# Precision of chiral-dispersive calculations of $\pi\pi$ scattering

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We calculate the combination $2a_0^{(0)}-5a_0^{(2)}$ (the Olsson sum rule) and the scattering lengths and effective ranges $a_1, a_2^{(I)}$ and $b_1, b_2^{(I)}$ dispersively (with the Froissart-Gribov representation) using, at low energy, the phase shifts for $\pi\pi$ scattering obtained by Colangelo, Gasser, and Leutwyler (CGL) from the Roy equations and chiral perturbation theory, plus experiment and Regge behavior at high energy, or directly, using the CGL parameters for $a$'s and $b$'s. We find mismatch, both among the CGL phases themselves and with the results obtained from the pion form factor. This reaches the level of several (2 to 5) standard deviations, and is essentially independent of the details of the intermediate energy region ($0.82 \leqslant E \leqslant 1.42$ GeV) and, in some cases, of the high energy behavior assumed. We discuss possible reasons for this mismatch, in particular in connection with an alternate set of phase shifts.

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## I. INTRODUCTION

In two remarkable recent papers, Ananthanarayan, Colangelo, Gasser and Leutwyler (ACGL) [1] and Colangelo, Gasser, and Leutwyler (CGL) [2] have used experimental information, analyticity and unitarity (in the form of the Roy equations [3]) and, in CGL, chiral calculations to two loops, to construct what is presented as a very precise $\pi\pi$ scattering amplitude at low energy, $E \equiv s^{1/2} \leqslant 0.8$ GeV.

There is little doubt that the small errors claimed by CGL, at the level of very few percent, follow from the Roy-chiral analysis, plus chiral perturbation theory (with the assumption of negligible higher order corrections), given the input scattering amplitude at high energy, say, for $E \gtrsim 1.42$ GeV. What is, however, not so clear is that the input selected by ACGL is unique, not even that it is the more physically acceptable one. The question then remains, what is the effect of changing this high energy input in the low energy $\pi\pi$ amplitude.

In the present paper we address ourselves to the matter of the consistency of the CGL $S$ matrix. To be precise, we evaluate the following quantities: the combination of $S0$, $S2$ scattering lengths $2a_0^{(0)}-5a_0^{(2)}$ (*Olsson sum rule*); the scattering length $a_1$ and effective range [4] $b_1$ in the $P$ wave; and the scattering lengths and effective ranges $a_2^{(I)}, b_2^{(I)}$ $I=0, 2$ for the $D0, D2$ waves. For the $a_1, b_1, a_2^{(I)}, b_2^{(I)}$ we use the Froissart-Gribov representation [5]. This presents two advantages. First of all, it was not verified in ACGL or CGL; therefore, it provides a novel test of the CGL phase shifts. Secondly, for $a_1, b_1$, and, to a lesser extent, for the $a_2^{(I)}$, the Froissart-Gribov representation is sensitive to the high energy scattering amplitude, precisely one of the features we want to probe. We then compare what we find with the values for $a$s and $b$s given by CGL themselves. For $a_1, b_1$ we also compare the CGL evaluations with the results of a direct fit of the $P$ wave to the pion form factor, this last a fully independent test.

The result of our calculations (Sec. IV) is that the solution of CGL is not consistent with the results from the fit to the pion form factor or with itself (if assuming a reasonable high energy Regge behavior) and the mismatch occurs essentially independently of the details of the intermediate energy ($0.82 \leqslant s^{1/2} \leqslant 1.42$ GeV) phase shifts we use, provided they fit experiment (Sec. IV D), and, in some cases, also of assumptions on the high energy ($s^{1/2} \geqslant 1.42$ GeV) behavior (Sec. IV E). For some of the quantities discussed above the disagreement reaches several (up to 5) standard deviations. For the $a_2^{(I)}$ the more striking discrepancy occurs for the combination $a_{0+} = 2/3[a_2^{(0)} - a_2^{(2)}]$. This is because it corresponds to a combination of only isospin 1,2 in the $s,u$ channels so the Froissat-Gribov integral is very accurate since the $S0$ wave, which is large and the one less well known, does not contribute. The chiral perturbation theory calculation for this quantity has also small errors since (to one loop) it only depends on one chiral Lagrangian constant, $\overline{l}_2$, see below. The mismatch is much less severe (below the $2\sigma$ level) for the ACGL results, the main reason being that their errors are at least three times as large as the CGL ones. We discuss in Sec. V the reasons for the CGL mismatch, which may be due to the use by CGL of an irrealistic high energy part of the scattering amplitude, which distorts their low energy ($s^{1/2} < 0.82$ GeV) phase shifts beyond the very small errors implied by their assumption of negligible higher chiral perturbative corrections.

Apart from these two sections, we present in Sec. II the Roy equations, in Sec. III the scattering amplitude we will use (including in particular a detailed discussion of the high energy pieces) and finish the paper with a summary and conclusions in Sec. VI.

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## II. THE ROY EQUATIONS

### A. Dispersion relations

The analyticity properties of the $\pi\pi$ scattering amplitude, $F(s,t)$, imply that we can write a Cauchy representation for it, fixing $t$ and allowing $s$ to be complex. For $s$ physical this reads

$$\mathrm{Re}\,F(s,t)=D(s,t)=\frac{1}{\pi}\mathrm{PP}\int_{4M_\pi^2}^{\infty}ds'\frac{A_s(s',t)}{s'-s}+\frac{1}{\pi}\int_{4M_\pi^2}^{\infty}ds'\frac{A_u(s',t)}{s'-u},\quad A(s,t)=\mathrm{Im}\,F(s,t). \tag{1}$$

(PP denotes Cauchy's principal part of the integral.)

Actually, and because, in some cases, the $A(s,t)$ grow linearly with $s$, Eq. (1) is divergent. This is repaired by subtractions; that is to say, by writing the Cauchy representation not for $F$ itself, but for $F(s,t)/(s-s_1)$ where $s_1$ is a convenient subtraction point, usually taken to coincide with a threshold. This introduces a function of $t$ in the equations [the value of $F(s,t)$ at $s=s_1$]; we leave it to the reader to rewrite our equations with the appropriate subtraction incorporated.

Let us separate out the high energy contribution, $s\geqslant s_h$ (we will fix $s_h$ later) to Eq. (1). We then have

$$D(s,t)=\frac{1}{\pi}\mathrm{PP}\int_{4M_\pi^2}^{s_h}ds'\frac{A_s(s',t)}{s'-s}+\frac{1}{\pi}\int_{4M_\pi^2}^{s_h}ds'\frac{A_u(s',t)}{s'-u}+V(s,t;s_h) \tag{2}$$

and

$$V(s,t;s_h)=\frac{1}{\pi}\int_{s_h}^{\infty}ds'\frac{A_s(s',t)}{s'-s}+\int_{s_h}^{\infty}ds'\frac{A_u(s',t)}{s'-u}; \tag{3}$$

we are assuming $s<s_h$. Both $D$ and the $A$ may be written in terms of the same set of phase shifts by expanding them [9], for fixed $s$ channel isospin $I$, as

$$A^{(I)}(s,t)=2\frac{2s^{1/2}}{\pi k}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)\frac{1}{\cot^2\delta_l^{(I)}(s)+1}, \tag{4}$$

$$D^{(I)}(s,t)=2\frac{2s^{1/2}}{\pi k}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)\frac{\cot\delta_l^{(I)}(s)}{\cot^2\delta_l^{(I)}(s)+1}. \tag{5}$$

One of the factors 2 above occurs because of the identity of pions; we work in the limit of exact isospin invariance.

These equations provide constraints for the phase shifts provided one knows (or has a reliable model) for the high energy term, $V(s,t;s_h)$. They enforce analyticity and $s\leftrightarrow u$ crossing symmetry.

### B. The Roy equations

Equations (1)–(5) look rather cumbersome. Roy [3] remarked that they appear simpler if we project them into partial waves, integrating over physical ($t\leqslant 0$) values of the cosine of the scattering angle: one finds the Roy equations

$$\frac{\cot\delta_l^{(I)}(s)}{\cot^2\delta_l^{(I)}(s)+1}=\sum_{l'=0}^{\infty}\int_{4M_\pi^2}^{s_h}ds'K_{ll'}(s,s')\frac{1}{\cot^2\delta_{l'}^{(I)}(s')+1}+V_l(s;s_h). \tag{6}$$

Here the kernels $K_{ll'}$ are known and the $V_l$ are the projections of $V$.

Equation (6) is valid in the simplified case we are considering here, i.e., without subtractions. If we had subtractions, the fixed $t$ dispersion relations would acquire an extra term, a function $g(t)$ [the value of $F(s_1,t)$ at the subtraction point]. This may be eliminated, using crossing symmetry, in favor of the $S$ wave scattering lengths. Equation (6) would be modified accordingly.

Let us rewrite the Roy equations in the form

$$\xi=\Phi(\xi,V) \tag{7}$$

where $\xi=\{\mathrm{Im}\,f_l\}_{l=0}^{\infty}$ stands for the set of imaginary parts of the partial waves, for $s\leqslant s_h$, and $\Phi$ is the functional that follows from Eq. (6). We can define a mapping,

$$\xi'\equiv\Phi(\xi,V) \tag{8}$$

and then the solution of the Roy equations is a fixed point of $\Phi$.

The relations Eq. (7) are highly nonlinear integral and matrix equations. Solutions are known to exist in some favorable cases; in fact, Atkinson [11] proved, even before the advent of Roy's equations, that, for any arbitrary $V(s,t;s_h)$ such that it is sufficiently smooth and decreasing at infinity, one can obtain, by iterating Eq. (8), a solution not only of the Roy equations, but of the full Mandelstam representation, and compatible with inelastic unitarity for all $s$ as well. Therefore, the solutions to the Roy equations are ambiguous in an unknown function, and the matter of what is an acceptable $V$ becomes crucial. This is particularly so because fulfillment of the Roy equations does not guarantee full analyticity and crossing; and it may happen that a given solution of the Roy equations is incompatible with other sum rules (as is the case for the CGL solution).

## III. THE SCATTERING AMPLITUDE

At low energy, say $s^{1/2}\leqslant 0.82$ GeV, the inelasticity in $\pi\pi$ scattering is known experimentally to be negligible; it is for these energies that the Roy equations Eq. (7) are to be solved. To do so we need as input the function $V$ or, equivalently, the imaginary part of the scattering amplitude for energies $s^{1/2}\geqslant 0.82$ GeV. In fact, for the Roy equations we need $\mathrm{Im}\,F(s,t)$ for $s$ physical and $t$ physical, $t\leqslant 0$. However, for other applications, we will require $\mathrm{Im}\,F(s,t)$ up to the edge of the Martin-Lehmann ellipse [12], $t\leqslant 4M_\pi^2$; our discussion will also cover this case. We now proceed with a discussion of the different waves and energy regions.

### A. The *S* and *P* waves for *E* below 0.82 GeV

Because we want to test the solution of CGL for the $\pi\pi$ $S$ matrix, we consider now the solution to the Roy equations, incorporating chiral perturbation theory to two loops, given there. The low energy $S0$,$S2$,$P$ waves are written by these authors as

$$\tan\delta_l^{(I)}(s)=k^{2l}\sqrt{1-4M_\pi^2/s}\{A_l^I+B_l^Ik^2/M_\pi^2+C_l^Ik^4/M_\pi^4 + D_l^Ik^6/M_\pi^6\}\frac{4M_\pi^2-s_{lI}}{s-s_{lI}}, \tag{9}$$

$k=\sqrt{s/4-M_\pi^2}$, and the values of the parameters, as given by CGL, Eq. (17.2), are

$$\begin{aligned}
&A_0^0=0.220,\quad B_0^0=0.268,\quad C_0^0=-0.0139,\quad D_0^0=-0.139/10^2,\quad s_{00}=36.77M_\pi^2\\
&A_2^0=-0.444/10,\quad B_2^0=-0.857/10,\quad C_2^0=-0.221/10^2,\quad D_2^0=-0.129/10^3,\quad s_{02}=-21.62M_\pi^2\\
&A_1=0.379/10,\quad B_1=0.140/10^4,\quad C_1=-0.673/10^4,\quad D_1=0.163/10^7,\quad s_1=30.72M_\pi^2.
\end{aligned} \tag{10}$$

These are the values of the phase shifts that we will use up to the energy $E=0.82$ GeV.

To test dispersion relations, either in the form of the Olsson relation or the Froissart-Gribov representation, we need also the values of the $S$,$P$ waves at intermediate energies ($0.82\leqslant E\leqslant 1.42$) and the values of the $D$,$F$ waves below 1.42 GeV, that we take from experiment; higher waves are presumably negligible. Moreover, we require $\mathrm{Im}\,F(s,t)$ for $s^{1/2}\geqslant 1.42$ GeV. This last we will obtain from Regge theory in Sec. III D, we now turn to the intermediate energy regions.

