Fuzzy Ginsparg-Wilson algebra: A solution of the fermion doubling problem

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The Ginsparg-Wilson algebra is the algebra underlying the Ginsparg-Wilson solution of the fermion doubling problem in lattice gauge theory. The Dirac operator of the fuzzy sphere is not afflicted with this problem. Previously, we have indicated that there is a Ginsparg-Wilson operator underlying it also in the absence of gauge fields and instantons. Here we develop this observation systematically and establish a Dirac operator theory for the fuzzy sphere with or without gauge fields, and always with the Ginsparg-Wilson algebra. There is no fermion doubling in this theory. The association of the Ginsparg-Wilson algebra with the fuzzy sphere is surprising as the latter is not designed with this algebra in mind. The theory reproduces the integrated $U(1)_A$ anomaly and index theory correctly.

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I. INTRODUCTION

A central task in the lattice approximation to quantum field theories (QFT's) is the treatment of chiral fermions. General theorems due to Nielsen and Ninomiya and others [1] reveal a serious obstruction to their rigorous formulation on a lattice. As the standard model involves chiral fermions, there is thus a fundamental difficulty with lattice approximations.

Years ago, Ginsparg and Wilson [2] proposed an approximate manner to overcome this difficulty. In the original formulation, it is based on a Dirac and chirality operator satisfying particular algebraic relations. In the continuum limit, anticommuting Dirac and chirality operators can be obtained therefrom. The Ginsparg-Wilson method is an effective tool in the theoretical analysis of lattice theories and reproduces important topological effects such as chiral anomalies in an approximate manner.

Fuzzy physics [3,4] concerns an approach to regulating QFT's which can be an alternative to lattice methods. It gives finite-dimensional matrix approximations to QFT's and incorporates ideas of noncommutative geometry [5]. It has a well-articulated theory of a Dirac operator for the fuzzy sphere which approximates the continuum Dirac operator very well and also reproduces the correct index theory and chiral anomaly. Subtle topological features such as instantons and complex structures can be formulated [6]. Chiral fermions too can be described with no fermion doubling [7]. For fuzzy $\mathbb{C}P^N$ models as well, the Dirac operator to the extent investigated [8] seems an excellent approximation to the continuum Dirac operator and capable of reproducing significant topological features of the continuum.

In a previous paper [7], we reported our joint work on the Dirac operator of [9] for the fuzzy sphere. Here we establish that the "free" fuzzy Dirac operator in the absence of instantons satisfies the defining relations of the Ginsparg-Wilson algebraic system. This result has a strong element of surprise as fuzzy physics is not consciously designed to satisfy such relations.

In this paper we review our previous work and extend it to cover gauge fields and instanton sectors. This extension has a new formulation of the Dirac operator on the fuzzy sphere and is based on an appropriate realization of the Ginsparg-Wilson algebra. This Dirac operator has several positive features. Its spectrum in the absence of gauge field fluctuations is precisely that in the continuum below a suitable angular momentum cutoff. There is no correction whatever to the spectrum below the cutoff. There is no fermion doubling, and chiral fermions can be effortlessly treated. The $U(1)_A$ anomaly in the integrated form is reproduced exactly. We have not looked at its local form, but its treatment in alternative approaches exists [10,11].

For other work applying the Ginsparg-Wilson approach to the fuzzy sphere, see [12].

While these are points in favor of our approach, it appears that the Ginsparg-Wilson approach, in either the lattice or fuzzy physics context, is not easy to adapt to numerical work. This is a serious difficulty and has to be overcome.

II. A REVIEW OF THE GINSPARG-WILSON ALGEBRA

We follow [4,7] in this presentation.

In its generality, the Ginsparg-Wilson algebra $\mathcal{A}$ can be defined as the unital * algebra over $\mathbb{C}$ generated by two *-invariant involutions Γ and Γ':

$$\mathcal{A}=\langle\Gamma,\Gamma' : \quad \Gamma^2=\Gamma'^2=\mathbb{1}, \quad \Gamma^*=\Gamma, \quad \Gamma'^*=\Gamma'\rangle, \tag{2.1}$$

* denoting the adjoint. The unity of $\mathcal{A}$ has been indicated by $\mathbb{1}$.

In any such algebra, we can define a Dirac operator

$$D'=\frac{1}{a}\Gamma(\Gamma+\Gamma'), \tag{2.2}$$

where a is the lattice spacing. It satisfies

$$D'^*=\Gamma\ D'\ \Gamma, \quad [\Gamma,D']_+=aD'\ \Gamma\ D'. \tag{2.3}$$

Equations (2.2) and (2.3) give the original formulation [2]. But they are equivalent to Eq. (2.1), since Eqs. (2.2) and (2.3) imply that

$$\Gamma' = \Gamma(aD') - \Gamma \tag{2.4}$$

is a $*$-invariant involution [13,14].

Each representation of Eq. (2.1) is a particular realization of the Ginsparg-Wilson algebra. Representations of physical interest are reducible.

