Chiral symmetry breaking solutions for QCD in the truncated Coulomb gauge

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In this paper we study the powerlike confining potentials r^{α}. The region of allowed α's is identified, the mass-gap equation is constructed for an arbitrary α and solved for several values of the latter, and the vacuum energy and the chiral condensate are calculated. The question of replica solutions to the mass-gap equation for such potentials is addressed, and it is demonstrated that the number of replicas is infinite for any α, as a consequence of the peculiar behavior of the quark self-energy in the infrared domain.

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I. INTRODUCTION

The problem of the spontaneous breaking of chiral symmetry (SBCS) and its relation to confinement is one of the cornerstones of QCD. Although the basic ideas of SBCS are already the subject of textbooks, this problem still lies at the crossroads of many studies of and approaches to QCD. In this paper we exploit the potential model for QCD, whose origins can be traced back to QCD in the truncated Coulomb gauge and which has proved to be successful in studies of the low-energy phenomena in QCD (see, for example, [1]). This class of models can be indicated as Nambu–Jona-Lasinio-(NJL-)type models [2] with a current-current quark interaction and the corresponding form factor coming from the bilocal gluonic correlator. The standard approximation in such types of models is to neglect the retardation and to approximate the gluonic correlator by a confining potential of a certain form. Powerlike potentials, which are the most natural candidates for the role of the confining force, are the subject of the present investigation. In the course of this paper, we re-examine the problem of SBCS for powerlike confining potentials $V(r)=K_0^{1+\alpha}r^{\alpha}$ with $\alpha\geqslant 0$, restrict the range of allowed α's and, for several values of the latter, find numerical solutions to the corresponding mass-gap equation, as well as the vacuum energy density and the chiral condensate for the chirally noninvariant vacuum of the theory, and study in detail the problem of the existence of replica solutions to the mass-gap equation for power like confining potentials.

The problem of the instability of the chirally invariant vacuum for powerlike confining potentials was studied in detail in the mid 1980s by the Orsay group [3], and this instability was proved for the range $0\leqslant\alpha<3$. For numerical studies, the harmonic oscillator type potential, $\alpha=2$, was chosen by these authors, as well as by the Lisbon group [4], and a set of results for the hadronic properties was obtained in the framework of the given model. In this paper, we study the mass-gap equation for an arbitrary value of α ranging from 0 to 2, with special attention paid to α's close to unity, since a linearly increasing potential is known to be preferred by phenomenology as the most successful candidate for the confining force, giving the correct Regge trajectory behavior, possessing a clear connection to the QCD string, and so on (see [5] and references therein). It is also claimed to be singled out by lattice calculations. We exclude the region $\alpha>2$ since the corresponding mass-gap equation diverges for such α's. On the other hand, it would be hard to justify the use of such a strong confining force in phenomenological models for QCD. We solve the mass-gap equation explicitly for several values of α from the allowed region and demonstrate that the chiral angle, the vacuum energy density, and the chiral condensate are smooth slow functions of the form of the confining potential, so that the results obtained for a potential of a given form—linear confinement being the most justified and phenomenologically successful choice — have a universal nature for any quark-quark kernel of such a type.

Following the set of recent publications devoted to possible multiple solutions for the chirally noninvariant vacuum in QCD [6,7] (see also [8], where a similar conclusion was made in a different approach), we address the question of replica existence for various power laws r^{α}. We find that for the whole range of allowed powers $0\leqslant\alpha\leqslant 2$, replica solutions do exist, similarly to the case of $\alpha=2$ studied in detail in [3,4]. We give the profiles of several replicas for the linear confinement and argue that the number of such solutions is infinite for any power α, including the weakest, logarithmic, potential which corresponds to $\alpha=0$. We argue that the source of replicas is the infrared behavior of the single-quark self-energy the dressed quark dispersive law $E(p)$ which, for small values of the quark momentum p, becomes a sharp negative function of p, thus enabling fast oscillations of the chiral angle with the frequency increasing with vanishing momentum p. Since this property of the quark dispersive law is expected to be an integral part of any confining interaction, we confirm the conclusion made in [7] that ‘‘across all these different quark kernels, the existence of vacuum replicas should constitute the rule rather than the exception.’’ We argue that, in real QCD, with the confining interaction flattening at large distances due to the effect of the string breaking, the number of replicas becomes finite. We find that the parameter of the SBCS given by the replica solutions decreases quickly with increasing number of nodes of the chiral angle, so that one has a well defined perturbative series in replicas and, therefore, taking account of only the first replica may be sufficient in many phenomenological applications (the details of the formalism that allows one to incorporate replicas into quark models can be found in Ref. [7]).

The paper is organized as follows. In the second section we give the necessary details of the formalism and derive the mass-gap equation for a powerlike confining potential, which is studied in detail in the third section, first, qualitatively, then quantitatively, and, finally, numerically. The mass-gap equation is solved numerically for several values of α and the chiral condensate and the excess of the vacuum energy density over the trivial solution are calculated for the solutions found. In the fourth section, devoted to replicas, we demonstrate how an infinite number of solutions to the mass-gap equation appears and explicitly build two replicas for linear confinement. Our conclusions are the subject of the last section.

