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New "square root" model of lepton family cyclic symmetry

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Following the newly formulated notion of form invariance of the neutrino mass matrix, a complete model of leptons is constructed. It is based on a specific unitary 3×3 matrix U in family space, such that U^2 is the simple discrete symmetry $\nu_e\to-\nu_e$, $\nu_\mu\leftrightarrow\nu_\tau$. Thus U also generates the cyclic group Z_4. The charged-lepton mass matrix is nearly diagonal while the neutrino mass matrix is of a form suitable for explaining maximal (large) mixing in atmospheric (solar) neutrino oscillations in the context of three nearly degenerate neutrino masses. Observable lepton flavor violation is predicted. Quarks may be treated in the same way as charged leptons.

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To understand the form of the neutrino mass matrix $\mathcal{M}_\nu$, a new idea has recently been proposed [1]. It is postulated that there is a specific 3×3 unitary matrix U with $U^n=1$ such that

$$U\mathcal{M}_\nu U^T=\mathcal{M}_\nu, \tag{1}$$

and for some $N<n$, the matrix U^N represents a well-defined discrete symmetry in the $\nu_{e,\mu,\tau}$ basis, for which the charged-lepton mass matrix $\mathcal{M}_l$ is diagonal. However, since each neutrino belongs to an $SU(2)_L\times U(1)_Y$ doublet, the corresponding left-handed charged leptons (e,μ,τ) must also transform under U. A complete theory must then reconcile the apparent contradictory requirement that $\mathcal{M}_\nu$ satisfies Eq. (1), but $\mathcal{M}_l$ does not. The resolution of this conundrum is in the soft and spontaneous breaking of the symmetry supported by U, as was done for example in the A_4 model [2–4] of degenerate neutrino masses. In this paper we show how it may be achieved with the specific unitary matrix

$$U=\begin{pmatrix} 0 & i/\sqrt{2} & -i/\sqrt{2} \\ i/\sqrt{2} & 1/2 & 1/2 \\ -i/\sqrt{2} & 1/2 & 1/2 \end{pmatrix}, \tag{2}$$

where

$$U^2=\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{3}$$

i.e. the simple discrete symmetry

$$\nu_e\to-\nu_e, \qquad \nu_\mu\leftrightarrow\nu_\tau. \tag{4}$$

The matrix U of Eq. (2) is thus a "square root" of this discrete symmetry. We see immediately also that $U^4=1$; hence our model is a specific realization of the cyclic group Z_4 as a family symmetry.

The most general Majorana mass matrix $\mathcal{M}_\nu$ is of the form

$$\mathcal{M}_\nu=\begin{pmatrix} A & D & E \\ D & B & F \\ E & F & C \end{pmatrix}. \tag{5}$$

Using Eqs. (1) and (2), we find that it becomes

$$\mathcal{M}_\nu=\begin{pmatrix} A & 0 & 0 \\ 0 & B & A+B \\ 0 & A+B & B \end{pmatrix}. \tag{6}$$

This mass matrix has eigenvalues A, $-A$, and $A+2B$, corresponding to the eigenstates ν_e, $(\nu_\mu-\nu_\tau)/\sqrt{2}$, and $(\nu_\mu+\nu_\tau)/\sqrt{2}$. We see immediately that $\nu_\mu-\nu_\tau$ mixing is maximal with

$$\Delta m^2_{atm}=(A+2B)^2-A^2=4B(A+B), \tag{7}$$

which is suitable for explaining atmospheric neutrino oscillations [5]. If $A\ll B$, we have the hierarchical structure of neutrino masses and $B=\sqrt{\Delta m^2_{atm}}/2\simeq0.025$ eV. On the other hand, if $B\ll A$, we have the more interesting scenario of three nearly degenerate neutrino masses, with the prediction that A is large enough to be measured by neutrinoless double beta decay. As for solar neutrino oscillations [6], we can obtain the large-mixing-angle solution by invoking flavor-changing radiative corrections [1,3,4]. Details will be presented in a later paragraph.

The leptonic Yukawa couplings of our model are given by

$$\mathcal{L}_Y=h_{ij}[\xi^0\nu_i\nu_j-\xi^+(\nu_il_j+l_i\nu_j)/\sqrt{2}+\xi^{++}l_il_j] + f^k_{ij}(l_i\phi^0_j-\nu_i\phi^-_j)l^c_k+\text{H.c.}, \tag{8}$$

where we have adopted the convention that all fermion fields are left handed, with their right-handed counterparts denoted by the corresponding (left-handed) charge-conjugate fields. We have also extended the Higgs sector of the standard model of particle interactions to include three doublets (ϕ_j^0,ϕ_j^-) and one very heavy triplet (ξ^{++},ξ^+,ξ^0). The electroweak $SU(2)_L\times U(1)_Y$ gauge symmetry will be spontaneously broken mainly by the vacuum expectation value of one particular Higgs doublet. As a result, the vacuum expectation values of the other two Higgs doublets can be naturally small [7] and that of the triplet even smaller [8].