In our work we choose

$$D = \frac{1}{a}(\Gamma + \Gamma') \tag{2.5}$$

instead of D' as our Dirac operator, as it is self-adjoint and has the desired continuum limit.

From Γ and Γ', we can construct the following elements of $\mathcal{A}$:

$$\Gamma_0 = \frac{1}{2}[\Gamma, \Gamma']_+, \tag{2.6}$$

$$\Gamma_1 = \frac{1}{2}(\Gamma + \Gamma'), \tag{2.7}$$

$$\Gamma_2 = \frac{1}{2}(\Gamma - \Gamma'), \tag{2.8}$$

$$\Gamma_3 = \frac{1}{2i}[\Gamma, \Gamma']. \tag{2.9}$$

Let us first look at the center $\mathcal{C}(\mathcal{A})$ of $\mathcal{A}$ in terms of these operators. It is generated by Γ_0 which commutes with Γ and Γ' and hence with every element of $\mathcal{A}$. Γ_i^2, $i=1,2,3$, also commute with every element of $\mathcal{A}$, but they are not independent of Γ_0. Rather,

$$\Gamma_1^2 = \frac{1}{2}(\mathbb{1} + \Gamma_0), \tag{2.10}$$

$$\Gamma_2^2 = \frac{1}{2}(\mathbb{1} - \Gamma_0), \tag{2.11}$$

$$\rightarrow \quad \Gamma_1^2 + \Gamma_2^2 = \mathbb{1} \tag{2.12}$$

$$\Gamma_0^2 + \Gamma_3^2 = \mathbb{1}. \tag{2.13}$$

Notice also that

$$[\Gamma_i, \Gamma_j]_+ = 0, \quad i,j = 1,2,3, \; i \neq j. \tag{2.14}$$

From now on by $\mathcal{A}$ we will mean a representation of $\mathcal{A}$.

The relations (2.10)–(2.13) contain spectral information. From Eq. (2.13) we see that

$$-1 \leqslant \Gamma_0 \leqslant 1, \tag{2.15}$$

where the inequality means that the eigenvalues of Γ_0 are accordingly bounded. By Eq. (2.10), this implies that the eigenvalues of Γ_1 are similarly bounded.

We now discuss three cases associated with Eq. (2.15).

Case 1. $\Gamma_0 = \mathbb{1}$. Call the subspace where $\Gamma_0 = \mathbb{1}$ V_{+1}. On V_{+1}, $\Gamma_1^2 = \mathbb{1}$ and $\Gamma_2 = \Gamma_3 = 0$ by Eqs. (2.10)–(2.13). This is the subspace of the top modes of the operator $|D|$.

Case 2. $\Gamma_0 = -\mathbb{1}$. Call the subspace where $\Gamma_0 = -\mathbb{1}$ V_{-1}. On V_{-1}, $\Gamma_2^2 = \mathbb{1}$ and $\Gamma_1 = \Gamma_3 = 0$ by Eqs. (2.10)–(2.13). This is the subspace of zero modes of the Dirac operator D.

Case 3. $\Gamma_0^2 \neq \mathbb{1}$. Call the subspace where $\Gamma_0^2 \neq \mathbb{1}$ V. On this subspace, $\Gamma_i^2 \neq 0$ for $i=1,2,3$ by Eqs. (2.9)–(2.12), and therefore

$$\mathrm{sgn}\,\Gamma_i = \frac{\Gamma_i}{|\Gamma_i|}, \qquad |\Gamma_i| = \text{positive square root of } \Gamma_i^2, \tag{2.16}$$

are well defined and by Eq. (2.14) generate a Clifford algebra on V:

$$[\mathrm{sgn}\,\Gamma_i, \mathrm{sgn}\,\Gamma_j]_+ = 2\delta_{ij}\mathbb{1}. \tag{2.17}$$

Consider Γ_2. It anticommutes with Γ_1 and D. Also,

$$\mathrm{Tr}\,\Gamma_2 = (\mathrm{Tr}_V + \mathrm{Tr}_{V_{+1}} + \mathrm{Tr}_{V_{-1}})\Gamma_2, \tag{2.18}$$

where the subscripts refer to the subspaces over which the trace is taken. These traces can be calculated:

$$\begin{aligned} \mathrm{Tr}_V\Gamma_2 &= \mathrm{Tr}_V(\mathrm{sgn}\,\Gamma_i)\Gamma_2(\mathrm{sgn}\,\Gamma_i) \qquad (i \text{ fixed}, \neq 2) \\ &= -\mathrm{Tr}_V\Gamma_2 \text{ by Eq. (2.17)} \\ &= 0, \end{aligned} \tag{2.19}$$

$$\mathrm{Tr}_{V_{+1}}\Gamma_2 = 0, \qquad \text{as } \Gamma_2 = 0 \text{ on } V_{+1}. \tag{2.20}$$

So

$$\mathrm{Tr}\,\Gamma_2 = \mathrm{Tr}_{V_{-1}}\Gamma_2 = \mathrm{Tr}_{V_{-1}}\left(\frac{\mathbb{1}+\Gamma_2}{2} - \frac{\mathbb{1}-\Gamma_2}{2}\right) = \text{index of } \Gamma_1. \tag{2.21}$$

