On α' corrections to D-brane solutions

Sebastian de Haro*
University of California, Los Angeles, Los Angeles, California 90095, USA

Annamaria Sinkovics†
Department of Physics, University of Wales Swansea, Singleton Park, Swansea, SA2 8PP, United Kingdom and Institute for Theoretical Physics, University of Amsterdam, Valckenierstraat 65, 1018XE Amsterdam, The Netherlands

Kostas Skenderis‡
Institute for Theoretical Physics, University of Amsterdam, Valckenierstraat 65, 1018XE Amsterdam, The Netherlands and Physics Department, Princeton University, Princeton, New Jersey 08544, USA

(Received 15 May 2003; published 3 September 2003)

We discuss the computation of the leading corrections to D-brane solutions due to higher derivative terms in the corresponding low energy effective action. We develop several alternative methods for analyzing the problem. In particular, we derive an effective one-dimensional action from which the field equations for spherically symmetric two-block brane solutions can be derived, show how to obtain first order equations, and discuss a few other approaches. We integrate the equations for extremal branes and obtain the corrections in terms of integrals of the evaluation of the higher derivative terms on the lowest order solution. To obtain completely explicit results one would need to know all leading higher derivative corrections which at present are not available. One of the known higher derivative terms is the R^4 term, and we obtain the corrections to the D3-brane solution due to this term alone. We note, however, that (unknown at present) higher terms depending on F_5 are expected to modify our result. We analyze the thermodynamics of brane solutions when such quantum corrections are present. We find that the R^4 term induces a correction to the tension and the electric potential of the D3-brane but not to its charge, and the tension is still proportional to the electric potential times the charge. In the near-horizon limit the corrected solution becomes $AdS_5\times S^5$ with the same cosmological constant as the lowest order solution but a different value of the (constant) dilaton.

DOI: 10.1103/PhysRevD.68.066001 PACS number(s): 11.25.Uv

I. INTRODUCTION

One can hardly overestimate the importance of supergravity solutions. The solutions describing the long-range fields associated with strings, D-branes and solitonic fivebranes have played an instrumental role in many advances in string theory. String dualities require the existence of certain solutions and conversely the pattern of supergravity solutions strongly hints of similar patterns and properties of the underlying microscopic theory. Furthermore, the interplay between the microscopic and the supergravity description of an object has been extremely fruitful. One of the most prominent examples is the case of black holes and their study in string theory. One can construct solutions describing black holes by superimposing (intersecting) ''elementary'' branes, i.e. fundamental strings, D-branes, etc. These objects have a well-defined description in string perturbation theory and, provided appropriate conditions hold, one can use this description to obtain results about black holes. For instance, such considerations led to a microscopic understanding of the black hole entropy for extremal black holes. Furthermore, such reasoning applied to D3 and other branes led to the anti–de Sitter/conformal field theory (AdS/CFT) correspondence, and generalizations thereof.

In all these studies, the *p*-brane solutions solve the field equations that follow from supergravity actions that involve up to two-derivative terms. These actions are the lowest order terms in the low-energy effective theories of string theories, and the latter are known to receive string corrections. The corrections appear as a series in α' and are higher derivative terms.

Given the importance of the *p*-brane solutions, one may ask how the solutions are modified by the higher dimensional terms. Any such modification will represent the leading stringy effects at low energies. It is known that some solutions do not receive any corrections. Examples of such solutions are maximally supersymmetric spacetimes such as flat space, and the $AdS_5\times S^5$ vacuum of type IIB supergravity, but also spacetimes with less supersymmetry such as pp-wave solutions [1]. These cases, however, are rather exceptional and generically one expects the solutions to receive corrections, see for example [2]. α' corrections to the near-horizon geometry of extremal and nonextremal D3-branes were studied in [1,3–5]. It was found that the $AdS_5\times S^5$ geometry is not corrected, but the nonextremal version is. Higher derivative corrections to near-horizon-NS5/little string theory thermodynamics have been considered in [6,7].

The precise form of the corrections may have implications in all problems involving *p*-brane solutions. For instance, the α' corrections to *p*-brane solutions will induce α' corrections to black hole solutions and their properties, such as their entropy formula. The explanation of such subleading

*Email address: sebas@physics.ucla.edu
†Email address: sinkovic@science.uva.nl
‡Email address: skenderi@science.uva.nl

terms in terms of a microscopic theory will then pose a new challenge to our understanding of black holes. In the context of the AdS/CFT correspondence, α' corrections are associated with subleading terms in the 't Hooft coupling expansion. Other applications involve the computation of α'-corrections to duality transformation rules. The higher derivative terms also become important near spacetime singularities where curvatures are large.

To compute the precise form of the corrected D-brane solutions, one would need the complete set of bosonic terms in the low-energy effective action at leading order. Higher derivative interactions can be computed by scattering amplitudes [8,9] or using sigma model techniques [10] (see [11] for a more complete list of references). However, apart from the well known R^4 term only a very few other terms are known, see [12–14] for recent discussions. One way to obtain further interactions terms is to find all terms that are related by supersymmetry to the known terms. In [11] we investigated in detail the possibility of constructing a superinvariant as a scalar superpotential term in type IIB superspace [15]. A linearized version of such a term was known to contain the R^4 term [16] leading to the expectation that such a superspace term contains all terms that are related to the R^4 by linearized supersymmetry. We have shown in [11], however, that a superinvariant based on a scalar superpotential does not exist. Finding the superinvariant associated to the R^4 term, and thus determining the complete set of interactions at leading order is still an open question. For the computation of the corrections to D-brane solutions one would need the full set of bosonic terms depending on the metric, dilaton and Ramond-Ramond (RR) fields.

In this paper we systematically analyze the computation of corrections to brane solutions. The computation consists of obtaining the corrected field equations, evaluating the terms that originate from the higher derivative terms on the lowest order solution and then integrating the resulting equations. We present several different methods to obtain the field equations. The straightforward determination of the field equations is possible but very laborious. A method that we find well-suited for this problem is the Palatini formalism. In this formalism the metric and the Christoffel symbols are considered as independent fields that are varied independently. The simplifications are due to the fact that one has to perform fewer partial integrations when deriving the field equations. This reduces the number of terms that participate in the field equations. This formulation, even though simpler than the direct computation, is still tedious.

A significant improvement is possible when one considers spherically symmetric solutions. In this case we derive an effective one-dimensional action that governs the field equations. This action may be thought of as the consistent reduction of the ten dimensional action over all coordinates but the radial one. The method developed can also be used more generally to derive consistent reductions in general. The field equations to be solved are second order differential equations. In the case where the lowest order solution is supersymmetric, we also derive associated first order equations that include the effects of the higher derivative terms (we present such an analysis for D3-branes, but similar considerations are applicable to other branes as well).

After the field equations are derived, we have to evaluate the higher derivative terms on the lowest order solution whose corrections we want to compute. This leads to r-dependent source terms in the field equations. To explicitly compute the source terms one needs to know the exact form of the higher derivative terms which is not known at present. Given such source terms, however, we succeeded in integrating the equations to obtain the corrections as integrals of the sources. When the higher derivative terms become available, these results would immediately lead to the exact form of the corrected solutions.

One of the cases that is under better control is the case of the D3-brane. In this case the lowest order solution has a constant dilaton and a self-dual five-form. This eliminates some of the possible interaction terms. For instance, higher derivative terms that depend on the derivatives of the dilaton will not contribute and thus they need not be considered. Even in the D3-brane case, however, there are possible yet undetermined interaction terms depending on the five-form RR field F_5 and derivatives thereof (the superpotential term mentioned above does contain such terms). In fact, our analysis indicates that such terms will contribute to the full form of the corrected D3-brane solutions. Noting this, we proceed by taking into account the corrections due to the R^4 term only. In this sense, the computation may be viewed as a toy model computation. We obtained the corrected solution in closed form. It has a nontrivial dilaton, is regular in the interior and approaches $AdS_5 \times S^5$ in the near-horizon limit.

In the presence of higher derivative interactions the standard formulas for the computation of the thermodynamic properties of the solutions are modified. We discuss in detail, following [17–19], how to do such computations. We find that the tension and the electric potential of the D3-brane renormalize, but the charge, temperature and entropy remain uncorrected. Despite the renormalization of the tension, we show that a Bogomol'nyi-Prasad-Sommerfield- (BPS-) type formula that relates the mass and the charge still holds. This formula follows from the integrated form of the first law of thermodynamics (Smarr formula). The renormalization of the mass is compensated by the renormalization of the electric potential.

Any correction to the mass of the D3-brane due to higher derivative terms has rather dramatic consequences: the mass of N_1 branes plus the mass of N_2 branes is higher than the mass of N_1+N_2 branes. This implies that there is a force between the branes and the branes will tend to coalesce together. This is opposite to what one expects from BPS branes. We take these results as a strong indication that the higher derivative terms contain F_5 dependent terms so that there are additional contributions to our computation.

One may expect that once the F_5 terms are included, the full extremal D3-brane solution will turn out to be uncorrected, but such a proof is still lacking. Such nonrenormalization will be consistent with the fact that the Kaluza-Klein- (KK-) monopole solution, which is connected to the D3-brane via dualities, does not receive corrections from the R^4 term. This follows from the fact that the corresponding sigma model is finite [20]. (Since the KK-monopole is a purely gravitational solution there is no issue of undetermined higher derivative interactions.) This argument, however, assumes that the duality rules will not introduce any α' corrections, but in general the T-duality rules are known to receive α' corrections, see for instance [21]. Another way to analyze this question would be to study Killing spinor equations but the corrections to supersymmetry rules due to the higher derivative corrections are also not yet available.

This paper is organized as follows. In the first three sections we analyze in detail the corrections to the D3-brane due to the R^4 term. In particular, in Sec. II we discuss the derivation of the corrected field equations. We present three methods: the direct derivation of the field equations, the application of the Palatini method and the derivation of an effective one-dimensional action. The analysis in this section holds for both extremal and nonextremal branes (but some of the explicit formulas apply only to extremal D3-branes). In Sec. III we restrict our attention to the extremal D3-brane and rewrite the equations of motion in first order form, which we then integrate to obtain the α' corrected solution. In Sec. IV we discuss in detail thermodynamics for higher derivative theories and apply the results to the corrected D3-brane solution. In Sec. V we discuss the corrections to extremal electric p-branes in D dimensional spacetimes. We conclude with a discussion of our results in Sec. VI. Finally in Appendixes A and B we give several results regarding the evaluation of the higher derivatives terms on lowest order solutions, and in Appendix C we present the most general D3-brane solution of the lowest order equations with a specific two-block ansatz.

II. EQUATIONS OF MOTION

The fields that participate in the D3-brane solution of type IIB supergravity are the metric g_{ij}, the dilaton ϕ, and the four-form gauge field $A_{i_1 \dots i_4}$. The terms in the classical type IIB supergravity action that only involve these fields, in the Einstein frame, read,[1]

$$I = -\frac{1}{16\pi G_N}\int d^{10}x\sqrt{-g}\left[R - \frac{1}{2}(\partial\phi)^2 - \frac{g_s^2}{4\cdot 5!}F_5^2\right] \tag{2.1}$$

where[2] $G_N = 8\pi^6 g_s^2 \alpha'^4$. The field equations derived from this action should be supplemented by the self-duality (SD) condition on F_5.

The leading higher derivative terms in the low energy effective action of type IIB string theory appear at order α'^3. The purely gravitational terms can be computed by the 4-point graviton scattering amplitudes [8] or a four-loop sigma model computation [10] and give rise to the well-known R^4 terms. To compute the corrections to the D3-brane solution we need to know the higher derivative terms that involve g_{ij}, ϕ and F_5.[3] As discussed in the Introduction, the complete set of such terms is not known at present. In principle, such terms can be computed by studying tree-level scattering amplitudes. One would need to compute up to 8-point functions in order to compute all 8-derivative terms in the effective action. Terms that depend on the RR-fields are more difficult to compute using the sigma model methods in the RNS formalism, but one could use sigma models in the pure spinor formalism [22,23] to perform a manifestly supersymmetric beta function computation, see [24] for such a computation.

We will proceed by considering only the effect of the R^4 term. This is what has been done in similar computations in most of the literature. We emphasize, however, that there is no *a priori* reason that the α'^3 terms can be truncated to only the R^4 term. In fact our results indicate that, at least for the computation of α' corrections to the D3-brane solution, the truncation is not consistent. We consider the following α'^3 corrections to Eq. (2.1),

$$I_W = -\frac{1}{16\pi G_N}\int d^{10}x\sqrt{-g}\,\gamma(\phi)W \tag{2.2}$$

where

$$\gamma(\phi) = \frac{1}{16}E_{3/2}(\phi)g_s^{3/2}\alpha'^3,$$

$$W = C^{imnj}C_{kmnl}C_i{}^{rsk}C^l{}_{rsj} + \frac{1}{2}C^{ijmn}C_{klmn}C_i{}^{rsk}C^l{}_{rsj}. \tag{2.3}$$

Notice that we used the field redefinition ambiguity [8,26] to reach a scheme where W depends only on the Weyl tensor. $E_{3/2}(\tau,\bar{\tau})$ is the nonholomorphic modular form of weight (0,0).[4] Here $\tau = \tau_1 + i\tau_2 = \chi + ie^{-\phi}$, where χ is the axion. In the following we set $\chi = 0$. The factor of $g_s^{3/2}$ in Eq. (2.3) is correlated with our conventions, see footnote 2. The dilaton dependence follows from supersymmetry and the $SL(2,Z)$ symmetry of type IIB string theory [16,25]. This behavior takes into account nonperturbative effects as well. At string tree-level $\gamma(\phi)|_{tree} = \frac{1}{8}\zeta(3)$.

[1] Our curvature conventions are $R_{ijk}{}^l = \partial_j\Gamma^l_{ik} - \Gamma^l_{ip}\Gamma^p_{jk} - (i\leftrightarrow j)$, $R_{ij} = R_{ikj}{}^k$, $R = g^{ij}R_{ij}$. The Weyl tensor is given by $C_{ijk}{}^l = R_{ijk}{}^l + \frac{1}{8}[\delta^l_i R_{jk} + g_{jk}R^l_i - \frac{1}{9}R\delta^l_i g_{jk} - (i\leftrightarrow j)]$.