II. THE MASS-GAP EQUATION

The chiral model that we use for our studies is given by a Hamiltonian with the current-current interaction parametrized by the bilocal correlator $K^{ab}_{\mu\nu}$,

$$H=\int d^3x\,\bar{\psi}(\vec{x},t)(-i\vec{\gamma}\cdot\vec{\nabla})\psi(\vec{x},t)+\frac{1}{2}\int d^3x d^3y J^a_\mu(\vec{x},t)K^{ab}_{\mu\nu}(\vec{x}-\vec{y})J^b_\nu(\vec{y},t), \tag{1}$$

where the quark current is $J^a_\mu(\vec{x},t)=\bar{\psi}(\vec{x},t)\gamma_\mu(\lambda^a/2)\psi(\vec{x},t)$, and the gluonic correlator is approximated by a potential,

$$K^{ab}_{\mu\nu}(\vec{x}-\vec{y})=g_{\mu0}g_{\nu0}\delta^{ab}V_0(|\vec{x}-\vec{y}|) \tag{2}$$

with

$$V_0(|\vec{x}|)=K_0^{\alpha+1}|\vec{x}|^\alpha. \tag{3}$$

In order to include the logarithmic potential in consideration an obvious modification of the potential is needed:

$$V_0(|\vec{x}|)\to\tilde{V}_0(|\vec{x}|)=K_0\left.\frac{(K_0|\vec{x}|)^\alpha-1}{\alpha}\right|_{\alpha\to0}=K_0\ln(K_0|\vec{x}|). \tag{4}$$

The model contains only one dimensional parameter the strength of the confining force K_0. For further convenience we shall consider a modified version of the potential (3) [3],

$$V_0(|\vec{x}|)=K_0^{\alpha+1}|\vec{x}|^\alpha e^{-m|\vec{x}|}, \tag{5}$$

where m plays the role of the regulator for the infrared behavior of the interaction. The limit $m\to0$ is understood.

The standard technique used in such models is the Bogoliubov-Valatin transformation from bare to dressed quarks parametrized by the chiral angle the main entity defining the chiral symmetry breaking, the structure of the BCS vacuum of the theory, as well as the properties of the hadronic states built over this vacuum [3,4]. For application of this technique to two-dimensional QCD [9], see the papers [10]. We choose the following parametrization:

$$\psi(\vec{x},t)=\sum_{\xi=\uparrow,\downarrow}\int\frac{d^3p}{(2\pi)^3}e^{i\vec{p}\vec{x}}[b_\xi(\vec{p},t)u_\xi(\vec{p})+d^\dagger_\xi(-\vec{p},t)v_\xi(-\vec{p})], \tag{6}$$

$$u(\vec{p})=\frac{1}{\sqrt{2}}[\sqrt{1+\sin\varphi_p}+\sqrt{1-\sin\varphi_p}\;(\vec{\alpha}\hat{\vec{p}})]u(0),$$

$$v(-\vec{p})=\frac{1}{\sqrt{2}}[\sqrt{1+\sin\varphi_p}-\sqrt{1-\sin\varphi_p}\;(\vec{\alpha}\hat{\vec{p}})]v(0), \tag{7}$$

$$b_\xi(\vec{p},t)=e^{iE_pt}b_\xi(\vec{p},0),\quad d_\xi(-\vec{p},t)=e^{iE_pt}d_\xi(-\vec{p},0), \tag{8}$$

where E_p [the shorthand notation for $E(p)$] stands for the dispersive law of the dressed quarks, and the chiral angle $\varphi(p)$ (we also use the shorthand notation φ_p for this) varies in the range $-\pi/2<\varphi_p\leqslant\pi/2$ with the boundary conditions $\varphi(0)=\pi/2$, $\varphi(p\to\infty)\to0$.

The Hamiltonian (1) normally arranged in the basis (8) splits into the vacuum energy and the quadratic and quartic parts in terms of the quark creation and annihilation operators. For the vacuum energy density one has

$$\mathcal{E}_{\rm vac}[\varphi]=\frac{1}{V}\langle0|TH[\varphi]|0\rangle=-\frac{g}{2}\int\frac{d^3p}{(2\pi)^3}(A(p)\sin\varphi_p+[B(p)+p]\cos\varphi_p), \tag{9}$$

where V is the three-dimensional volume; the degeneracy factor g counts the number of independent quark degrees of freedom,

$$g=(2s+1)N_CN_f, \tag{10}$$

with $s=\frac{1}{2}$ being the quark spin; the number of colors, N_C, is put to 3, and the number of light flavors, N_f, is 2. Thus we find that $g=12$. The auxiliary functions $A(p)$ and $B(p)$ are defined as

$$A(p)=\frac{1}{2}C_F\int\frac{d^3k}{(2\pi)^3}V_0(\vec{p}-\vec{k})\sin\varphi_k, \tag{11}$$

$$B(p)=p+\frac{1}{2}C_F\int\frac{d^3k}{(2\pi)^3}\;(\hat{\vec{p}}\cdot\hat{\vec{k}})V_0(\vec{p}-\vec{k})\cos\varphi_k, \tag{12}$$

where $C_F=\frac{4}{3}$ is the $SU(3)_C$ Casimir operator in the fundamental representation. The actual form of the chiral angle is such that the quadratic part of the normally ordered Hamiltonian diagonalizes, or, alternatively, the vacuum energy takes its minimal value. The corresponding equation

$$\frac{\delta\mathcal{E}_{\rm vac}[\varphi]}{\delta\varphi_p}=0, \tag{13}$$

known as the mass-gap equation, reads

$$A(p)\cos\varphi_p-B(p)\sin\varphi_p=0. \tag{14}$$

For the generalized power like potential (5) one can find

$$A(p)=-C_F\Gamma(\alpha+1)\frac{K_0^{\alpha+1}}{p}\times\int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{k\cos\{(\alpha+1)\arctan[(k-p)/m]\}}{[m^2+(k-p)^2]^{(\alpha+1)/2}}\sin\varphi_k, \tag{15}$$

$$B(p)=p-C_F\Gamma(\alpha)\frac{K_0^{\alpha+1}}{p^2}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\times\Bigg[\frac{\alpha pk\cos\{(\alpha+1)\arctan[(k-p)/m]\}}{[m^2+(k-p)^2]^{(\alpha+1)}/2}-\frac{\cos\{(\alpha-1)\arctan[(k-p)/m]\}}{(\alpha-1)[m^2+(k-p)^2]^{(\alpha-1)/2}}-\frac{(k-p)\sin\{\alpha\arctan[(k-p)/m]\}}{[m^2+(k-p)^2]^{\alpha/2}}\Bigg]\cos\varphi_k, \tag{16}$$