Following Fujikawa [14], we can use Γ_2 as the generator of chiral transformations. It is not involutive on $V \oplus V_{+1}$:

$$\Gamma_2^2 = \mathbb{1} - \frac{\mathbb{1}+\Gamma_0}{2}. \tag{2.22}$$

But this is not a problem for fuzzy physics. In the fuzzy model below, in the continuum limit, $\Gamma_0 \to -\mathbb{1}$ on all states with $|D| \leqslant$ a fixed "energy" E_0 independent of a (and is $-\mathbb{1}$ on V_{-1} where $D=0$). We can see this as follows. $\Gamma_1 = aD$, so that if $|D| \leqslant E_0$, $\Gamma_1 \to 0$ as $a \to 0$. Hence by Eqs. (2.10)–(2.12), $\Gamma_0 \to -\mathbb{1}$ and $\Gamma_2^2 \to \mathbb{1}$ on these levels.

There are of course states, such as those of V_{+1}, on which Γ_2^2 does not go to $\mathbb{1}$ as $a \to 0$. But their (Euclidean) energy diverges and their contribution to functional integrals vanishes in the continuum limit.

We can interpret Eq. (2.22) as follows. The chiral charge of levels with $D\neq 0$ gets renormalized in fuzzy physics. For levels with $|D|\leqslant E_0$, this renormalization vanishes in the naive continuum limit.

We note that the last feature is positive: it resolves a problem in previous work [15], where all the top modes had to be projected out because of the insistence that chirality squares to $\mathbb{1}$ on V_{+1}; see below.

For Dirac operators of maximum symmetry, Γ_0 is a function of the conserved total angular momentum $\vec{J}$ as we shall show. It increases with $\vec{J}^2$ so that V_{+1} consists of states of maximum $\vec{J}^2$. This maximum value diverges as $a\to 0$, as the general argument above shows.

III. FUZZY MODELS

A. The basic algebra

The algebra for the fuzzy sphere characterized by cutoff $2L$ is the full matrix algebra $\mathrm{Mat}(2L+1)\equiv M_{2L+1}$ of $(2L+1)\times(2L+1)$ matrices. On M_{2L+1}, the $SU(2)$ Lie algebra acts either on the left or on the right. Call the operators for left action L_i^L and those for right action L_i^R. We have

$$L_i^L a=L_i a,\quad L_i^R a=aL_i,\quad a\in M_{2L+1},$$
$$[L_i^L,L_j^L]=i\epsilon_{ijk}L_k^L,\quad [L_i^R,L_j^R]=-i\epsilon_{ijk}L_k^R,$$
$$(L_i^L)^2=(L_i^R)^2=L(L+1)\mathbb{1}, \tag{3.1}$$

where L_i is the standard matrix for the ith component of the angular momentum in the $(2L+1)$-dimensional irreducible representation (IRR). The orbital angular momentum which becomes $-i(\vec{r}\wedge\vec{\nabla})_i$ as $L\to\infty$ is

$$\mathcal{L}_i=L_i^L-L_i^R,\qquad \mathcal{L}_i a=[L_i,a]. \tag{3.2}$$

As $L\to\infty$, both $\vec{L}^L/L$ and $\vec{L}^R/L$ approach the unit vector $\hat{x}$ with commuting components:

$$\frac{\vec{L}^{L,R}}{L}\underset{L\to\infty}{\longrightarrow}\hat{x},\qquad \hat{x}\cdot\hat{x}=1,\qquad [\hat{x}_i,\hat{x}_j]=0. \tag{3.3}$$

$\hat{x}$ labels a point on the sphere S^2 in the continuum limit.

B. The fuzzy Dirac operator (no instantons or gauge fields)

Consider $M_{2L+1}\otimes\mathbb{C}^2$. $\mathbb{C}^2$ is the carrier of the spin 1/2 representation of $SU(2)$ with generators $\frac{1}{2}\sigma_i$, $\sigma_i=$ Pauli matrices. We can couple its spin 1/2 and the angular momentum L of L_i^L to the value $L+1/2$. If $(1+\Gamma)/2$ is the corresponding projector, then [7,16]

$$\Gamma=\frac{\vec{\sigma}\cdot\vec{L}^L+1/2}{L+1/2}. \tag{3.4}$$

Γ is a self-adjoint involution,

$$\Gamma^*=\Gamma,\qquad \Gamma^2=\mathbb{1}. \tag{3.5}$$

There is likewise the projector $(\mathbb{1}+\Gamma')/2$ coupling the spin 1/2 of $\mathbb{C}^2$ and the right angular momentum $-L_i^R$ to $L+1/2$, where

$$\Gamma'=\frac{-\vec{\sigma}\cdot\vec{L}^R+1/2}{L+1/2}=\Gamma'^*,\quad \Gamma'^2=\mathbb{1}. \tag{3.6}$$

The algebra $\mathcal{A}$ is generated by Γ and Γ'.