[2] Notice that we use the convention of leaving a factor of g_s in Newton's constant. This means that our "Einstein frame" is related to the string frame by $\tilde{g}_E = e^{-(\phi-\phi_\infty)/2}g_{st} = g_s^{1/2}g_E$, where g_E is the true Einstein metric and $e^{\phi_\infty} = g_s$. Under S-duality g_E is invariant, but $\tilde{g}_E \to g_s^{-1}\tilde{g}_E$.

[3] We assume throughout this work that the fields that are zero in the lowest order solution remain zero after the α' corrections are taken into account. This would be correct if all higher derivative terms are at least quadratic in the fields that are zero at lowest order.

[4] Explicitly, $E_{3/2}(\tau,\bar{\tau}) = \Sigma_{(m,n)\neq(0,0)}(\tau_2^{3/2}/|m+n\tau|^3)$, where (m,n) denotes the greatest common divisor of the integers m and n. A nonholomorphic form $F^{(w,\hat{w})}$ of weight $(w,\hat{w})$ transforms as $F^{(w,\hat{w})} \to F^{(w,\hat{w})}(c\tau+d)^w(c\bar{\tau}+d)^{\hat{w}}$ under the $SL(2,Z)$ transformation $\tau \to (a\tau+b)/(c\tau+d)$.

The equations of motion of type IIB supergravity in the Einstein frame, restricted on these fields, and including the corrections from Eq. (2.2), read,

$$E_{ij}\equiv R_{ij}-\frac{1}{2}g_{ij}R-\frac{1}{2}\left[\partial_i\phi\partial_j\phi-\frac{1}{2}g_{ij}(\partial\phi)^2\right]$$
$$-\frac{g_s^2}{96}\left(F_{il_1\ldots l_4}F_j{}^{l_1\ldots l_4}-\frac{1}{10}g_{ij}F_5^2\right)$$
$$+\left(w_{ij}-\frac{1}{2}g_{ij}\gamma(\phi)W\right)$$
$$=0 \tag{2.4}$$

$$E\equiv\Box\phi-\gamma_\phi(\phi)W=0 \tag{2.5}$$

$$F_5=\star F_5 \tag{2.6}$$

where[5]

$$\gamma_\phi=\frac{\partial\gamma}{\partial\phi}=-\frac{1}{32}\alpha'^3 g_s^{3/2}(D_0+\bar{D}_0)E_{3/2} \tag{2.7}$$

and w_{ij} is defined by

$$\int d^{10}x\sqrt{g}\,\gamma(\phi)\,\delta W=\int d^{10}x\sqrt{g}\,\delta g^{ij}w_{ij} \tag{2.8}$$

and is given in Appendix A. Using the fact that the Weyl tensor is Weyl invariant one can show that

$$g^{ij}w_{ij}=4\gamma(\phi)W. \tag{2.9}$$

Notice that the self-duality equation (2.6) is expected to receive corrections from the α'^3 terms that depend on F_5. The reason is the following. An F_5-dependent α' correction will give rise, upon variation with respect to the gauge field, to the equation

$$\frac{1}{\sqrt{-g}}\frac{1}{2\cdot 4!}\partial_l(\sqrt{-g}F^{li_1\cdots i_5})+\gamma w_A{}^{i_1\cdots i_5}=0 \tag{2.10}$$

where $w_A{}^{i_1\cdots i_5}(g,A,\phi)$ is the variation of the extra term with respect to the gauge field. Suppose now that the self-duality condition holds. The first term in Eq. (2.10) would then vanish by itself and we obtain,

$$w_A{}^{i_1\cdots i_5}(g,A,\phi)=0. \tag{2.11}$$

We thus find a new equation arising at order α'^3.[6] The higher derivative terms, however, should only correct the lowest order equations, not introduce new equations. Any new equations would generically make the system of equations inconsistent. It follows that if the higher derivative terms are F_5 dependent, the self-duality condition will have to be deformed. In other words, it should be a combination of the 5-form field strength with other fields that is self-dual not the 5-form by itself. Notice that any superinvariant based on the dilaton superfield will contain F_5 dependent terms [11]. Since in this work we only take into account Eq. (2.2) the self-duality equation holds at order α'^3 as well. This is one point where the complete analysis is expected to deviate from the analysis presented here.

We look for perturbative solutions in α' of these equations of motion. The general ansatz we consider is

$$ds^2=e^a((-fdt^2+d\vec{x}^2)+e^h(f^{-1}dr^2+r^2d\Omega_5^2)) \tag{2.12}$$

where the functions a, h and f depend only on the radius r. Extremal solutions have $f=1$, but we will keep f arbitrary for the time being, and set $f=1$ at a later stage. The self-duality condition is solved by

$$F_{tabcr}=16\pi N\alpha'^2\epsilon_{abc}e^{-2h}r^{-5},$$
$$F_{m_1\ldots m_5}=16\pi N\alpha'^2\epsilon_{m_1\ldots m_5} \tag{2.13}$$

where a,b,c are spatial worldvolume coordinates, $m_1\ldots m_5$ are indices on the S^5 directions and ϵ_{abc} and $\epsilon_{m_1\ldots m_5}$ are the volume densities on flat R^3 and on the unit five-sphere, S^5, respectively.

The lowest order equations of motion admit the solution,

$$e^{-2a_0}=e^{h_0}=1+\frac{\ell^4}{r^4},\quad e^{\phi_0}=g_s,\quad f=1,\quad \ell^4=4\pi g_s N\alpha'^2 \tag{2.14}$$

where the subscript in a_0, h_0, and ϕ_0 indicates that this is the lowest order solution. The solution describes the long range field of N D3-branes. Removing the "1" from the harmonic function yields $AdS_5\times S^5$, the near-horizon limit of the D3-branes.

Our objective is to obtain a solution of the equations of motion (2.4) perturbatively in α', i.e. we will look for solutions

$$a=a_0+\gamma a_1,\quad h=h_0+\gamma h_1,\quad \phi=\phi_0+\gamma_\phi\phi_1,\quad f=1. \tag{2.15}$$

To obtain a_1, h_1 and ϕ_1 one may substitute the ansatz (2.12) with the coefficients in Eq. (2.15) to the field equations (2.4) and keep only the terms of order α'^3. The computation involves evaluating the order α'^3 terms in Eq. (2.4) on the lowest order solution (2.14). We now present a few different formulations of the problem.

A. Direct computation

This is the straightforward approach where one first obtains w_{ij} by varying the new term in the 10D action and then

[5] $D_w=i(\tau_2\partial/\partial\tau-iw/2)$ and $\bar{D}_{\hat{w}}=-i(\tau_2\partial/\partial\bar{\tau}+i\hat{w}/2)$ are modular covariant derivatives that map a modular form of weight $(w,\hat{w})$ to another one of a different weight, $D_wF^{(w,\hat{w})}=F^{(w+1,\hat{w}-1)}$, $\bar{D}_{\hat{w}}F^{(w,\hat{w})}=F^{(w-1,\hat{w}+1)}$.

[6] This conclusion would be avoided if there is a higher derivative term that depends on F_5 and is not zero on-shell (with respect to the lowest order equations), but whose variation vanishes on-shell. As far as we can tell, this cannot happen.

substitutes the lowest order solution. Both of these steps are straightforward but very tedious. The general expression for w_{ij} is given in Appendix A. The evaluation of the corrections on the lowest order solution is also very tedious because the expressions involve tensors with complicated index contractions. A useful observation is that one can use the symmetries of the Weyl tensor to rewrite Eq. (2.3) in the following compact form:

$$W = B_{ijkl}(2B^{iklj} - B^{lijk}) \tag{2.16}$$

where

$$B_{ijkl} = C^{m}{}_{ijn}C^{n}{}_{lkm}. \tag{2.17}$$

This tensor is symmetric under a pair interchange and under simultaneous permutation of the first two and last two indices,

$$B_{ijkl} = B_{klij}, \quad B_{ijkl} = B_{jilk}. \tag{2.18}$$

The use of MATHEMATICA was instrumental in obtaining the final equations. We will present these equations after presenting two alternative methods for performing the computation.

B. Palatini formalism

There is an alternative method to derive field equations that is particularly useful in higher derivative actions. We outline it here because it is completely general and can be used when there are no symmetries that one can employ to derive a simple form of the action (as we do in the next section). Furthermore, this method is simpler than the direct derivation of the equations of motion described in the previous section. In this method one constructs a Palatini action that is first rather than second order in derivatives, and has the metric and the covariant derivative (or, equivalently, the Christoffel symbols) as independent variables (see [27] for an elementary exposition):

$$I[g,\Gamma] = \int d^{10}x\sqrt{g}\left(R[g,\Gamma] - \frac{1}{2}(\partial\phi)^2 - \frac{g_s^2}{4\cdot 5!}F_5^2 + \gamma(\phi)W[g,\Gamma]\right). \tag{2.19}$$

In deriving the equations of motion, let us vary the connection first. This gives

$$\delta_i^{(j}\nabla_l g^{k)l} - \nabla_i g^{jk} + (g^{jk}\delta_i^p - \delta_i^{(j}g^{k)p})(\Gamma^l_{lp} - \partial_p \log\sqrt{g}) + W_i^{jk} = 0, \tag{2.20}$$

which at lowest order in α' implies the usual compatibility condition between the metric and the connection. Standard manipulations yield

$$\Gamma^k_{0ij} = \frac{1}{2}g_0^{kl}(\partial_i g_{0jl} + \partial_j g_{0ik} - \partial_l g_{0ij}). \tag{2.21}$$

We also find the following solution at next order:

$$\Gamma^k_{1ij} = -\frac{1}{18}(\delta_i^k W_{jl}{}^l + \delta_j^k W_{il}{}^l) + \frac{1}{2}(W_j{}^k{}_i + W_i{}^k{}_j - W^k{}_{ij})$$

$$\Gamma^j_{1ij} = -\frac{1}{9}W_{ij}{}^j, \quad g^{jk}\Gamma^i_{1jk} = -\frac{1}{9}W_j{}^{ij}. \tag{2.22}$$

The right-hand side of the above formulas should be read as being evaluated in the lowest order metric. W_i^{jk} is such that

$$\int \gamma(\phi)\delta W = \int \delta g^{ij}W_{ij} + \int \delta\Gamma^i_{jk}W_i^{jk}. \tag{2.23}$$

These tensors satisfy the following identities:

$$g^{ij}W_{ij} = 4\gamma(\phi)W$$

$$W^j{}_{ji} = \frac{1}{2}W_{ij}{}^j. \tag{2.24}$$

By explicit computation one finds that W_i^{jk} is given by the covariant derivative of a tensor that is cubic in the Weyl tensor, although we will not give the explicit expression here. Symbolically, W_i^{jk} has the structure $W\sim\nabla[\gamma(\phi)CB]$ where B is the tensor defined in the previous section. At the end of the day, the combination that appears in the equation of motion is given in terms of a *single* scalar function of r when computed for the lowest order solution, but we will not report the details here.

The remaining equations for the metric and matter fields are also easily derived. In particular, deriving the equation of motion for the metric is much simpler than in the second order formalism. One obtains equations of motion where the Ricci tensor depends on both g and Γ. One then expands this in the above solutions to obtain the standard form of the Einstein equations:

$$R_{ij} - \frac{1}{2}\partial_i\phi\partial_j\phi - \frac{g_s^2}{96}F^2_{5\,ij} + \left[W_{ij} - \frac{3}{8}g_{ij}\gamma(\phi)W\right] + \frac{1}{2}\nabla_k[W_{ji}{}^k + W_{ij}{}^k - W_{ij}{}^k] = 0$$

$$\Box\phi - \gamma_\phi W = 0, \tag{2.25}$$

which are supplemented with the self-duality condition. These equations are equivalent to the ones found by direct computation, but their derivation is simplified.

C. Effective 1D action

We show in this section that for spherically symmetric solutions, there is an effective one-dimensional action that yields the same field equations as Eq. (2.4) evaluated on the ansatz (2.12). To obtain the one-dimensional action we start from the variation of the ten-dimensional action,

$$\delta I=\int d^{10}x\sqrt{g}[\delta g^{ij}E_{ij}+\delta\phi E], \tag{2.26}$$

where we have substituted the solution (2.13) of the self-duality equation (2.6) in E and E_{ij}. We now use the ansatz (2.12) to express δg^{ij} in terms of $\delta a, \delta h$, and δf. This yields

$$\delta I=\int d^{10}x\sqrt{g}\bigg[-\delta a(g^{ij}E_{ij})+\frac{\delta f}{f}(g^{rr}E_{rr}-g^{tt}E_{tt}) -\delta h(g^{rr}E_{rr}+g^{mn}E_{mn})+\delta\phi E\bigg]. \tag{2.27}$$

Since all the fields depend only on the radial variable, one can now perform all integrations but the radial one. The resulting variations can be integrated again to a one-dimensional action,

$$I_{1\mathrm{D}}=\int dr e^{4a+2h}\frac{r^5}{\ell^5}\bigg[\frac{1}{6}\bigg(64fa''+40fh''+148fa'^2 +168fa'h'+50fh'^2+3f\phi'^2+400\,\frac{fa'}{r}+240\,\frac{fh'}{r} +200\,\frac{f}{r^2}+6f''+64f'a'+37f'h'+80\,\frac{f'}{r}-\frac{120}{r^2}\bigg) +\frac{8\ell^8}{r^{10}e^{4(a+h)}}-\gamma(\phi)e^{a+h}W\bigg] \tag{2.28}$$

where we have discarded an overall (infinite) volume factor. W is given by Eq. (2.3) evaluated on Eq. (2.12). It is a function of a, h, f and their first two radial derivatives. The explicit expression is given in Appendix A.

