Duality for symmetric second rank tensors. II. The linearized gravitational field

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The construction of dual theories for linearized gravity in four dimensions is considered. Our approach is based on the parent Lagrangian method previously developed for the massive spin-two case, but now considered for the zero mass case. This leads to a dual theory described in terms of a rank two symmetric tensor, analogous to the usual gravitational field, and an auxiliary antisymmetric field. This theory has an enlarged gauge symmetry, but with an adequate partial gauge fixing it can be reduced to a gauge symmetry similar to the standard one of linearized gravitation. We present examples illustrating the general procedure and the physical interpretation of the dual fields. The zero mass case of the massive theory dual to the massive spin-two theory is also examined, but we show that it only contains a spin-zero excitation.

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I. INTRODUCTION

In preceding papers we have discussed the construction of dual theories for massive fields in a Lagrangian framework, and in particular we fully developed the case of a massive spin two theory [1,2]. The purpose of this paper is to extend the analysis to the massless case, focusing on the linearized gravitational field in four dimensions.

Let us recall the procedure. Starting from a second order Lagrangian, the first step is to construct a first order Lagrangian, with a particular structure defined by the kinetic term. It contains the derivative of the original field times a new auxiliary variable, which corresponds to the field strength of the original theory. The key recipe to construct the dual theory is to introduce a point transformation in the configuration space for the auxiliary variable which involves the completely antisymmetric tensor $\epsilon^{\mu\nu\sigma\tau}$ and leads to the first order parent Lagrangian. From the latter, both the original and the dual theories are obtained. In fact, the equations of motion for the auxiliary variables take us back to our starting action. Alternatively, we can eliminate the original field from the parent Lagrangian, using its equations of motion, thus obtaining the dual theory which is equivalent to the original one through the transformation defined by such equations of motion.

Using the well known example of massive scalar-tensor duality we will revise the main steps of the procedure summarized above, to point out the new features that will appear in the massless case. Such a duality corresponds to the equivalence between a free scalar field φ, with field strength $f_\mu=\partial_\mu\varphi$, and an antisymmetric potential $B_{\mu\nu}$, the Kalb-Ramond field, with field strength $H_{\mu\nu\sigma}=\partial_\mu B_{\nu\sigma}+\partial_\nu B_{\sigma\mu}+\partial_\sigma B_{\mu\nu}$ [3–5]. Starting from the standard second order Lagrangian for φ we derive a first order Lagrangian

$$L(\varphi,L^\mu)=L^\mu\partial_\mu\varphi-\tfrac{1}{2}L^\mu L_\mu-\tfrac{1}{2}m^2\varphi^2+J\varphi. \tag{1}$$

To construct the dual theory we introduce the point transformation $L_\mu=\epsilon_{\mu\nu\rho\sigma}H^{\nu\rho\sigma}$ for the variable L_μ, which leads to a new first order Lagrangian

$$L(\varphi,H_{\nu\sigma\tau})=H_{\nu\sigma\tau}\epsilon^{\mu\nu\sigma\tau}\partial_\mu\varphi+3H_{\nu\sigma\tau}H^{\nu\sigma\tau}-\tfrac{1}{2}m^2\varphi^2+J\varphi. \tag{2}$$

This turns out to be the parent Lagrangian from which both theories (original and dual) can be obtained. On the one hand, using the equation of motion for $H_{\nu\sigma\tau}$ we get

$$H^{\nu\sigma\tau}(\varphi)=\tfrac{1}{6}\epsilon^{\nu\sigma\tau\mu}\partial_\mu\varphi, \tag{3}$$

which takes us back to the original second order Lagrangian for φ after it is substituted in Eq. (2). On the other hand, we can eliminate the field φ from Lagrangian (2) using its own equation of motion

$$m^2\varphi=-\partial_\mu\epsilon^{\mu\nu\sigma\tau}H_{\nu\sigma\tau}+J. \tag{4}$$

In this way we obtain the new theory

$$L(H_{\nu\sigma\tau})=\tfrac{1}{2}(\epsilon^{\mu\nu\sigma\tau}\partial_\mu H_{\nu\sigma\tau})^2+3m^2H_{\nu\sigma\tau}H^{\nu\sigma\tau}-J\epsilon^{\mu\nu\sigma\tau}\partial_\mu H_{\nu\sigma\tau}+\tfrac{1}{2}J^2, \tag{5}$$

which is equivalent to the original one through transformation (4). This is a singular Lagrangian for the massive field $H_{\nu\sigma\tau}$, which is equivalent to a scalar field of mass m.