where, for the sake of convenience, we continued the integral to negative values of k, assuming $\cos\varphi_{-k},=-\cos\varphi_k$, $\sin\varphi_{-k}=\sin\varphi_k$ (the most natural realization of these conditions can be achieved in terms of some even function m_p, such that $\sin\varphi_p=m_p/\sqrt{p^2+m_p^2}$, $\cos\varphi_p=p/\sqrt{p^2+m_p^2}$, which plays the role of the effective mass of the quark). Consequently, the mass-gap equation (14) takes the form

$$p^3\sin\varphi_p=C_FK_0^{\alpha+1}\Gamma(\alpha)\int_{-\infty}^{\infty}\frac{dk}{2\pi}\times\Bigg\{\frac{\alpha pk\cos\{(\alpha+1)\arctan[(k-p)/m]\}}{[m^2+(k-p)^2]^{(\alpha+1)/2}}\times\sin[\varphi_p-\varphi_k]+\Bigg(\frac{\cos\{(\alpha-1)\arctan[(k-p)/m]\}}{(\alpha-1)[m^2+(k-p)^2]^{(\alpha-1)/2}}+\frac{(k-p)\sin\{\alpha\ \arctan[(k-p)/m]\}}{[m^2+(k-p)^2]^{\alpha/2}}\Bigg)\times\cos\varphi_k\sin\varphi_p\Bigg\}, \tag{17}$$

and this is the main object of our studies.

III. INVESTIGATION OF THE GENERAL FORMULA

A. Qualitative analysis

As the first step in studies of the general formula (17), we perform its simple qualitative analysis. Using the techniques described in [6], we assume that a solution $\varphi^{(0)}(p)$ to this equation exists, and the vacuum energy is minimal on this solution. If the function $\varphi^{(0)}(p/A)$ with an arbitrary stretching parameter $0\leqslant A<\infty$ is substituted into the vacuum energy (9), then the function $\mathcal{E}_{\rm vac}(A)$ must reveal a minimum for $A=1$. Moreover, the corresponding minimum should lie lower than the one for the trivial solution $\varphi(p)\equiv 0$, when there is no dressing of quarks and the chiral symmetry is unbroken. If the regulator m is removed from the vacuum energy functional, then the strength of the potential remains the only dimensional parameter in the theory, so that, after a proper rescaling of the integration variables, one arrives at the simple formula, for an arbitrary number of spatial dimensions d,

$$\mathcal{E}_{\rm vac}(A)=C_1A^{d+1}+C_2K_0^{\alpha+1}A^{d-\alpha}, \tag{18}$$

where C_1 and C_2 are two constants independent of A which are interrelated by the constraint $\partial\mathcal{E}_{\rm vac}(A)/\partial A_{|A=1}=0$. The first term in Eq. (18) comes from the kinetic energy, the second term is due to the interaction. The following four situations are possible: (i) $0<\alpha<d$, (ii) $\alpha>d$, and two boundary cases, (iii) $\alpha=d$, and (iv) $\alpha=0$. In the first case the vacuum energy has a double-well form with two minima: trivial for $A=0$ and nontrivial for $A=1$. The difference $\mathcal{E}_{\rm vac}(A=1)-\mathcal{E}_{\rm vac}(A=0)$ is negative, so that the chirally nonsymmetric nontrivial solution is energetically preferable. For the second case one has an interaction term in Eq. (18) containing negative powers of A and, as a result, the trivial solution, with unbroken chiral symmetry and which corresponds to $A=0$, possesses an infinite energy and therefore does not exist. In the meantime, a nontrivial solution with $A=1$ may still be present. The boundary case of $\alpha=d$ leads to a logarithmic dependence of the vacuum energy on the parameter A,

$$\mathcal{E}_{\rm vac}(A)=C_1A^{d+1}+C_2K_0^{d+1}\ln\frac{A}{K_0}, \tag{19}$$

so that qualitatively the same conclusion holdsthe theory possesses only a chirally nonsymmetric phase. Two-dimensional QCD [9] is an example of a theory with such a logarithmic dependence (see [6] for the details).

Finally, for the case (iv), that is, for the logarithmic potential (4), one has

$$\mathcal{E}_{\rm vac}(A)=C_1A^{d+1}+C_2K_0A^d\ln\frac{A}{K_0}, \tag{20}$$

where the logarithmic growth of the energy, when approaching the trivial solution limit $A=0$, is canceled by the power factor A^d, so that both chirally symmetric and nonsymmetric solutions coexist in this case, similarly to other potentials with $0<\alpha<d$.

Thus we conclude that once the power of the potential reaches a critical value equal to the number of spatial dimensions the behavior of the theory changes drastically, the chirally symmetric phase being swept off. Meanwhile, the qualitative analysis performed above ignored the problem of convergence of the integrals in the expression for the vacuum energy and in the corresponding mass-gap equation. It also fails to answer the question of how many solutions to the mass-gap equation exist. In what follows we turn to the quantitative and numerical analyses of these problems.

B. Quantitative analysis

Now we turn to a detailed analysis of the mass-gap equation (17), such as the problem of convergence, the allowed region for the α's, the dependence on the regulator m, and so on.

First of all, one can easily check that the case $\alpha=0$ causes no difficulties—the right-hand side (RHS) vanishes if the limit $\alpha\to 0$ is taken naively, whereas to arrive at the mass-gap equation for the logarithmic potential one has to divide the RHS by α [see Eq. (4)], which leads to a finite result after taking the limit $\alpha\to 0$.

For $\alpha=1$ the divergent term proportional to $1/(\alpha-1)$ vanishes on the RHS of Eq. (17), since the cosine of the chiral angle is odd. An accurate expansion of this term for $\alpha\to 1$ brings about logarithmic terms.