Notice that this derivation of the effective action guarantees that all solutions of the 1D action are solutions of the 10D action. In other words, the reduction from 10D to 1D is consistent. What is crucial is that the number of independent functions appearing in the ansatz (2.12) is equal to the number of equations one gets by evaluating Eq. (2.4) on the Eq. (2.12). For the problem at hand this number is four even when $f=1$, so even in this case one must first proceed with general f and then set $f=1$. The method presented here can be used more generally in order to provide consistent reductions of the higher dimensional theories. One should contrast this method with the most common practice to substitute an ansatz in the action and then reduce. This latter does not guarantee a consistent reduction.

It is instructive to rewrite Eq. (2.28) in terms of the variables used in the reduction of the type IIB supergravity over S^5. Such a reduction was presented in [4]. Using their variables the one-dimensional action reads

$$I_{1\mathrm{D}}=-\int dr\sqrt{g_5}\bigg[R_5-\frac{1}{2}g_5^{rr}(\partial_r\phi)^2-\frac{40}{3}g_5^{rr}(\partial_r\nu)^2 -V(\nu)+\gamma(\phi)e^{-(10/3)\nu}W\bigg] \tag{2.29}$$

$$V(\nu)=\frac{1}{\ell^2}[8e^{-(40/3)\nu}-20e^{-(16/3)\nu}] \tag{2.30}$$

where g_5 denotes the determinant of the five-dimensional metric g_{5mn} given in Eq. (2.32). The fields appearing in this action are related to the a, h and f by

$$\nu(r)=\frac{1}{2}(a+h)+\log\frac{r}{\ell} \tag{2.31}$$

$$ds_5^2=g_{5mn}dx^m dx^n=e^{(1/3)(8a+5h)}\bigg(\frac{r}{\ell}\bigg)^{10/3} \times\bigg(-fdt^2+\delta_{ab}dx^a dx^b+\frac{e^h}{f}dr^2\bigg). \tag{2.32}$$

This can be shown using the standard reduction formula

$$ds_{10}^2=e^{-(10/3)\nu}g_{5mn}dx^m dx^n+e^{2\nu}\ell^2 d\Omega_5^2 \tag{2.33}$$

and matching with the ansatz in Eq. (2.12). The dimensionful parameter l is proportional to the Planck length and is introduced into the ansatz on dimensional grounds.

The equations of motion that follow from Eq. (2.28) with f set equal to one, $f=1$, are given by

$$18a''+10h''+36a'^2+36a'h'+10h'^2+\phi'^2+\frac{90a'}{r}+\frac{50h'}{r} +\gamma w_a=0 \tag{2.34}$$

$$10\bigg(a''+\frac{1}{2}h''\bigg)+2a'^2+20\bigg(a'+\frac{1}{2}h'\bigg)^2+\frac{1}{2}\phi'^2 +\frac{50}{r}\bigg(a'+\frac{1}{2}h'\bigg)-\frac{8\ell^8}{r^{10}e^{4(a+h)}}+\gamma w_h=0 \tag{2.35}$$

$$8a''+5h''-4a'^2-4a'h'+\phi'^2+\frac{5h'}{r}+\gamma w_f=0 \tag{2.36}$$

$$\phi''+\bigg(4a'+2h'+\frac{5}{r}\bigg)\phi'-2\gamma_\phi w_\phi=0 \tag{2.37}$$

where γw_a is the variation of α'^3 term in action (2.28) with respect to a, etc. We give in Appendix A the explicit form of W as a function of a, h, f, and their derivatives. From there one may derive w_a etc. The evaluation of the corrections on the lowest order solution (2.14) is given by

$$w_a = w_\phi = -14400\,\frac{\ell^{16}}{r^{24}e^{(19/2)h_0}}$$

$$w_h = -\frac{4800\ell^{12}}{r^{28}e^{(19/2)h_0}}(112\ell^8 - 249\ell^4 r^4 + 84r^8)$$

$$w_f = \frac{28800\ell^{12}}{r^{28}e^{(19/2)h_0}}(14\ell^8 - 35\ell^4 r^4 + 10r^8). \tag{2.38}$$

Notice that the metric in Eq. (2.12) depends on a only through an overall conformal factor. It follows that the Weyl tensor does not depend on a, and that W depends on it only through the inverse metrics used in contracting indices. This explains why w_a is equal to w_ϕ.

We have explicitly verified that Eqs. (2.34)–(2.37) are equivalent to the equations one obtains by evaluating Eq. (2.4) on the ansatz (2.12), as discussed in Sec. II A. This remains true even when f is not set equal to one. This is a nice check, especially on w_a, w_h, w_f and w_ϕ, as the organization of the two computations is rather different. In the next section we present yet another reformulation in terms of first order equations.

III. FIRST ORDER SYSTEM

The D3-brane solution is half supersymmetric [28]. This implies that there must be an equivalent first order formulation of the field equations when the ansatz for the solution is consistent with supersymmetry. In this section we set $f=1$, and present such a reformulation. A (somewhat different) discussion of first order equations appeared in [14].

Let us first consider the effective action without the α' correction. The potential in Eq. (2.30) has an AdS critical point at $\nu=0$. This critical point is stable as it is maximally supersymmetric. It follows that the potential $V(\nu)$ admits a "superpotential" $\mathcal{W}$ such that the AdS critical point is a critical point of $\mathcal{W}$ [29]. Indeed, one finds that the potential (2.30) can be rewritten as

$$V(\nu) = \frac{3}{10}\left(\frac{\partial \mathcal{W}_\pm}{\partial \nu}\right)^2 - \frac{16}{3}\mathcal{W}_\pm^2 \tag{3.1}$$

$$\mathcal{W}_\pm = \frac{1}{\ell}\left[e^{-(20/3)\nu} \pm \frac{5}{2}e^{-(8/3)\nu}\right]. \tag{3.2}$$

The formula for the potential (3.1) coincides with the one in [29] after the differences in conventions are taken into account. The AdS critical point is also a critical point of $\mathcal{W}_-$. We shall henceforth consider only $\mathcal{W}_-$, which we shall denote $\mathcal{W}$, and only add a few comments about $\mathcal{W}_+$.

A simple Bogomol'nyi argument implies that the theory admits BPS domain wall solutions [30,31]

$$ds_5^2 = e^{2c(\rho)}\eta_{ab}dx^a dx^b + d\rho^2$$

$$\nu = \nu(\rho)$$

$$\phi = \phi(\rho) \tag{3.3}$$

where $c(\rho)$ and $\nu(\rho)$ are solutions of the first order equations

$$\partial_\rho \nu = \frac{3}{20}\frac{\partial \mathcal{W}}{\partial \nu}, \quad \partial_\rho c = -\frac{2}{3}\mathcal{W}, \quad \partial_\rho \phi = 0. \tag{3.4}$$

One can verify that solutions of the first order system solve the second order equations. The first order equations also follow from the requirement that the "Killing spinor" equations

$$\left(D_\mu + \frac{1}{\sqrt{15}}\mathcal{W}\Gamma_\mu\right)\epsilon = 0,$$

$$\left(\Gamma^\mu \partial_\mu \nu - \frac{3}{5\sqrt{2}}\frac{\partial \mathcal{W}}{\partial \nu}\right)\epsilon = 0, \quad \Gamma^\mu \partial_\mu \phi \epsilon = 0 \tag{3.5}$$

admit solutions for nonzero spinor ϵ [30]. In the context of supergravity these are the variations of the gravitino and dilatino, and the solutions of the first order equations are supersymmetric solutions.

The coordinate transformation,

$$r = r(\rho), \quad \frac{d\rho}{dr} = \left(\frac{r}{l}\right)^{5/3} e^{4/3[a(r)+h(r)]}, \tag{3.6}$$

can bring the metric (2.32) to the form (3.3). Furthermore, $a(r)$ and $h(r)$ are related to $c(\rho)$ and $\nu(\rho)$ in a simple way,

$$\nu(\rho) = \frac{1}{2}\{a[r(\rho)] + h[r(\rho)]\} + \log\frac{r(\rho)}{l} \tag{3.7}$$

$$c(\rho) = \frac{1}{3}\left\{4a[r(\rho)] + \frac{5}{2}h[r(\rho)] + 5\log\frac{r(\rho)}{l}\right\}. \tag{3.8}$$

It follows that one can obtain first order equations for a and h by substituting Eqs. (3.7) and (3.8) in Eq. (3.4). One obtains,

$$\partial_r a + \partial_r h + \frac{2l^4}{r^5}e^{-2(a+h)} = 0,$$

$$4\partial_r a + \frac{5}{2}\partial_r h + \frac{2l^4}{r^5}e^{-2(a+h)} = 0, \quad \partial_r \phi = 0. \tag{3.9}$$

These are exactly the equations that follow from the analysis of supersymmetry in ten dimensions [28].

Before we move on to consider the modification due to α' corrections we note that had we considered the superpotential $\mathcal{W}_+$, we would have ended up with a solution of the form (2.14) but with

$$e^{h_0}=1-\frac{l^4}{r^4}, \quad r^4>l^4, \quad e^{h_0}=-1+\frac{l^4}{r^4}, \quad r^4<l^4. \tag{3.10}$$

This solution has a curvature singularity at $r=l$, and is related to the standard D3-brane solution by analytic continuation to imaginary r.

A. α' corrections to the first order system

We now discuss the extension of the analysis to include the α' corrections. Ideally one would like to write the effective action as a sum and/or differences of squares and then read off the α'-corrected first order equations. Such a rewriting should be possible because of supersymmetry. However, the complexity of W for general h, a and ϕ makes such an exercise rather formidable. Furthermore, as we discussed, our action is not complete since further relevant bosonic terms may be present and such additional terms may be necessary in order to rewrite the action as a sum of squares.

We proceed by adding order α'^3 terms in Eq. (3.9) and demand that the solutions of the first order system solve the second order equations (2.34)–(2.37),

$$a'+h'+\frac{2l^4}{r^5}e^{-2(a+h)}=\gamma j_1(r) \tag{3.11}$$

$$4a'+\frac{5}{2}h'+\frac{2l^4}{r^5}e^{-2(a+h)}=\gamma j_2(r) \tag{3.12}$$

$$\phi'=2\gamma_\phi j_3(r) \tag{3.13}$$

where the prime indicates a derivative with respect to r. This yields

$$j_1=2\left(1+\frac{10}{rh_0'}\right)b_1-\frac{1}{2h_0'}(w_h+w_f-w_a),$$

$$j_2=5\left(1+\frac{4}{rh_0'}\right)b_1-\frac{1}{2h_0'}(w_h+w_f-w_a),$$

$$j_3=\frac{1}{r^5}\int^r dr' r'^5 w_\phi+\frac{C_1}{r^5} \tag{3.14}$$

where $b_1\equiv a_1'+\frac{1}{2}h_1'$ satisfies

$$b_1'+\frac{9}{r}b_1=\frac{1}{10}(w_f-w_a). \tag{3.15}$$

Notice that supersymmetry demands that $b_0=0$ to lowest order. There are nonsupersymmetric solutions of the lowest order second order equations (2.34)–(2.37), including nonsupersymmetric solutions with $b_0=0$, as we discuss in Appendix C, but we shall not consider them here.

Once w_f, w_a and w_h are computed using the lowest order solution, Eq. (3.15), for b_1 the equation for j_3 can be easily integrated. b_1 in turn gives the source terms j_1, and j_2. The integration constants are fixed by requiring that solution is asymptotically flat and regular at the horizon.

B. α'-corrected solution

Using the w's in Eq. (2.38) one can easily compute the sources,

$$j_1(r)=C_0\frac{3\ell^4+5r^4}{r^9\ell^4}-\frac{16\ell^{12}}{2431r^{43}e^{(17/2)h_0}}[768\ell^{24}+7808\ell^{20}r^4+35360\ell^{16}r^8+93840\ell^{12}r^{12}+161330\ell^8r^{16}+3658655\ell^4r^{20}-3500640r^{24}]$$

$$j_2(r)=\frac{5C_0}{r^5\ell^4}-\frac{320\ell^{12}}{2431r^{39}e^{(17/2)h_0}}[64\ell^{20}+544\ell^{16}r^4+2040\ell^{12}r^8+4420\ell^8r^{12}+133705\ell^4r^{16}-109395r^{20}]$$

$$j_3(r)=\frac{C_1}{r^5}-\frac{160\ell^{16}}{2431r^{39}e^{(17/2)h_0}}(128\ell^{16}+1088\ell^{12}r^4+4080\ell^8r^8+8840\ell^4r^{12}+12155r^{16}). \tag{3.16}$$

The integration constants C_0 and C_1 can be fixed by requiring that the terms on the right-hand side of Eqs. (3.11)–(3.13) are small compared to the terms on the left-hand side for all r. This implies that j_1, j_2 and j_3 should be at most the same order as the terms on the left-hand side. Near $r=0$ the terms on the left-hand side behave as $1/r$. On the other hand, j_1 behaves as $1/r^9$ and j_2 and j_3 as $1/r^4$. This can be remedied by choosing appropriately the integration constants,

$$C_0=\frac{2^{12}l^2}{2431}, \quad C_1=\frac{5}{l^4}C_0. \tag{3.17}$$

This is a nontrivial result since the number of terms that we need to set to zero is greater than the integration constants we have. The same values of integration constants follow by requiring that the solution we present in the next section is smooth at the horizon.

We note that the j_1 and j_2 are such that they cannot be absorbed into a α'-modification of the superpotential $\mathcal{W}$. To check this one may rewrite Eq. (3.11) in the coordinate system (3.6). Let us call $J_1(\rho)$ and $J_2(\rho)$ the sources that appear on the right-hand side of the first and second equations in Eq. (3.4). One may absorb $J_2(\rho)$ into $\mathcal{W}$ by $\mathcal{W}'=\mathcal{W}-\frac{3}{2}\gamma J_2$. In order for this transformation to also remove the source J_1 the following relation should hold:

$$\frac{\partial J_2}{\partial\rho}+\frac{2}{3}\frac{\partial\nu}{\partial\rho}J_1=0. \tag{3.18}$$

A direct computation shows that this is not satisfied, but we note that there are (unexpected) cancellations between the two terms. Had we been able to absorb the sources into a modified superpotential, we would conclude that the form of the supersymmetry rules in Eq. (3.5) is not modified at order α'^3, so these results may be taken to indicate that there are new terms in the supersymmetry transformation rules at order α'^3. We should add, however, that given that we only consider a part of the complete effective action such a conclusion is premature.