We now assume that $\mathcal{L}_Y$ is invariant under the transformation

$$(\nu,l)_i\to U_{ij}(\nu,l)_j,\quad l_k^c\to l_k^c, \tag{9}$$

$$(\phi^0,\phi^-)_i\to U_{ij}(\phi^0,\phi^-)_j,\quad (\xi^{++},\xi^+,\xi^0)\to(\xi^{++},\xi^+,\xi^0). \tag{10}$$

This means

$$U^ThU=h,\quad U^Tf^kU=f^k, \tag{11}$$

resulting in

$$h=\begin{pmatrix} a & 0 & 0\\ 0 & b & a+b\\ 0 & a+b & b\end{pmatrix},$$

$$f^k=\begin{pmatrix} a_k & d_k & -d_k\\ -d_k & b_k & a_k+b_k\\ d_k & a_k+b_k & b_k\end{pmatrix}. \tag{12}$$

Note that h has no d terms because it has to be symmetric.

The neutrino mass matrix $\mathcal{M}_\nu$ is then given by

$$\mathcal{M}_\nu=2h\langle\xi^0\rangle, \tag{13}$$

whereas the charged-lepton mass matrix $\mathcal{M}_l$ linking l_i to l_k^c is

$$\begin{pmatrix} a_1v_1+d_1(v_2-v_3) & a_2v_1+d_2(v_2-v_3) & a_3v_1+d_3(v_2-v_3)\\ -d_1v_1+b_1(v_2+v_3)+a_1v_3 & -d_2v_1+b_2(v_2+v_3)+a_2v_3 & -d_3v_1+b_3(v_2+v_3)+a_3v_3\\ d_1v_1+a_1v_2+b_1(v_2+v_3) & d_2v_1+a_2v_2+b_2(v_2+v_3) & d_3v_1+a_3v_2+b_3(v_2+v_3)\end{pmatrix}, \tag{14}$$

where $v_i\equiv\langle\phi_i^0\rangle$. Assume $d_k,b_k\ll a_k$ and $v_{1,3}\ll v_2$, then all elements in the first, second, and third rows are of order dv_2+av_1, bv_2+av_3, and av_2, respectively. It is clear that they may be chosen to be of order m_e, m_μ, and m_τ, in which case $\mathcal{M}_l$ will become nearly diagonal by simply redefining the l_k^c basis. The mixing matrix V_L in the l_i basis (such that $V_L\mathcal{M}_l\mathcal{M}_l^\dagger V_L^\dagger$ is diagonal) will be very close to the identity matrix with off-diagonal terms of order m_e/m_μ, m_e/m_τ, and m_μ/m_τ. This construction then allows us to consider $\mathcal{M}_\nu$ to be in the basis (ν_e,ν_μ,ν_τ) to a very good approximation.

We now understand why it is sensible [1] to consider Eq. (1) as a condition on $\mathcal{M}_\nu$. The key lies in the fact that $\mathcal{M}_\nu$ comes from neutrino couplings to a single field ξ^0 which is invariant under U, whereas $\mathcal{M}_l$ comes from couplings to $\phi_{1,2,3}^0$ which are not. Let the scalar trilinear coupling of ξ to $\Phi_2\Phi_2$ be μ, then [8]

$$\langle\xi^0\rangle\simeq\frac{-\mu v_2^2}{M_\xi^2}, \tag{15}$$

which shows clearly that neutrino masses may be of order 1 eV or less if $M_\xi^2/\mu\sim10^{13}$ GeV. Similarly, $v_{1,3}$ can be small compared to v_2 if $M_2^2<0$ but the $M_{1,3}^2$ terms in the Higgs potential are large and positive, in which case [7]

$$v_1\simeq\frac{-\mu_{12}^2v_2}{M_1^2},\quad v_3\simeq\frac{-\mu_{23}^2v_2}{M_3^2}, \tag{16}$$

where μ_{12}^2 and μ_{23}^2 are the coefficients of the $\Phi_1^\dagger\Phi_2$ and $\Phi_2^\dagger\Phi_3$ terms, respectively.