Before doing so, however, we want to emphasize that, in the present paper, we do not deal with the matter of the consistency of the fits for the $S$,$P$ waves between 0.82 and 1.42 GeV that we will give in the section, or of those for the $D$,$F$ waves. These fits [Eqs. (11)–(22)] are merely a convenient way to summarize the experimental data; our results would change very little if we had instead used a spline interpolation for the experimental phase shifts. We will discuss this further in Sec. IV D 1, where we will show that the discrepancy will remain essentially unchanged provided we demand a resemblance to the data (allowing for a large uncertainty) in this intermediate region.

### B. The *S*, *P* waves between 0.82 and 1.42 GeV

For the $S0$ wave in the region between the 0.82 GeV and $\bar{K}K$ threshold we use the parametrization, obtained by fitting experimental data [13] (as in [8]),

$$\cot\delta_0^{(0)}(s)=\frac{s^{1/2}}{2k}\frac{M_\pi^2}{s-\frac{1}{2}M_\pi^2}\frac{M_\sigma^2-s}{M_\sigma^2}\left\{B_0+B_1\frac{\sqrt{s}-\sqrt{s_0-s}}{\sqrt{s}+\sqrt{s_0-s}}+B_2\left[\frac{\sqrt{s}-\sqrt{s_0-s}}{\sqrt{s}+\sqrt{s_0-s}}\right]^2\right\}, \tag{11}$$

$$s_0^{1/2}=2M_K,\quad \chi^2/\mathrm{DOF}=11.1/(19-4),$$

$$M_\sigma=806\pm 21,\quad B_0=21.91\pm 0.62,\quad B_1=20.29\pm 1.55,$$

$$B_2=22.53\pm 3.48,\quad a_0^{(0)}=(0.226\pm 0.015)M_\pi^{-1}.$$

The solution depends on the value of $\delta_0^{(0)}(M_K^2)$ we impose in the fit. In Eq. (11) we took that following from the more recent measurements of $K_{2\pi}$ decay [14] $\delta_0^{(0)}(M_K^2)=41.5\pm 3°$. Another possibility is to average this with the older determination [14], thus imposing the value $\delta_0^{(0)}(M_K^2)=43.3\pm 2.3°$; this we will discuss in Sec. V B.

Solution Eq. (11) is, up to $s^{1/2}=0.84$ GeV, similar to the CGL one, Eqs. (9) and (10); see Fig. 1. We will use the CGL solution up to 0.82 GeV slightly above their nominal maximum range, $s^{1/2}=0.80$ GeV, and Eq. (11) between 0.82 and 0.96 GeV.

For the $S2$ wave between 0.82 GeV and 1.42 GeV we use the phase shift obtained by fitting experimental data and including the requirement $a_0^{(2)}=0.044\pm 0.003M_\pi^{-1}$ (this last follows from the analysis of CGL):

$$\cot\delta_0^{(2)}(s)=\frac{s^{1/2}}{2k}\frac{M_\pi^2}{s-2z_2^2}\left\{B_0+B_1\frac{\sqrt{s}-\sqrt{s_0-s}}{\sqrt{s}+\sqrt{s_0-s}}\right\},$$

$$s_0^{1/2}=1.45\ \text{GeV},\quad \chi^2/\mathrm{DOF}=16.1/(18-2),$$

$$B_0=-115\pm 4,\quad B_1=-106\pm 3,$$

$$z_2=139.57\ \text{MeV (fixed)}. \tag{12}$$

This actually corresponds to $a_0^{(2)}=-0.0457\pm 0.0074$. One can allow $z_2$ to vary by 8 MeV, and still be within $1\sigma$ of the minimum, but we will not do so here.

Then we have the $P$ wave between 0.82 GeV and 1.0 GeV. Here we fit the pion form factor, including $e^+e^-$ and $\tau$ decay data. There are now two possibilities: the first one is

$$\cot\delta_1(s)=\frac{s^{1/2}}{2k^3}(M_\rho^2-s)\left\{B_0+B_1\frac{\sqrt{s}-\sqrt{s_0-s}}{\sqrt{s}-\sqrt{s_0-s}}\right\},$$

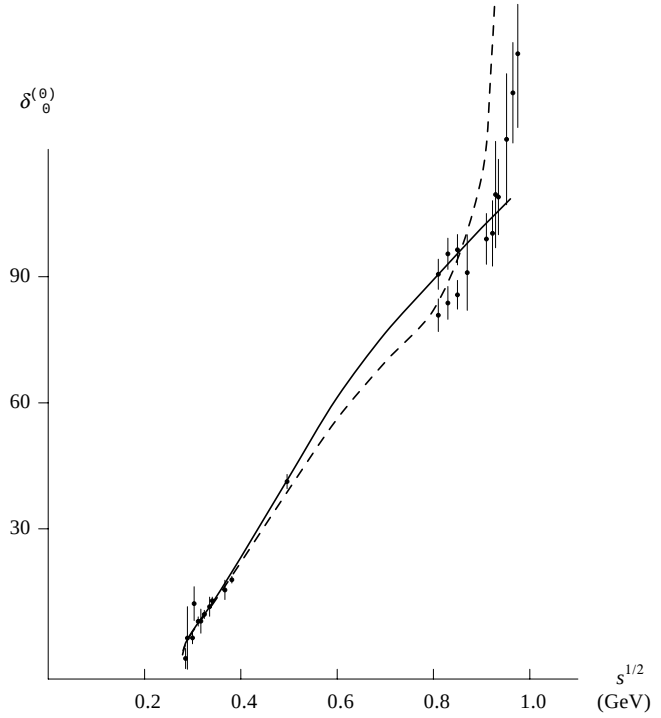


FIG. 1. The $I=0$, $S$-wave phase shifts corresponding to Eq. (11) (continuous line) and Colangelo, Gasser, and Leutwyler [2] (dashed line).

$$s_0^{1/2}=1.05\ \text{GeV},\quad \chi^2/\text{DOF}=1.3,$$

$$M_\rho=772.3\pm0.5\ \text{MeV},\quad B_0=1.060\pm0.005,$$

$$B_1=0.24\pm0.06,$$

$$(0.82\ \text{GeV}\leqslant s^{1/2}\leqslant 1.0\ \text{GeV}). \quad (13)$$

In particular, for the low energy parameters, this gives

$$a_1=(40.6\pm1.3)\times10^{-3}M_\pi^{-3},$$

$$b_1=(4.18\pm0.43)\times10^{-3}M_\pi^{-5}. \quad (14)$$

This result is obtained from the fit to the pion form factor, with only statistical experimental errors taken into account, performed in [15]. If we also take systematic normalization errors into account, Eq. (13) is replaced by

$$\cot\delta_1(s)=\frac{s^{1/2}}{2k^3}(M_\rho^2-s)\left\{B_0+B_1\frac{\sqrt{s}-\sqrt{s_0-s}}{\sqrt{s}+\sqrt{s_0-s}}\right\},$$

$$s_0^{1/2}=1.05\ \text{GeV},\quad \chi^2/\text{DOF}=1.1,$$

$$M_\rho=773.5\pm0.85\ \text{MeV},\quad B_0=1.071\pm0.007,$$

$$B_1=0.18\pm0.05$$

$$(0.82\ \text{GeV}\leqslant s^{1/2}\leqslant 1.0\ \text{GeV}) \quad (15)$$

and now

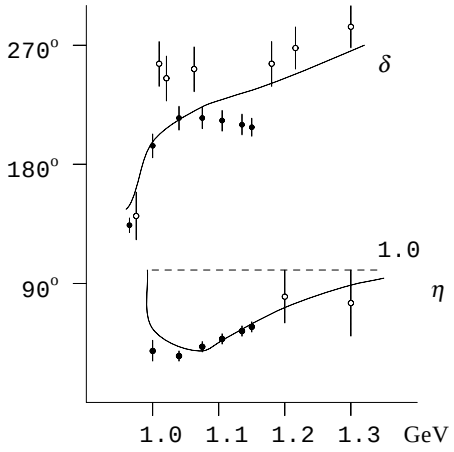


FIG. 2. Fits to the $I=0$, $S$-wave phase shift and inelasticity from 960 to 1350 MeV. Also shown are the data points from solution 1 of Protopopescu *et al.* [17] (black dots) and some data of Grayer *et al.* [18] (open circles).

$$a_1=(38.6\pm1.2)\times10^{-3}M_\pi^{-3},$$

$$b_1=(4.47\pm0.29)\times10^{-3}M_\pi^{-5}. \quad (16)$$

We will consider both possibilities, but the calculations of dispersive and Froissart-Gribov integrals will be made with Eq. (15), for definiteness. If using Eq. (13) the differences would be minute [16].

We next turn to the $S0$, $P$ waves in the higher energy regions, but still $E\leqslant1.42$ GeV. For the $S0$ wave between $\bar{K}K$ threshold, 0.992 GeV, and 1.42 GeV, we use a semi-empirical formula that fits reasonably well the existing data [18,17] from $s^{1/2}\geqslant0.96$ GeV to 1.50 GeV:

$$\cot\delta_0^{(0)}(s)=c_0\frac{(s-M_\sigma^2)(M_f^2-s)|k_2|}{M_f^2 s^{1/2}k_2^2},\quad k_2=\frac{\sqrt{s-4M_K^2}}{2},$$

$$\eta=1-\left(c_1\frac{k_2}{s^{1/2}}+c_2\frac{k_2^2}{s}\right)\frac{M'^2-s}{s},$$

$$(0.992\leqslant s^{1/2}\leqslant1.42\ \text{GeV}),\quad c_0=1.36\pm0.05,$$

$$c_1=6.7\pm0.15,\quad c_2=-17.6\pm0.7,$$

$$M_K=496\ \text{MeV},\quad M_\sigma=0.802\ \text{GeV},$$

$$M_f=1.32\ \text{GeV},\quad M'=1.5\ \text{GeV}. \quad (17)$$

Note that, for inelastic scattering, we define our parameters so that, in general,

$$\text{Im}\,\hat{f}_1^{(I)}(s)=\frac{\eta_l^{(I)}}{1+\cot^2\delta_l^{(I)}(s)}+\frac{1-\eta_l^{(I)}}{2}. \quad (18)$$

In the elastic region, $\eta_l^{(I)}(s)=1$. The fit to the data following from Eq. (17) is shown in Fig. 2.

Finally, for the $P$ wave between 1 GeV and 1.42 GeV, we use an empirical formula, obtained adding a resonance (with mass 1.45 GeV) to a nonresonant background:

$$\mathrm{Im}\,\hat{f}_1(s)=\frac{1}{1+[\lambda+1.1k_2/s^{1/2}]^2}+\mathrm{BR}\frac{M_{\rho'}^2\Gamma^2}{(s-M_{\rho'}^2)^2+M_{\rho'}^2\Gamma^2}$$

$$(1.0\leqslant s^{1/2}\leqslant 1.42\ \ \mathrm{GeV}),$$

$$M_{\rho'}=1.45\ \ \mathrm{GeV},\quad \Gamma=0.31\ \ \mathrm{GeV},\quad \lambda=2.6\pm 0.2,$$

$$\mathrm{BR}=0.25\pm 0.05. \tag{19}$$

Note that the effect of the $\rho(1450)$ is very small, as will be clear in our various evaluations below.

### C. The *D,F* waves below 1.42 GeV

We take these waves as given (from threshold to 1.42 GeV) by the fits of [8], with inelasticity added for the $D0$ wave and, for the $F$ wave, including also the tail of the $\rho_3$ resonance. Moreover, we have required (for compatibility with the CGL analysis) that the corresponding scattering lengths agree within errors with those given in CGL; that is to say, we include the CGL values, weighted with their errors, in the fits for $D2$, $F$ (for the $D0$ wave it is not necessary, as there are enough precise experimental data). For the $D0$ wave we thus write

$$\cot\delta_2^{(0)}(s)=\frac{s^{1/2}}{2k^5}(M_{f_2}-s)M_\pi^2\left\{B_0+B_1\frac{\sqrt{s}-\sqrt{s_0-s}}{\sqrt{s}+\sqrt{s_0-s}}\right\},$$

$$s_0^{1/2}=1.430\ \ \mathrm{GeV},$$

$$M_{f_2}=1270\ \ \mathrm{MeV},\quad B_0=23.7\pm 0.7,\quad B_1=22.9\pm 2.7,$$

$$\eta=1-2\times 0.15\frac{2[k/k(M_{f_2}^2)]^{10}}{1+[k/k(M_{f_2}^2)]^{20}}. \tag{20}$$

The inelasticity on the $f_2$ is taken from the Particle Data Tables. Equation (20) corresponds to $a_2^{(0)}=(15\pm 3.5)\times 10^{-4}\ M_\pi^{-5}$ against CGL's value $(17.5\pm 0.3)\times 10^{-4}\ M_\pi^{-5}$. For the $D2$ wave [19],

$$\cot\delta_2^{(2)}(s)=\frac{s^{1/2}}{2k^5}\frac{M_\pi^4 s}{4(M_\pi^2+\Delta^2)-s}\left\{B_0+B_1\frac{\sqrt{s}-\sqrt{s_0-s}}{\sqrt{s}+\sqrt{s_0-s}}\right\},$$

$$s_0^{1/2}=1.43\ \ \mathrm{GeV},$$

$$B_0=(2.33\pm 0.17)\times 10^3,\quad B_1=(-0.39\pm 0.75)\times 10^3,$$

$$\Delta=90\pm 11\ \ \mathrm{MeV}. \tag{21}$$

Now $a_2^{(2)}=(1.6\pm 0.4)\times 10^{-4}\ M_\pi^{-5}$ [CGL's value: $(1.7\pm 0.13)\times 10^{-4}\ M_\pi^{-5}$].