The fuzzy Dirac operator of Grosse *et al.* [9] is

$$D=\frac{1}{a}(\Gamma+\Gamma')=\frac{2}{a}\Gamma_1=\vec{\sigma}\cdot(\vec{L}^L-\vec{L}^R)+1,\quad a=\frac{1}{L+1/2}. \tag{3.7}$$

Thus the Dirac operator is in this case an element of the Ginsparg-Wilson algebra $\mathcal{A}$.

We can calculate Γ_0 in terms of $\vec{J}=\vec{\mathcal{L}}+\vec{\sigma}/2$:

$$\Gamma_0=\frac{a^2}{2}\left[\vec{J}^2-2L(L+1)-\frac{1}{4}\right]. \tag{3.8}$$

Thus the eigenvalues of Γ_0 increase monotonically with the eigenvalues $j(j+1)$ of $\vec{J}^2$, starting with a minimum for $j=1/2$ and attaining a maximum of 1 for $j=2L+1/2$.

Γ_2 is the chirality. It anticommutes with D. For fixed j, as $L\to\infty$, $\Gamma_0\to-\mathbb{1}$ and $\Gamma_2^2=\mathbb{1}$, as expected. In fact, Γ_2 in the naive continuum limit is the standard chirality for fixed j. As $L\to\infty$, $\Gamma_2\to\sigma\cdot\hat{x}$. As mentioned earlier, use of Γ_2 as chirality resolves a difficulty addressed elsewhere [7,15], where $\mathrm{sgn}(\Gamma_2)$ was used as the chirality. That necessitates projecting out V_{+1} and creates a very inelegant situation.

Finally, we note that there is a simple reconstruction of Γ and Γ' from their continuum limits [17]. If $\vec{x}$ is not normalized, $\vec{\sigma}\cdot\hat{x}=(\vec{\sigma}\cdot\vec{x}/|\vec{\sigma}\cdot\vec{x}|),|\vec{\sigma}\cdot\vec{x}|\equiv|[(\vec{\sigma}\cdot\vec{x})^2]^{1/2}|$. As $\vec{x}$ can be represented by $\vec{L}^L$ or $\vec{L}^R$ in fuzzy physics, natural choices for Γ and Γ' are $\mathrm{sgn}(\vec{\sigma}\cdot L^L)$ and $-\mathrm{sgn}(\vec{\sigma}\cdot L^R)$. The first operator is $+1$ on vectors having $\vec{\sigma}\cdot\vec{L}^L>0$ and -1 if instead $\vec{\sigma}\cdot\vec{L}^L<0$. But if $(\vec{L}^L+\vec{\sigma}/2)^2=(L+1/2)(L+3/2)$, then $\vec{\sigma}\cdot\vec{L}^L=L>0$, while if $(\vec{L}^L+\vec{\sigma}/2)^2=(L-1/2)(L+1/2)$, $\vec{\sigma}\cdot\vec{L}^L=-(L+1)<0$. Γ is $+1$ on the former states and -1 on the latter states. Thus

$$\mathrm{sgn}(\vec{\sigma}\cdot\vec{L}^L)=\Gamma, \tag{3.9}$$

and similarly

$$\mathrm{sgn}(\vec{\sigma}\cdot\vec{L}^R)=-\Gamma'. \tag{3.10}$$

We omit the calculation of the spectrum of D as it has been done before (see [9,7] and references there). We emphasize that this spectrum agrees completely with the spectrum of the continuum Dirac operator, except at the $j=(2L+1/2)$ level.

C. The fuzzy gauged Dirac operator (no instanton fields)

We adopt the convention that gauge fields are built from operators on $\mathrm{Mat}(2L+1)$ which act by left multiplication.

For $U(k)$ gauge theory, we start from $\mathrm{Mat}(2L+1)\otimes\mathbb{C}^k$. The fuzzy gauge fields A_i^L are $k\times k$ matrices $[(A_i^L)_{mn}]$ where each entry is the operator of left multiplication by $(A_i)_{mn} \in \mathrm{Mat}(2L+1)$ on $\mathrm{Mat}(2L+1)$. A_i^L thus acts on $\xi=(\xi_1,\ldots,\xi_k)$, $\xi_i\in$ $\mathrm{Mat}(2L+1)$, according to

$$(A_i^L\xi)_m=(A_i)_{mn}\xi_n. \tag{3.11}$$

The gauge-covariant derivative is then

$$\nabla_i(A^L)=\mathcal{L}_i+A_i^L=L_i^L-L_i^R+A_i^L. \tag{3.12}$$

Note how only the left angular momentum is augmented by a gauge field.

