l^{EE} .

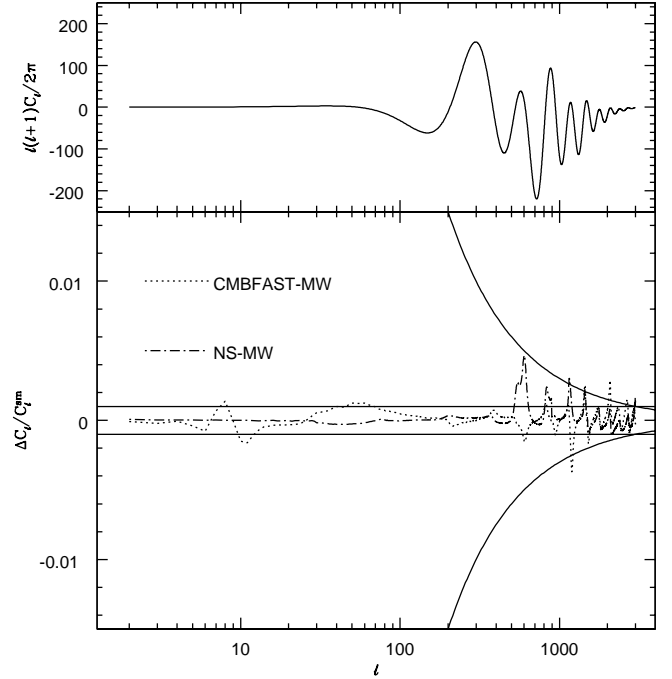


FIG. 4. Same as Fig. 2 for C_l^{TE} . At zero crossings of C_l^{TE} the relative error is ill-defined, so we compare to a smoothed version, where the smoothing is $\Delta l = 50$. The plotted sampling variance limit $1 \pm 3/l$ is a lower limit to the actual sampling variance, as discussed in the text.

power at low l with a contribution from $z < z_{\text{reion}} \sim 10-20$. Our reionization model comparisons show a better agreement. In any case, in this regime even a 1% error is a factor of at least 10 lower than the sampling variance and thus irrelevant.

Finally, Fig. 4 shows the temperature-polarization cross-correlation C_l^{TE} . The relative error is ill-defined at zero crossings of C_l^{TE} . For this reason we compare to a smoothed version of C_l^{TE} , smoothing over $\Delta l = 50$. The agreement is again very good, no worse than for C_l^{TT} or C_l^{EE} . Note that the sampling variance for C_l^{TE} at a fixed l is given by $\delta C_l^{TE}/C_l^{TE} \sim \sqrt{2[l(1 + C_l^{TT}C_l^{EE}/(C_l^{TE})^2)]}$ [53], which is always larger than the corresponding limits for C_l^{TT} and C_l^{EE} , so our plotted sampling variance limit is a conservative lower limit. We find a similar level of agreement when comparing the absolute errors without smoothing, which are also at the level of 0.1%.

III. CONCLUSIONS

We have performed a comparison of 3 current high accuracy linear perturbation theory codes. The initial agreement was at 1% level, while the final one was at 0.1% level, an order of magnitude improvement. The same 0.1% accuracy is also found for polarization and its cross-correlation with temperature. For the dark matter power spectrum the agreement is also at 0.1% level or even better. It seems unlikely that we will ever need better accuracy than this both for the CMB and for the matter power spectrum. The theoretical predictions of the CMB and matter power spectra are thus

well under control, at least for the codes and models used in the current comparison. We note that the modifications needed to upgrade the CMBFAST code to 0.1% accuracy have been implemented in version 4.3, which is available from www.cmbfast.org. As a caveat we note that the open/closed model implementation remains at 1% level and that the accuracies of lensing, massive neutrinos and dark energy remain to be explicitly verified. Some of these comparisons are currently in progress.