The $D$ phases are depicted in Figs. 3 and 4.

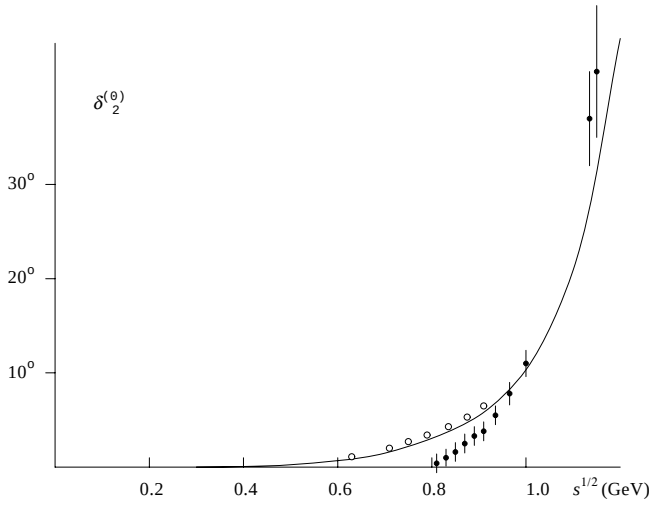


FIG. 3. Fits to the $I=0$, $D$-wave phase shift. Also shown are the data points from solution 1 of Protopopescu *et al.* [17] (black dots) and some data of Estabrooks and Martin [18] (open circles).

Finally, for the $F$ wave we write a background plus a Breit-Wigner. The background is obtained fitting low energy; the resonance is the $\rho_3$ with its properties taken from the Particle Data Tables:

$$\mathrm{Im}\,\hat{f}_3(s)=\frac{1}{1+\cot^2\delta_3}+\left(\frac{k}{k(M_{\rho_3})}\right)^{14}\times\mathrm{BR}\frac{M_{\rho_3}^2\Gamma^2}{(s-M_{\rho_3}^2)^2+M_{\rho_3}^2\Gamma^2}, \tag{22}$$

$$\cot\delta_3(s)=\frac{s^{1/2}}{2k^7}M_\pi^6\left\{B_0+B_1\frac{\sqrt{s}-\sqrt{s_0-s}}{\sqrt{s}+\sqrt{s_0-s}}\right\},$$

$$s_0^{1/2}=1.5\ \ \mathrm{GeV},$$

$$M_{\rho_3}=1.69\ \ \mathrm{GeV},\quad \Gamma=0.161\ \ \mathrm{GeV},\quad \mathrm{BR}=0.24,$$

$$B_0=(1.07\pm 0.03)\times 10^5,\quad B_1=(1.35\pm 0.03)\times 10^5.$$

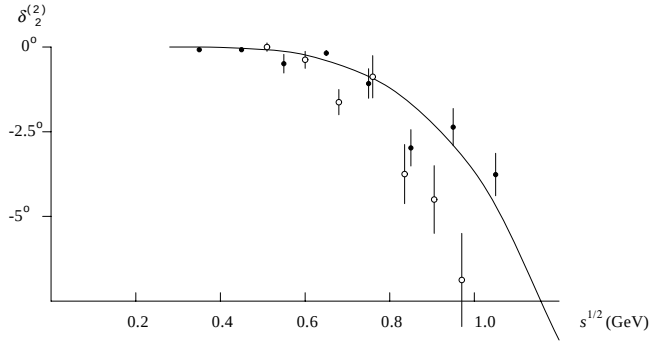


FIG. 4. Fits to the $I=2$, $D$-wave phase shift. Also shown are the data points of Losty *et al.* (open circles) and from solution A of Hoogland *et al.* (black dots) [20].

Here $a_3=(7.0\pm0.8)\times10^{-5}\ M_\pi^{-7}$; the value reported in CGL is $(5.6\pm0.2)\times10^{-5}\ M_\pi^{-7}$.

### D. High energy: The Regge picture

As we will discuss in Sec. V, the experimental phase shift analyses become unreliable as soon as the inelasticity is large; for $\pi\pi$ scattering, this occurs at and above $E\sim1.4$ GeV. Fortunately, Regge pole theory provides an input for high energy scattering; we will now briefly describe those of its features that are of interest to us. Before starting with the details, however, it is perhaps worthwhile to remark that Regge theory is as much part of QCD as, say, chiral perturbation theory; in fact, Regge theory is probably of more general validity than QCD. By using Regge formulas we are thus not introducing extra assumptions. The only debatable point is when is Regge theory applicable; QCD only specifies $s\gg\Lambda^2$, $s\gg|t|$. Fortunately, factorization allows us to relate $\pi\pi$ to $\pi N$ and $NN$ cross sections. From this, and the fact that Regge formulas and experimental cross sections for $\pi\pi$ scattering agree (within errors) around $s^{1/2}=1.4$ GeV, as shown in Figs. 6 and 8 below, we will conclude that Regge formulas are applicable at and above these energies; specifically, we will use them above $E=1.42$ GeV. We now turn to a brief discussion of the details.

Consider the collision of two hadrons, $A+B\to A+B$. According to Regge theory, the high energy scattering amplitude, at fixed $t$ and large $s$, is governed by the exchange of complex, composite objects (known as *Regge poles*) related to the resonances that couple to the $t$ channel. Thus, for isospin 1 in the $t$ channel, high energy scattering is dominated by the exchange of a ''Reggeized'' $\rho$ resonance. If no quantum number is exchanged, we say that the corresponding Regge pole is the vacuum, or a Pomeranchuk Regge pole; this name is often shortened to *Pomeron*. In a QCD picture, the Pomeron (for example) will be associated with the exchange of a gluon ladder between two partons in particles *A,B* (Fig. 5). The corresponding formalism has been developed by Dokshitzer, Gribov, Lipatov, and other Russian physicists in the 1970s, and is related to the so-called Altarelli-Parisi, or DGLAP, mechanism in deep inelastic scattering [21].

One of the useful properties of Regge theory is *factorization* [21]; it can be proved from general properties of Regge theory [22]. Factorization states that, for example, the imaginary part of the scattering amplitude $F_{A+B\to A+B}(s,t)$ can be written as a product

$$\mathrm{Im}\,F_{A+B\to A+B}(s,t)\underset{\substack{s\to\infty\\ t\ \text{fixed}}}{\simeq} f_A(t)f_B(t)(s/\hat{s})^{\alpha_R(t)}. \tag{23}$$

Here $\hat{s}$ is a constant, usually taken to be 1 GeV$^2$ (we will do so here); the functions $f_A, f_B$ depend on the corresponding particles (if we had external currents, also on their virtuality), but the power $(s/\hat{s})^{\alpha_R(t)}$ is universal and depends only on the quantum numbers exchanged in channel $t$. The exponent $\alpha_R(t)$ is the Regge trajectory associated with the quantum numbers in channel $t$ and, for small $t$, may be considered linear:

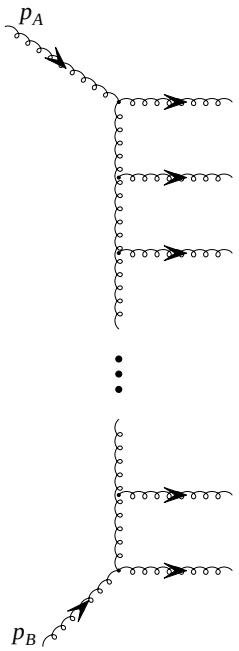


FIG. 5. Cut Pomeron ladder exchanged between the partons $p_A$ and $p_B$ in hadrons *A,B*. The emitted gluons will materialize into a shower of particles. The cross section is proportional to the square of the cut ladder.

$$\alpha_R(t)\underset{t\sim0}{\simeq}\alpha_R(0)+\alpha_R' t. \tag{24}$$

For the $\rho$ and Pomeron pole, fits to high energy processes give

$$\alpha_\rho(0)=0.52\pm0.02,\quad \alpha_\rho'=1.01\ \ \mathrm{GeV}^{-2},$$
$$\alpha_P(0)=1,\quad \alpha_P'=0.11\pm0.03\ \ \mathrm{GeV}^{-2}. \tag{25}$$

The Regge parameters taken here are essentially those in the global fit 1a of Rarita *et al.* [23]; for $\alpha_\rho(0)$, however, we take the value $0.52\pm0.02$ which is more consistent with recent determinations based on deep inelastic scattering [24]. The results depend very little on this.

Let us consider the imaginary part of the $\pi N$ or $NN$ scattering amplitudes (here by $NN$ we also understand $\bar{N}N$). We have

$$\mathrm{Im}\,F_{NN}^{(I_t)}(s,t)\simeq[f_N^{(I_t)}(t)]^2(s/\hat{s})^{\alpha_R(t)},$$
$$\mathrm{Im}\,F_{\pi N}^{(I_t)}(s,t)\simeq f_\pi^{(I_t)}(t)f_N^{(I_t)}(t)(s/\hat{s})^{\alpha_R(t)}. \tag{26}$$

For $I_t=1$, $R=\rho$; for $I_t=0$, $R=P$ (the Pomeron). Therefore, using factorization, we find

$$\mathrm{Im}\,F_{\pi\pi}^{(I_t)}(s,t)\simeq[f_\pi^{(I_t)}(t)]^2(s/\hat{s})^{\alpha_R(t)}. \tag{27}$$

The functions $f_i^{(I_t)}(t)$ depend exponentially on $t$ for small $t$ and may be written, approximately, as [25]

$$f_i^{(I_t=0)}(t)=\sigma_i(P)e^{bt},$$

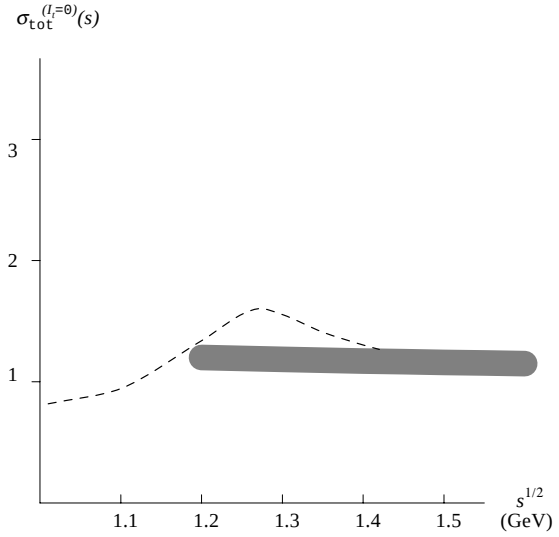


FIG. 6. The average cross section $\frac{1}{3}[2\sigma_{\pi^0\pi^+}+\sigma_{\pi^0\pi^0}]$, which is pure $I_t=0$, arbitrarily normalized. Broken line: experimental cross section. Note that the bump here, as the larger bumps in Figs. 5 and 6, is due to the coincidence of two resonances, $f_0(1270), f_2(1370)$, mostly elastic, around $s^{1/2}\sim 1.3$ GeV. Thick gray line: Regge formula Eq. (29). The thickness of the line covers the error in the theoretical value of the Regge residue.

$$f_i^{(I_t=1)}(t)=\sigma_i(\rho)\,\frac{1+\alpha_\rho(t)}{1+\alpha_\rho(0)}[(1+1.48)e^{bt}-1.48],$$

$$b=(2.38\pm 0.20)\ \ \mathrm{GeV}^{-2}. \tag{28}$$

The exponent $b$ appears to be the same for rho, Pomeron, and $P'$, within errors [23].