The Hermiticity condition on A_i^L is

$$(A_i^L)^*=A_i^L, \tag{3.13}$$

where

$$((A_i^L)^*\xi)_m=(A_i^*)_{nm}\xi_n, \tag{3.14}$$

$(A_i^*)_{nm}$ being the Hermitian conjugate of $(A_i)_{nm}$. To simplify the notation we shall limit ourselves to the $U(1)$ case in the following. The corresponding field strength F_{ij} is defined by

$$[(L+A)_i^L,(L+A)_j^L]=i\epsilon_{ijk}(L+A)_k^L+iF_{ij}. \tag{3.15}$$

There is a further point to attend to. We need a gauge-invariant condition which in the continuum limit eliminates the component of A_i normal to S^2. There are different such conditions, the following one being due to [18]:

$$(L_i^L+A_i^L)^2=(L_i^L)^2=L(L+1). \tag{3.16}$$

This condition is gauge invariant, and looks simple, although it represents a rather complicated quadratic equation among matrices in $\mathrm{Mat}(2L+1)$. For large L it gives

$$[x_i^L,A_i^L]_++\frac{(A_i^L)^2}{L}=0. \tag{3.17}$$

A_i^L is to remain bounded as $L\to\infty$. Also $x_i^L\to\hat{x}_i$, the unit normal to the sphere at $\hat{x}$. So in the limit, if $A_i^L\to A_i$, $\hat{x}\cdot\vec{A}(\hat{x})=0$, as required.

The Ginsparg-Wilson system can be introduced as follows. As Γ squares to $\mathbb{1}$, there are no zero modes for Γ and hence for $\vec{\sigma}\cdot\vec{L}^L+1/2$. But from Eqs. (3.15),(3.16)

$$\left(\vec{\sigma}\cdot(\vec{L}^L+\vec{A}^L)+\frac{1}{2}\right)^2=\left(L+\frac{1}{2}\right)^2-\frac{1}{2}\epsilon_{ijk}\sigma_iF_{ij}, \tag{3.18}$$

which shows that for generic $\vec{A}^L$ its gauged version $\vec{\sigma}\cdot(\vec{L}^L+\vec{A}^L)+\frac{1}{2}$ also has no zero modes unless we choose $\vec{A}^L$ such that the norm of $\epsilon_{ijk}\sigma_iF_{ij}$ grows like L^2, which is unphysical. Hence we can set

$$\Gamma(A^L)=\frac{\vec{\sigma}\cdot(\vec{L}^L+\vec{A}^L)+1/2}{|\vec{\sigma}\cdot(\vec{L}^L+\vec{A}^L)+1/2|},\quad \Gamma(A^L)^*=\Gamma(A^L),$$

$$\Gamma(A^L)^2=\mathbb{1}. \tag{3.19}$$

It is the gauged involution that reduces to $\Gamma=\Gamma(0)$ for zero $\vec{A}^L$.

As for the second involution $\Gamma'(A^L)$, we can set

$$\Gamma'(A^L)=\Gamma'(0)\equiv\Gamma'. \tag{3.20}$$

On following Eqs. (2.6)–(2.9), these idempotents generate the Ginsparg-Wilson algebra with operators $\Gamma_\lambda(A^L)$, where $\Gamma_\lambda(0)=\Gamma_\lambda$.

The operators $\vec{L}^{L,R}$ do not individually have continuum limits as their squares $L(L+1)$ diverge as $L\to\infty$. In contrast $\vec{\mathcal{L}}$ and $\vec{A}^L$ do have continuum limits. This was remarked earlier on for the latter, while $\vec{\mathcal{L}}$ just becomes orbital angular momentum.

To see more precisely how $D(A^L)$, the Dirac operator for the gauge field A^L [$D(0)$ being D of Eq. (3.7)], and $\Gamma_2(A^L)$ behave in the continuum limit, we can use Eqs. (3.18) to derive the expansions

$$\frac{1}{|\vec{\sigma}\cdot(\vec{L}^L+\vec{A}^L)+1/2|}=\frac{2}{\sqrt{\pi}}\int_0^\infty ds e^{-s^2[\vec{\sigma}\cdot(\vec{L}^L+\vec{A}^L)+1/2]^2}$$
$$=\frac{1}{L+1/2}+\frac{1}{4\left(L+\frac{1}{2}\right)^3}\epsilon_{ijk}\sigma_iF_{jk}+\cdots, \tag{3.21}$$

$$D(A^L)=(2L+1)\Gamma_1(A^L)$$
$$=\vec{\sigma}\cdot(\vec{L}^L-\vec{L}^R+\vec{A}^L)+1+\frac{\vec{\sigma}\cdot(\vec{L}^L+\vec{A}^L)+1/2}{4(L+1/2)^2}\epsilon_{ijk}\sigma_iF_{jk}$$
$$+\cdots,$$

$$\Gamma_2(A^L)=\frac{\vec{\sigma}\cdot(\vec{L}^L+\vec{A}^L)+1/2}{2(L+1/2)}-\frac{-\vec{\sigma}\cdot\vec{L}^R+1/2}{2(L+1/2)}$$
$$+\frac{\vec{\sigma}\cdot(\vec{L}^L+\vec{A}^L)+1/2}{8(L+1/2)^3}\epsilon_{ijk}\sigma_iF_{jk}+\cdots. \tag{3.22}$$

So in the continuum limit $D(A^L)\to\vec{\sigma}\cdot(\vec{\mathcal{L}}+\vec{A})+1$ and $\Gamma_2(A)\to\vec{\sigma}\cdot\hat{x}$, exactly as we want.