Now let us check the largest value of α that does not lead to divergences in the mass-gap equation. When the regulator m tends to zero, the first term in the large curly brackets in Eq. (17), formally, is the most singular term for $k\sim p$, and it can be written as

$$\int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{pk\cos\{(\alpha+1)\arctan[(k-p)/m]\}}{[m^2+(k-p)^2]^{(\alpha+1)/2}}\sin[\varphi_p-\varphi_k]. \tag{21}$$

In the region $k\sim p$ the integrand admits an expansion in the powers $(k-p)^n$:

$$\frac{\cos\{(\alpha+1)\arctan[(k-p)/m]\}}{2\pi[m^2+(k-p)^2]^{(\alpha+1)/2}}\Big\{p^2(k-p)\varphi_p' + (k-p)^2\Big[p\varphi_p'+\frac{1}{2}p^2\varphi_p''\Big]\dots\Big\}, \tag{22}$$

where the first term in the large curly brackets is odd and, therefore, it vanishes in the integral in k around $k=p$. The remaining expression, as well as other terms on the RHS of Eq. (17), behaves as $|k-p|^{1-\alpha}$, and the corresponding contribution to the mass-gap equation converges for $\alpha<2$, which is the upper limit of the range of valid powers α. For such α's,

$$\arctan\frac{k-p}{m}\underset{m\to 0}{\to}\frac{\pi}{2}\mathrm{sgn}(k-p)+O(m), \tag{23}$$

and the regulator m can be removed from the mass-gap equation. For the case of $\alpha=2$ the entire RHS of Eq. (17) vanishes after the substitution (23). To be more precise, one has to keep the next the terms in the expansion (23), with positive powers of the regulator m, reproducing the well-known representation of the delta function

$$\delta(p-k)=\lim_{m\to 0}\frac{1}{\pi}\frac{m}{m^2+(k-p)^2}$$

and its derivatives. We shall consider this special case separately.

Notice that the proof of the chirally symmetric vacuum instability given in [3] was based on a consideration of the self-energy functional $F(\vec{p})=\int[d^3k/(2\pi)^3]V_0(\vec{p}-\vec{k})(\hat{\vec{p}}\hat{\vec{k}})\propto 1/p^{\alpha}$, which gives an infrared divergent contribution to integrals in d^3p if $\alpha\geqslant 3$. In the meantime, as demonstrated above, the requirement of finiteness of the mass-gap equation imposes a stronger restriction on α: $\alpha\leqslant 2$.

One encounters no more difficulties for α's within the interval $0\leqslant\alpha<2$ and, when the regulator m is removed, one arrives at the mass-gap equation in the ultimate form:

$$p^3\sin\varphi_p=C_FK_0^{\alpha+1}\Gamma(\alpha+1)\sin\frac{\pi\alpha}{2}\times\int_{-\infty}^{\infty}\frac{dk}{2\pi}\left\{\frac{pk\sin[\varphi_k-\varphi_p]}{|p-k|^{\alpha+1}}+\frac{\cos\varphi_k\sin\varphi_p}{\alpha-1}\left[\frac{1}{|p-k|^{\alpha-1}}-\frac{1}{\lambda^{\alpha-1}}\right]\right\}. \tag{24}$$

We introduced the term $1/\lambda^{\alpha-1}$, with an arbitrary mass parameter λ, in order to emphasize the convergence of the integral for $\alpha=1$. This extra term does not contribute to the integral due to the parity of $\cos\varphi_k$.

In particular, the mass-gap equation for the logarithmic potential follows from Eq. (24) in the limit $\alpha\to 0$, if the proper modification of the potential, given in Eq. (4), is applied:

$$p^3\sin\varphi_p=\frac{C_FK_0}{4}\int_{-\infty}^{\infty}\frac{dk}{|p-k|}[pk\sin(\varphi_k-\varphi_p)-(p-k)^2\cos\varphi_k\sin\varphi_p]. \tag{25}$$

For the case of $\alpha=1$, that is, for linear confinement, the formula well-known in the literature is readily reproduced (see, for example, [6]):

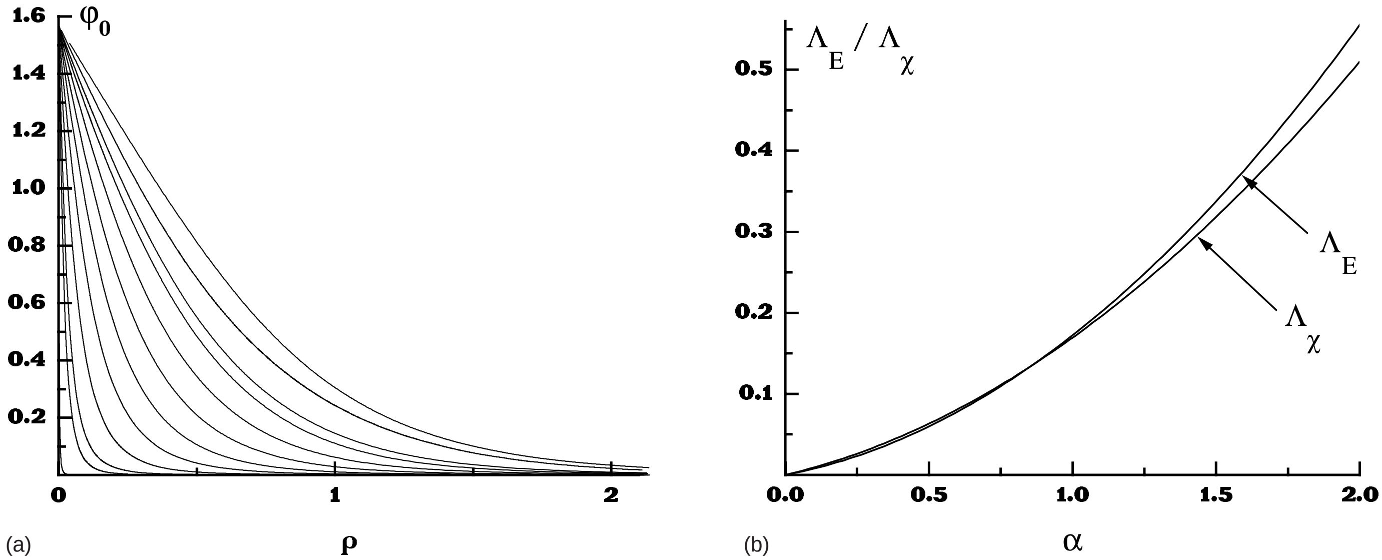

FIG. 1. The solutions to the mass-gap equation (24) for $\alpha=0.1$, 0.3, 0.5, 0.7, 0.9, 1.0, 1.1, 1.3, 1.5, 1.7, 1.9, and 2.0 (a); (the curves localized closer to the origin correspond to smaller α's), the mass parameters defining the chiral condensate, as $\langle\bar{q}q\rangle=-\Lambda_\chi^3$, and the excess of the vacuum energy density over the trivial vacuum, for two quark flavors and three colors, as $\Delta\mathcal{E}_{\rm vac}=-\Lambda_{\mathcal{E}}^4$, (b). All dimensional quantities are given in units of K_0.