Knowing the sources, it is straightforward to integrate the first order equations (3.11)–(3.13). Taking the sum of the first two equations one obtains a differential equation for $a+h/2$ which can be easily integrated. Feeding back one solves for a_1 and h_1. The integration of Eq. (3.13) is equally straightforward. All integration constants are set to zero by requiring that the solution be asymptotically flat. The result is

$$h_1=-\frac{1024(3\ell^8+9\ell^4r^4+10r^8)}{2431\ell^2r^{12}e^{h_0}}-\frac{32\ell^{12}}{2431r^{38}e^{(15/2)h_0}}\times[-96\ell^{20}-912\ell^{16}r^4-3910\ell^{12}r^8-22355\ell^8r^{12}-97240\ell^4r^{16}+218790r^{20}]$$

$$a_1=\frac{1024(2\ell^8+5\ell^4r^4+5r^8)}{2431\ell^2r^{12}e^{h_0}}+\frac{8\ell^{12}}{2431r^{38}e^{(15/2)h_0}}\times[-256\ell^{20}-2304\ell^{16}r^4-9384\ell^{12}r^8-47600\ell^8r^{12}-197795\ell^4r^{16}+486200r^{20}]$$

$$\phi_1=-\frac{10240}{2431l^2r^4}+\frac{160\ell^{16}}{2431r^{34}e^{(15/2)h_0}}[64\ell^{12}+408\ell^8r^4+1020\ell^4r^8+1105r^{12}]. \tag{3.19}$$

The corrections are smooth at $r=0$, and the choice of integration constants was crucial for this property.

Let us consider the near horizon limit of the solution. Following [32] we consider the limit

$$\alpha'\to 0,\quad \frac{r}{\alpha'}\ \text{fixed},\quad g_s\ \text{fixed}. \tag{3.20}$$

In this limit we find that

$$\gamma h_1=-\gamma a_1=\frac{1}{N^{3/2}}\frac{E_{3/2}(g_s)}{2\cdot 2431\pi^{3/2}},\qquad \gamma_\phi\phi_1=-180\frac{\gamma_\phi}{\gamma}(\gamma h_1). \tag{3.21}$$

It is intriguing that even though we ignored higher derivative terms that depend on F_5 the near horizon limit is still $AdS_5\times S^5$, just as one would expect for the "true" D3-brane solution [1,3]. This may indicate that F_5 dependent terms can be ignored in the near-horizon limit. Recall that in the near-horizon limit F_5 is proportional to the volume form both in the AdS_5 and the S^5 directions. One may verify using the results in [11] that the F_5 dependent terms of the dilaton superfield vanish in this case. This is an additional indication that our results are exact in this case.

Notice that the AdS radius does not receive corrections but the string coupling constant does. The choice of the integration constants in Eq. (3.17) is crucial for the limit (3.20) to exist.

IV. THERMODYNAMICS OF CORRECTED SOLUTIONS

In this section we discuss the thermodynamics of the corrected solution. The quantities of interest are the mass density,[7] the temperature, the entropy and the charge density of the solution. One may use either Euclidean of Lorentzian methods to study thermodynamics. In the present case the self-duality of the lowest order solution presents an additional complication in the Euclidean computation since one needs to understand the proper analytic continuation of the self-duality condition. We will follow a Lorentzian analysis and adapt the method of Wald [17–19] for the problem at hand.

Recall that the entropy, mass and charge of a black hole satisfy the first law of thermodynamics which in integrated form (Smarr formula) reads

$$TS=M-vQ. \tag{4.1}$$

Here T is the Hawking temperature, S is the entropy, M is the mass, Q is the charge and v the corresponding potential. Extremal black holes have zero temperature $T=0$ (and quite often zero entropy as well) so the Smarr formula implies

$$M=vQ. \tag{4.2}$$

In the context of supersymmetric black holes, this relation originates from the supersymmetry algebra. The case of D-branes is exactly analogous, but the appropriate quantities are now densities. One may wrap the spatial worldvolume coordinates of the brane on a torus (or some other compact manifold) and reduce over that manifold to obtain a black hole in lower dimensions. For instance, the D3-brane can be viewed as a 7D black hole after reduction over the spatial worldvolume coordinates. Our analysis will be done from the ten-dimensional point of view.

In the presence of higher derivative terms, the extremal D3-brane still has zero temperature (as we verify below), so a relation of the form (4.2) should still hold since Eq. (4.1) follows from first law alone. Since the charge of the D3-brane is quantized one might expect that Eq. (4.2) would imply that the mass does not renormalize. We find, however, that things are more subtle and both the mass and the potential v renormalize.

Given a gravitational system described by an action I one may compute the gravitational energy as follows. Let us consider a spacetime M and denote by ∂M_∞ its asymptotic infinity which is considered as its boundary. We first require

[7]Notice that we use interchangeably the terminology "mass density" and "tension." With abuse of terminology will also sometimes just call "mass" the tension, and "charge" the charge density. It will be clear, however, from the context which quantity we are discussing.

that the theory, subject to appropriate boundary conditions, has a well-defined variational problem, i.e. all boundary terms in the variation of the action should vanish automatically so that the bulk field equations are true extrema of the action. In gravitational systems this requires the addition of boundary terms B,

$$I=\int_M L-\int_{\partial M_\infty} B. \tag{4.3}$$

Under a variation we have

$$\delta L=(\text{field equations})+d\Theta(\Phi,\delta\Phi) \tag{4.4}$$

where Φ denotes collectively all fields. In order for the variational problem to be well-defined B and Θ should be related by

$$\delta\int_{\partial M_\infty} B=\int_{\partial M_\infty}\Theta(\Phi,\delta\Phi). \tag{4.5}$$

In pure gravity $B=2K$, where K is the trace of the second fundamental form. In more general theories B may contain additional terms.

The action (4.3) is invariant under diffeomorphisms. This implies that there is a corresponding Noether current,

$$\mathbf{J}=\mathbf{\Theta}(\Phi,\mathcal{L}_\xi\Phi)-i_\xi\mathbf{L} \tag{4.6}$$

where ξ^a is a vector that generates the diffeomorphism, $\mathcal{L}_\xi$ is the Lie derivative along ξ^a, i_ξ is the inner derivative [when acting on a n-form it produces a $(n-1)$-form by contracting its first index by ξ^a] and we use form notation [$\mathbf{J}$ is a $(d-1)$-form, $\mathbf{L}$ is a d-form etc.]. When the field equations are satisfied $\mathbf{J}$ is closed, $d\mathbf{J}=0$, and one can construct locally a $(d-2)$-form $\mathbf{Q}$ such that $\mathbf{J}=d\mathbf{Q}$. The Hamiltonian that provides the dynamics generated by ξ^a is given by

$$H=\int_C \mathbf{J}-\int_{\partial M_\infty} i_\xi\mathbf{B} \tag{4.7}$$

where C is a Cauchy surface. On-shell this evaluates to a surface term

$$H=\int_{\partial M_\infty}(\mathbf{Q}-i_\xi\mathbf{B}). \tag{4.8}$$

The gravitational energy is now defined by taking ξ to be a timelike Killing vector. In general, this expression is divergent so a suitable subtraction should be employed. In asymptotically AdS spacetimes one may incorporate in $\mathbf{B}$ covariant boundary counterterms [33], but in asymptotically flat spacetimes such universal covariant local counterterms do not exist [34]. We (implicitly) use the background subtraction method below.

Let us now consider the theory based on the action (2.1). Following similar steps as in [18] one finds that the mass density of a D3-brane solution is given by

$$\mu=\frac{M}{V}=\frac{1}{16\pi G_N}\int(\partial_m h_{pm}-\partial_p h_j^j)dS_5^p \tag{4.9}$$

where h_{ij} is the deviation of the metric from the Minkowski metric and V is the volume of the spatial worldvolume directions. The integration is over the sphere at asymptotic infinity in transverse space. The index j runs over all spatial indices and p and m only over the transverse coordinates. Formula (4.9) generalizes the ADM formula to apply to p-brane spacetimes [35]. One may rewrite this formula as a Komar-like mass formula,

$$M=\frac{1}{8\pi G_N}\int_{S_\infty}\epsilon_{a_1\ldots a_8bc}\nabla^{[b}\xi^{c]} \tag{4.10}$$

where S_∞ is the spacelike surface at infinity enclosing the brane. We wrap the spatial worldvolume coordinates of the brane on a torus of volume v, so that $S_\infty=T^3\times S^5$. Static spacetimes of the form (2.12) have a timelike Killing vector

$$\xi=\xi^i\frac{\partial}{\partial x^i},\quad \xi^t=1,\quad \xi^i=0, i\neq t. \tag{4.11}$$

A straightforward computation yields

$$\nabla^r\xi^t=\frac{1}{2}g^{rr}g^{tt}\partial_r g_{tt}. \tag{4.12}$$

One may now substitute this expression in Eq. (4.10) to obtain the mass of the solution.

The temperature T associated with a spacetime is equal to $T=\kappa/2\pi$ where κ is the surface gravity. The latter can be shown to be equal to [27]

$$\kappa^2=-\frac{1}{2}\nabla^a\xi^b\nabla_a\xi_b=-\frac{1}{4}g^{tt}g^{rr}g_{tt,r}^2 \tag{4.13}$$

where in the last step we used Eq. (4.12).

The entropy of a solution can be computed using the $(d-2)$-form $\mathbf{Q}$ introduced earlier,

$$S=\int_{\mathcal{H}}\mathbf{Q}=-\pi\int_{\mathcal{H}}\mathrm{d}\Sigma_{ij}Q^{ij} \tag{4.14}$$

where $\mathrm{d}\Sigma_{ij}=\mathrm{d}^8x\sqrt{h}\epsilon_{ij}$ is the surface element defined over the horizon $\mathcal{H}$, with h the induced metric on the horizon. Q_{ij} is related to the Noether charge as discussed above and (after fixing ambiguities with certain choices) is given by

$$Q^{ij}=-2\mathcal{L}^{ijkl}\nabla_k\xi_l+4\nabla_k\mathcal{L}^{ijkl}\xi_l. \tag{4.15}$$

$\mathcal{L}^{ijkl}$ is the variation of the action with respect to the Riemann tensor and ξ is the timelike Killing vector. In the case where the action contains only the Einstein-Hilbert term, the result gives the well-known Bekenstein-Hawking formula, $S=A/4$. The derivation of Eq. (4.15) assumes a nondegenerate horizon $(\kappa\neq 0)$. It was successfully applied, however (in a context similar to ours), to extremal black holes as well [36].[8] We will assume that this formula remains valid for extremal black holes.

Finally the electric charge density of the solution is given by

$$q=\frac{g_s}{\sqrt{16\pi G_N}}\int_{S^5}*F_5. \tag{4.16}$$

The prefactor is due to the normalization of the F_5 terms in Eq. (2.1). The magnetic charge $\tilde{q}$ is given by a similar integral that involves F_5. In general, the electric and magnetic charges satisfy the Dirac quantization condition [37]

$$q\tilde{q}=2\pi n \tag{4.17}$$

where n is an integer. For dyons this formula is modified and it does not by itself lead to a quantization condition for self-dual solutions with $q=\tilde{q}$, such as the D3-brane solution which we discuss [28]. The exact quantization condition for D3-branes is determined in string theory by string dualities (see for instance Sec. 3 of [38]). With the normalizations as in Eq. (2.14) the charge q of a single self-dual brane comes out to be $q=\sqrt{2\pi}$ [see Eq. (4.19)], which agrees with the naive application of Eq. (4.17) with $n=1, q=\tilde{q}$.

A. Lowest order solution

Before we proceed to incorporate the α' corrections let us discuss the lowest order D3-brane solution. In this case the metric is given in Eq. (2.14), and the mass density can be easily calculated [using either Eq. (4.9) or Eq. (4.10)] to be

$$\mu=\frac{N}{(2\pi)^3 g_s \alpha'^2} \tag{4.18}$$

where we used $G_N=8\pi^6 g_s^2\alpha'^4$. The charge density of the solution is given by

$$q=\sqrt{2\pi}N, \tag{4.19}$$

where the factor of $\sqrt{2\pi}$ is discussed below Eq. (4.17). It is straightforward to use the formulas given above to compute the entropy and the temperature of the solution. The result is that both of them are equal to zero. So one expects a formula of the form (4.2) and we indeed find

$$\mu=\frac{1}{\sqrt{16\pi G_N}}q. \tag{4.20}$$

This is the BPS formula derived in [28]. Let us now derive this relation in a way that will be useful when we consider the corrected solution.

Using Stokes' theorem one can express the surface integral in Eq. (4.10) in terms of a volume integral,

$$M=-\frac{1}{4\pi G_N}\int_\Sigma \epsilon_{a_1\ldots a_9 b}R^b_c\xi^c+\frac{1}{8\pi G}\int_{\mathcal{H}}\epsilon_{a_1\ldots a_8 bc}\nabla^{[b}\xi^{c]} \tag{4.21}$$

where Σ is a spacelike hypersurface that extends from the horizon to spatial infinity, and the last term is a surface integral over the horizon (which also involves the worldvolume T^3). To derive this one needs to use

$$\nabla_j\nabla^j\xi^i=-R^i_j\xi^j \tag{4.22}$$

which holds for Killing vectors. The integral over the horizon may be evaluated using our explicit metric and it vanishes. In general, this term gives the entropy contribution in the first law.