Going back to $\mathcal{M}_\nu$ of Eq. (6), we now consider how solar neutrino oscillations may arise in the 2×2 submatrix spanning ν_e and $(\nu_\mu-\nu_\tau)/\sqrt{2}$, i.e.

$$\mathcal{M}=\begin{pmatrix} A & 0\\ 0 & -A\end{pmatrix}. \tag{17}$$

Consider the most general radiative corrections to $\mathcal{M}$, i.e.

$$R=\begin{pmatrix} r_{11} & r_{12}\\ r_{12}^* & r_{22}\end{pmatrix}, \tag{18}$$

then $\mathcal{M}$ becomes

$$(1+R)\mathcal{M}(1+R^T)\simeq A\begin{pmatrix} 1+2r_{11} & r_{12}^*-r_{12}\\ r_{12}^*-r_{12} & -1-2r_{22}\end{pmatrix}. \tag{19}$$

In terms of the full radiative correction matrix [3,4]

$$r_{11}=r_{ee},\quad r_{22}=\tfrac{1}{2}(r_{\mu\mu}+r_{\tau\tau})-\mathrm{Re}(r_{\mu\tau}), \tag{20}$$

$$r_{12}^*-r_{12}=-2i\,\mathrm{Im}(r_{12})=-i\sqrt{2}[\mathrm{Im}(r_{e\mu})-\mathrm{Im}(r_{e\tau})]. \tag{21}$$

Thus the radiatively corrected $\mathcal{M}$ has eigenvalues

$$m_{1,2}=A\{1+r_{11}+r_{22}\mp\sqrt{(r_{11}-r_{22})^2+4[\mathrm{Im}(r_{12})]^2}\} \tag{22}$$

corresponding to the eigenvectors $\nu_e\cos\theta - i(\nu_\mu - \nu_\tau)\sin\theta/\sqrt{2}$ and $\nu_e\cos\theta + i(\nu_\mu - \nu_\tau)\cos\theta/\sqrt{2}$, respectively, where

$$\tan\theta = \frac{r_{11}-r_{22}+\sqrt{(r_{11}-r_{22})^2+4[\mathrm{Im}(r_{12})]^2}}{2|\mathrm{Im}(r_{12})|}, \quad (23)$$

with $r_{22}-r_{11}>0$. Since $\tan^2\theta\simeq 0.46$ is desirable [9] for understanding solar neutrino oscillations [6], flavor changing ($r_{12}\neq 0$) and flavor nonuniversal ($r_{11}\neq r_{22}$) interactions are required. Specific examples have already been proposed [1,3,4]. As for Δm^2_{sol}, it is given here by

$$\Delta m^2_{sol} = m_2^2 - m_1^2 \simeq 4A^2\sqrt{(r_{11}-r_{22})^2+4[\mathrm{Im}(r_{12})]^2}. \quad (24)$$

For $|r|\sim 10^{-3}$ which is a typical size for radiative corrections and $\Delta m^2_{sol}\simeq 6.9\times 10^{-5}$ eV^2 [9], $|A|\sim 0.13$ eV is then obtained. (The recent Wilkinson microwave anisotropy result implies [10] an upper bound of 0.23 eV on $|A|$ from cosmological considerations.) Given that $|A|$ is also the effective neutrino mass measured in neutrinoless double beta decay with a present upper bound of about 0.4 eV, this is an encouraging prediction for future experiments [11]. If $A \simeq 0.13$ eV, then using Eq. (7), we find $B\simeq 0.0048$ eV and $B/A\simeq 0.037$ which is of the same order as $m_\mu/m_\tau\simeq 0.059$, as suggested by $\mathcal{M}_l$ of Eq. (14).

With flavor changing radiative corrections, the U_{e3} entry of the neutrino mixing matrix becomes nonzero. It is given here by

$$U_{e3}\simeq -i\frac{[\mathrm{Re}(r_{e\mu})+\mathrm{Re}(r_{e\tau})]A}{\sqrt{2}B}, \quad (25)$$

which may be as large as the experimental upper bound [12] of 0.16 in magnitude. It is also purely imaginary so that *CP* violation is maximal [13] in this model.

Going back to the Yukawa couplings of the leptons to the 3 Higgs doublets given by Eq. (12) and assuming the hierarchy $d_k \ll b_k \ll a_k$ and taking the limit that only v_2 is nonzero, we have $\mathcal{M}_l$ of Eq. (14) simply given by

$$\mathcal{M}_l \simeq v_2\begin{pmatrix} d_1 & d_2 & d_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{pmatrix}, \quad (26)$$

whereas Φ_1 and Φ_3 couple to $l_i l_j^c$ according to

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ -d_1 & -d_2 & -d_3 \\ d_1 & d_2 & d_3 \end{pmatrix}, \quad \begin{pmatrix} -d_1 & -d_2 & -d_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}, \quad (27)$$

respectively. After rotating $\mathcal{M}_l$ of Eq. (26) in the l_j^c basis to define the state corresponding to τ, we see immediately from Eq. (27) that the dominant coupling of Φ_1 is $(m_\tau/v_2)e\tau^c$ and that of Φ_3 is $(m_\tau/v_2)\mu\tau^c$. Other couplings are at most of order m_μ/v_2 in this model, and some are only of order m_e/v_2. We thus have a natural understanding of the smallness of flavor changing decays in the leptonic sector, even though they should be present and are potentially observable.