From Eqs. (26) and (27) we can deduce the relations among the cross sections

$$\frac{\sigma_{\pi\pi\to\mathrm{all}}}{\sigma_{\pi N\to\mathrm{all}}}=\frac{\sigma_{\pi N\to\mathrm{all}}}{\sigma_{NN\to\mathrm{all}}},$$

and from these relations one can obtain the parameters $\sigma_\pi$ in Eq. (28) in terms of the known $\pi N$ and $NN$ cross sections. Using this, we can write explicit formulas for $\pi\pi$ scattering with exchange of isospin $I_t=0$ in the $t$ channel

$$\mathrm{Im}\,F^{(I_t=0)}(s,t)\underset{\substack{s\to\infty\\ t\ \text{fixed}}}{\simeq}\left\{1+0.24\sqrt{\frac{\hat{s}}{s}}\right\}\sigma_\pi(P)e^{bt}(s/\hat{s})^{\alpha_P(0)+\alpha_P' t}, \tag{29}$$

and we have added empirically the subleading contribution, proportional to $\sqrt{\hat{s}/s}$, of the so-called $P'$ pole (associated with the $f_2$ resonance) that is necessary at the lowest energy range (see Fig. 6). For $I_t=1$,

$$\mathrm{Im}\,F^{(I_t=1)}_{\pi\pi\to\pi\pi}(s,t)\underset{\substack{s\to\infty\\ t\ \text{fixed}}}{\simeq}\mathrm{Im}\,F^{(\rho)}(s,t)+\mathrm{Im}\,F^{(\mathrm{Bk})}(s,t),$$

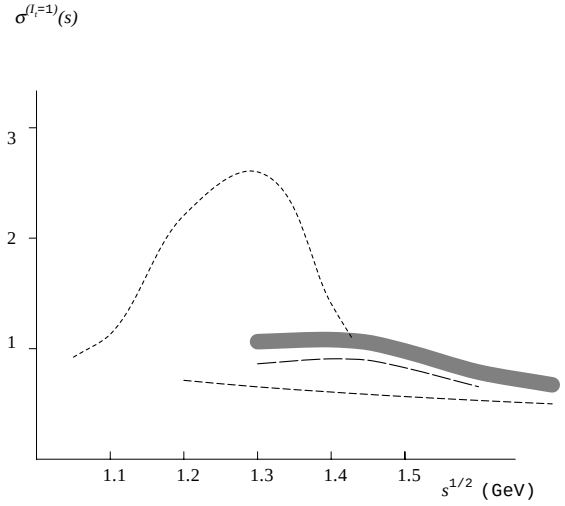


FIG. 7. The cross section $\sigma^{(I_t=1)}$, for isospin 1 in the $t$ channel, arbitrarily normalized. The dotted line is experiment; the short-dashed line the rho exchange Regge theory. The long-dashed line is obtained by adding to this the $\rho(1450)$ contribution. Finally, the thick gray line includes also the background Regge piece. The thickness of this line is equal to the error due *only* to the rho Regge residue (the total error for the full theoretical formula used in the text, that includes errors due to $\rho(1450)$ and the background Regge piece, is some 20% larger).

$$\mathrm{Im}\,F^{(\rho)}(s,t)=\sigma_\pi(\rho)\,\frac{1+\alpha_\rho(t)}{1+\alpha_\rho(0)}[(1+1.48)e^{bt}-1.48]\times(s/\hat{s})^{\alpha_\rho(0)+\alpha_\rho' t},$$

$$\mathrm{Im}\,F^{\mathrm{Bk}}(s,t)=(0.4\pm 0.1)\left(\frac{\hat{s}}{s}\right)^{1/2}\mathrm{Im}\,F^{(\rho)}(s,t). \tag{30}$$

We have added a background (Bk) contribution to the isospin 1 amplitude; this should be considered purely empirical and is adjusted so that the asymptotic formula joins smoothly the experimental amplitude at low energy, within errors; see Fig. 7.

From Eq. (28) and the known cross sections for $\pi N, NN$ scattering we have

$$\sigma_\pi(P)=3.0\pm 0.3,\quad \sigma_\pi(\rho)=0.84\pm 0.10, \tag{31}$$

where the errors are obtained by considering the dispersion of the values of the parameters in [23], and increasing the result by 50%, which should cover amply the uncertainty on the point where one joins experimental and asymptotic formulas (that here we have taken to be 1.42 GeV) as well as errors in the parameters we have taken fixed.

It is important to note that the Regge parameters in the fit of Rarita *et al.* are obtained by global fit to $\pi N$, $NN$, and $\bar{N}N$ data for small momentum transfer and for c.m. kinetic energies in the region between 1 GeV and 6 GeV (approximately), which is the region of interest for us here as the contribution to the various integrals above this energy is negligible. The results of Rarita *et al.* are still the best available as indeed there are essentially no new data in that energy

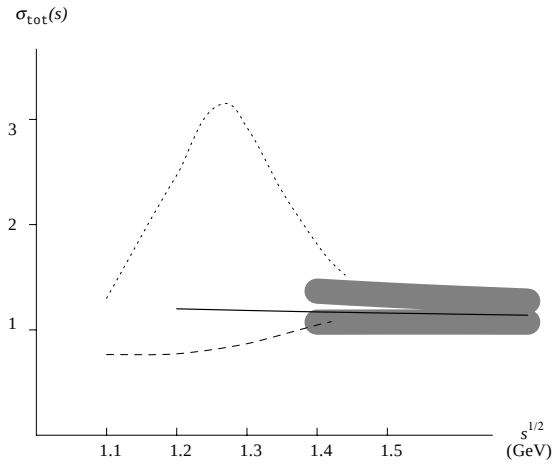


FIG. 8. The cross sections $\sigma(\pi^0\pi^+)$ (dashed line), $\sigma(\pi^0\pi^0)$ (dotted line), and the Pomeron plus $P'$ (continuous line). The thick gray bands are obtained including the $I=2$ exchange contributions. Their thickness corresponds only to the error of the Pomeron piece.

range. We have, on the other hand, verified that the cross sections are compatible with the corresponding values as given in the more recent editions of the Particle Data Tables.

We will treat the errors in the various Regge parameters as uncorrelated. In fact, the leading Regge amplitudes (Pomeron and rho) are uncorrelated; there is some correlation with, respectively, the $I_t=2$ exchange (see below) and the Bk piece for the $I_t=1$ exchange, because they have been fixed by fitting the sum to the pion cross sections. Since this only affects subleading pieces this would only have a minute influence in the results (in fact, they would slightly decrease the overall error due to the Regge contributions), and anyway the variations are substantially smaller than the 50% increase in the errors of the Regge residues with which we have made our evaluations.

For each individual process $\pi^0\pi^+,\pi^0\pi^0$, we have to incorporate the amplitude for exchange of isospin $I_t=2$ in the $t$ channel, which would be due to the double rho exchange. This cannot be obtained from factorization, since $\pi N$ or $NN$ do not contain such amplitude. We use an empirical formula,

$$\mathrm{Im}\,F^{(I_t=2)}(s,t)=C_2 e^{-bt}[\mathrm{Im}\,F^{(\rho)}(s,t)]^2\left(\frac{\hat{s}}{s}\right),$$

$$C_2=0.8\pm 0.2, \quad (32)$$

and we have obtained the constant $C_2$ by fitting the difference between the experimental $\pi^0\pi^0$ and $\pi^0\pi^+$ total cross sections at $s^{1/2}=1.42$ GeV, and the Pomeron plus $P'$ values; see Fig. 8.

The dependence of our results on $\mathrm{Im}\,F^{(\mathrm{Bk})},\mathrm{Im}\,F^{(I_t=2)}$ is very slight (for the second, with the exception of the $b_2^{(I)}$).

We now add a few words on the matter of when one may apply formulas like Eqs. (29)–(32). From the QCD, DGLAP version of the Pomeron, we expect the following pattern to occur: in the region $|t|\ll s$, $s\gg\Lambda^2$ (with $\Lambda\sim 0.3$ GeV the QCD parameter) the ladder exchange mechanism will start to dominate the collision $A+B$. We then will have the onset of the Regge regime with, at the same time, a large increase of inelasticity and a smoothing of the total cross section according to the behavior in Eqs. (29)–(31).

For $\pi N,NN$ scattering this occurs as soon as one is beyond the region of elastic resonances; in fact (as can be seen in the cross section summaries in the Particle Data Tables) as soon as the kinetic energy or laboratory momentum is above 1 to 1.2 GeV. For $\pi\pi$ we thus expect the Regge description to be valid for the corresponding energies, that is to say, for $s^{1/2}\gtrsim 1.4$ GeV. Indeed, around $s^{1/2}\sim 1.4$ GeV it is still possible to calculate the $\pi\pi$ scattering amplitudes reliably from experimental phase shifts and indeed they agree, within a 10%, with the Regge expressions in the $\pi^0\pi$ cases; see Figs. 6 and 8. Moreover, the experimental inelasticity for $\pi\pi$ around 1.4 GeV, ~20%, also agrees with the value of the inelasticity measured at the same energies for $\pi N$ or $NN$ scattering.

For the $I_t=1$ amplitude, and because it is a difference between large amplitudes, the influence of resonances may be expected to extend to higher energies. Indeed, we see in Fig. 7 that agreement between experiment and the Regge expression (within errors) around 1.4 GeV requires adding the resonance $\rho(1450)$, as in Eq. (19). We will do so in our calculations. Thus, for all $\pi\pi$ amplitudes we will assume the Regge formula [eventually adding the $\rho(1450)$ contribution] to be valid for $s^{1/2}\geqslant 1.42$ GeV.

As is clear from this minireview, the reliability of the Regge calculation of high energy pion-pion scattering cannot go beyond an accuracy of ~10%, even for small $t$. The deviations off simple Regge behavior are expected to be much larger for large $|t|$, because the counting rules of QCD imply a totally different behavior for fixed $t/s$. This is one of the problems involved in using, e.g., the Roy equations that require integration up to $-t\sim s\sim 1.7$ GeV$^2$, where the Regge picture fails completely (we expect instead the Brodsky-Farrar behavior, $\sigma_{\text{fixed cos }\theta}\sim s^{-5}$). However, for forward dispersion relations or the Froissart-Gribov representation we will work only for $t=0$ or $t=4M_\pi^2$ for which the largest variation, that of $e^{bt}$, is still small, since $b\times(t=4M_\pi^2)\simeq 0.19$. So we expect no large error due to departure off linearity [26] for the exponent in $f_i^{(I_t)}(t)$ or for the Regge trajectories, $\alpha_R(t)$.

## IV. OLSSON'S SUM RULE AND THE FROISSART-GRIBOV CALCULATION OF $a_1,b_1,a_2^{(I)},b_2^{(I)}$

### A. The Olsson sum rule

The Olsson sum rule is simply a forward dispersion relation for the amplitude $F^{(I_t=1)}$ with isospin 1 in the $t$ channel, evaluated at threshold. Expressing $F^{(I_t=1)}(4M_\pi^2,0)$ in terms of the scattering lengths, this reads

$$2a_0^{(0)}-5a_0^{(2)}=D_{\mathrm{O}},\quad D_{\mathrm{O}}=3M_\pi\int_{4M_\pi^2}^{\infty}ds\,\frac{\mathrm{Im}\,F^{(I_t=1)}(s,0)}{s(s-4M_\pi^2)}. \quad (33)$$

In terms of isospin in the $s$ channel,

$$F^{(I_t=1)}(s,t)=\tfrac{1}{3}F^{(I_s=0)}(s,t)+\tfrac{1}{2}F^{(I_s=1)}(s,t)-\tfrac{5}{6}F^{(I_s=2)}(s,t), \tag{34}$$

the $F^{(I_s)}$ are normalized by

$$\mathrm{Im}\,F^{(I_s)}(s,t)=2\,\frac{2s^{1/2}}{\pi k}\sum_l\,(2l+1)P_l(\cos\theta)\mathrm{Im}\,\hat{f}_l^{(I_s)}(s). \tag{35}$$

Substituting in the right-hand side above the *S,P* phases of CGL up to 0.82 GeV, the phases (as given in the parametrizations of Sec. III B) for the same at intermediate energies ($0.82\leqslant E\leqslant 1.42$ GeV), the *D,F* phases from Eqs. (20)–(22), the tail of the $\rho(1450)$ resonance between 1.42 and 1.6 GeV, and the $\rho$ plus background Regge parameters of Sec. III D we find, for $2a_0^{(0)}-5a_0^{(2)}$ in units of $M_\pi$,

| CGL, direct | CGL, dispersive | |
|---|---|---|
| | $0.400\pm 0.007$ | (CGL *S,P*, $s^{1/2}\leqslant 0.82$ GeV) |
| | $0.146\pm 0.004$ | [Rest, $s^{1/2}\leqslant 1.42$ GeV (incl., *D,F* below 0.82 GeV)] |
| | $0.073\pm 0.010$ | (Regge, $\rho$, $s^{1/2}\geqslant 1.42$ GeV) |
| | $0.010\pm 0.003$ | (Regge, Bk; $s^{1/2}\geqslant 1.42$ GeV) |
| | $0.005\pm 0.001$ | $\rho(1450)$, $1.42\leqslant s^{1/2}\leqslant 1.6$ GeV |
| $0.663\pm 0.007$ | $0.635\pm 0.014$ | (Total, dispersive). |

(36)

By ''direct'' we mean the value of the corresponding quantity (in our case, $2a_0^{(0)}-5a_0^{(2)}$) as given in CGL. By ''dispersive'' we understand that we have used the dispersive formula, $D_{\mathrm{O}}$ in Eq. (33), to calculate the same quantity. The ''Rest'' are the contributions of the *D,F* waves below 1.42 GeV, plus the *S,P* waves between 0.82 and 1.42 GeV. Of this ''Rest,'' the largest contribution comes from the $D0$ and $P$ waves.

The error in the CGL *S,P* piece below 0.82 GeV we obtain by varying the *A,B,C,D* parameters in Eqs. (9) and (10) according to the formulas given by ACGL (in their Appendix). It is almost identical to the error given for the whole of the direct quantity itself. We will discuss in some detail the discrepancy between the direct and dispersive determinations of this quantity $2a_0^{(0)}-5a_0^{(2)}$ as the situation for the other *a*'s and *b*'s to be considered below will be very similar.