$$p^3\sin\varphi_p=C_FK_0^2\int_{-\infty}^{\infty}\frac{dk}{2\pi}\left\{\frac{pk}{(p-k)^2}\sin[\varphi_k-\varphi_p]-\ln\frac{|p-k|}{\lambda}\cos\varphi_k\sin\varphi_p\right\} \quad (26)$$

$$=C_FK_0^2\int_0^{\infty}\frac{dk}{2\pi}\left\{\frac{4p^2k^2}{(p^2-k^2)^2}\sin[\varphi_k-\varphi_p]-\left[\frac{2pk}{(p+k)^2}+\ln\left|\frac{p-k}{p+k}\right|\right]\cos\varphi_k\sin\varphi_p\right\}.$$

To complete our investigation, let us consider the case of the harmonic oscillator potential, $\alpha=2$. The Fourier transform of the potential $V(r)=K_0^3r^2$ is the Laplacian of the three-dimensional delta function, so that the resulting mass-gap equation becomes differential [3]:

$$p^3\sin\varphi_p=\frac{1}{2}C_FK_0^3[p^2\varphi_p''+2p\varphi_p'+\sin2\varphi_p]. \quad (27)$$

Formally, Eqs. (17), (24) remain valid for $-1<\alpha<0$, and an ultraviolet divergence is encountered for $\alpha=-1$, that is, for the Coulomb potential. We disregard this region since the resulting force fails to be confining. Thus the valid confining potentials, in momentum space, range symbolically as $(2\pi)^3K_0\delta^{(3)}(\vec{p})<V(\vec{p})\leqslant(2\pi)^3K_0^3\Delta\delta^{(3)}(\vec{p})$, that is, from a constant to the harmonic oscillator potential, respectively.

For numerical investigation of the mass-gap equation (24) it is convenient to evaluate analytically the contribution of the strip $|k-p|<\lambda$ to the integral on the RHS,

$$I_\lambda=\frac{C_FK_0^{\alpha+1}}{2\pi}\frac{\lambda^{2-\alpha}}{2-\alpha}\Gamma(\alpha+1)\sin\frac{\pi\alpha}{2}[p^2\varphi_p''+2p\varphi_p'+\sin2\varphi_p], \quad (28)$$

where this integral admits an extra contribution if the width of the strip is chosen different from λ. For $\alpha=2$ the dependence of the strip integral (28) on λ disappears, and the full mass-gap equation (27) is readily reproduced as a consequence of the delta functional form of the Fourier transform of the harmonic oscillator potential.

C. Numerical analysis

In this subsection we present the results of numerical studies of the mass-gap equation (24) with $0\leqslant\alpha\leqslant2$. The equation for the harmonic oscillator potential (27) is studied in detail in the literature, so the interested reader can find the details, for example, in Refs. [3,4]. In Fig. 1(a) we plot the profile of the nontrivial solutions $\varphi_0(p)$ to the mass-gap equation (24) for several values of α. The solution for $\alpha=2$, found in Refs. [3,4], is also depicted for the sake of completeness. In Fig. 1(b) we present the results for the chiral condensate $\langle\bar{q}q\rangle=-(3/\pi^2)\int_0^\infty dpp^2\sin\varphi_0(p)\equiv-\Lambda_\chi^3$ and for the excess of the vacuum energy $\Delta\mathcal{E}_{\rm vac}=\mathcal{E}_{\rm vac}[\varphi_0]-\mathcal{E}_{\rm vac}[\varphi\equiv0]\equiv-\Lambda_\varepsilon^4$ over the trivial vacuum as functions of α. From Fig. 1(b) one can see that $\Lambda_\chi\approx\Lambda_{\mathcal{E}}$ in the whole range of allowed α's, and that their dependence on the form of the potential is smooth, solutions for α around unity being quite close to one another [see Fig. 1(a)]. Thus we conclude that the concrete form of the confining potential does not play a crucial role in the physics of chiral symmetry breaking, resulting in only minor numerical changes. In particular, one can see that the qualitative behavior of the solution is very stable against deviations of the potential from the purely linear form, usually adopted in phenomenological models and declared to be confirmed by lattice calculations. Therefore, at least SBCS and the spectrum of low-lying hadrons will not be strongly affected if the behavior of the confining potential is slightly changed, deviating from linearity, as suggested in [11].

IV. THE REPLICAS

As argued in a sequence of recent papers [6,7], it is possible that ‘‘same ultraviolet behavior, for instance for the quark propagator, bifurcates to different solutions when we go to the low-energy domain’’ in QCD, and such a replica was discovered for a phenomenology inspired potential. In addition, the whole infinite tower of excited solutions for the mass-gap equation (27) for the harmonic oscillator potential was found in Ref. [3] and also confirmed in [6]. With the general form of the mass-gap equation (17) and (24), we are in the position to investigate the problem of replica existence for various powerlike confining potentials. We find that *any* powerlike potential r^α, with the power $0\leqslant\alpha\leqslant 2$, maintains replicas. In [3] a detailed analysis was performed for the harmonic oscillator potential mass-gap equation (27) and the existence of an infinite tower of solutions was proved analytically. We failed to repeat this analysis for the general form of the mass-gap equation (24) since, in contrast to Eq. (27), Eq. (24) is integral, with the coefficients tuned to provide overall convergence of the integral, but leaving no hope of using any expansions under the integral. Instead, let us use an approximate method in order to demonstrate how replica solutions occur for the mass-gap equation (24). To this end we use the parametrization of the chiral angle through the effective quark mass and introduce a new function ψ_p:

$$\sin\varphi_p=m_pD_p,\quad \cos\varphi_p=pD_p,\quad \psi_p=p\sin\varphi_p,$$
$$D_p^{-1}\equiv\sqrt{p^2+m_p^2}. \tag{29}$$