To evaluate the volume term we now use the Einstein equation,

$$R_{ij}=\frac{1}{2}\partial_i\phi\partial_j\phi+\frac{g_s^2}{96}F_{il_1\ldots l_4}F_j{}^{l_1\ldots l_4}. \tag{4.23}$$

We further note that ξ generates an isometry of the solution, so

$$\xi^i\nabla_i\phi=0,\quad \xi^i\nabla_i A^{j_1\ldots j_4}+4(\nabla^{[j_1}\xi_k)A^{|k|j_2j_3j_4]}=0. \tag{4.24}$$

Inserting Eq. (4.23) in Eq. (4.21) we get a term that depends on the dilaton and a term that depends on F. The former yields a vanishing contribution upon using Eq. (4.24). The latter can be manipulated as follows:

$$\begin{aligned}\xi^i F_{il_1\ldots l_4}F^{jl_1\ldots l_4}=&-4\nabla_{l_1}(\xi^i A_{il_2l_3l_4}F^{jl_1l_2l_3l_4})\\&+[\xi^i\nabla_i A_{l_1l_2l_3l_4}\\&+4(\nabla_{[l_1}\xi^k)A_{|k|l_2l_3l_4]}]F^{jl_1l_2l_3l_4}\\&-4\xi^i A_{il_2l_3l_4}\nabla_{l_1}F^{jl_1\ldots l_4}.\end{aligned} \tag{4.25}$$

The last two terms vanish due to the F-field equation and the invariance of the solution (4.24). We finally get

$$M=-\frac{g_s^2}{96\pi G_N}\int_{S_\infty\cup\mathcal{H}}\epsilon_{tra_1\ldots a_8}\xi^i A_{il_1l_2l_3}F^{trl_1l_2l_3}. \tag{4.26}$$

One may integrate F_{t123r} to obtain A_{t123r},

$$A_{t123}=\frac{1}{g_s}(e^{-h_0}-1) \tag{4.27}$$

where the constant part was chosen such that A_{t123} vanishes asymptotically. It follows then that $\xi^i A_{i123}|_{\mathcal{H}}=-1/g_s$. In general, one can change the asymptotic value of A_{t123} by performing a gauge transformation. This will modify the value of $\xi^i A_{i123}$ at the horizon. The combination

$$v=\xi^i A_{i123}|_{S_\infty}-\xi^i A_{i123}|_{\mathcal{H}}=-\frac{1}{g_s} \tag{4.28}$$

[8]We thank Bernard de Wit for discussions about this point.

is the associated electric potential and is gauge invariant.[9] It is this gauge invariant combination that couples to the electric charge q (notice that one may use Stokes' theorem to show that $\int_{\mathcal{H}} *F = \int_{S_\infty} *F)$. It follows that

$$\mu = \frac{1}{\sqrt{16\pi G_N}}(-g_s v)q \tag{4.30}$$

which is equal to Eq. (4.20) upon using Eq. (4.28).

In the present case we were able to explicitly manipulate the bulk integral in Eq. (4.21) into boundary terms. When we include the α' corrections, however, similar manipulations involving the higher derivative term become increasing complicated. Instead of using Eq. (4.25) in order to manipulate the bulk integral one could also just use the explicit solution to evaluate the bulk integral. Since the dilaton is constant the first term on the right-hand side drops out. The contribution of F_5 can also be computed straightforwardly since the integral can be computed by elementary means. These manipulations lead to the same result (4.20), but now the contribution of the charges was computed via a bulk integral.

B. Corrected solution

The α' corrected D3-brane solution in the Einstein frame is given by

$$\begin{aligned} ds^2 &= e^{-1/2h_0}(1+\gamma a_1)(-dt^2+d\vec{x}^2)+e^{1/2h_0} \\ &\quad \times[1+\gamma(a_1+h_1)](dr^2+r^2 d\Omega_5^2) \\ e^{\phi} &= g_s(1+\gamma_\phi \phi_1) \\ F_{tabcr} &= 16\pi N\alpha'^2 \epsilon_{abc} e^{-2h_0}(1-2\gamma h_1)r^{-5}, \\ F_{m_1\dots m_5} &= 16\pi N\alpha'^2 \epsilon_{m_1\dots m_5} \end{aligned} \tag{4.31}$$

where the e^{h_0} is given in Eq. (2.14) and a_1, h_1 and ϕ_1 are given in Eq. (3.19). We should emphasize that this solution would be the true corrected D3-brane only if the part of the effective action relevant for this problem consisted of only Eqs. (2.1) and (2.2). However, as we discussed earlier, it is likely that additional terms that depend on F_5 are relevant.

To compute the mass of the solution one has to take into account that the action has been modified by the addition of the term (2.2), so the mass formula should also be modified accordingly. The discussion in the beginning of this section outlines the steps that are necessary in order to compute the new mass formula. This computation is technically rather complex because of the complicated tensor contractions in W. We will use instead the following shortcut. We will start from Eq. (4.10) and use Stokes' theorem to rewrite it as in Eq. (4.21). We should emphasize that the starting point does not represent the entire mass of the corrected solution since it does not properly take into account that the mass formula has been modified. We now use the field equations,

$$R_{ij} = \frac{1}{2}\partial_i\phi\partial_j\phi + \frac{g_s^2}{96}F_{il_1\dots l_4}F_j{}^{l_1\cdots l_4} + \left(\frac{3}{8}\gamma(\phi)Wg_{ij} - w_{ij}\right). \tag{4.32}$$

The first two contributions can be analyzed as in the previous section. The last contribution represents an additional gravitational contribution. It should thus be combined with the term on the left-hand side to yield the mass of the solution. We thus propose as a mass formula

$$\begin{aligned} M = &\frac{1}{8\pi G_N}\int_{S_\infty} \epsilon_{a_1\dots a_8 bc}\nabla^{[b}\xi^{c]} \\ &+ \frac{1}{4\pi G_N}\int_{\Sigma} \epsilon_{da_1\cdots a_9}\left(\frac{3}{8}\gamma(g_s)Wg^d{}_e - w_e^d\right)\xi^e. \end{aligned} \tag{4.33}$$

The logic here is similar to the one discussed in the last paragraph of the previous section: one could either rewrite the bulk integral as a surface integral or just directly compute the bulk integral.

The result for the mass is

$$\mu = \mu_0\left(1+\gamma(g_s)\frac{5\times 2^{10}}{2431}\frac{1}{l^6}\right) = \mu_0\left(1+\frac{1}{N^{3/2}}\frac{40E_{3/2}(g_s)}{2431\pi^{3/2}}\right) \tag{4.34}$$

where $\mu_0 = N/(2\pi)^3 g_s\alpha'^2$ is the mass density of the lowest order solution. In this result the three terms in Eq. (4.33) contribute to the correction with relative weights $-1,3/2,0$.

The form of the correction in Eq. (4.34) follows by dimensional analysis and the fact that the lowest order solution depends on the parameters of the solution only via l^4. The detailed form of the higher derivative term only determines the numerical coefficient. In particular, if the numerical coefficient is nonzero, as we find in Eq. (4.34), then the mass of N_1+N_2 branes is less than the mass of N_1 branes plus the mass of N_2 branes. This follows from the inequality

$$\frac{1}{\sqrt{N_1+N_2}} < \frac{1}{\sqrt{N_1}} + \frac{1}{\sqrt{N_2}}. \tag{4.35}$$

[9]On a curved spacetime one may define the electric and magnetic part of a field strength as

$$\mathbf{E} = i_\xi \mathbf{F}, \quad \mathbf{B} = i_\xi *\mathbf{F} \tag{4.29}$$

here $\mathbf{E}$ and $\mathbf{B}$ are four-forms in our case, i_ξ is the inner derivative and ξ is a timelike Killing vector. For self-dual solutions, $\mathbf{E}=\mathbf{B}$. When the field equations and Bianchi identity hold, $d\mathbf{F} = d*\mathbf{F} = 0$, one finds that $d\mathbf{E} = d\mathbf{B} = 0$, so locally there are electric and magnetic potentials $\mathbf{E} = d\mathbf{v}$, $\mathbf{B} = d\tilde{\mathbf{v}}$, respectively. In the case at hand, the electric potential is related with the gauge field as $\mathbf{v} = i_\xi \mathbf{A}$. One may show in general that $\mathbf{v}$ is constant at the horizon and the difference between its asymptotic value and the constant value at the horizon is gauge invariant.

It follows that energetically the branes would prefer to coalesce to form a single group. Thus there should be a force acting on the two sets of branes. This is opposite to what one expects from BPS configurations, where the branes should not feel any force. We believe that after taking into account the effect of the (presently unknown) F_5 dependent higher derivative terms the mass of the D3-brane solution will not renormalize.

With the definition of mass in Eq. (4.33) one may proceed as in the previous section to derive

$$M=-\frac{g_s^2}{96\pi G_N}\int_{S_\infty\cup\mathcal{H}}\epsilon_{tra_1\ldots a_8}\xi^i A_{il_1l_2l_3}F^{trl_1l_2l_3}. \tag{4.36}$$

One may integrate F_{t123r} to obtain A_{t123},

$$A_{t123}=\frac{1}{g_s}(e^{-h_0}-1)-\frac{32\pi N\gamma(g_s)\alpha'^2}{2431l^2r^{40}e^{9h_0}}(256r^{28}e^{7h_0}(3l^4+5r^4)+16l^{14}r^2e^{1/2h_0}(-48l^{16}-352l^{12}r^4-1037l^8r^8+1700l^4r^{12}+19890r^{16})) \tag{4.37}$$

and from here we obtain

$$v=\xi^iA_{i123}|_{S_\infty}-\xi^iA_{i123}|_{\mathcal{H}}=v_0\left(1+\gamma(g_s)\frac{5\times 2^{10}}{2431}\frac{1}{l^6}\right) \tag{4.38}$$

where $v_0=-1/g_s$ is the value of the electric potential for the lowest order solution. The remaining computation is exactly the same as the one in the previous paragraph, and we end up with

$$\mu=\frac{1}{\sqrt{16\pi G_N}}(-g_sv)q. \tag{4.39}$$

The charge density of the solution retains its lowest order value, as is required by charge quantization. We thus find that even though the mass of the solution renormalizes and the charge does not renormalize, a BPS-type formula still holds. This is possible because the electric potential renormalizes.

One can understand this behavior as follows. In the absence of corrections to the gauge field equation, the formula for the charge, $q\sim\int *dA$, remains uncorrected. Since q does not renormalize and * renormalizes (since it depends on the metric), A has to renormalize in such a way that the combined corrections to * and A cancel each other. So unless the gauge field equation is corrected the electric potential will renormalize. Then the first law (4.39) can be used to infer that the mass renormalizes. As we argued above, however, any correction to the mass would imply that the branes feel a force. This strongly indicates that the gauge field equations, and therefore the self-duality condition of F_5 receives corrections such that at the end the mass of the brane does not renormalize.

One may easily check that the temperature and the entropy remain equal to zero. For the temperature this follows upon using Eq. (4.13). It goes to zero as r, as in lowest order solution, but the coefficient of r receives corrections. For the entropy we use Eqs. (4.14), (4.15). The corrections to the entropy vanish as r^{15} (after factoring out the behavior of the temperature).

V. OTHER BRANES

Corrections to other R and NS D-brane solutions can be analyzed as in the D3-brane case. Analogously to the 3-brane case, a 1D effective action may be derived. The system of second order equations can be integrated by introducing variables suggested by the lowest order supersymmetry relations. As in the D3 case, these equations contain source terms evaluated on the lowest order solution. Once the complete source terms are known, the corrected solutions can be derived.

A. Equations of motion

The D dimensional action in the Einstein frame relevant for general brane solutions is given by

$$S=-\frac{1}{16\pi G}\int d^Dx\sqrt{-g}\left(R-\frac{4}{D-2}(\partial\phi)^2-\frac{1}{2(p+2)!}e^{\tilde{a}\phi}F_{p+2}^2+\gamma e^{-12/(D-2)\phi}W\right) \tag{5.1}$$

where

$$\gamma=\frac{1}{8}\zeta(3)\alpha'^3g_s^{12/(D-2)},\quad \tilde{a}=\frac{2(D-2p-4)}{D-2}-a_{NS} \tag{5.2}$$

$a_{NS}=2$ for NS branes but zero otherwise. The dilaton factor in front of W is that of a tree-level string correction. W is expected to depend on F_{p+2} and on its covariant derivatives as well as on the covariant derivatives of the dilaton. As discussed in the Introduction, this expression is not known at present, so in our analysis we will keep W arbitrary.

The equations of motion from the above action are

$$E_{ij}=R_{ij}-\frac{1}{2}g_{ij}R-\frac{1}{2}\left(\partial_i\phi\partial_j\phi-\frac{1}{2}g_{ij}(\partial\phi)^2\right)-\frac{e^{\tilde{a}\phi}}{2(p+2)!}\left((p+2)F_{il_1\ldots l_{p+1}}F_j^{\;l_1\ldots l_{p+1}}-\frac{1}{2}g_{ij}F_{p+2}^2\right)+\gamma\left(w_{ij}-\frac{1}{2}g_{ij}e^{-[12/(D-2)]\phi}W\right)=0 \tag{5.3}$$

$$E=\nabla^2\phi-\frac{\tilde{a}e^{\tilde{a}\phi}}{2(p+2)!}F_{p+2}^2+\gamma w_\phi=0 \tag{5.4}$$

$$E^{i_1\cdots i_{p+1}}=\frac{1}{\sqrt{-g}}\frac{1}{(p+1)!}\partial_l(\sqrt{-g}e^{\tilde{a}\phi}F^{li_1..i_{p+1}})+\gamma w_A{}^{i_1..i_{p+1}}=0 \tag{5.5}$$

where w_{ij} is defined by

$$\int d^Dx\sqrt{g}e^{-[12/(D-2)]\phi}\delta W=\int d^Dx\sqrt{g}\delta g^{ij}w_{ij} \tag{5.6}$$

and w_ϕ and w_A denote the variation of $e^{-[12/(D-2)]\phi}W$ with respect to the dilaton and the gauge field, respectively.