Using Eq. (27), we see that the decays $\tau^-\to e^-e^+e^-$ and $\tau^-\to e^-e^+\mu^-$ may proceed through ϕ_1^0 exchange with coupling strengths of order $m_\mu m_\tau/v_2^2 \simeq (g^2/2)(m_\mu m_\tau/M_W^2)$, whereas the decays $\tau^-\to\mu^-\mu^+\mu^-$ and $\tau^-\to\mu^-\mu^+e^-$ may proceed through ϕ_3^0 exchange also with coupling strengths of the same order. We estimate the order of magnitude of these branching fractions to be

$$B\sim\left(\frac{m_\mu^2 m_\tau^2}{M_{1,3}^4}\right)B(\tau\to\mu\nu\nu)\simeq 6.1\times 10^{-11}\left(\frac{100\ \ \mathrm{GeV}}{M_{1,3}}\right)^4, \quad (28)$$

which is well below the present experimental upper bound of the order of 10^{-6} for all such rare decays [14].

The decay $\mu^-\to e^-e^+e^-$ may also proceed through ϕ_1^0 with a coupling strength of order m_μ^2/v_2^2. Thus

$$B(\mu\to eee)\sim\frac{m_\mu^4}{M_1^4}\simeq 1.2\times 10^{-12}\left(\frac{100\ \mathrm{GeV}}{M_1}\right)^4, \quad (29)$$

which is at the level of the present experimental upper bound of 1.0×10^{-12}. The decay $\mu\to e\gamma$ may also proceed through ϕ_3^0 exchange (provided that $\mathrm{Re}\phi_3^0$ and $\mathrm{Im}\phi_3^0$ have different masses) with a coupling of order $m_\mu m_\tau/v_2^2$. Its branching fraction is given by [2]

$$B(\mu\to e\gamma)\sim\frac{3\alpha}{8\pi}\frac{m_\tau^4}{M_{eff}^4}, \quad (30)$$

where

$$\frac{1}{M_{eff}^2}=\frac{1}{M_{3R}^2}\left(\ln\frac{M_{3R}^2}{m_\tau^2}-\frac{3}{2}\right)-\frac{1}{M_{3I}^2}\left(\ln\frac{M_{3I}^2}{m_\tau^2}-\frac{3}{2}\right). \quad (31)$$

Using the experimental upper bound [15] of 1.2×10^{-11}, we find $M_{eff}>164$ GeV.

The new physics which generates the flavor-changing radiative corrections discussed earlier also results in flavor-changing decays such as $\mu\to e\gamma$ and $\tau\to\mu\gamma$. These have been discussed previously [1,3,4]. Although the soft supersymmetry breaking sector may provide such flavor-changing corrections [3], it is much simpler to use the nonsupersymmetric mechanism [4] which employs charged scalar singlets χ_k^+ with couplings of the form $f_{ij}^k(\nu_i l_j - l_i\nu_j)\chi_k^+$. As shown previously [1,4], for χ_k^+ masses of order 1 TeV, all flavor-changing leptonic decays due to their exchange are suppressed below present experimental bounds.

In the quark sector, if we use the same 3 Higgs doublets for the corresponding Yukawa couplings, the resulting *up* and *down* mass matrices will be of the same form as Eq. (14). Because the quark masses are hierarchical in each sector, we will also have nearly diagonal mixing matrices as in the case of the charged leptons. This provides a qualitative understanding in our model of why the charged-current mixing matrix linking *up* quarks to *down* quarks has small off-diagonal entries.

Once ϕ_1^0 or ϕ_3^0 is produced, its dominant decay will be to $\tau^\pm e^\mp$ or $\tau^\pm \mu^\mp$ if each couples only to leptons. If they also couple to quarks (and are sufficiently heavy), then the dominant decay products will be $t\bar{u}$ or $t\bar{c}$ together with their conjugates. As for ϕ_2^0, it will behave very much as the single Higgs doublet of the standard model, with mostly diagonal couplings to fermions.

In conclusion we have constructed a complete theory of leptons where the neutrino mass matrix $\mathcal{M}_\nu$ may be derived from the requirement that $U\mathcal{M}_\nu U^T = \mathcal{M}_\nu$, where U is a specific 3×3 unitary matrix in family space such that U^2 is the simple discrete symmetry $\nu_e \to -\nu_e$, $\nu_\mu \leftrightarrow \nu_\tau$. We obtain three nearly degenerate neutrino masses with maximal mixing for atmospheric neutrino oscillations. Solar neutrino oscillations are induced by flavor-changing radiative corrections. As a result, neutrinoless double beta decay is predicted to occur at the 0.1 eV range. There are also three Higgs doublets in this model, two of which have dominant flavor-changing couplings proportional to m_τ and may be easily observed at future colliders.

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