The reason the analysis of the discrepancy is not straightforward is that both determinations are strongly correlated, as they both depend on the same parameters. The direct determination is obtained from the parameters *A,B,C,D* in CGL [as given in Eq. (3.1) here], which describe in particular the *S* waves. So we should really write

$$[2a_0^{(0)}-5a_0^{(2)}]^{\text{“direct”}}_{A,B,C,D}.$$

The integrals in the dispersive determination contain the contributions of the *S,P* waves up to 0.82 GeV, which are given by the same *A,B,C,D*, so one also has

$$[2a_0^{(0)}-5a_0^{(2)}]^{\text{“dispersive},S,P\ \text{CGL”}}_{A,B,C,D}.$$

Now, it is clear that if we try to change the parameters *A,B,C,D* in the direct determination to, for example, decrease the value of $a_0^{(0)}$ to bring this closer to the dispersive value, the same change in the *A,B,C,D* will affect the integral over the $S0$ wave in $[2a_0^{(0)}-5a_0^{(2)}]^{\text{“dispersive},S,P\ \text{CG”}}_{A,B,C,D}$, making this smaller and therefore nullifying to a large extent the improvement.

What one has to do to solve this problem is to consider the difference $\Delta=2a_0^{(0)}-5a_0^{(2)}-D_{\mathrm{O}}$ and vary here the parameters *A,B,C,D*. Then we find the value

$$\Delta=0.027\pm 0.011,$$

that is to say, a $2.5\sigma$ discrepancy.

This procedure will also be followed for the Froissart-Gribov sum rules, where the correlation in the CGL analysis is transmitted in part by the common chiral perturbation theory parameters $\bar{l}_i$. We will discuss more about errors in Secs. IV D and V A.

### B. The Froissart-Gribov representation: $a_1, b_1$

By projecting the dispersion relation Eq. (1) (or a derivative with respect to $t$ of it) over the $l$th partial wave in the $t$ channel, at $t=4M_\pi^2$, one finds the Froissart-Gribov representation

$$a_l=\frac{\sqrt{\pi}\Gamma(l+1)}{4M_\pi(l+3/2)}\int_{4M_\pi^2}^{\infty}ds\,\frac{\mathrm{Im}\,F(s,4M_\pi^2)}{s^{l+1}},$$

$$b_l=\frac{\sqrt{\pi}\Gamma(l+1)}{2M_\pi(l+3/2)}\int_{4M_\pi^2}^{\infty}ds\left\{\frac{4\,\mathrm{Im}\,F'_{\cos\theta}(s,4M_\pi^2)}{(s-4M_\pi^2)s^{l+1}}-\frac{(l+1)\mathrm{Im}\,F(s,4M_\pi^2)}{s^{l+2}}\right\}, \tag{37}$$

$\mathrm{Im}\,F'_{\cos\theta}\equiv(\partial/\partial\cos\theta_s)\mathrm{Im}\,F$. For amplitudes with fixed isospin in the $t$ channel, an extra factor 2 (due to identity of particles) has to be added to the left-hand side; so we have, for example,

$$2a_l^{(I=1)}=\frac{\sqrt{\pi}\Gamma(l+1)}{4M_\pi\Gamma(l+3/2)}\int_{4M_\pi^2}^{\infty}ds\,\frac{\mathrm{Im}\,F^{(I_t=1)}(s,4M_\pi^2)}{s^{l+1}}. \tag{38}$$

With the same type of calculation as for the Olsson sum rule, and with the same definitions, we now find, in units of $10^{-3}\times M_\pi^{-3}$,

| $a_1$,CGL, direct | CGL, Froissart-Gribov | TY (St.) | TY (St.+Sys.) | |
|---|---|---|---|---|
| | $18.5\pm0.2$ (CGL $S,P$, $s^{1/2}\leqslant 0.82$ GeV) | | | |
| | $9.1\pm0.3$ (Rest, $s^{1/2}\leqslant 1.42$ GeV) | | | |
| | $8.1\pm1.1$ (Regge, $\rho$) | | | |
| | $1.0\pm0.3$ (Regge, Bk) | | | |
| | $0.3\pm0.1$, $\rho(1450)$ | | | |
| $37.9\pm0.5$ | $37.1\pm1.3$ (Total, Froissart-Gribov) | $40.6\pm1.4$ | $38.6\pm1.2$. | (39) |

Here, and for $b_1$, we profit from the existence of an independent determination of the $P$ wave parameters, using the pion form factor data both in the timelike and in the spacelike regions [15] denoted by TY. From this we have chosen two values: from the fit taking into account only the statistical errors in the various data sets (St.), as in Eq. (13) here; or taking also into account the systematic normalization errors (St.+Sys.), as in Eq. (15).

The distance between the direct evaluation and the one with the Froissart-Gribov calculation is now $0.6\sigma$, and there is also acceptable overlap with the TY (St.+Sys.) figure.

For the quantity $b_1$ we have, in units of $10^{-3}\times M_\pi^{-5}$,

| $b_1$,CGL, direct | CGL, Froissart-Gribov | TY (St.) | TY (St.+Sys.) | |
|---|---|---|---|---|
| | $-0.92\pm0.05$ (CGL $S,P$, $s^{1/2}\leqslant 0.82$ GeV) | | | |
| | $1.02\pm0.04$ (Rest, $s^{1/2}\leqslant 1.42$ GeV) | | | |
| | $5.33\pm0.86$ (Regge, $\rho$) | | | |
| | $0.55\pm0.16$ (Regge, Bk) | | | |
| | $0.01\pm0.0$, $\rho(1450)$ | | | |
| $5.67\pm0.13$ | $5.99\pm0.88$ (Total, Froissart-Gribov) | $4.18\pm0.43$ | $4.47\pm0.29$. | (40) |

Here the Regge contribution is particularly important because the lower energy pieces cancel almost completely. The numbers labeled TY, as before, refer to what one obtains from the fit to the pion form factor. We remark that this last is a very robust determination in that it is obtained by fitting some 210 points from several independent experiments, is independent of high energy assumptions and it covers spacelike as well as timelike momenta: thus, the values of the threshold parameters are obtained by *interpolation*, notoriously more stable than extrapolations.

There is no inconsistency between the direct and Froissart-Gribov numbers for the CGL calculation, but they are both too large by almost $4\sigma$ compared to even the more favorable value, TY (St.+Sys.), following from the pion form factor.

### C. The Froissart-Gribov representation: $a_2^{(I)}, b_2^{(I)}$; $I=0,2$

We first calculate the two combinations of scattering lengths $a_{0+}=\frac{2}{3}[a_2^{(0)}-a_2^{(2)}]$ and $a_{00}=\frac{2}{3}[a_2^{(0)}+2a_2^{(2)}]$. They correspond to the $s$-channel amplitudes

$$F_{\pi^0\pi^+}=\tfrac{1}{2}F^{(I_s=1)}+\tfrac{1}{2}F^{(I_s=2)},\quad F_{\pi^0\pi^0}=\tfrac{1}{3}F^{(I_s=0)}+\tfrac{2}{3}F^{(I_s=2)}. \tag{41}$$

The only important difference with the cases in the previous section is that the dominant high energy part is given now by the Pomeranchuk trajectory (instead of the rho) and its importance is small because the integrals converge faster. We find, in units of $10^{-4}\times M_\pi^{-5}$,

| $a_{0+}$,CGL, direct | CGL, Froissart-Gribov | |
|---|---|---|
| | $8.43\pm0.09$ (CGL $S,P$, $s^{1/2}\leqslant 0.82$ GeV) | |
| | $1.84\pm0.05$ (Rest, $s^{1/2}\leqslant 1.42$ GeV) | |
| | $0.68\pm0.07$ (Regge, $I_t=0$) | |
| | $-0.06\pm0.02$ (Regge, $I_t=2$) | |
| | $0.04\pm0.01$, [$\rho(1450)$] | |
| $10.53\pm0.10$ | $10.94\pm0.13$ (Total, Froissart-Gribov). | (42) |

In finding the error of the direct value, $(10.53\pm0.10)\times10^{-4}M_\pi^{-5}$, it is important to take into account the strong correlations of the errors of the $a_2^{(0)}, a_2^{(2)}$. To do this, we use Eq. (14.4) in ACGL to calculate directly the quantity $a_{0+}$. The difference between the direct and Froissart-Gribov values, with correlations taken into account, as we did in the case of the Olsson sum rule, is now

$$0.41\pm0.09,$$

so that the discrepancy reaches the $4\sigma$ level.

In the same units, $10^{-4}\times M_\pi^{-5}$, we have

| $a_{00}$, CGL, direct | CGL, Froissart-Gribov |
|---|---|
| | $11.73\pm0.32$ [CGL $S,P$, $s^{1/2}\leqslant0.82$ GeV] |
| | $1.91\pm0.04$ (Rest, $s^{1/2}\leqslant1.42$ GeV) |
| | $0.68\pm0.07$ (Regge, $I_t=0$) |
| | $0.12\pm0.04$ (Regge, $I_t=2$) |
| $13.94\pm0.32$ | $14.44\pm0.33$ (Total, Froissart-Gribov). |

(43)

We have also taken into account the correlations *à* la ACGL to evaluate the error of the direct number. The difference between direct and Froissart-Gribov (F.-G.) values for CGL are, with correlations taken into account, of

$$0.49\pm0.09,$$

i.e., a $5\sigma$ discrepancy.

Finally, we present the results for $b_{0+}=\frac{2}{3}[b_2^{(0)}-b_2^{(2)}]$ and $b_{00}=\frac{2}{3}[b_2^{(0)}+2b_2^{(2)}]$, both in units of $10^{-4}\times M_\pi^{-7}$:

| $b_{0+}$, CGL, direct | CGL, Froissart-Gribov |
|---|---|
| | $-0.331\pm0.015$ (CGL $S$, $P$, $s^{1/2}\leqslant0.82$ GeV) |
| | $0.04\pm0.00$ (Rest, $s^{1/2}\leqslant1.42$ GeV) |
| | $0.12\pm0.02$ (Regge, $I_t=0$) |
| | $-0.05\pm0.02$ (Regge, $I_t=2$) |
| $-0.189\pm0.016$ | $-0.233\pm0.036$ (Total, Froissart-Gribov). |

(44)

The contribution of the resonance $\rho(1450)$ is now negligible. For the difference between the direct and Froissart-Gribov result we have

$$0.044\pm0.026,$$

that is to say, almost a $2\sigma$ discrepancy. For $b_{00}$,

| $b_{00}$, CGL, direct | CGL, Froissart-Gribov |
|---|---|
| | $-6.90\pm0.22$ (CGL $S$, $s^{1/2}\leqslant0.82$ GeV) |
| | $0.07\pm0.01$ (Rest, $s^{1/2}\leqslant1.42$ GeV) |
| | $0.12\pm0.02$ (Regge, $I_t=0$) |
| | $0.10\pm0.05$ (Regge, $I_t=2$) |
| $-6.72\pm0.22$ | $-6.62\pm0.23$ (Total, Froissart-Gribov). |

(45)

For $b_{00}$ the direct result and the one following from the Froissart-Gribov representation differ by $2\ \sigma$:

$$0.10\pm0.05.$$

However, one cannot take this or the discrepancy for $b_{0+}$ as seriously as in the previous cases. This is so because of the large (relative) size of the contribution of the $I_t=2$ exchange piece, proportional to the derivative with respect to $t$ of an expression we have obtained purely empirically by fitting at $t=0$.

### D. How significant are the discrepancies?

In the present section we investigate whether the inconsistencies we have found can be eliminated (or to what extent they can be made less severe) by altering the non-CGL part of the dispersive, or Froissart-Gribov calculations. We will do so in two steps. First, we will consider what happens if we alter the pieces labeled ''Rest'' in Eqs. (36)–(45); then we will address the question of what can be done at high energy ($s^{1/2}\geqslant1.42$ GeV).

#### *1. The region between 0.82 and 1.42 GeV*

We start with the first question that we discuss in detail for the Olsson sum rule since the results for the Froissart-

Gribov calculations are very similar. We then consider the following set of drastic modifications of our calculations: For the $S0$ wave, and $0.82 \leqslant E \leqslant 0.992$ GeV we may replace Eq. (11) by the CGL parametrization, Eqs. (9) and (10). For the $S0$ wave and $0.992 \leqslant E \leqslant 1.42$ GeV, where it is poorly known, we allow $\delta_0^{(0)}$ to vary between the two extreme values $\pi$ and $3\pi/2$. For the $S2$ wave, we multiply by 3 the errors given in Eq. (12). For the $P$ wave, and $1 \leqslant E \leqslant 1.42$ GeV, we change the elasticity of the $\rho(1450)$ resonance by 50% (up and down). For the $D0$ wave, that supplies the more important contribution to ''Rest,'' we consider the effect of taking the $f_2(1270)$ resonance to be purely elastic, or 30% inelastic. The remaining contributions to ''Rest'' are so small that we need not worry about them.