The main remaining concern are the physical assumptions that enter into the calculations. These have been scrutinized by many workers over the past decade (see, e.g., Ref. [11]), which gives us some confidence that there cannot be too many physical processes that have been overlooked by now. The principal concern at the moment is the accuracy of the recombination calculation. The original treatment [54,55] has been revisited in Ref. [44]. It was found that the original work by Peebles was remarkably accurate, but there were some improvements at the level of a few percent. For example, it was shown that HeI recombination cannot be well described by the Saha equation approximation and that the Boltzmann equilibrium assumption for the higher levels of hydrogen was not sufficiently accurate. The latter can be approximated by a fudge factor added to the previous treatment. These changes lead to a few percent differences in the CMB spectrum and were implemented into the RECFAST routine [44]. While we have no reason to suspect the accuracy of these calculations it is also not obvious that it is at the 0.1% which will be needed for the next generation of CMB experiments. Thus the accuracy of the CMB spectrum calculations may well be limited by the treatment of the recombination and it would thus be useful to revisit this issue to assess the level of remaining uncertainty. If this proves to be larger than

the upcoming experimental sensitivity then a possible approach would be to parametrize the uncertainty in the recombination physics and to reduce the uncertainty directly from the observations. As a simple example, if hydrogen recombination rate is uncertain then one could treat the fudge factor mentioned above as a free parameter that one could determine directly from CMB observations. It is an open question at present how uncertainty in the physics governing recombination feeds into measurements or reconstructions which rely on the CMB damping tail.

The new generation of the CMB experiments under construction or planning will achieve a subpercent accuracy on several cosmological parameters. Of special importance are the parameters related to the shape of the primordial power spectrum, which should be determined with exquisite precision. Such measurements will allow high precision tests of early universe models such as various models of inflation. This can, however, only be achieved if theoretical predictions match the observations in accuracy. It is comforting to know that the numerical precision of linear calculations is not among the worries for the future of high precision cosmology.

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- [1] G.F. Smoot *et al.*, *Astrophys. J. Lett.* **396**, L1 (1992).
 - [2] C. L. Bennett *et al.*, *Astrophys. J., Suppl.* **148**, 1 (2003).
 - [3] D.N. Spergel *et al.*, *astro-ph/0302209*.
 - [4] B.S. Mason *et al.*, *Astrophys. J.* **591**, 540 (2003).
 - [5] C.L. Kuo *et al.*, *astro-ph/0212289*.
 - [6] D.J. Eisenstein, W. Hu, and M. Tegmark, *Astrophys. J.* **518**, 2 (1999).
 - [7] W. Hu and M. Tegmark, *Astrophys. J. Lett.* **514**, L65 (1999).
 - [8] R. Mandelbaum, P. McDonald, U. Seljak, and R. Cen, *astro-ph/0302112*.
 - [9] E. V. Linder, *astro-ph/0210217*.
 - [10] M. White, *Astrophys. J.* **506**, 495 (1998).
 - [11] W. Hu, D. Scott, N. Sugiyama, and M. White, *Phys. Rev. D* **52**, 5498 (1995).
 - [12] U. Seljak and M. Zaldarriaga, *Astrophys. J.* **469**, 437 (1996).
 - [13] E.M. Lifshitz, *J. Phys. USSR* **10**, 116 (1946).
 - [14] J.M. Bardeen, *Phys. Rev. D* **22**, 1882 (1980).
 - [15] H. Kodama and M. Sasaki, *Prog. Theor. Phys. Suppl.* **78**, 1 (1984).
 - [16] P.J.E. Peebles and J.T. Yu, *Astrophys. J.* **162**, 815 (1970).
 - [17] R.A. Sunyaev and Y.B. Zeldovich, *Astrophys. Space Sci.* **7**, 3 (1970).
 - [18] R.K. Sachs and A.M. Wolfe, *Astrophys. J.* **147**, 73 (1967).
 - [19] M. Rees and D.W. Sciama, *Nature (London)* **511**, 61 (1968).
 - [20] J. Silk, *Astrophys. J.* **151**, 459 (1968).
 - [21] A.G. Doroshkevich, Y.B. Zel'Dovich, and R.A. Sunyaev, *Sov. Astron.* **22**, 523 (1978).
 - [22] M.L. Wilson and J. Silk, *Astrophys. J.* **243**, 14 (1981).
 - [23] N. Vittorio and J. Silk, *Astrophys. J. Lett.* **285**, L39 (1984).
 - [24] J.R. Bond and G. Efstathiou, *Astrophys. J. Lett.* **285**, L45 (1984).
 - [25] J.R. Bond and G. Efstathiou, *Mon. Not. R. Astron. Soc.* **226**, 655 (1987).
 - [26] J.A. Holtzman, *Astrophys. J., Suppl.* **71**, 1 (1989).
 - [27] S. Dodelson and J.M. Jubas, *Phys. Rev. Lett.* **70**, 2224 (1993).
 - [28] R. Stompor, *Astron. Astrophys.* **287**, 693 (1994).
 - [29] M.L. Wilson, *Astrophys. J.* **273**, 2 (1983).
 - [30] N. Gouda, N. Sugiyama, and M. Sasaki, *Prog. Theor. Phys.* **85**, 1023 (1991).
 - [31] R. Crittenden, J.R. Bond, R.L. Davis, G. Efstathiou, and P.J. Steinhardt, *Phys. Rev. Lett.* **71**, 324 (1993).
 - [32] U. Seljak and E. Bertschinger, in *Present and Future of the Cosmic Microwave Background*, edited by J. L. Sanz, E. Martinez-Gonzalez, and L. Cayon (Springer-Verlag, Berlin, 1994), p. 164.