The equations of motion admit both electric and magnetic solutions [39–41]. Here we will consider only electric branes, but similar methods apply to magnetic branes as well. For an electric brane solutions we take the ansatz

$$ds^2=e^a\left(-dt^2+\sum_{a=1}^{d-1}(dx^a)^2+e^h(dr^2+r^2d\Omega_{q+1}^2)\right)$$

$$A_{t\alpha_1\ldots\alpha_p}=\epsilon_{\alpha_1\ldots\alpha_p}c(r). \tag{5.7}$$

The field equations evaluated on this ansatz give

$$a''+\frac{(D-2)}{2}a'^2+\frac{q}{2}a'h'+(q+1)\frac{a'}{r}-e^{\tilde{a}\phi}\frac{q}{D-2}K^2+\gamma\tilde{w}_t=0 \tag{5.8}$$

$$(D-1)a''+(q+1)h''-\frac{d}{2}a'h'+\phi'^2+\frac{(q+1)}{r}a'+\frac{(q+1)}{r}h'-e^{\tilde{a}\phi}\frac{q}{D-2}K^2+\gamma\tilde{w}_r=0 \tag{5.9}$$

$$a''+h''+\frac{(D-2)}{2}a'^2+\frac{1}{2}(2q+d)a'h'+\frac{1}{2}qh'^2+(2q+d+1)\frac{a'}{r}+(2q+1)\frac{h'}{r}+e^{\tilde{a}\phi}\frac{d}{D-2}K^2+\gamma\tilde{w}_m=0 \tag{5.10}$$

$$\phi''+\left(\frac{(D-2)}{2}a'+\frac{q}{2}h'+\frac{(q+1)}{r}\right)\phi'+\frac{\tilde{a}}{2}e^{\tilde{a}\phi}K^2+\gamma\tilde{w}_\phi=0 \tag{5.11}$$

$$\frac{1}{(p+1)!}e^{-(D/2)a-(q+2)h-\tilde{a}\phi}r^{-(q+1)}\partial_r\times(e^{(D/2-p-2)a+1/2qh+\tilde{a}\phi}r^{(q+1)}k(r))+\gamma\tilde{w}_A=0 \tag{5.12}$$

where $d=p+1$ and $q=D-p-3$ and

$$\tilde{w}_t=2e^h\left[\frac{1}{D-2}(-g^{kl}w_{kl}+e^{-[12/(D-2)]\phi}W)g_{tt}+w_{tt}\right] \tag{5.13}$$

$$\tilde{w}_r=-2\left[\frac{1}{D-2}(-g^{kl}w_{kl}+e^{-[12/(D-2)]\phi}W)g_{rr}+w_{rr}\right] \tag{5.14}$$

$$\tilde{w}_m=-2e^{a+h}\left[\frac{1}{D-2}(-g^{kl}w_{kl}+e^{-[12/(D-2)]\phi}W)g_{mm}+w_{mm}\right]g^{mm} \tag{5.15}$$

$$\tilde{w}_\phi=g_{rr}w_\phi \tag{5.16}$$

$$\tilde{w}_A=e^{-\tilde{a}\phi}w_A \tag{5.17}$$

$$K^2=k(r)^2e^{-da},\quad k=\frac{1}{p+2}c' \tag{5.18}$$

and

$$w_A{}^{i_1\cdots i_{p+1}}=w_A\epsilon_{i_1\cdots i_{p+1}}. \tag{5.19}$$

The same equations also hold for magnetic solutions. One only has to exchange $q\leftrightarrow d$ and take $\tilde{a}\to-\tilde{a}$.

The lowest order equation admits the following electric solution [39–41]:

$$e^{h_0}=\left[1+\left(\frac{l}{r}\right)^q\right]^{4/\Delta},\quad a_0=\frac{-q}{D-2}h_0,$$

$$\phi_0=\frac{\tilde{a}}{2}h_0,\quad k(r)=Qe^{[-\tilde{a}\phi_0-(q/2)h_0+(d+1-D/2)a_0]}r^{-(q+1)},$$

$$\Delta=\tilde{a}^2+\frac{2dq}{D-2},\quad Q^2=\frac{4q^2}{\Delta}l^{2q}, \tag{5.20}$$

where Q is the charge of the solutions.

We would like to obtain perturbative solutions

$$a=a_0+\gamma a_1$$

$$h=h_0+\gamma h_1$$

$$\phi=\phi_0+\gamma\phi_1$$

$$c=c_0+\gamma c_1. \tag{5.21}$$

Before we proceed to integrate the equations we will present a 1D effective action from which the field equations (5.8)–(5.12) can be derived.

B. Effective action

As in the case of the D3-brane, the 1D effective action is most naturally written in terms of $(d+1)$ dimensional fields. The $(d+1)$ dimensional action may be viewed as the reduction of the D-dimensional theory over the sphere S^{q+1} in transverse infinity and is given by

$$I_{d+1}=\int dr\sqrt{-g_{d+1}}\bigg[R_{d+1}-\frac{1}{2}g^{rr}_{d+1}\phi'^2 -\frac{(q+1)(D-2)}{(d-1)}g^{rr}_{d+1}\nu'^2+\frac{1}{2}g^{rr}_{d+1}\frac{e^{\tilde{a}\phi-da}}{f(p+2)^2}c'(r)^2 -V(\nu)\bigg] \tag{5.22}$$

where

$$V(\nu)=-\frac{q(q+1)}{l^2}e^{-[2(D-2)/(d-1)]\nu}. \tag{5.23}$$

The metric g_{d+1} and the scalar field ν are related to a, h and f by

$$g_{d+1,ij}dx^i dx^j=e^{\alpha}\left(-fdt^2+\sum_{p=1}^{d-1}dx_p^2+\frac{e^h}{f}dr^2\right)$$
$$\alpha=\frac{1}{d-1}\left((D-2)a+(q+1)h+2(q+1)\log\frac{r}{l}\right)$$
$$\nu=\frac{1}{2}\left(a+h+2\log\frac{r}{l}\right). \tag{5.24}$$

This can be shown by using the standard reduction formula

$$ds_D^2=e^{-2[(q+1)/(d-1)]\nu}g_{d+1,ij}dx^i dx^j+e^{2\nu}l^2 d\Omega^2_{q+1} \tag{5.25}$$

and matching with Eq. (5.7).

The α' correction can be incorporated by adding to the action the following term:

$$I_W=\gamma\int dr\sqrt{-g_{d+1}}e^{-[2(q+1)/(d-1)]\nu-[12/(D-2)]\phi}\times W[r,a,h,\phi,f,c]. \tag{5.26}$$

To connect the D-dimensional equations with the equations of the effective action we note that

$$\delta g^{tt}=-\left(\delta a+\frac{\delta f}{f}\right)g^{tt},\quad \delta g^{rr}=\left(-(\delta a+\delta h)+\frac{\delta f}{f}\right)g^{rr}$$
$$\delta g^{mn}=-(\delta a+\delta h)g^{mn},\quad \delta A_{i_1\dots i_{p+1}}=\epsilon_{i_1\dots i_{p+1}}\delta c. \tag{5.27}$$

Substituting in the D-dimensional action we obtain

$$\delta I=\int d^Dx\sqrt{-g}\bigg[-E_{ij}g^{ij}\delta a+\frac{\delta f}{f}(g^{rr}E_{rr}-g^{tt}E_{tt})-(g^{rr}E_{rr}+g^{mn}E_{mn})\delta h+E\,\delta\phi+E_{[j}{}^{ji_1\dots i_{p+1}}\epsilon_{i_1\dots i_{p+1}]}\bigg]$$
$$=\int d^Dx\sqrt{-g}[E_a\delta a+E_f\delta f+E_h\delta h+E_c\delta c]. \tag{5.28}$$

Since the solutions of interest depend on only r one may integrate over all directions but r. Integrating the variation to an action we get Eq. (5.22).

From Eq. (5.28) we can read off the relation between the D and 1D field equations:

$$E_a=-E_{ij}g^{ij}$$
$$E_f=\frac{1}{f}(g^{rr}E_{rr}-g^{tt}E_{tt})$$
$$E_h=-(g^{rr}E_{rr}+g^{mn}E_{mn})$$
$$E_\phi=E$$
$$E_c=E^{i_1\dots i_{p+1}}\epsilon_{i_1\dots i_{p+1}}. \tag{5.29}$$

The sources $\tilde{w}$ in Eqs. (5.8)–(5.12) can be expressed as a combination of the source terms in the a,h,f,ϕ, c equations of motion:

$$w_{a,h,f,\phi,c}=\frac{1}{\sqrt{-g}}\frac{\delta}{\delta(a,h,f,\phi,c)}(\sqrt{-g}e^{-[12/(D-2)]\phi}W) \tag{5.30}$$

with the result

$$\tilde{w}_t=\frac{2e^{(a+h)}}{d(D-2)}[qw_a-(D-2)w_h]$$
$$\tilde{w}_r=\frac{2e^{(a+h)}}{d(D-2)}[qw_a-(D-2)(w_h+dw_f)]$$
$$\tilde{w}_m=\frac{2e^{(a+h)}}{d(q+1)(D-2)}\{(D-2)[(1+d)w_h+dw_f]-[q+d(q+2)]w_a\}$$
$$\tilde{w}_\phi=e^{(a+h)}w_\phi$$
$$w_A=\frac{1}{(p+1)!}w_c. \tag{5.31}$$

C. Integrating the equations

In this section we integrate the equations assuming that the source terms $\tilde{w}$ are known. It is convenient to introduce the following variables:

$$b = a + \frac{q}{D-2}h$$

$$\Phi = \phi - \frac{\tilde{a}}{2}h. \tag{5.32}$$

These combinations are motivated by the fact that the supersymmetry of the lowest order solution implies that b and Φ vanish on the lowest order solution. Equations (5.8)–(5.11) depend on the gauge field only through the combination K^2. It is convenient to introduce the notation

$$\overline{K}^2 = e^{\tilde{a}\phi}K^2$$

$$\overline{K}^2 = \overline{K}_0{}^2 + \gamma \overline{K}_1 \tag{5.33}$$

where $\overline{K}_0$ is the lowest order value of $\overline{K}$ and $\overline{K}_1$ depends on the corrections. In terms of these variables Eqs. (5.8)–(5.11) become

$$b_1'' - \frac{q}{D-2}h_1'' - \frac{q}{2}b_1'h_0' + \frac{(q+1)}{r}b_1' - \frac{q(q+1)}{(D-2)}\frac{h_1'}{r} - \frac{q}{D-2}\overline{K}_1 + \tilde{w}_t = 0 \tag{5.34}$$

$$(D-1)b_1'' + \frac{d}{D-2}h_1'' - \frac{d}{2}b_1'h_0' + \frac{\Delta}{2}h_0'h_1' + \frac{(q+1)}{r}b_1' + \frac{(q+1)d}{(D-2)}\frac{h_1'}{r} + \tilde{a}\Phi_1'h_0' - \frac{q}{D-2}\overline{K}_1 + \tilde{w}_r = 0 \tag{5.35}$$

$$b_1'' + \frac{d}{D-2}h_1'' + \frac{d}{2}b_1'h_0' + (2q+d+1)\frac{b_1'}{r} + \frac{d(q+1)}{(D-2)}\frac{h_1'}{r} + \frac{d}{(D-2)}\overline{K}_1 + \tilde{w}_m = 0 \tag{5.36}$$

$$\Phi_1'' + \frac{(q+1)}{r}\Phi_1' + \frac{\tilde{a}}{2}\left(h_1'' + \frac{(D-2)}{2}b_1'h_0' + \frac{(q+1)}{r}h_1'\right) + \frac{\tilde{a}}{2}\overline{K}_1 + \tilde{w}_\phi = 0. \tag{5.37}$$

Consider the linear combination of equations $d(5.34) + q(5.36)$. The resulting equation can be integrated,

$$b_1' = \frac{1}{(D-2)}\frac{1}{r^{2q+1}}\left[-\int r^{2q+1}(d\tilde{w}_t + q\tilde{w}_m) + C_0\right]. \tag{5.38}$$

Let us introduce

$$y = \frac{\tilde{a}}{2}b_1 + \frac{q}{(D-2)}\Phi_1. \tag{5.39}$$

The linear combination of equation $\tilde{a}a/2(5.34) + q/(D-2)(5.37)$ can be integrated as

$$y' = \frac{1}{r^{q+1}}\left[-\int r^{q+1}\left(\frac{\tilde{a}}{2}\tilde{w}_t + \frac{q}{(D-2)}\tilde{w}_\phi\right) + Y_0\right]. \tag{5.40}$$

Integrating b_1' and y' once more one gets Φ_1,

$$\Phi_1 = \frac{(D-2)}{q}\left(y - \frac{\tilde{a}}{2}b_1\right). \tag{5.41}$$

Finally, given b_1 and y, one may integrate the linear combination [Eq. (5.35)–Eq. (5.34)] to obtain

$$h_1' = \frac{1}{e^{(\Delta/2)h_0}r^{q+1}}\left\{\int r^{q+1}e^{(\Delta/2)h_0}\left[(\tilde{w}_t - \tilde{w}_r) - \left((D-2)b_1'' + \frac{(q-d)}{2}b_1'h_0' + \tilde{a}\Phi_1'h_0'\right)\right] + C_3\right\} \tag{5.42}$$

which can be further integrated to yield h_1.

Having obtained a_1, h_1 and ϕ_1 one then substitutes to the gauge field equation (5.12) to obtain c_1. Finally there is one further equation among Eqs. (5.8)–(5.11) to satisfy (we only used three linear combinations to obtain a_1, h_1 and ϕ_1). This is expected to follow from the other equations up to a constant because of the Bianchi identity. This final equation will thus impose a condition among the integration constants, as in the case of the D3-brane.

VI. CONCLUSIONS

We have studied in this paper the computation of quantum corrections to brane solutions. These corrections are driven by the leading higher derivative corrections to the string theory effective action. The corrections were computed perturbatively in α', i.e. the lowest order solution was substituted into the leading higher derivative term of the effective action and then the resulting equations were integrated to obtain the corrected solution. The straightforward application of this procedure, i.e. the direct computation of the corrected equations, is very tedious, basically due to the complicated tensor structure of the higher derivative terms. We completed the computation in this manner but we also developed several alternative methods for analyzing the problem.