The alterations just discussed are rather extreme; nevertheless, their effects are of no relevance. They produce the following extra errors (we give the central value of each term as well):

$$S0,\ 0.82 \leqslant s^{1/2} \leqslant 0.992\ \text{GeV}: \quad 0.026^{+0.0}_{-0.006},$$

$$S0,\ 0.992 \leqslant s^{1/2} \leqslant 1.42\ \text{GeV}: \quad 0.018^{+0.005}_{-0.013},$$

$$S2,\ 0.82 \leqslant s^{1/2} \leqslant 1.42\ \text{GeV}: \quad -0.022 \pm 0.004,$$

$$P,\quad 1.0 \leqslant s^{1/2} \leqslant 1.42\ \text{GeV}: \quad 0.024 \pm 0.005,$$

$$D0,\quad s^{1/2} \leqslant 1.42\ \text{GeV}: \quad 0.055 \pm 0.001.$$

Including these increased errors we get that, for the Olsson sum rule, the result for the ''Rest'' changes according to

$$\text{Rest:}\quad 0.145 \pm 0.004 \rightarrow 0.145^{0.009}_{-0.016},$$

and, for the whole dispersive result, we now get

$$\text{Total:}\quad 0.631 \pm 0.013 \rightarrow 0.631^{+0.015}_{-0.019},$$

i.e., practically no change at all in the upper error bar. The mismatch of the Olsson sum rule merely gets reduced from 2.5- to $2\sigma$.

#### *2. The high energy region, $s^{1/2} \geqslant 1.42$ GeV*

Once we have verified that the inconsistencies between the CGL direct and dispersive calculations of low energy parameters cannot be due to the contributions of the intermediate energy region, we turn to the high energy ($s^{1/2} \geqslant 1.42$) piece. Then, we relax the condition of factorization for the $\rho$ and Pomeron Regge residues (but we do not change the others). We treat them now as free parameters, describing an *effective* scattering amplitude, to see under which conditions one can reconcile the direct and Froissart-Gribov (or dispersive) evaluations for the scattering lengths and effective range, in the CGL-like analysis. Starting with the isospin 1 case, we thus write

$$\operatorname{Im} F_{\text{eff}}^{(\rho)}(s,t) \underset{\substack{s\to\infty \\ t\ \text{fixed}}}{\simeq} \lambda \sigma_\pi(\rho) \frac{1+\alpha_\rho(t)}{1+\alpha_\rho(0)} [(1+1.48)e^{bt} - 1.48] \times (s/\hat{s})^{\alpha_\rho(0)+\alpha'_\rho t},$$

that is to say, we modulate the $\rho$ amplitude in Eq. (30) by the constant $\lambda$. We then fix $\sigma_\pi(\rho) = 0.85$, and treat $\lambda$ as a free parameter. We then find that overlap between the direct and dispersive determinations for the quantity $2a_0^{(0)} - 5a_0^{(2)}$ involved in the Olsson sum rule would require $\lambda = 1.4$, which is well outside expectations and, moreover, this spoils the overlap for $a_1, b_1$, which become inconsistent at the 2 to 2.5 $\sigma$ level.

For the $a_2^{(I)}$ the situation is even more transparent. Consider, for example the quantity $a_{00}$, Eq. (43). Integrating only to 0.82, with the CGL phases, we find 11.73, which is the bulk of the result. Even if the errors of what we call ''Rest'' were underestimated by a factor 3, and this ''Rest'' would be 1.79 (instead of 1.91), adding it one would get at least $13.52 \pm 0.33$ for the contribution below 1.42 GeV. The direct result, with the CGL values of the $a_l^I$, is 13.94. To get agreement, one would require the high energy, $E > 1.42$ (Regge) estimate to be wrong by a factor 2, very difficult to believe. And it would be no good: the same Pomeron that contributes to $a_{00}$ contributes to $a_{0+}$ and to the $b_{0+}, b_{00}$. The disagreement would be shifted to the $b_{0+}, b_{00}$, which would then be wrong by about $4\sigma$, and $a_{0+}$ would still be wrong by almost $2\sigma$. As for the proverbial square peg in the round hole, trying to fit a corner only makes others worsen more sharply.

## V. DISCUSSION OF THE ACGL AND CGL ANALYSES

### A. Possible cause of the distortion of the CGL solution

In this section we try to ascertain the reasons for the troubles that seem to afflict the CGL analysis. This is particularly important because, although ACGL or CGL did not verify the Froissart-Gribov relations, they *did* check relations similar to the Olsson sum rule. It follows that the reasons for the discrepancies must be due to the high energy input. Here you have two regions: between 0.82 and 1.42 GeV (more or less) the inelasticity is low, and, as we have shown, one can trust the experimental phase shifts. Even if they have systematic errors, these will likely not be large and they will just produce a slight fluctuation of the solution of the Roy equations, as we have shown explicitly in Sec. IV D 1 that it occurs for our evaluations.

The difficult region, however, is for $s^{1/2}$ above 1.42 GeV. Between 1.42 and 2 GeV, CGL presumably use the phase shifts of [18] and, above 2 GeV, a Regge-type formula. We start the discussion with the region $1.42 \leqslant s^{1/2} \leqslant 2$ GeV. Here inelasticity is very high, and the phase shifts and inelasticity parameters cannot be determined reliably, at the level of accuracy required [27]. Of course, you can always give numbers that fit the experimentally observed moments in peripheral two-pion production; but so will other, in some cases very different values of $\delta$s and $\eta$s. In the energy region 1.4

$\leqslant s^{1/2} \leqslant 2$ GeV, the phase shifts and inelasticities all stem from a single set of experiments and are likely to disagree with reality by much more than their nominal errors. In fact, this can be seen to occur for the S wave even at lower energy: as soon as the $\bar{K}K$ channel opens, the Cern-Munich phase shifts [18] disagree violently with the Berkeley [17] ones. This emphasizes the dangers of relying on a single experiment for the phase shifts, as one has to do already for $s^{1/2} \geqslant 1.2$ GeV.

It is not difficult to see how different phases may give similar results, for the elastic cross section. For example, consider the *elastic* $\pi\pi$ cross section, in the *P* wave: in both cases (Cern-Munich and Particle Data Tables results) it is small. In the Cern-Munich one, because $\sin^2 \delta_1$ is small; in the other because $\eta$ is small. Unfortunately, the *imaginary* parts of the inelastic amplitudes are very different; contrary to the Cern-Munich results, in the PDG case it would be large, at least around the resonances, because of the contribution of the inelastic channels. The converse (i.e., overestimate of the total cross section) may, of course, also happen. In fact, the cases mentioned are just examples of an ambiguity (over and above that due to experimental errors) proved to exist quite generally in [29], and which is likely to be large as soon as you have important inelastic channels open.

Now, CGL (following Pennington [30]) take the Cern-Munich phase shifts, that probably contain large and unknown systematic errors, and impose sum rules [e.g., the sum rules (B.6,7), (C.2) in ACGL], following from low energy crossing symmetry, to fix the Regge parameters at energies $E > 2$ GeV. Not surprisingly, CGL (and Pennington [31]) get irrealistic Regge parameters (as realized by CGL themselves); for example, ACGL and CGL get a Pomeron with a width of the diffraction peak which is *s* independent, and twice the standard value (at low *s*), and a residue much smaller than what factorization implies. In fact, we will show in the Appendix explicit calculations of two sum rules [in particular of the sum rule (B.7), one of the crossing sum rules that Pennington and ACGL use] which are perfectly satisfied by a standard Regge amplitude, with factorization for the rho and Pomeron trajectories, provided one uses Regge asymptotics from $s^{1/2} = 1.42$ GeV.

According to CGL this deviation from conventional Reggeistics is not important because the influence of the high energy region ($s^{1/2} \geqslant 1.42$ GeV) into their low energy ($s^{1/2} \leqslant 0.82$ GeV) phase shifts is very slight. However, and as we have shown in the present paper, inconsistencies show up as soon as one considers sum rules—like the Froissart-Gribov sum rules—that are sensitive to the high energy behavior of the amplitudes.

From the previous analysis it thus follows that CGL start, in the Roy equations, from a *V* with incorrect Regge behavior and dubious phase shifts above 1.42 GeV. Let us call this *V* (Wrong R). CGL run this through the Roy equations Eq. (8) and find a solution, $\xi$(Wrong R). Now, this solution is not horrendous because experimental low energy data, chiral perturbation theory and crossing sum rules force you to have the errors in Regge parameters and cross sections compensating, to a certain extent, in what regards their low energy effects.

Indeed, the independence on the low energy partial waves on the high energy amplitudes used is approximately true for the ACGL results, where the mismatch that occurs if using the correct Regge asymptotics stays below the $2\sigma$ level. However, for the CGL results, the use of chiral perturbation theory (with neglect of higher order corrections) has the dual effect of highly correlating the various low energy parameters and excessively decreasing the errors. Thus, for example, the value for the quantity $a_{+0}$ that follows from the Froissart-Gribov representation, $10.94 \pm 0.13$ (in units of $10^{-4} M_\pi^{-5}$) is displaced $4\sigma$ from the value following directly from the parameters of CGL, $10.53 \pm 0.10$. Now, $a_{+0}$ is directly related to the chiral constant $\bar{l}_2$, $a_{+0} = [\bar{l}_2 - 27/20]/720\pi^3 f_\pi^4 M_\pi$. Hence a variation of $a_{+0}$ implies a corresponding variation of $\bar{l}_2$ that destabilizes all the quantities that depend on it in a chiral perturbative analysis; in particular, the low energy *S* and *P* waves. As we have shown in the present paper, inconsistencies show up in the CGL scattering amplitude (with standard Regge parameters) as soon as one considers sum rules that, like the Froissart-Gribov or Olsson ones, are sensitive to the high energy behavior. What the inconsistencies found in the previous section show is that the distortion is several times larger than the nominal CGL error bars.

### B. A tentative alternate solution

In support of the idea that the effects discussed in the previous section are indeed the cause of the mismatches in the CGL *S* matrix, we have calculated the Olsson sum rule and the quantities $a_l$, $b_1$ using now, for $s^{1/2} \leqslant 0.82$ GeV, the results of the fit, wave by wave, reported in Sec. 7.6 of [8]. For the wave $S0$ we take now the fit obtained imposing the value $\delta_0^{(0)}(M_K^2) = 43.3 \pm 2.3°$, and with only three parameters [32]; we then have

$$\cot \delta_0^{(0)}(s) = \frac{s^{1/2}}{2k} \frac{M_\pi^2}{s - \frac{1}{2} M_\pi^2} \frac{M_\sigma^2 - s}{M_\sigma^2} \left\{ B_0 + B_1 \frac{\sqrt{s} - \sqrt{s_0 - s}}{\sqrt{s} + \sqrt{s_0 - s}} \right\},$$

$$B_0 = 21.04, \quad B_1 = 6.62, \quad M_\sigma = 782 \pm 24 \text{ MeV},$$

$$\frac{\chi^2}{\text{DOF}} = \frac{15.7}{19 - 3}, \quad a_0^{(0)} = (0.230 \pm 0.010),$$

$$\delta_0^{(0)}(M_K) = 41.0° \pm 2.1°. \quad (46)$$

The errors of the $B_i$ are strongly correlated; uncorrelated errors are obtained if replacing the $B_i$ by the parameters *x*,*y* with

$$B_0 = y - x, \quad B_1 = 6.62 - 2.59x. \tag{47}$$

Then,

$$y = 21.04 \pm 0.75, \quad x = 0 \pm 2.4. \tag{48}$$

The solution is shown, compared to the CGL phase, in Fig. 9. We then integrate Eqs. (46), (47), and (48) up to $E = 0.82$ GeV and with Eq. (23) from the $0.82s$ to the $\overline{K}K$ threshold. For $S2,P$ we take the same fits as before, specifically, Eqs. (12) and (15).

We find the following results, in units of $M_\pi$:

| Olsson | direct | dispersive | |
|---|---|---|---|
| | $0.691 \pm 0.042$ | $0.659 \pm 0.020$ | (49) |

[here "direct" means that we take the values following from the fits in Eq. (12) and Eqs. (46)–(48)]. Moreover, and also in units of $M_\pi$,

| $a_1$ | direct (TY, St.) | direct (TY, St.+Sys.) | Froissart-Gribov | |
|---|---|---|---|---|
| | $(40.6 \pm 1.4) \times 10^{-3}$ | $(38.6 \pm 1.2) \times 10^{-3}$ | $(37.9 \pm 1.4) \times 10^{-3}$ | (50) |

and

| $b_1$ | direct (TY, St.) | direct (TY, St.+Sys.) | Froissart-Gribov | |
|---|---|---|---|---|
| | $(4.18 \pm 0.43) \times 10^{-3}$ | $(4.47 \pm 0.29) \times 10^{-3}$ | $(5.69 \pm 0.96) \times 10^{-3}$. | (51) |

The tag "direct" now refers to the values of [15], with only statistical errors (St) or including also systematic errors (St.+Sys.). Thus, we find agreement at the $1\sigma$ level in all three cases, for $a_1$, $b_1$, with the TY (St.+Sys.) solution. With the same parameters we find, for the $D$ waves, and with the help of the Froissart-Gribov representation, the values

$$a_{0+} = (10.60 \pm 0.17) \times 10^{-4} M_\pi^{-5},$$

$$a_{00} = (14.99 \pm 0.68) \times 10^{-4} M_\pi^{-5} \tag{52}$$

and

$$b_{0+} = (-0.170 \pm 0.083) \times 10^{-4} M_\pi^{-7},$$

$$b_{00} = (-6.91 \pm 0.47) \times 10^{-4} M_\pi^{-7}. \tag{53}$$

This is compatible with what we found for the $a_2^{(I)}$ by a direct fit to the experimental data in Sec. III C within the rather large errors of these last values.