Following the same parent Lagrangian approach we have also constructed a family of dual theories for the massive Fierz-Pauli field $h_{\mu\nu}$ in terms of the fields $T_{(\mu\nu)\rho}$ satisfying $T_{(\mu\nu)\rho}=-T_{(\nu\mu)\rho}$ and $T_{(\mu\nu)}{}^{\nu}=0$. The cyclic identity

$$T_{(\mu\nu)\rho}+T_{(\nu\rho)\mu}+T_{(\rho\mu)\nu}=0, \tag{6}$$

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which selects $T_{(\mu\nu)\rho}$ in the spin two irreducible representation, was not assumed as a starting point and arose as a dynamical result via the equations of motion [1,2].

We now turn to the massless case. Here a very important difference appears, which we still illustrate in the scalar field context. In this case the equation of motion (4) of φ becomes a constraint on $H_{\nu\sigma\tau}$

$$\partial_\mu \epsilon^{\mu\nu\sigma\tau} H_{\nu\sigma\tau} = J. \tag{7}$$

Out of the sources, where $\partial_\mu \epsilon^{\mu\nu\sigma\tau} H_{\nu\sigma\tau} = 0$, this constraint tells us that the field $H_{\nu\sigma\tau}$ can be considered as a field strength with an associated potential

$$H_{\nu\sigma\tau} = \partial_\nu B_{\sigma\tau} + \partial_\sigma B_{\tau\nu} + \partial_\tau B_{\nu\sigma}. \tag{8}$$

In the region where $J \neq 0$ this is not valid. Hence it is not possible to give a global solution for $B_{\sigma\tau}$ because, using Eq. (8), the left-hand side (LHS) of Eq. (7) is always zero while the right-hand side (RHS) might be non-null in a given domain. The problem is similar to that of finding the electromagnetic potential for a magnetic monopole.

To deal with this situation we introduce a Dirac-type string singularity $f^\mu(x)$ defined by [6]

$$f^\mu(x) = \int_C^x d\xi^\mu \delta^{(4)}(\xi), \quad \partial_\mu f^\mu(x) = \delta^{(4)}(x). \tag{9}$$

The path C begins at infinity, ends up at the point x and can be chosen as a straight line if considered convenient. Thus we can write a particular solution to Eq. (7) as

$$\epsilon^{\mu\nu\sigma\tau} H_{\nu\sigma\tau}(x) = \int (dy) f^\mu(x-y) J(y), \tag{10}$$

with the general solution being

$$H_{\nu\sigma\tau} = \partial_\nu B_{\sigma\tau} + \partial_\sigma B_{\tau\nu} + \partial_\tau B_{\nu\sigma} + \frac{1}{6}\epsilon_{\nu\sigma\tau\rho} \times \int (dy) f^\rho(x-y) J(y), \tag{11}$$

in terms of the potential field and the one dimensional singular string.

The Lagrangian for the potential $B_{\alpha\beta}$ is obtained by substituting Eq. (11) into Eq. (5), which produces the corresponding equations of motion.

The duality transformations are expressed in terms of the following nonlocal relation between the field φ, describing the original zero-mass scalar theory, and the potential $B_{\alpha\beta}$, which is obtained through the comparison of $H^{\mu\nu\rho}$ in Eqs. (3) and (11)

$$\frac{1}{6}\epsilon_{\nu\sigma\tau\mu}\partial^\mu \varphi = \partial_\nu B_{\sigma\tau} + \partial_\sigma B_{\tau\nu} + \partial_\tau B_{\nu\sigma} + \frac{1}{6}\epsilon_{\nu\sigma\tau\rho} \times \int (dy) f^\rho(x-y) J(y). \tag{12}$$

Thus, in the case of massless theories the first order equation of motion for the original variable becomes a constraint, i.e. it looks like a Bianchi identity, which states that the dual field can now be considered as a field strength with an associated potential plus a non-local contribution. Solving the constraint we obtain the dual theory, in which this potential becomes the basic field. Both theories arise from the same parent Lagrangian and represent the same physics. This procedure strongly resembles the electric-magnetic duality of the Maxwell theory. In fact, it is a generalization of the well known p-form duality to arbitrary tensorial massless fields [7].