It is remarkable that even in the presence of gauge field, there is the operator

$$\Gamma_0(\vec{A}^L)=\frac{1}{2}[\Gamma(\vec{A}^L),\Gamma'(\vec{A}^L)]_+, \tag{3.23}$$

which is in the center of $\mathcal{A}$. It assumes the role of $\vec{J}^2$ in the presence of $\vec{A}^L$. In the continuum limit, it has the following meaning: $\mathrm{sgn}[D(A^L)]$ and $\Gamma_2(A^L)$ generate a Clifford algebra in that limit and the Hilbert space splits into a direct sum of subspaces, each carrying its IRR. $\Gamma_0(A^L)$ is a label for these subspaces.

IV. THE BASIC INSTANTON COUPLING

The instanton sectors on S^2 correspond to $U(1)$ bundles thereon. The connection on these bundles is not unique. Those with maximum symmetry have a particular simplicity and are therefore important for analysis.

In a similar way, on S_F^2, there are projective modules which in the algebraic approach substitute for sections of bundles [5,4,19,6]. There are particular connections on these modules with maximum symmetry and simplicity. In this section we build the Ginsparg-Wilson system for such connections. The Dirac operator is then also simple. It has zero modes which are responsible for the axial anomaly. Their presence will also be shown by simple reasoning.

To build the projective module for Chern number $2T$, $T>0$, introduce $\mathbb{C}^{2T+1}$ carrying the angular momentum T representation of $SU(2)$. Let T_α, $\alpha=1,2,3$, be the angular momentum operators in this representation with standard commutation relations. Let $\mathrm{Mat}(2L+1)^{2T+1}\equiv\mathrm{Mat}(2L+1)\otimes\mathbb{C}^{2T+1}$. We let $P^{(L+T)}$ be the projector coupling left angular momentum operators $\vec{L}^L$ with $\vec{T}$ to produce maximum angular momentum $L+T$. Then the projective module $P^{(L+T)}\mathrm{Mat}(2L+1)^{2T+1}$ is the fuzzy analogue of sections of $U(1)$ bundles on S^2 with Chern number $2T>0$ [6]. If instead we couple $\vec{L}^L$ and $\vec{T}$ to produce the least angular momentum $(L-T)$ using the projector $P^{(L-T)}$, $P^{(L-T)}\mathrm{Mat}(2L+1)^{2T+1}$ corresponds to Chern number $-2T$ (we assume that $L\geqslant T$).

We go about as follows to set up the Ginsparg-Wilson system. For Γ we now choose

$$\Gamma^{\pm}=\frac{\vec{\sigma}\cdot(\vec{L}^L+\vec{T})+1/2}{L\pm T+1/2}. \tag{4.1}$$

The domain of $\Gamma^{\pm}$ is $P^{(L\pm T)}\mathrm{Mat}(2L+1)^{2T+1}\otimes\mathbb{C}^2$ with σ acting on $\mathbb{C}^2$. On this module $(\vec{L}^L+\vec{T})^2=(L\pm T)(L\pm T+1)$ and $(\Gamma^{\pm})^2=\mathbb{1}$.

As for Γ', we choose it to be the same as in Eq. (3.6).

$\Gamma^{\pm}$ and Γ' generate the new Ginsparg-Wilson system. The operators Γ_λ are defined as before, as also is the new Dirac operator $D^{(L\pm T)}=(2/a)\Gamma_1$. For $T>0$ it is convenient to choose

$$a=\frac{1}{\sqrt{(L+1/2)(L\pm T+1/2)}}. \tag{4.2}$$

A. Mixing of spin and isospin

The total angular momentum $\vec{J}$ which commutes with $P^{(L\pm T)}$ and hence acts on $P^{(L\pm T)}\mathrm{Mat}(2L+1)\otimes\mathbb{C}^2$ is not $\vec{L}^L-\vec{L}^R+\vec{\sigma}/2$, but $\vec{L}^L+\vec{T}-\vec{L}^R+\vec{\sigma}/2$. The addition of $\vec{T}$ here is the algebraic analogue of the "mixing of spin and isospin" [20]. Such a term is essential in $\vec{J}$ since $\vec{L}^L-\vec{L}^R+\vec{\sigma}/2$, not commuting with $P^{(L\pm T)}$, would not preserve the modules. It is interesting that a mixing of "spin and isospin" already occurs in our finite-dimensional matrix model and does not need noncompact spatial slices and spontaneous symmetry breaking.

B. The spectrum of the Dirac operator

The spectrum of Γ_1 and $D^{(L\pm T)}$ can be derived simply by angular momentum addition, confirming the results of Sec. II.