The first alternative method is the extension of the Palatini formalism to higher derivative theories. In this method the metric and the Christoffel symbols are considered as independent fields. The main advantage of this method is that it reduces the number of partial integrations needed in order to derive the field equations. This is a significant simplification because each factor of the Riemann tensor requires two partial integrations in the standard derivation of the field equations, so for higher derivative theories that depend on R^p, where R^p denotes p Riemann tensors contracted in some way, one gains at least p^2 terms when using the Palatini method. Furthermore, the organization of the computation is more transparent.

Even with the simplifications of the Palatini method, however, the computations are still very laborious. Things simplify enormously if one is studying spherically symmetric solutions. In this case we derived an effective one-dimensional action that governs the field equations, as we now describe. We start by substituting in the variation of the ten-dimensional action, the ansatz for the metric and the matter fields. Since by assumption we are considering a spherically symmetric solution, the resulting expression depends only on the radial coordinate r, and one may thus trivially integrate over all coordinates but r. After discarding an overall (infinite) volume factor, the result can be integrated back to a one-dimensional action where the fields are the functions that appear in the original ansatz. By construction the solutions of the one-dimensional field equations automatically solve the field equations of the original theory. Provided that the number of independent functions in the original ansatz is equal to the number of field equations one gets by substituting the ansatz in the original field equations, this method guarantees that the lower dimensional theory is a consistent truncation of the higher dimensional one (i.e., all solutions of the lowest order equations lift to solutions of the higher dimensional theory).

In the cases we study in this paper, the 1D action has the most transparent parametrization in terms of fields that appear in an intermediate step. In the brane solutions one can parametrize the transverse space in terms of polar coordinates. The intermediate theory is obtained by reducing over the sphere at infinity. In the context of near-horizon geometries, this is the sphere that appears in the near-horizon limit. We have derived an effective 1D action for all D-branes (the explicit formulas are for electric branes, but the magnetic case can be obtained along similar lines). These results can be used to study α' corrections to extremal and nonextremal branes, but we only studied extremal branes in this paper.

In the case of the D3-brane we further derive first order equations. The existence of such first order equations follows from the fact that the potential of the intermediate theory (obtained by the reduction over the S^5 at infinity) admits an AdS critical point (since a particular solution of our equations is the $AdS_5 \times S^5$ solution). This implies that the lowest equations admit a superpotential, and we indeed show that this is the case. The inclusion of the α' corrections modifies the first order equations by the addition of source terms.

The main limitation in our considerations comes from the fact that the complete set of the leading higher derivative terms is not yet available. Provided that we are supplied with these terms, we show how to integrate the equations in all cases to obtain the corrections to the solutions. The corrections are given in terms of integrals of the evaluation of the higher derivative terms on the lowest order solution.

The case of the D3-brane is under better control because the dilaton is constant, so higher derivative terms depending on derivatives of the dilaton do not contribute. To compute the correction one still needs to know the higher derivative terms that depend on F_5, but these terms are not known at present. Such terms are expected to be present because they are present in the dilaton superfield [11].

These terms would also lead to a modification of the self-duality equation of F_5, as discussed in the Introduction. We proceed by considering the effect of only the R^4 term, so one may consider this computation as a ‘‘toy’’ model computation. Including the R^4 only, we explicitly integrate the equations and obtain the corrected solution for this case. We find that the integration constants may be adjusted so that the solution is asymptotically flat and regular in the interior. This is a nontrivial result because the integration constants at our disposal are less than the number of terms that are diverging in the near-horizon limit. In turns out, however, that the coefficient of these terms is appropriately correlated and a smooth limit exists. In the near-horizon limit the solution becomes $AdS_5 \times S^5$ with the same value of the cosmological constant but a different (constant) value of the dilaton than the lowest order solution. The fact that the cosmological constant is uncorrected is due to a cancellation.

We used the general method of Wald [17–19] to analyze the thermodynamics of the corrected solutions. In the presence of higher derivative terms the ADM mass formula and other thermodynamic quantities receive corrections and we discuss how to obtain the new formulas. In particular, we computed all thermodynamics quantities of the corrected D3-brane solution. We find that the temperature and the entropy remain equal to zero, the charge is uncorrected but the mass and the electric potential renormalize. For solutions with zero temperature or entropy the first law of thermodynamics in integrated form (Smarr formula) implies that the mass density μ, charge density q and electric potential v are related by

$$\mu = -\frac{g_s}{\sqrt{16\pi G_N}} v q \tag{6.1}$$

where the factor depending on Newton’s constant is conventional. The lowest order D3-brane solution satisfies Eq. (6.1) with $\mu \sim q \sim N$, where N is the number of D3-branes and $v \sim g_s^{-1}$. In the corrected solution q remains uncorrected but μ and v renormalize such that Eq. (6.1) still holds.

We emphasize again that in computing the corrections to the D3-brane solution we did not take into account (presently unknown) higher derivative terms that depend on F_5. Such terms will modify the field equations (in particular the self-duality condition for F_5) as well as the formula for the charges of the solution (similarly to the fact that the R^4 term modifies the ADM mass formula). A simple argument that uses dimensional analysis and the form of the lowest order solution shows that any (positive) correction to the mass density would imply that it is energetically favored for D3-branes to coalesce together rather than remain separated. This contradicts the no-force condition and strongly suggests that half supersymmetric D3-brane solutions do not renormalize. This in turn suggests that the F_5 terms will make a significant contribution. To properly address this issue the exact knowledge of the F_5-dependent higher derivative terms will be required.

In this paper we studied corrections to extremal branes. Even though the exact form of the higher derivative terms are not known, we succeeded in integrating the equations in general. Some of our results, such as the effective one-dimensional action, hold for nonextremal branes as well. It would be interesting to investigate whether the nonextremal equations can be similarly integrated in general. Other generalizations of our analysis involve studying corrections to intersecting brane configurations. It will be interesting to see if the simple intersection rules generalize when α' corrections are included. This study will be relevant for obtaining α' corrections to black hole configurations.

ACKNOWLEDGMENTS

K.S. and A.S. are supported by NWO. This material is also based upon work supported by the National Science Foundation under Grant Nos. PHY-9802484 and PHY-0099590.

APPENDIX A: EXPLICIT EXPRESSIONS FOR *W* AND ITS VARIATION

In this appendix we outline the details of the direct computation of the variation of the α'^3 term in the action for arbitrary p-branes, and present some results for specific lowest order solutions.

We will be completely general in that we will consider a generic p-brane in D dimensions, with an ansatz for the metric of the form:

$$ds^2 = e^a[(-f dt^2 + d\vec{x}_p^2) + e^h(f^{-1}dr^2 + r^2 d\Omega_d^2)], \quad (A1)$$

where $a=a(r)$ and $h=h(r)$, and p and d satisfy $p+d=D-2$.

In a background with these symmetries the nonzero components of the Weyl tensor are

$$C^a{}_{cbd} = Q(\delta^a_b g_{cd} - \delta^a_d g_{bc})$$

$$C^t{}_{atb} = S g_{ab}$$

$$C^r{}_{arb} = T g_{ab}$$

$$C^m{}_{anb} = U \delta^m_n g_{ab}$$

$$C^t{}_{rtr} = V g_{rr}$$

$$C^r{}_{mrn} = X g_{mn}$$

$$C^t{}_{mtn} = Y g_{mn}$$

$$C^p{}_{mqn} = Z(\delta^p_q g_{mn} - \delta^p_n g_{mq}). \quad (A2)$$

The explicit expressions for the functions $Q,\dots,Z$ are

$$Q = -\frac{e^{-a-h}}{4(D-1)(D-2)r^2}[2rf'(4d - rh' + 2drh') + 4(d - d^2 + r^2 f'') + df(4drh' + (d-1)r^2h'^2 + 4(d-1+r^2h'')]$$

$$S = -\frac{e^{-a-h}}{4(D-1)(D-2)r^2}[4d - 4d^2 + rf'(10d - 2Dd - 3rh' + 5drh' + Drh' - dDrh') + 6r^2f'' - 2Dr^2f'' + df(4drh' + (d-1)r^2h'^2 + 4(d-1 + r^2h''))]$$

$$T = \frac{e^{-a-h}}{4(D-1)(D-2)r^2}[-4d + 4d^2 + rf'(-10d + 2Dd + 3rh' - 5drh' - Drh' + Ddrh') - 6r^2f'' + 2Dr^2f'' + df([4 - 4d - 2(1+2d-D)rh' - (d-1)r^2h'^2 - 6r^2h'' + 2Dr^2h''])]$$

$$U = -\frac{e^{-a-h}}{4(D-1)(D-2)r^2}[2rf'(2 + 4d - 2D + 2drh' - Drh') - 4(d^2 - 1 + D - Dd - r^2f'') + f(4d^2 - 4 + 4D - 4Dd + 2(-1 + 2d + 2d^2 + D - 2dD)rh' + (d-1)(1+d-D)r^2h'^2 + 2r^2h'' + 4dr^2h'' - 2Dr^2h'')]$$

$$V = \frac{e^{-a-h}}{4(D-1)(D-2)r^2}[(D-3)rf'(4d - 2rh' + 2drh' + Drh') + 2(-2d + 2d^2 - 6r^2f'' + 5Dr^2f'' - D^2r^2f'') + df[4 - 4d - 2(1+2d-D)rh' - (d-1)r^2h'^2 - 6r^2h'' + 2Dr^2h'']]$$

$$X = -\frac{e^{-a-h}}{4(D-1)(D-2)r^2}[4 - 4d^2 - 4D + 4Dd + rf'(8 + 10d - 10D - 2Dd + 2D^2 + rh' + 5drh' - 4Drh' - dDrh' + D^2rh') + 6r^2f'' - 2Dr^2f'' + (1+d-D)f \times[-4 + 4d + 2(1+2d-D)rh' + (d-1)r^2h'^2 + 6r^2h'' - 2Dr^2h'']]$$

$$Y=\frac{e^{-a-h}}{4(D-1)(D-2)r^2}\{-4(d-1)(1+d-D)(-1+f) -2[4+5d+D(-5-d+D)]rf'+2[1-2d(d+1) +(2d-1)D]rfh'-[1+5d+D(-4-d+D)]r^2f'h' +[3-(d-4)d+(d-3)D]r^2fh'^2+2(D-3)r^2f'' -2(1+2d-D)r^2f(h'^2+h'')\}$$

$$Z=-\frac{e^{-a-h}}{4(D-1)(D-2)r^2}[2(rf'(4+4d-4D+rh' +2drh'-2Drh')-2(d+d^2-D-2dD+D^2 -r^2f''))+(1+d-D)f(4(1+d-D)rh' +(d-D)r^2h'^2+4(d-D+r^2h''))]. \quad (A3)$$

The gravitational α'^3 correction W in terms of these variables can be expressed as

$$W=p(p-1)[T^4+S^4+4(S^3+T^3)Q+3(p-2)Q^4+4Q^2(T^2 +S^2+dU^2)+dU^4+4dU^3Q]+d(d-1)[X^4+Y^4 +4(X^3+Y^3)Z+3(d-2)Z^4+4Z^2(X^2+Y^2+pU^2) +pU^4+4pU^3Z]+2p[T^2S^2+V^2T^2+V^2S^2+2(V^2TS +S^2VT+T^2VS)]+2d[X^2Y^2+V^2X^2+V^2Y^2+2(V^2XY +Y^2VX+X^2VY)]+2dp[X^2T^2+S^2Y^2+U^2(X^2+Y^2 +T^2+S^2)+2(S^2UY+Y^2SU+T^2UX+X^2TU+U^2TX +U^2SY)]. \quad (A4)$$

We stress that these formulas are valid for arbitrary p and D. In Appendix B we will give the results for specific extremal and nonextremal D3-brane solutions.

Next we compute the variation of W. Let us define

$$\int d^Dx\sqrt{g}e^{-[12/(D-2)]\phi}\delta W=\int d^Dx\sqrt{g}\delta g^{ij}w_{ij} =\int d^Dx\sqrt{g}\delta g_{ij}w^{ij}. \quad (A5)$$

Notice that $w^{ij}=-g^{ik}g^{jl}w_{kl}$. Explicit computation of w_{ij} gives

$$w^{ij}=w_1^{ij}+\omega^{ij} \quad (A6)$$

where

$$w_1^{ij}=\frac{1}{2}e^{-[12/(D-2)]\phi}[(-4B_{mk}{}^{ln}+4B^{ln}_{mk} -3B_m{}^{ln}{}_k)C^{mip}{}_nC^{kj}{}_{pl}-B_{mk}{}^{li}B^{mjk}{}_l+(i\leftrightarrow j)] \quad (A7)$$

$$\omega^{ij}=\frac{1}{4}\Bigg[\nabla_n\nabla_m(D^{inmj}+D^{ijmn}-D^{nimj})-\frac{1}{(D-2)}\Big(\nabla_l\nabla^i(d^{jl} +d^{lj})-g^{ij}\nabla_m\nabla_nd^{nm}-\nabla^2d^{ij}+2(R^i_mD^j{}_l{}^{lm}-R^m_nD_m{}^{ijn}) +\frac{2}{(D-1)}((R^{ij}-\nabla^i\nabla^j+\nabla^2g^{ij})D_{mn}{}^{mn}-RD_l{}^{ilj})\Big) +(i\leftrightarrow j)\Bigg] \quad (A8)$$

and we have defined

$$B_{ijkl}=C^m{}_{ijn}C^n{}_{lkm}$$

$$D_j{}^{lki}=e^{-[12/(D-2)]\phi}[(2B_{jm}{}^{ni}+3B_j{}^{ni}{}_m-2B^{ni}_{jm})C^{mlk}{}_n -B^{lni}{}_mC^m{}_j{}^k{}_n-B^{ln}_{mj}C^m{}_n{}^{ki}]-(k\leftrightarrow i)$$

$$d^{ij}=D_l{}^{ilj}-D^i{}_l{}^{lj}. \quad (A9)$$

APPENDIX B: EXPLICIT FORM OF THE CORRECTIONS FOR EXTREMAL AND NONEXTREMAL D3-BRANES

In this appendix we give explicit formulas for W and its variation for thermal AdS, and for the extremal and nonextremal D3-brane. These solutions satisfy the constraint $2a_0'+h_0'=0$ (which is a necessary but not sufficient condition for supersymmetry), and we have

$$e^{h_0}=\kappa_0+\left(\frac{\ell}{r}\right)^{d-1}$$

$$f=1-\left(\frac{r_0}{r}\right)^{d-1}$$

$$e^{\phi_0}=g_s. \quad (B1)$$

1. Thermal $AdS_5\times S^5$

The AdS limit can simply be taken by setting $\kappa_0=0$. The scalars $Q,\ldots,Z$ are all given in terms of a single function Ψ. We get:

$$S=T=-Q=-\Psi$$

$$V=3\Psi$$

$$\Psi=\frac{r_0^4}{r^4\ell^2} \quad (B2)$$

and

$$W=180\Psi^4, \quad (B3)$$

which agrees with the expression in [4].