It is also convenient to use the following integral:

$$\int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{\psi_k-\psi_p}{|k-p|^{\alpha+1}} =-\frac{1}{2\Gamma(\alpha+1)\sin(\pi\alpha/2)}\int_{-\infty}^{\infty}dx|x|^\alpha\psi_x e^{ipx}. \tag{30}$$

Then the mass-gap equation (24) can be rewritten in a simple physically transparent form:

$$[2E(\hat{p})+C_FK_0^{\alpha+1}|x|^\alpha]\psi_x=0, \tag{31}$$

where the operator $E(\hat{p})$ has the meaning of the quark self-energy and is given by the following expression:

$$E(p)=\frac{1}{D_p}-\frac{C_FK_0^{\alpha+1}}{p^2D_p}\Gamma(\alpha+1)\sin\frac{\pi\alpha}{2}\times\int_{-\infty}^{\infty}\frac{dk}{2\pi}\left[\frac{p^2D_p-k^2D_k}{|p-k|^{\alpha+1}}+\frac{pkD_pD_k}{(\alpha-1)|p-k|^{\alpha-1}}\right]. \tag{32}$$

The even function $m(p)$ takes its maximal value at $p=0$ and then decreases rapidly. Thus, for momenta larger than some $p_0\sim m(0)$, Eq. (31) can be linearized by putting $D_p^{-1}\approx|p|$ in the expression (32). Then the integral on the RHS of Eq. (32) is easily evaluated, and the self-energy takes the form

$$E(p)\approx|p|-\frac{2C_FK_0^{\alpha+1}\Gamma(\alpha+1)\sin(\pi\alpha/2)}{\pi\alpha(2-\alpha)|p|^\alpha}. \tag{33}$$

Notice that for *any confining potential*, $\alpha\geqslant 0$, the second term on the RHS of the expression (33) dominates in the low-momentum region, bringing a large negative contribution to the quark self-energy [the corresponding term becomes logarithmic, $\sim K_0\ln(K_0/|p|)$, for the potential (4)]. This general feature of the quark self-energy in a confining potential has been discussed in the literature several times (see, for example, [3,4], or [10], where the case of two-dimensional QCD is discussed in detail), and it is known to play a crucial role for the properties of the theory. In addition, replicas exist in the theory also due to this feature of the quark self-energy $E(p)$. In order to demonstrate this let us write the linearized mass-gap equation (31) in the form of a Schrödinger-type equation in momentum space,

$$\left[C_FK_0^{\alpha+1}|\hat{x}|^\alpha+2|p|-\frac{4C_FK_0^{\alpha+1}\Gamma(\alpha)\sin(\pi\alpha/2)}{\pi(2-\alpha)|p|^\alpha}\right]\psi_p =\varepsilon\psi_p, \tag{34}$$

and notice that the linearization suppresses the first, positive, term on the RHS of Eq. (32) and enhances the second, negative, term. As a result, we are interested in the *odd* [see the definition (29)] eigenstates of the linear equation (34) with *negative* eigenvalues. Each such state indicates a solution of the full nonlinear mass-gap equation (24) and can be used as the starting anzatz for the iterative numerical method to solve the latter. Therefore, in order to find the number of solutions to the mass-gap equation (24), it is sufficient to count the odd eigenstates, with negative eigenvalues, of the linear equation (34). If the Bohr-Sommerfeld quantization procedure is applied directly to Eq. (34), then the quasiclassical integral $I_{\rm WKB}=\int_{p_{\min}}^{p_{\max}}x(p)dp$ diverges logarithmically at $p=0$, where the chiral angle is no longer small and the approximation $D_p^{-1}\approx|p|$ obviously fails. In this region, the sharp behavior of the self-energy is smeared by the effective quark mass $\mu\sim m_p(p\to 0)$, which plays the role of the effective regulator, so that we modify Eq. (34) accordingly:

$$\left[C_F K_0^{\alpha+1}|\hat{x}|^\alpha+2\sqrt{p^2+\mu^2}-\frac{4C_F K_0^{\alpha+1}\Gamma(\alpha)\sin(\pi\alpha/2)}{\pi(2-\alpha)(p^2+\mu^2)^{\alpha/2}}\right]\psi_p$$
$$=\varepsilon\psi_p. \tag{35}$$

The spectrum of eigenstates of Eq. (35) starts, for $n=0$, at the bottom of the deep well described by the effective potential $V(p)=2E(p)$, $\varepsilon_0\approx V(p=0)\sim -K_0^{1+\alpha}/\mu^\alpha$, and, for some $n_{\max}$, reaches zero from below. Then the spectrum continues for positive eigenvalues up to infinity. Since the last negative eigenenergy $\varepsilon_{n_{\max}}$ is small, we expand the quasiclassical integral accordingly and find

$$\varepsilon_n \underset{n\approx n_{\max}}{\approx} K_0\left(\frac{C_F\Gamma(1/2)\Gamma((4-\alpha)/2)}{\Gamma((1+\alpha)/2)}\right)^{1/(1+\alpha)}$$
$$\times\frac{\Gamma([1/\alpha(1+\alpha)])}{\Gamma(1/\alpha)\Gamma(\alpha(1+\alpha))}\left[\pi n\left(\frac{\sqrt{\pi}\Gamma((4-\alpha)/2)}{2^\alpha\Gamma((1+\alpha)/2)}\right)^{1/\alpha}\right.$$
$$\left.-\ln\frac{K_0}{\mu}\right], \tag{36}$$

where we expressed $\sin(\pi\alpha/2)$ through the Euler Gamma functions as

$$\sin\frac{\pi\alpha}{2}=2^{\alpha-1}\frac{\Gamma(1/2)\Gamma((1+\alpha)/2)}{\Gamma(\alpha)\Gamma(2-\alpha/2)}. \tag{37}$$