Naively one could think that another possibility to generate a massless dual theory for the linearized gravity is to take $m=0$ in the massive $T_{(\mu\nu)\sigma}$ Lagrangian of Ref. [2]. We explore this possibility in the Appendix, with negative results. The Dirac analysis shows that such a theory describes only a spin zero excitation. This result corrects our previous preliminary calculation of the number of degrees of freedom for the massless theory reported in Refs. [1,2], which erroneously stated that this number was two.

The construction of dual theories is usually based on a kinematical perspective where the basic dual fields are assigned to associated representations of the Poincaré group [8,9]. Some dynamical realizations of duality have also been considered in the framework of four dimensional higher derivatives theories of gravity [10] and in other gravitational theories [11]. Our approach is based on a Lagrangian basis, where the auxiliary fields are not in irreducible representations to begin with, but the ensuing Lagrangian constraints warrant that the dynamics develops in an adequate reduced space, with a well defined spin content.

The paper is organized as follows. In the next section we apply the dualization scheme to the Fierz-Pauli theory and obtain the dual description in terms of two tensors, a symmetric one, $\tilde{h}_{\mu\nu}$, and an antisymmetric one, $\omega_{\mu\nu}$. Section III contains the analysis of the gauge symmetries of the dual theory, which clarifies the physical meaning of the dual fields. In Sec. IV we consider two examples which illustrate the construction and the effect of the dual transformations: (i) the field describing polarized gravitational waves and (ii) the gravitational field produced by a point mass at rest. The last section contains a summary and some comments on the work. Finally, in the Appendix we discuss the massless theory for the $T_{(\mu\nu)\sigma}$ field.

II. THE MASSLESS SPIN 2 FIELD PARENT LAGRANGIAN

The parent Lagrangian for $m=0$ is [see Eq. (39) of Ref. [2] with $a=e^2=1/4$]

$$L = \tfrac{1}{8} T_{(\mu\nu)\sigma} T^{(\mu\nu)\sigma} + \tfrac{1}{4} T_{(\mu\nu)\sigma} T^{(\mu\sigma)\nu} + \tfrac{1}{2} T_{(\mu\nu)\sigma} \epsilon^{\mu\nu\alpha\beta} \partial_\alpha h^\sigma_\beta + h^{\alpha\beta}\Theta_{\alpha\beta}, \tag{13}$$

where the source $\Theta_{\alpha\beta}$ is symmetric, $\Theta_{\alpha\beta} = \Theta_{\beta\alpha}$, and conserved, $\partial^\alpha \Theta_{\alpha\beta} = 0$. The field $T_{(\mu\nu)\rho}$ has zero trace, $T_{(\mu\nu)}{}^\mu = 0$. From the equation of motion for $T_{(\mu\nu)\sigma}$ we can solve $T_{(\mu\nu)\sigma}$ in terms of h^σ_β,

$$T_{(\mu\nu)\beta}=-\epsilon_{\mu\nu}{}^{\alpha\sigma}\partial_\alpha h_{\sigma\beta}+\epsilon_{\mu\nu\beta\lambda}(\partial_\alpha h^{\alpha\lambda}-\partial^\lambda h^\alpha_\alpha), \quad (14)$$

which indeed has a null trace. Plugging back expression (14) in Lagrangian (13) one obtains a Lagrangian for $h^{\sigma\kappa}$, which is the linearized Einstein Lagrangian

$$\mathcal{L}=-\partial_\mu h^{\mu\nu}\partial_\alpha h^\alpha_\nu+\tfrac{1}{2}\partial^\alpha h_{\mu\nu}\partial_\alpha h^{\mu\nu}+\partial_\mu h^{\mu\nu}\partial_\nu h^\alpha_\alpha -\tfrac{1}{2}\partial_\alpha h^\mu_\mu\partial^\alpha h^\nu_\nu+h^{\alpha\beta}\Theta_{\alpha\beta}, \quad (15)$$

as we proved in a previous paper [2].