2. Extremal and nonextremal D3-branes

For the extremal D3-brane $r_0=0$, the result is

$$Q=S=0$$

$$T=V=5\chi$$

$$U=Y=-\chi$$

$$X=-4\chi$$

$$Z=2\chi$$

$$\chi=\frac{\ell^4\kappa_0}{r^6}e^{-(5/2)h_0} \tag{B4}$$

and

$$W=28800\chi^4. \tag{B5}$$

In the nonextremal case, W has a more complicated form:

$$\begin{aligned}W=&\frac{60}{r^{16}(\ell^4+\kappa_0 r^4)^{10}}[3\ell^{32}r_0^{16}+32\kappa_0\ell^{28}r^4r_0^{16}+219\kappa_0^8r^{32}r_0^{16}\\&+12\kappa_0^7\ell^4r^{28}r_0^{12}(62r^4+23r_0^4)+2\kappa_0^2\ell^{24}r^8r_0^8(12r^8\\&-12r^4r_0^4+83r_0^8)+4k_0^3\ell^{20}r^{12}r_0^8(64r^8-58r^4r_0^4+131r_0^8)\\&+2\kappa_0^6\ell^8r^{24}r_0^8(612r^8-4r^4r_0^4+371r_0^8)\\&+16\kappa_0^5\ell^{12}r^{20}r_0^4(60r^{12}-14r^8r_0^4+28r^4r_0^8+53r_0^{12})\\&+2\kappa_0^4\ell^{16}r^{16}(240r^{16}-480r^{12}r_0^4+832r^8r_0^8-464r^4r_0^{12}\\&+515r_0^{16})].\end{aligned} \tag{B6}$$

We also give here the variation of W obtained from Eq. (A8) for the extremal D3-brane. One finds that the off-diagonal components are zero and the diagonal ones are equal to

$$w^{tt}=-\frac{4800\kappa_0^3\ell^{12}}{r^{28}e^{10h_0}}(56\ell^8-123\kappa_0\ell^4r^4+42\kappa_0^2r^8)g^{tt}$$

$$w^{aa}=-\frac{4800\kappa_0^3\ell^{12}}{r^{28}e^{10h_0}}(56\ell^8-123\kappa_0\ell^4r^4+42\kappa_0^2r^8)g^{aa}$$

$$w^{rr}=-\frac{9600\kappa_0^3\ell^{12}}{r^{28}e^{10h_0}}(7\ell^8-9\kappa_0\ell^4r^4+6\kappa_0^2r^8)g^{rr}$$

$$w^{mm}=\frac{1920\kappa_0^3\ell^{12}}{r^{28}e^{10h_0}}(119\ell^8-267\kappa_0\ell^4r^4+90\kappa_0^2r^8)g^{mm} \tag{B7}$$

where a runs over the spatial worldvolume coordinates and m over the coordinates of the sphere.

APPENDIX C: THE MOST GENERAL SOLUTION OF THE LOWEST ORDER EQUATIONS

In this appendix we present the most general solution of the lowest order equations (2.34)–(2.37). An analysis of this system has also been presented in [42]. Let us define

$$\alpha=2a'+h', \quad \beta=h', \quad \gamma=\phi'. \tag{C1}$$

Consider the following linear combinations of the equations

$$\frac{1}{5}[(2.34)-(2.36)]: \quad \alpha'+\frac{9}{r}\alpha+2\alpha^2=0 \tag{C2}$$

$$(2.37): \quad \gamma'+\gamma\left(2\alpha+\frac{5}{r}\right)=0 \tag{C3}$$

$$-\frac{4}{5}\left[(2.34)-\frac{9}{4}(2.36)\right]:$$

$$\beta'+\frac{5}{r}\beta+\beta^2-\frac{36}{r}\alpha-9\alpha^2+\gamma^2=0. \tag{C4}$$

The solution of these equations should still satisfy Eq. (2.35).

Equations (C2), (C3), (C4) can be integrated by elementary means. Let us first consider the special case

$$\alpha=0\Rightarrow h+2a=c_1. \tag{C5}$$

Then we get

$$\gamma=\frac{4c_2}{r^5}\Rightarrow\phi=c_3-\frac{c_2}{r^4} \tag{C6}$$

$$\beta=-\frac{4c_2}{r^5}\tan\left(c_4-\frac{c_2}{r^4}\right)\Rightarrow h=c_5+\log\cos\left(c_4-\frac{c_2}{r^4}\right). \tag{C7}$$

Inserting in Eq. (2.35) we get

$$c_2e^{c_1+c_5}=\pm l^4. \tag{C8}$$

Requiring that the solution is asymptotically flat and the dilaton approaches 1 asymptotically fixes

$$c_1=0, \quad e^{-c_5}=\cos c_4, \quad c_3=1. \tag{C9}$$

We thus finally get the solution

$$\begin{aligned}ds^2=&\left(\frac{\cos\left(c_4-\frac{l^4\cos c_4}{r^4}\right)}{\cos c_4}\right)^{-1/2}(-dt^2+d\vec{x}^2)\\&+\left(\frac{\cos\left(c_4-\frac{l^4\cos c_4}{r^4}\right)}{\cos c_4}\right)^{1/2}(dr^2+r^2d\Omega_5^2)\end{aligned}$$

$$\phi=1\mp\frac{l^4\cos c_4}{r^4} \tag{C10}$$

and the self-dual F_5 is given in Eq. (2.13). The $\mp$ sign in the dilaton is related to the two sign choices in Eq. (C8). Neither the metric nor the five-form depend on these signs. The reason is that one can change the relative sign in $c_4 - l^4 \cos c_4/r^4$ by taking $c_4 \rightarrow -c_4$. This does not affect the metric and the five-form because this combination appears inside the cosine. The standard supersymmetric D3-brane solution is obtained by the limit $\cos c_4 \rightarrow 0$.

Equation (C2) admits more general solutions than Eq. (C5). The most general solution of Eq. (C2) is

$$\alpha = \frac{4}{(r^8 d_0 - 1)r} \Rightarrow h + 2a = d_1 + \frac{1}{2}\log\left(d_0 - \frac{1}{r^8}\right). \quad \text{(C11)}$$

Here and in the following we assume $d_0 > 0$, but a similar analysis can be done for $d_0 < 0$. Inserting the solution of α in Eq. (C3) and integrating we obtain

$$\gamma = \frac{d_2 r^3}{d_0 r^8 - 1} \Rightarrow \phi = d_3 + \frac{d_2}{8\sqrt{d_0}} \log \frac{\sqrt{d_0}\, r^4 - 1}{\sqrt{d_0}\, r^4 + 1}. \quad \text{(C12)}$$

Equation (C4) becomes

$$\beta' + \frac{5}{r}\beta + \beta^2 + \frac{64 d_0 D r^6}{(d_0 r^8 - 1)^2} = 0 \quad \text{(C13)}$$

where $D = (d_2^2 - 144 d_0)/64 d_0$. This can be solved as follows. Let us define

$$H = e^h, \quad \rho = \frac{1}{2}\left(\frac{1}{\sqrt{d_0} r^4} + 1\right). \quad \text{(C14)}$$

In terms of these variables Eq. (C13) becomes

$$\partial_\rho^2 H + \frac{D}{\rho^2 (1-\rho)^2} H = 0. \quad \text{(C15)}$$

The most general solution of this equation is

$$H = d_4 \rho^{\alpha_+} (1-\rho)^{\alpha_-} + d_5 \rho^{\alpha_-} (1-\rho)^{\alpha_+} \quad \text{(C16)}$$

where $\alpha_\pm = \frac{1}{2}(1 \pm \sqrt{1 - 4D})$. The exponents are real for $D \leqslant 1/4$. The case $D = 1/4$ is a special case since in this case $\alpha_+ = \alpha_-$. In this case the second independent solution involves a logarithm. One should still impose Eq. (2.35) which should relate the integration constants to the scale l. Asymptotic flatness fixes $d_4 + d_5 = 2, d_1 = -\frac{1}{2}\log d_0, d_3 = 1$.

[1] R. Kallosh and A. Rajaraman, Phys. Rev. D **58**, 125003 (1998).
[2] C.G. Callan, R.C. Myers, and M.J. Perry, Nucl. Phys. **B311**, 673 (1989).
[3] T. Banks and M.B. Green, J. High Energy Phys. **05**, 002 (1998).
[4] S.S. Gubser, I.R. Klebanov, and A.A. Tseytlin, Nucl. Phys. **B534**, 202 (1998).
[5] J. Pawelczyk and S. Theisen, J. High Energy Phys. **09**, 010 (1998).
[6] T. Harmark and N.A. Obers, Phys. Lett. B **485**, 285 (2000).
[7] M. Berkooz and M. Rozali, J. High Energy Phys. **05**, 040 (2000).
[8] D.J. Gross and E. Witten, Nucl. Phys. **B277**, 1 (1986).
[9] D.J. Gross and J.H. Sloan, Nucl. Phys. **B291**, 41 (1987).
[10] M.T. Grisaru, A.E. van de Ven, and D. Zanon, Phys. Lett. B **173**, 423 (1986); M.T. Grisaru and D. Zanon, *ibid.* **177**, 347 (1986).
[11] S. de Haro, A. Sinkovics, and K. Skenderis, Phys. Rev. D **67**, 084010 (2003).
[12] S. Frolov, I.R. Klebanov, and A.A. Tseytlin, Nucl. Phys. **B620**, 84 (2002).
[13] K. Peeters, P. Vanhove, and A. Westerberg, Class. Quantum Grav. **19**, 2699 (2002).
[14] S. Frolov and A.A. Tseytlin, Nucl. Phys. **B632**, 69 (2002).
[15] P.S. Howe and P.C. West, Nucl. Phys. **B238**, 181 (1984).
[16] M.B. Green and M. Gutperle, Nucl. Phys. **B498**, 195 (1997).
[17] R.M. Wald, Phys. Rev. D **48**, 3427 (1993).
[18] V. Iyer and R.M. Wald, Phys. Rev. D **50**, 846 (1994).
[19] V. Iyer and R.M. Wald, Phys. Rev. D **52**, 4430 (1995).
[20] L. Alvarez-Gaume and D.Z. Freedman, Commun. Math. Phys. **80**, 443 (1981); P.S. Howe and G. Papadopoulos, Nucl. Phys. **B289**, 264 (1987).
[21] N. Kaloper and K.A. Meissner, Phys. Rev. D **56**, 7940 (1997).
[22] N. Berkovits, J. High Energy Phys. **04**, 018 (2000).
[23] N. Berkovits and P.S. Howe, Nucl. Phys. **B635**, 75 (2002).
[24] J. de Boer and K. Skenderis, Nucl. Phys. **B481**, 129 (1996); hep-th/9611066.
[25] M.B. Green and S. Sethi, Phys. Rev. D **59**, 046006 (1999); B. Pioline, Phys. Lett. B **431**, 73 (1998).
[26] A.A. Tseytlin, Phys. Lett. B **176**, 92 (1986).
[27] R.M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984).
[28] M.J. Duff and J.X. Lu, Phys. Lett. B **273**, 409 (1991).
[29] P.K. Townsend, Phys. Lett. **148B**, 55 (1984).
[30] K. Skenderis and P.K. Townsend, Phys. Lett. B **468**, 46 (1999).
[31] O. DeWolfe, D.Z. Freedman, S.S. Gubser, and A. Karch, Phys. Rev. D **62**, 046008 (2000).
[32] J. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998); Int. J. Theor. Phys. **38**, 1113 (1998).
[33] For a review see K. Skenderis, Int. J. Mod. Phys. A **16**, 740 (2001).

[34] K. Skenderis, Class. Quantum Grav. **19**, 5849 (2002).
[35] A. Dabholkar, G. Gibbons, J.A. Harvey, and F. Ruiz Ruiz, Nucl. Phys. **B340**, 33 (1990).
[36] G. Lopes Cardoso, B. de Wit, and T. Mohaupt, Phys. Lett. B **451**, 309 (1999).
[37] R.I. Nepomechie, Phys. Rev. D **31**, 1921 (1985); C. Teitelboim, Phys. Lett. **167B**, 69 (1986).
[38] K. Skenderis, in *Towards Quantum Gravity*, edited by J. Kowalski-Glikman, Lecture Notes in Physics Vol. 541 (Springer-Verlag, Heidelberg, 2000), p. 325.
[39] G.T. Horowitz and A. Strominger, Nucl. Phys. **B360**, 197 (1991).
[40] M.J. Duff and J.X. Lu, Nucl. Phys. **B416**, 301 (1994).
[41] M.J. Duff, R.R. Khuri, and J.X. Lu, Phys. Rep. **259**, 213 (1995).
[42] B. Zhou and C.J. Zhu, hep-th/9905146.