On the $P^{(L\pm T)}\mathrm{Mat}(2L+1)^{2T+1}$ modules, $(\vec{L}^L+\vec{T})^2$ has the fixed values $(L\pm T)(L\pm T+1)$, and

$$(\Gamma_1)^2=\frac{1}{[2(L\pm T)+1](2L+1)}\Bigg[\Bigg(\vec{L}^L+\vec{T}-\vec{L}^R+\frac{1}{2}\vec{\sigma}\Bigg)^2+\frac{1}{4}-T^2\Bigg], \tag{4.3}$$

$$\Gamma^{\pm}=\frac{[\vec{L}^L+\vec{T}+(1/2)\vec{\sigma}]^2-(L\pm T)(L\pm T+1)-1/4}{(L\pm T)+1/2}, \tag{4.4}$$

$$\Gamma'=\frac{[-\vec{L}^R+(1/2)\vec{\sigma}]^2-L(L+1)-1/4}{L+1/2}. \tag{4.5}$$

Comparing Eq. (4.3) with Eq. (2.10) we see that the "total angular momentum" $(\vec{J})^2=(\vec{L}^L+\vec{T}-\vec{L}^R+\frac{1}{2}\vec{\sigma})^2$ is linearly related to $\Gamma_0=\frac{1}{2}[\Gamma^{\pm},\Gamma']_+$. The eigenvalues $(\gamma_1)^2$ of $(\Gamma_1)^2$ are determined by those of $(\vec{J})^2$; call them $j(j+1)$.

For $j=j_{max}=L\pm T+L+\frac{1}{2}$ we have $(\Gamma_1)^2=1$, so this is V_{+1}, and the degeneracy is $2j_{max}+1=2(2L\pm T+1)$. The maximum value of j can be achieved only if

$$\left(\vec{L}^L+\vec{T}+\frac{1}{2}\vec{\sigma}\right)^2=\left(L\pm T+\frac{1}{2}\right)\left(L\pm T+\frac{3}{2}\right),$$

$$\left(-\vec{L}^R+\frac{1}{2}\vec{\sigma}\right)^2=\left(L+\frac{1}{2}\right)\left(L+\frac{3}{2}\right). \tag{4.6}$$

Replacing these values in Eqs. (4.4),(4.5) we see that on V_{+1} we have $\gamma_1=1$ and $\Gamma_2=0$.

The case $T=0$ has been treated before [9,6,7]. So we here assume that $T>0$. In that case, for either module $j_{min}=T-\frac{1}{2}$, which gives an eigenvalue $(\gamma_1)^2=0$ with degeneracy $2T$; we are in V_{-1}, the space of the zero modes. To realize this minimum value of j we must have

$$\left(\vec{L}^L+\vec{T}+\frac{1}{2}\vec{\sigma}\right)^2=\left(L\pm T\mp\frac{1}{2}\right)\left(L\pm T\mp\frac{1}{2}+1\right),$$

$$\left(-\vec{L}^R+\frac{1}{2}\vec{\sigma}\right)^2=\left(L\pm\frac{1}{2}\right)\left(L\pm\frac{1}{2}+1\right). \tag{4.7}$$

Replacing these values in Eqs. (4.4),(4.5) we find that on the corresponding eigenstates $\Gamma_2 = \mp 1$: they are all either chiral left or chiral right. These are the results needed by continuum index theory and axial anomaly.

For $j_{min} < j < j_{max}$, that is, on V, we have $0 < (\gamma_1)^2 < 1$, and by Eq. (2.12) $\Gamma_2 \neq 0$. Since $[\Gamma_1, \Gamma_2]_+ = 0$, to each state ψ such that $\Gamma_1 \psi = \gamma_1 \psi$ corresponds a state $\psi' = \Gamma_2 \psi$ such that $\Gamma_1 \psi' = -\gamma_1 \psi'$.

For any value of j we can write $j = n + T - \frac{1}{2}$ with $n = 0, 1, \ldots, 2L+1$ when the projector is $P^{(L+T)}$, and $n = 0, 1, \ldots, 2(L-T)+1$ when the projector is $P^{(L-T)}$, while correspondingly

$$(\gamma_1)^2 = \frac{n(n+2T)}{[2(L \pm T)+1](2L+1)}. \tag{4.8}$$

With the choice (4.2) for a this gives for the squared Dirac operator the eigenvalues $\rho^2 = n(n+2T)$. This spectrum agrees *exactly* with what one finds in the continuum [21], except at the top value of n. Such a result is true also for $T = 0$ [7,6]. For the top value of n, $\Gamma_2 = 0$, and we get only the eigenvalue $\gamma_1 = 1$, whereas in the continuum, $\Gamma_2 \neq 0$, and both eigenvalues $\gamma_1 = \pm 1$ occur. This result [7,6], valid also for $T = 0$, has been known for a long time.

Finally, we can check that, summing the degeneracies of the eigenvalues we have found, we get exactly the dimension of the corresponding module. In fact,

$$2T + 2\sum_{n=1}^{2L}\left[2\left(n+T-\frac{1}{2}\right)+1\right] + 2(2L+T+1) = 2(2L+1)[2(L+T)+1],$$

$$2T + 2\sum_{n=1}^{2(L-T)}\left[2\left(n+T-\frac{1}{2}\right)+1\right] + 2(2L-T+1) = 2(2L+1)[2(L-T)+1]. \tag{4.9}$$

We show below that the axial anomaly on S_F^2 is stable against perturbations compatible with the chiral properties of the Dirac operator and is hence a "topological" invariant.