The large error, and the separation in the central values in the Olsson sum rule, Eq. (49), is due to the fact that the data do not fix with sufficient accuracy the $a_0^{(2)}$ scattering length, which provides most of the error in the direct number. In fact, as is known, one can use the Olsson sum rule to refine the parameters of the $S2$ wave; if we do so, fixing all other parameters to their central values (within errors) and include the Olsson sum rule in the fit to the $S2$ wave we find

$$\cot \delta_0^{(2)}(s) = \frac{s^{1/2}}{2k} \frac{M_\pi^2}{s - 2z_2^2} \left\{ B_0 + B_1 \frac{\sqrt{s} - \sqrt{s_0 - s}}{\sqrt{s} + \sqrt{s_0 - s}} \right\},$$

$$s_0^{1/2} = 1.45 \text{ GeV}, \quad \chi^2/\text{DOF} = 17.2/(19-2),$$

$$B_0 = -118 \pm 2.5, \quad B_1 = -105 \pm 2.5,$$

$$z_2 = 139.57 \text{ MeV} \quad \text{(fixed)}. \tag{54}$$

Then one has $a_0^{(2)} = -0.0428 \pm 0.0022$ and Eq. (49) becomes

| Olsson | direct | dispersive | |
|---|---|---|---|
| | $0.671 \pm 0.023$ | $0.663 \pm 0.018$. | (55) |

The rest of the relations Eqs. (50)–(53) improve slightly, and the $D$ wave scattering lengths also change a little:

$$a_1 = 38.0 \pm 1.2 \times 10^{-3} M_\pi^{-3}, \quad b_1 = 5.64 \pm 0.96 \times 10^{-3} M_\pi^{-5},$$

$$a_{0+} = (10.51 \pm 0.15) \times 10^{-4} M_\pi^{-5},$$

$$a_{00} = (14.89 \pm 0.65) \times 10^{-4} M_\pi^{-5}. \tag{56}$$

It should be noted that the error here for $a_{0+}$ is at the edge of the region of credibility, as indeed it is of the order of magnitude of electromagnetic corrections which the analysis does not take into account. This value of $a_{0+}$ implies, at one loop level, a very precise value for the chiral perturbation theory parameter [33] $\overline{l}_2$ of

$$\overline{l}_2 = 5.97 \pm 0.07.$$

Of course the agreement in Eqs. (49), (50), (51) and (55) is not enough to guarantee that the new solution is consistent; to prove that, one would have to check the whole set of dispersion relations and crossing constraints, something that will be the subject of a separate paper. But it clearly suggests that the CGL solution fails to pass the tests because it is distorted. This can also be inferred by comparing the CGL solution for the $S2$ wave with Eq. (12) as in Fig. 10, where we show the CGL and Eq. (12) together. While both fit the data below 0.82 GeV [Eq. (12) gives actually a slightly better fit even there], the distortion of the CGL solution above that energy is suggestive. A similar pattern is found in Figs. 1 and

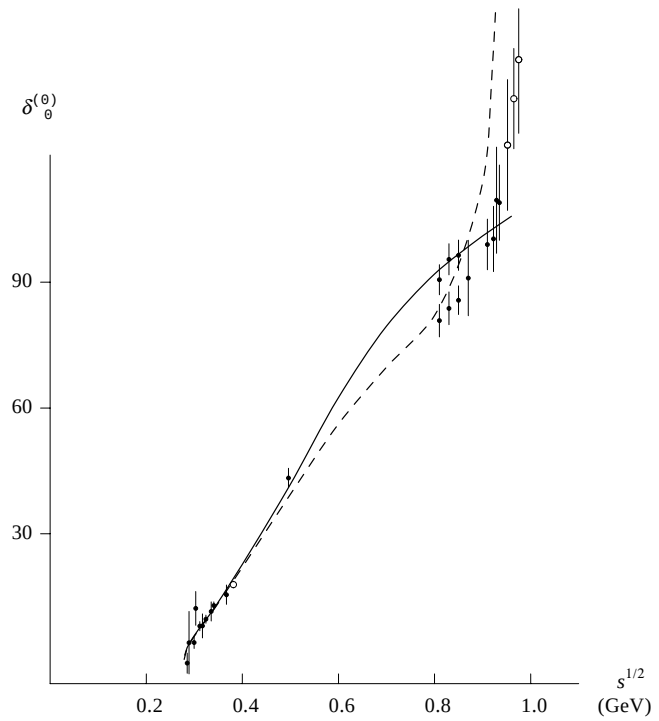


FIG. 9. The $I=0$, $S$-wave phase shifts corresponding to Eqs. (46)–(48) (continuous line) and Colangelo, Gasser, and Leutwyler [2] (dashed line). Some experimental points are also shown.

9. This very much suggests that the CGL fit is a forced fit, biased by a reflection of a faulty high energy scattering amplitude.

## VI. SUMMARY AND CONCLUSIONS

We have checked a number of tests of the low energy ($s^{1/2}\leqslant 0.82$ GeV) $S0$, $S2$, and $P$ wave phase shifts given in [2] by Colangelo, Gasser, and Leutwyler, based on two-loop chiral perturbation theory plus the Roy equations with a certain high energy ($s^{1/2}\geqslant 1.42$ GeV) input. We have shown that, if we used the values for this high energy piece that follow from Regge theory, then the Olsson sum rule and the combinations of scattering lengths $a_{0+}=\frac{2}{3}[a_2^{(0)}-a_2^{(2)}]$, $a_{00}=\frac{2}{3}[a_2^{(0)}+2a_2^{(2)}]$ show mismatch by as much as $4\sim 5\sigma$. We have discussed in detail why we think that the discrepancy is inherent to the *low energy* ($s^{1/2}\leqslant 0.82$ GeV) CGL phases. Thus, in Sec. IV D 1 we have shown that even rather drastic alterations of the middle energy region, $0.82\leqslant s^{1/2}\leqslant 1.42$ do not alter the inconsistencies.

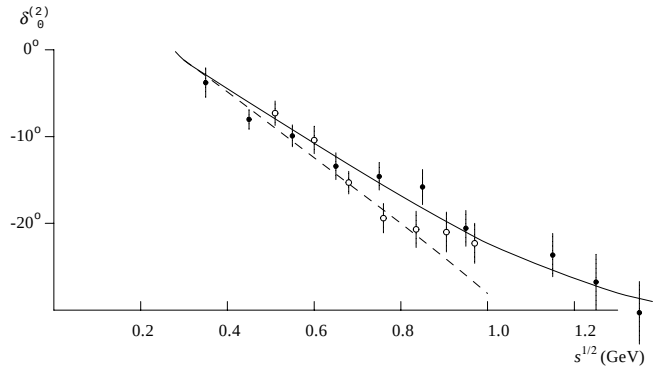


FIG. 10. The $I=2$, $S$-wave phase shifts corresponding to Eq. (3.3) (continuous line) and Colangelo, Gasser, and Leutwyler [2] (dashed line). Also shown are the data points of Losty *et al.* (open circles) and from solution A of Hoogland *et al.* (black dots) [20].

With respect to the higher energy region ($s^{1/2}\geqslant 1.42$ GeV), the situation is such that, if one tries to modify the Regge piece to fit the Olsson sum rule (say) then not only the alteration (40% to 100%) is much more than what one can reasonably expect, but the lack of consistency is shifted to $a_1, b_1$. A similar phenomenon—in fact, even more pronounced—occurs with $a_{0+}$ and $a_{00}$. This we discussed in detail in Sec. IV D 2, where it is clear that the mismatch is due to the low energy CGL input. Moreover, the value of the quantity $b_1$ remains displaced by $4\sigma$ from what one gets from a fit to the pion form factor.

It should be borne in mind that we are talking here about disagreements at the level of a few percent; so, if one is prepared to shift the central values of CGL by up to $2\sigma$, and double their errors, the inconsistencies disappear. This is what happens, for example, in the analysys of ACGL, where the errors are from 3 to 10 times larger than those in CGL. Nevertheless, at the level of precision claimed by CGL, the disagreements are real. We have argued that they are probably due to an irrealistic high energy ($s^{1/2}\geqslant 1.42$ GeV) input, which distorts the low energy phase shifts. In support of this we have shown that a direct fit to data, including fully analyticity constraints, for the $P,S0,S2$ waves (in the case of the last, requiring also consistency of the Olsson sum rule to decrease its errors) plus a high energy input given by orthodox Regge theory, produces a different set of compatible low energy phase shifts and high energy scattering amplitude. This set is formed by the phase shifts given in Eq. (15), Eqs. (46)–(48) and Eq. (54), and is in fact similar to that of CGL, but is slightly displaced and its errors are slightly larger; so, for example, the quantity $\delta_0^{(0)}(M_K^2)-\delta_0^{(2)}(M_K^2)$, important for kaon decays, changes according to

$$\delta_0^{(0)}(M_K^2)-\delta_0^{(2)}(M_K^2)=47.7\pm 1.5^\circ \quad \text{(CGL)}$$

$$\rightarrow 48.4\pm 2.1^\circ \quad \text{(our solution)}.$$

A fact that should be mentioned here is that Descotes *et al.* [34] have, in a recent article, found a solution whose central values differ from that of CGL by almost 2(CGL) standard deviations and in fact point in the direction of our tentative alternate solution here. In particular, they have, in units of $M_\pi$,

$$a_0^{(0)}=0.228\pm 0.012,$$

$$a_0^{(2)}=-0.0382\pm 0.0038 \quad \text{(Descotes } \textit{et al.}\text{)}.$$

Their errors are also more like what we have in our alternate solution. Note, however, that whether or not the alternate solutions turn out to be consistent has nothing to do with the

consistency of the CGL solution: this last fails independently of the failure or success of the novel one.

Analyticity determines the real part of the scattering amplitude in terms of its imaginary part. However, to get the real part you need to know the imaginary part up to infinity. Now, if the imaginary part is wrong at high energy and yet the dispersion relation (or Roy equations) are satisfied, it necessarily follows that one must have made a compensating error in the low energy imaginary part. In other words: you have fallen into a spurious solution. The fact that the solution is spurious should be manifest as soon as one devises a test that gives a *different* weight to high and low energy pieces: this is exactly what we do in our paper, for the CGL solution, with the help of the Froissart-Gribov representations.

*Note added in proof*. After this article was sent to the publisher, a preprint appeared (see Ref. [36]) in which some of the conclusions (but not the calculations) of our work are contested. We do not think it necessary to alter our paper on account of the work of Caprini *et al.*; we plan to present a discussion of it in a separate article.

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## APPENDIX

In this appendix we discuss briefly the reason for the unorthodox Reggeistics chosen in ACGL, CGL, following Pennington [30], inasmuch as it has a bearing on our subject matter here. These authors set up crossing sum rules [Eqs. (B7), (C2) in ACGL], which relate high and low energy, and conclude that they are satisfied only if, in particular, the Pomeron residue is about 1/3 of the value implied by factorization.

Contrarily to the conclusion of ACGL, however, we will show by explicit calculation of a typical sum rule that, if one assumes orthodox Regge behavior from $s^{1/2} \geqslant 1.42$ GeV, the low energy phase shifts are perfectly compatible with the value of the Regge residues implied by factorization. This will cinch the proof that, as discussed in Sec. V A, the Reggeistics of ACGL are very likely due to compensation of the unrealistic phase shifts used between $1.42 \leqslant s^{1/2} \leqslant 2$ GeV.

Specifically, we will consider the sum rule (B7) in ACGL; since it is independent of the $S$ and $P$ waves, it constitutes an independent test of the Regge structure. It may be written as

$$J \equiv \int_{4M_\pi^2}^{\infty} ds \left\{ \frac{4\,\mathrm{Im}\,F'^{(0)}(s,0) - 10\,\mathrm{Im}\,F'^{(2)}(s,0)}{s^2(s-4M_\pi^2)^2} - 6(3s-4m_\pi^2)\,\frac{\mathrm{Im}\,F'^{(1)}(s,0) - \mathrm{Im}\,F^{(1)}(s,0)}{s^2(s-4M_\pi^2)^3} \right\} = 0. \tag{A1}$$

Here $F'^{(I)}(s,t) = \partial F^{(I)}(s,t)/\partial \cos\theta$, and the index $I$ refers to isospin in the $s$ channel.