Notice that in actuality the regulator μ itself is not a constant, but rapidly decreases with increasing n, and it is such that all eigenvalues of Eq. (35) vanish for all n's. Then the corresponding eigenfunctions ψ_n provide, according to the definition (29), the solutions to the exact mass-gap equation (24). From Eq. (36) one can easily see that

$$\mu_n(\alpha)=K_0\exp(-C_\alpha\pi n),\quad C_\alpha=\left[\frac{\sqrt{\pi}\Gamma((4-\alpha)/2)}{2^\alpha\Gamma((1+\alpha)/2)}\right]^{1/\alpha}. \tag{38}$$

Using a more accurate expansion in Eq. (36), one can find the correction δ_α to the leading regime (38), $\mu_n(\alpha)=K_0\exp[-C_\alpha(\pi n+\delta_\alpha)]$, as was done in [3]. Unlike the leading logarithmic term, this correction is sensitive to the concrete form of the regularization of the potential $V(p)$, and we do not give it here.

From the formula (38) we conclude that, for any α, the mass-gap equation (24) supports an infinite number of solutions which, in the low-momentum region ($p\ll K_0$), behave as

$$\varphi_n(p\sim 0)=\frac{\pi}{2}-\frac{p}{K_0}e^{C_\alpha(\pi n+\delta_\alpha)}+\cdots, \tag{39}$$

then oscillate n times, and, finally, approach zero for infinite momentum. The constant C_α is a monotonically decreasing

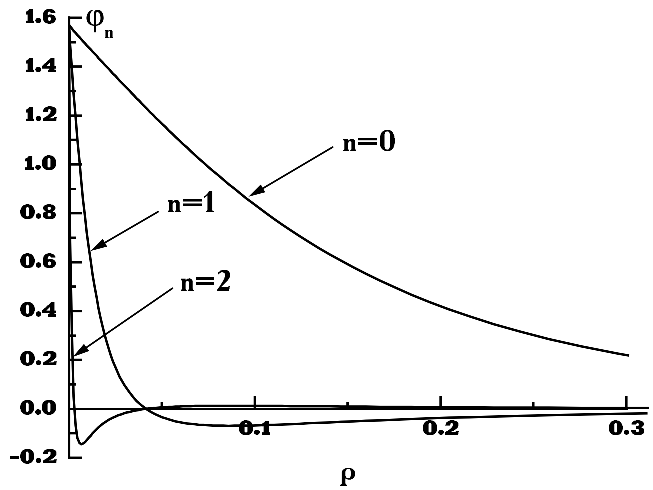

FIG. 2. The first three solutions to the mass-gap equation (26) corresponding to the ground state BCS vacuum ($n=0$) and to the first two replica states ($n=1,2$). The momentum p is given in units of K_0.

function of α in the whole interval $0\leqslant\alpha\leqslant 2$ and it varies from $C_0=\frac{1}{2}\exp\{\frac{1}{2}[\gamma-1-\psi(\frac{1}{2})]\}\approx 1.08$ to $C_2=1/\sqrt{2}\approx 0.71$. Numerically, the same behavior of the solutions to the mass-gap equation (24) is observed for smaller α's and higher n's the chiral angle becomes steeper at the origin, in accordance with the obvious identification $\varphi'_{n|p=0}=-1/\mu_n\propto\exp(C_\alpha[\pi n+\delta_\alpha])$ [see Figs. 1(a) and 2].

The formula (38) is approximate since the dependence on μ in Eq. (35) reproduces only the gross features of the quark self-energy, whereas the exact dependence of the self-energy on the chiral angle is, in turn, a consequence of the mass-gap equation, and the problem becomes self-consistent. To estimate the accuracy of the formula (38) we continue it to $\alpha=2$ and compare our result with the result found in [3]:

$$C_2=\frac{1}{\sqrt{2}}\approx 0.71,\quad C_2(\text{Ref. [3]})=\frac{2}{\sqrt{7}}\approx 0.76,$$

that is, the error is about 6–7%.

As an indirect confirmation of the conclusion made above let us mention that no critical phenomena are observed in the mass-gap equation for any α, as well as in the limit $\alpha\to 2$, so that all properties of the mass-gap equation, including the infinite number of solutions, may be continued from $\alpha=2$ to smaller α's.

In Fig. 2 we give the profiles of the ground state, as well as of the first two replica solutions for the linear confinement, $\alpha=1$. Solutions with larger numbers of nodes are also available for numerical study, but the method has to be very precise since each new solution possesses oscillations squeezed to zero and seen only with the help of methods with a sufficiently high resolution. According to the formula (39), the behavior of these solutions at the origin can be approximated as

$$\varphi_n(p\sim 0)=\frac{\pi}{2}-\frac{p}{K_0}\exp\left[\frac{\pi}{4}(\pi n+\delta_1)\right]+\cdots, \qquad (40)$$

so that $\mu_{n+1}(1)/\mu_n(1)\approx\exp(\pi^2/4)\sim 10$, that is, the parameter of SBCS decreases about ten times for each successive replica.

In Ref. [7] a method is proposed which allows one to take the vacuum replicas into account in quark models. The generalization of this approach to the case of many replicas is straightforward and assumes a summation over the contributions of all replicas. On the other hand, since the mass parameter of the solution, for example, μ_n, decreases fast (approximately ten times for linear confinement) for each successive solution, then one has a sort of perturbative series a the well defined convergence parameter. Therefore it may be sufficient to consider only the first replica, neglecting the contribution of higher replicas, starting from the second one, which are hard to distinguish numerically from the trivial solution $\varphi_p\equiv 0$. To have a good phenomenological description of quarkonia one should supply the purely confining potential with the short-range Coulomb interaction and, possibly, with a constant term in order to fit for the right value of the chiral condensate (an attempt to evaluate this constant from first principles was undertaken in [12]). Such a potential was considered in [6,7], and the solutions for the ground state BCS vacuum as well as for one vacuum replica were found numerically.