V. GAUGING THE DIRAC OPERATOR IN INSTANTON SECTORS

The operator $\vec{\mathcal{L}} + \vec{T}$ commutes with $P^{(L\pm T)}$ and hence preserves the projective modules. It is important to preserve this feature on gauging as well. So the gauge field $\vec{A}^L$ is taken to be a function of $\vec{L}^L + \vec{T}$ (which remains bounded as $L \to \infty$). For $L \to \infty$, it becomes a function of x. The limiting transversality of $\vec{T} + \vec{A}^L$ can be guaranteed by imposing the condition

$$(\vec{L}^L + \vec{T} + \vec{A}^L)^2 = (\vec{L}^L + \vec{T})^2 = (L \pm T)(L \pm T + 1) \tag{5.1}$$

which generalizes Eq. (3.16).

We can now construct the Ginsparg-Wilson system using

$$\Gamma(A^L) = \frac{\sigma \cdot (\vec{L}^L + \vec{T} + \vec{A}^L) + 1/2}{|\sigma \cdot (\vec{L}^L + \vec{T} + \vec{A}^L) + 1/2|} \tag{5.2}$$

and the Γ' of Eq. (3.6), $\Gamma(0)$ being Γ of Eq. (4.1). $\sigma \cdot (\vec{L}^L + \vec{T}) + 1/2$ has no zero modes, and therefore Eq. (5.2) is well defined for generic $\vec{A}^L$. We can now use Sec. II to construct the Dirac theory.

We have a continous number of Ginsparg-Wilson algebras labeled by $\vec{A}^L$. For each, Eq. (2.21) holds:

$$\operatorname{Tr} \Gamma_2(A^L) = n(A^L). \tag{5.3}$$

Here, as $n(A^L) \in \mathbb{Z}$, it is in fact a constant by continuity. The index of the Dirac operator and the axial anomaly (5.3) are thus independent of $\vec{A}^L$ as previously indicated.

The expansions (3.18)–(3.22) are easily extended to the instanton sectors and imply the continuum limit of $D^{(L\pm T)}(\vec{A}^L)$ and chirality $\Gamma_2(\vec{A}^L)$:

$$D^{(L\pm T)}(\vec{A}^L) \to \vec{\sigma} \cdot (\vec{\mathcal{L}} + \vec{T} + \vec{A}) + 1,$$

$$\Gamma_2(A^L) \to \vec{\sigma} \cdot \hat{x}. \tag{5.4}$$

Chirality is thus independent of the gauge field in the limiting case, where the $\vec{A}^L$ dependence is suppressed by $1/L$ factors [cf. Eq. (3.22)], but not otherwise.

VI. REMARKS

The Ginsparg-Wilson system developed above can be generalized to any number of products of S_F^2. For example, consider $S_F^2 \otimes S_F^2$. Its algebra is $\mathrm{Mat}(2L+1) \otimes_{\mathbb{C}} \mathrm{Mat}(2L'+1)$, where L, and L' can differ. There is a Ginsparg-Wilson system for each factor with its Γ and Γ'. Denote them (with or without instantons and/or gauge fields) by $\Gamma(1), \Gamma'(1), \Gamma(2)$, and $\Gamma'(2)$. The Γ and Γ' for $S_F^2 \otimes S_F^2$ are

$$\Gamma = \frac{\Gamma(1) + \Gamma_3(1)\Gamma(2)}{|\sqrt{1+\Gamma_3(1)^2}|}, \qquad \Gamma' = \frac{\Gamma'(1) + \Gamma_3(1)\Gamma'(2)}{|\sqrt{1+\Gamma_3(2)^2}|},$$

$$\Gamma_3(1) = \frac{1}{2i}[\Gamma(1), \Gamma'(1)], \qquad \Gamma_3(2) = \frac{1}{2i}[\Gamma(2), \Gamma'(2)]. \tag{6.1}$$

They square to unity since $[\Gamma(j), \Gamma_3(j)]_+ = 0$. Since the denominators in Eq. (6.1) commute with the operators in the numerators, there is no ordering problem in these equations.

Generalizations of the present investigation to fuzzy spaces such as $\mathbb{C}P_F^N$ await future work.

We have already treated the integrated $U(1)_A$ anomaly. Its local form has not been treated in the present approach; see, however, [10–12]. As for gauge anomalies, the central and familiar problem is that noncommutative algebras allow gauging only by the particular groups $U(N)$, and that too by their particular representations [10]. This is so in a naive approach. There are clever methods to overcome this problem on the Moyal planes [22] using the Seiberg-Witten map [23], but they too have failed us for the fuzzy spaces. Thus gauge anomalies can be studied for fuzzy spaces only in a very limited manner, but even this is yet to be done. More elaborate issues like anomaly cancellation in a fuzzy version of the standard model have to wait until the above mentioned problem is solved.

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