- [33] S. Dodelson, E. Gates, and A. Stebbins, *Astrophys. J.* **467**, 10 (1996).
- [34] M. Zaldarriaga and U. Seljak, *Phys. Rev. D* **55**, 1830 (1997).
- [35] M. Kamionkowski, A. Kosowsky, and A. Stebbins, *Phys. Rev. D* **55**, 7368 (1997).
- [36] W. Hu and M. White, *Phys. Rev. D* **56**, 596 (1997).
- [37] U. Seljak, *Astrophys. J.* **463**, 1 (1996).
- [38] M. Zaldarriaga and U. Seljak, *Phys. Rev. D* **58**, 023003 (1998).
- [39] S. Cole and G. Efstathiou, *Mon. Not. R. Astron. Soc.* **239**, 195 (1989).
- [40] W. Hu, *Phys. Rev. D* **62**, 043007 (2000).
- [41] M. White and D. Scott, *Astrophys. J.* **459**, 415 (1996).
- [42] A. Lewis, A. Challinor, and A. Lasenby, *Astrophys. J.* **538**, 473 (2000).
- [43] M. Zaldarriaga and U. Seljak, *Astrophys. J., Suppl.* **129**, 431 (2000).
- [44] S. Seager, D.D. Sasselov, and D. Scott, *Astrophys. J. Lett.* **523**, L1 (1999).
- [45] S. Seager, D.D. Sasselov, and D. Scott, *Astrophys. J., Suppl.* **128**, 407 (2000).
- [46] M.S. Turner and M. White, *Phys. Rev. D* **56**, 4439 (1997).
- [47] R.R. Caldwell, R. Dave, and P.J. Steinhardt, *Phys. Rev. Lett.* **80**, 1582 (1998).
- [48] N. Sugiyama and N. Gouda, *Prog. Theor. Phys.* **88**, 803 (1992).
- [49] N. Sugiyama, *Astrophys. J., Suppl.* **100**, 281 (1995).
- [50] W. Hu, U. Seljak, M. White, and M. Zaldarriaga, *Phys. Rev. D* **57**, 3290 (1998).
- [51] M. Doran, astro-ph/0302138.
- [52] E. Bertschinger, astro-ph/9506070.
- [53] U. Seljak, *Astrophys. J.* **482**, 6 (1997).
- [54] P.J.E. Peebles, *Astrophys. J.* **153**, 1 (1968).
- [55] Y.B. Zeldovich, V.G. Kurt, and R.A. Syunyaev, *Zh. Eksp. Teor. Fiz.* **55**, 278 (1968).
- [56] <http://bolo.berkeley.edu/apexsz/>
- [57] <http://astro.uchicago.edu/spt/>
- [58] <http://www.hep.upenn.edu/~angelica/act/act.html>