V. CONCLUSIONS

In this paper we complete the study of power like potentials $K_0^{\alpha+1}r^\alpha$ from the point of view of SBCS and the number of nontrivial solutions of the mass-gap equation. We use a constructive method to prove the chirally symmetric vacuum instability for such potentials, solving the mass-gap equation explicitly and calculating the vacuum energy for the corresponding solution. We establish the region of allowed powers of α for such confining potentials that lead to a convergent mass-gap equation and a finite excess of the vacuum energy density over the trivial solution with unbroken chiral symmetry. Thus, using simple qualitative arguments, we demonstrate that for $0\leqslant\alpha<d$, d being the number of spatial dimensions, at least two (chirally symmetric and nonsymmetric) solutions should exist, whereas for $\alpha\geqslant d$ the trivial solution possesses an infinite energy density and disappears. In the meantime, a restriction for the parameter α exists, $0\leqslant\alpha\leqslant 2$, which comes from the fact that the corresponding mass-gap equation should be convergent. We find numerical solutions to the mass-gap equation for various values of α from the allowed region and, for the solutions found, evaluate the vacuum energy density and the chiral condensate, which are given by the same scale $\Lambda_\varepsilon\sim\Lambda_\chi$ and turn out to be slow functions of the parameter α.

We address the question of the existence of the second, third, and higher chirally nonsymmetric solutions of the mass-gap equation for powerlike confining potentials and find that any potential with $0\leqslant\alpha\leqslant 2$ supports such solutionsthe replicaswhich is rather the rule for confining potentials and comes from the peculiar behavior of the quark self-energy in the infrared domain. We find that the number of such replicas is infinite for any α and estimate the slopes of the solutions to the mass-gap equation at the origin.

Thus we dare predict the existence of replicas regardless of the explicit form of the confinement and of the details of the model used in calculations. In real QCD, with light quark flavors and the quark-quark potential flattening at large distances due to the effect of QCD string breaking (such an effect can be taken into account in the interquark potential through a coordinate dependence of the effective string tension [13]), the number of replicas is expected to be finite. Indeed, the distance at which the string starts to break, L, will play the role of the infrared regulator in a formula similar to Eq. (35), instead of μ. Therefore, as follows from Eq. (36), $n_{\rm max}\sim\ln(K_0L)\sim 1$, where we consider $L\sim K_0^{-1}$. In other words, confinement becomes less ''binding'' due to string breaking and the corresponding mass-gap equation supports fewer solutions''bound states.'' A more detailed analysis of the QCD inspired interaction, including the proper string dynamics, from the point of view of replicas is in progress now and will be the subject of future publications.

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[1] P. Bicudo, S. Cotanch, F. Llanes-Estrada, P. Maris, E. Ribeiro, and A. Szczepaniak, Phys. Rev. D **65**, 076008 (2002).

[2] Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1961).

[3] A. Amer, A. Le Yaouanc, L. Oliver, O. Pene, and J.-C. Raynal, Phys. Rev. Lett. **50**, 87 (1983); A. Le Yaouanc, L. Oliver, O. Pene, and J.-C. Raynal, Phys. Lett. **134B**, 249 (1984); Phys. Rev. D **29**, 1233 (1984).

[4] P. Bicudo and J. E. Ribeiro, Phys. Rev. D **42**, 1611 (1990); **42**, 1625 (1990); **42**, 1635 (1990); P. Bicudo, Phys. Rev. Lett. **72**, 1600 (1994); Phys. Rev. C **60**, 035209 (1999).

[5] S. L. Adler and A. C. Davis, Nucl. Phys. **B244**, 469 (1984); Y. L. Kalinovsky, L. Kaschluhn, and V. N. Pervushin, Phys. Lett. B **231**, 288 (1989); P. Bicudo, J. E. Ribeiro, and J. Rodrigues, Phys. Rev. C **52**, 2144 (1995); R. Horvat, D. Kekez, D. Palle, and D. Klabucar, Z. Phys. C **68**, 303 (1995); Yu. A. Simonov, Yad. Fiz. **60**, 2252 (1997) [Phys. At. Nucl. **60**, 2069 (1997)]; N. Brambilla and A. Vairo, Phys. Lett. B **407**, 167 (1997); Yu. A. Simonov and J. A. Tjon, Phys. Rev. D **62**, 014501 (2000); P.

Bicudo, N. Brambilla, E. Ribeiro, and A. Vairo, Phys. Lett. B **442**, 349 (1998); F. J. Llanes-Estrada and S. R. Cotanch, Phys. Rev. Lett. **84**, 1102 (2000).
[6] P. J. A. Bicudo, A. V. Nefediev, and J. E. F. T. Ribeiro, Phys. Rev. D **65**, 085026 (2002).
[7] A. V. Nefediev and J. E. F. T. Ribeiro, Phys. Rev. D **67**, 034028 (2003).
[8] A. A. Osipov and B. Hiller, Phys. Lett. B **539**, 76 (2002).
[9] G. 't Hooft, Nucl. Phys. **B75**, 461 (1974).
[10] I. Bars and M. B. Green, Phys. Rev. D **17**, 537 (1978); Ming Li, *ibid.* **34**, 3888 (1986); Ming Li, L. Wilets, and M. C. Birse, J. Phys. G **13**, 915 (1987); Yu. S. Kalashnikova and A. V. Nefediev, Usp. Fiz. Nauk **172**, 378 (2002) [Physica (Utrecht) **45**, 347 (2002)].
[11] D. Diakonov and V. Petrov, Phys. Scr. **61**, 536 (2000).
[12] Yu. A. Simonov, Phys. Lett. B **515**, 137 (2001).
[13] A. M. Badalian, B. L. G. Bakker, and Yu. A. Simonov, Phys. Rev. D **66**, 034026 (2002).