We will separate $J$ into a low energy and a high energy piece:

$$J = J_{\mathrm{l.e.}} + h_{\mathrm{h.e.}}, \quad J_{\mathrm{l.e.}} = \int_{4M_\pi^2}^{s_h} ds \ldots, \quad J_{\mathrm{h.e.}} = \int_{s_h}^{\infty} ds \ldots .$$

The low energy piece, $J_{\mathrm{l.e.}}$, only contains contributions of waves $D$ and higher. Since these waves are only known with (relatively) large errors [35], it is (generally speaking) very dangerous to draw conclusions about the high energy integral, $J_{\mathrm{h.e.}}$, from the experimental value of the low energy piece, $J_{\mathrm{l.e.}}$. Nevertheless, we will show that, if we choose $s_h = 1.42^2$ GeV$^2$, then we find perfect consistency, within errors. In this calculation we will first neglect the contributions of exchange of $I=2$ and of the background to rho exchange, both of dubious status and substantially smaller than the Pomeron and rho exchange pieces, but we keep the $P'$. Using the parametrizations of Sec. III C for the $D$,$F$ waves we find, in units of $M_\pi^{-6}$,

$$J_{\mathrm{l.e.}}(D \ \text{waves}) = 1.222 \times 10^{-4},$$

$$J_{\mathrm{l.e.}}(F \ \text{wave}) = -0.076 \times 10^{-4}$$

so that, including the errors,

$$J_{\mathrm{l.e.}} = (1.15 \pm 0.05) \times 10^{-4}. \tag{A2}$$

For the high energy piece, expanding in amplitudes with definite isospin in the $t$ channel, and with the numbers in Sec. III D for the Pomeron and rho contributions, we get

$$J_{\mathrm{h.e.}}(\text{Pomeron}) = -1.093 \times 10^{-4},$$

$$J_{\mathrm{h.e.}}(\rho) = 0.034 \times 10^{-4},$$

i.e.,

$$J_{\mathrm{h.e.}} = (-1.06 \pm 0.17) \times 10^{-4}. \tag{A3}$$

Thus, we have cancellation between Eqs. (A2) and (A3), within errors: there is no reason to justify departure off the expected Regge behavior.

To finish this appendix, we comment a little on the $P'$ and on the inclusion of the $I_t = 2$ contribution. Because the high energy part of the sum rule Eq. (A1) is mostly given by the $t$ derivative of the even isospin amplitudes, a more precise evaluation than the one carried here would require that we replace the $P'$ contribution of Eq. (29) by a more accurate formula. Unfortunately, the characteristics of this Regge pole are poorly known; see [23]. If we take for the the $P'$ trajectory a formula like that of the $\rho$, then Eq. (A3) is replaced by

$$J_{\mathrm{h.e.}}(\text{with corrected } P') = (-1.2 \pm 0.2) \times 10^{-4}.$$

Including also the $I_t = 2$ contribution, as given in Eq. (32), we would find

$$J_{\text{h.e.}}(\text{with corrected } P', \text{ and including } I_t=2) = (-0.5\pm 0.3)\times 10^{-4}. \quad \text{(A4)}$$

This still cancels the low energy piece, Eq. (A2), but only at the $2\sigma$ level. This discrepancy cannot be taken seriously, because of the uncertainties in the $P'$ trajectory and because the $t$ slope in formula Eq. (32) is little more than guesswork.

The second sum rule is obtained by profiting from the threshold behavior to write an unsubtracted forward dispersion relation for the quantity $F^{(I_s=1)}(s,0)/(s-4M_\pi^2)$. This gives the relation

$$\frac{6M_\pi}{\pi}a_1=\frac{1}{\pi}\int_{M_\pi^2}^{\infty}ds\,\frac{\text{Im}\,F^{(I_s=1)}(s,0)}{(s-4M_\pi^2)^2} + \frac{1}{\pi}\sum_I\, C_{1I}^{(su)}\int_{M_\pi^2}^{\infty}ds\,\frac{\text{Im}\,F^{(I)}(s,0)}{s^2}, \quad \text{(A5)}$$

which is known at times as the (second) Olsson sum rule. The index $I$ refers to isospin in the $s$ channel and $C_{1I}^{(su)}$ are the $s-u$ crossing matrix elements. Cancelling $a_1$ with the Froissart-Gribov expression for this quantity and substituting the $C_{1I}^{(su)}$, we find the result

$$I\equiv\int_{M_\pi^2}^{\infty}ds\,\frac{\text{Im}\,F^{(I_t=1)}(s,0)-\text{Im}\,F^{(I_t=1)}(s,0)}{s^2} - \int_{M_\pi^2}^{\infty}ds\,\frac{8M_\pi^2[s-2M_\pi^2]}{s^2(s-4M_\pi^2)^2}\,\text{Im}\,F^{(I_s=1)}(s,0)\equiv I_1+I_2=0. \quad \text{(A6)}$$

The contributions of the $S$ waves cancel in Eq. (A6), so only the $P$,$D$ and $F$ waves contribute (as usual, we neglect waves $G$ and higher). At high energy, $I_2$ contributes little since the corresponding integral converges rapidly: most of the high energy contribution comes from the first term, dominated by rho exchange. We will use units so that $M_\pi=1$ and obtain the following results:

$$I(\text{low energy, } P \text{ wave})=(-2.80\pm 0.31)\times 10^{-2},$$

$$I(\text{low energy, } D0+D2 \text{ waves})=(0.56\pm 0.03)\times 10^{-2},$$

$$I(\text{low energy, } F \text{ wave})=(0.01\pm 0.00)\times 10^{-2},$$

$$I(\text{high energy},\rho)=(2.41\pm 0.37)\times 10^{-2},$$

$$I(\text{high energy},I=0)=-(0.17\pm 0.02)\times 10^{-2},$$

$$I(\text{high energy},I=2)=-(0.02\pm 0.01)\times 10^{-2}.$$

By ''low energy'' we understand the contributions from energies below 1.42 GeV, where we use phase shifts and inelasticities to calculate the scattering amplitudes, and ''high energy'' is above 1.42 GeV, where a Regge description is employed. The final result for the sum rule is

$$I=(0.016\pm 0.37)\times 10^{-2},$$

i.e., complete cancellation of low and high energy contributions.

The remarkable fulfillment of these sum rules show the incorrectness of the assertions found in ACGL, CGL: both for Pomeron and rho, standard Regge behavior for $\pi\pi$ scattering is perfectly consistent with crossing symmetry provided one imposes it systematically for energies above 1.42 GeV.


[1] B. Ananthanarayan *et al.*, Phys. Rep. **353**, 207 (2001).

[2] G. Colangelo, J. Gasser, and H. Leutwyler, Nucl. Phys. **B603**, 125 (2001).

[3] S. M. Roy, Phys. Lett. **36B**, 353 (1971).

[4] Actually, $b_1$ is not the effective range, though it is related to it. We use the definitions of ACGL and CGL for the $a$ and $b$, except that we take the dimensions of $a_l$ to be $M_\pi^{-(2l+1)}$. Here $M_\pi$ is the charged pion mass, $M_\pi\simeq 139.57$ MeV.

[5] The method of the Froissart-Gribov representation to calculate scattering lengths and effective ranges was introduced in [6,7]. It is also discussed in some detail in [8].

[6] F. P. Palou and F. J. Ynduráin, Nuovo Cimento A **19**, 245 (1974).

[7] F. P. Palou, J. L. Sánchez-Gómez, and F. J. Ynduráin, Z. Phys. A **274**, 161 (1975).

[8] F. J. Ynduráin, ''Low energy pion interactions,'' FTUAM02-28, hep-ph/0212282.

[9] We are actually simplifying in that Eqs. (2) and (3) should take into account the different isospin structure of $s$ and $u$ channels, which the reader may find in, e.g. the text of Martin, Morgan, and Shaw [10].

[10] B. R. Martin, D. Morgan, and G. Shaw, *Pion-Pion Interactions in Particle Physics* (Academic, New York, 1976).

[11] D. Atkinson, Nucl. Phys. **B7**, 375 (1968); **B23**, 397 (1970).

[12] For analyticity properties of $\pi\pi$ scattering see for example [10].

[13] The (slight) differences with some of the parameters in [8] occur because now we are using $M_\pi=m_{\pi^+}=139.57$ MeV instead of the average pion mass, 138 MeV, and we are also essentially eliminating from the fit the data for energies above 0.96 MeV. The change in the $\chi^2$/DOF of the $I=0$ $S$ wave corrects an error there. We send to this reference for details on the fitting procedure.

[14] For the more recent determination, see A. Aloisio *et al.*, Phys. Lett. B **538**, 21 (2002); the older one is from P. Pascual and F. J. Ynduráin, Nucl. Phys. **B83**, 362 (1974).

[15] J. F. de Trocóniz and F. J. Ynduráin, Phys. Rev. D **65**, 093001 (2002).

[16] The fact that the errors for $a_1,b_1$ are smaller when using Eq. (15) than when using Eq. (13), which at first sight appears counterintuitive, can be understood as follows. The errors in the parameters $B_0,M_\rho$ in Eq. (15) are larger than those in Eq.

(13)—as suggested by intuition. The error in $B_1$, however, is smaller, and it is this quantity that influences most $b_1$ (the error in $a_1$ stays essentially constant). Including systematic errors makes the determinations of the pion from factor in the time-like and spacelike regions more compatible one with another, and this allows a more precise determination of low energy parameters.

[17] S. D. Protopopescu *et al.*, Phys. Rev. D **7**, 1279 (1973).

[18] B. Hyams *et al.*, Nucl. Phys. **B64**, 134 (1973); G. Grayer *et al.*, *ibid.* **B75**, 189 (1974); see also the analysis of the same experimental data in P. Estabrooks and A. D. Martin, *ibid.* **B79**, 301 (1974).

[19] Equation (21) corrects a mistake in the corresponding wave in [8].

[20] M. J. Losty *et al.*, Nucl. Phys. **B69**, 185 (1974); W. Hoogland *et al.*, *ibid.* **B126**, 109 (1977).

[21] M. Gell-Mann, Phys. Rev. Lett. **8**, 263 (1962); V. N. Gribov and I. Ya. Pomeranchuk, *ibid.* **8**, 343 (1962); for more references in general Regge theory, see V. D. Barger and D. B. Cline, *Phenomenological Theories of High Energy Scattering* (Benjamin, New York, 1969); for references to the QCD analysis, F. J. Ynduráin, *The Theory of Quark and Gluon Interactions* (Springer, Berlin, 1999).

[22] In potential theory the proof can be made mathematically rigorous; in relativistic theory, it follows from extended unitarity or, in QCD, in the DGLAP formalism, as is intuitively obvious from Fig. 3.

[23] W. Rarita *et al.*, Phys. Rev. **165**, 1615 (1968).

[24] Fits to deep inelastic scattering processes, and references to previous literature, may be found in the book by Ynduráin in [21].

[25] Consistency requires a more complicated form for the residue functions $f_i^{(I_t)}(t)$; see [6,23]. For the small values of $t$ in which we are interested, our expressions are sufficiently accurate.

[26] In the case of the rho trajectory, exact linearity would imply $\alpha'_\rho(0)=1/2(M_\rho^2-M_\pi^2)\simeq 0.87\ \mathrm{GeV}^{-2}$, not far from the value $1.01\ \mathrm{GeV}^{-2}$ that the actual fits give, and which we have used here.

[27] This unreliability is reflected, for example, in the Particle Data Tables (e.g., the edition of [28]), where no number is given for the branching ratios of $\pi\pi$ resonances with masses at or above 1.2 GeV [with the exception of the $\rho_3(1690)$] and even the $S0$ phase around the $f_0(980)$ has a dubious status: this last due to the ambiguity caused by the inelastic $\bar{K}K$ channel. In fact, the resonances that appear in $\pi\pi$ production are not the same that one finds in $e^+e^-$, $\tau$ decay or $J/\psi$ decay, and the inelasticities in both cases are also quite different.

[28] Particle Data Group, D. E. Groom *et al.*, Eur. Phys. J. C **15**, 1 (2000).

[29] D. Atkinson, G. Mahoux, and F. J. Ynduráin, Nucl. Phys. **B54**, 263 (1973); **B98**, 521 (1975).

[30] M. R. Pennington, Ann. Phys. (N.Y.) **92**, 164 (1975).

[31] This should not be taken as a criticism of the work of Pennington; at that time the data, very poor, did indeed suggest possible deviations of Regge behavior for $\pi\pi$. On the other hand, the fact that QCD also implies factorization was of course unknown.

[32] This fit is actually a refinement of that of Eq. (7.6.2) in [8]; more details about this will be presented in a separate publication.

[33] J. Gasser and H. Leutwyler, Ann. Phys. (N.Y.) **158**, 142 (1984).

[34] S. Descotes, N. H. Fuchs, L. Girlanda, and J. Stern, Eur. Phys. J. C **24**, 469 (2002).

[35] These errors are particularly large, and uncertain, above 1.3 GeV, where inelasticity begins to be important. For example, the error on the $D0$ wave contribution to $J_{\mathrm{l.e.}}$ due to a 50% change in the inelasticity of the $f_2(1270)$ resonance is as large as the nominal error due to only the errors in the parameters in Eq. (20).

[36] I. Caprini, G. Colangelo, J. Gasser, and H. Leutwyler, following paper, Phys. Rev. D **68**, 074006 (2003).