The corresponding equations of motion for $h_{\mu\nu}$ are the linearized Einstein equations

$$\partial^\alpha\partial_\alpha h_{\mu\nu}+\partial_\mu\partial_\nu h^\alpha_\alpha-(\partial_\mu\partial_\alpha h^\alpha_\nu+\partial_\nu\partial_\alpha h^\alpha_\mu)-\eta_{\mu\nu}(\partial^\alpha\partial_\alpha h^\beta_\beta -\partial_\alpha\partial_\beta h^{\alpha\beta})=\Theta_{\mu\nu}, \quad (16)$$

which clearly show that h^σ_β is a spin 2 massless field. This Lagrangian has the gauge symmetry

$$h_{\mu\nu}\to h_{\mu\nu}+\partial_\mu\varepsilon_\nu+\partial_\nu\varepsilon_\mu . \quad (17)$$

On the other hand the Euler-Lagrange equation for the Lagrange multiplier $h^{\mu\nu}$ in Eq. (13) reduces to a simple constraint

$$\partial_\sigma(\epsilon^{\alpha\beta\sigma\nu}T_{(\alpha\beta)}{}^\mu+\epsilon^{\alpha\beta\sigma\mu}T_{(\alpha\beta)}{}^\nu)=\Theta^{\mu\nu}. \quad (18)$$

Substituting expression (14) in the above equation we recover the equation of motion (16) for $h_{\mu\nu}$.

We face here a situation similar to the one already encountered for the scalar field. From constraint (18) we are able to introduce a potential for the field $T_{(\alpha\beta)}{}^\mu$ only outside of the sources. Therefore there is no global solution for a potential. In analogy with the scalar field, we choose the particular solution $\overline{T}_{(\alpha\beta)}{}^\mu$ of Eq. (18) as

$$(\epsilon^{\alpha\beta\sigma\nu}\overline{T}_{(\alpha\beta)}{}^\mu+\epsilon^{\alpha\beta\sigma\mu}\overline{T}_{(\alpha\beta)}{}^\nu)(x)=\int(dy)f^\sigma(x-y)\Theta^{\mu\nu}(y), \quad (19)$$

which leads to

$$\overline{T}_{(\alpha\beta)}{}^\mu=-\frac{1}{6}\epsilon_{\alpha\beta\sigma\nu}\int(dy)f^\sigma(x-y)\Theta^{\mu\nu}(y). \quad (20)$$

Next, we find the solution $\widetilde{T}_{(\alpha\beta)}{}^\nu$ to the homogeneous equation associated with Eq. (18),

$$\partial_\sigma(\epsilon^{\alpha\beta\sigma\nu}\widetilde{T}^\mu_{(\alpha\beta)}+\epsilon^{\alpha\beta\sigma\mu}\widetilde{T}^\nu_{(\alpha\beta)})=0. \quad (21)$$

The above equation implies that the symmetric part of the tensor $k^{\mu\nu}=\partial_\sigma\epsilon^{\alpha\beta\sigma\nu}\widetilde{T}^\mu_{(\alpha\beta)}$ is zero. Furthermore, this tensor has a vanishing divergence $\partial_\nu k^{\mu\nu}=0$, and thus it can be written as $k^{\mu\nu}=\epsilon^{\mu\nu\sigma\delta}\partial_\sigma A_\delta$, leading to

$$\partial_\sigma(\epsilon^{\alpha\beta\sigma\nu}\widetilde{T}^\mu_{(\alpha\beta)}-\epsilon^{\alpha\beta\sigma\mu}\widetilde{T}^\nu_{(\alpha\beta)})=2\epsilon^{\mu\nu\sigma\delta}\partial_\sigma A_\delta . \quad (22)$$

In fact, Eq. (22) is a linear equation for $\widetilde{T}^\mu_{(\alpha\beta)}$, whose solution consists of the general solution for the homogeneous equation plus a particular solution for the complete one. The homogenous equation corresponding to Eq. (22) tells us that $\widetilde{T}^\mu_{(\alpha\beta)}$ is a closed 2-form for each μ, while a particular solution is given by $\widetilde{T}^\mu_{(\alpha\beta)}=\delta^\mu_\alpha A_\beta-\delta^\mu_\beta A_\alpha$. Thus the general solution for $\widetilde{T}_{(\alpha\beta)\mu}$ in Eq. (22) is

$$\widetilde{T}_{(\alpha\beta)\mu}=(\eta_{\mu\alpha}A_\beta-\eta_{\mu\beta}A_\alpha)+(\partial_\alpha B_{\mu\beta}-\partial_\beta B_{\mu\alpha}), \quad (23)$$

where the tensor $B_{\mu\beta}$ is not necessarily symmetric. It can be expressed in terms of its symmetric and antisymmetric parts, $\widetilde{h}_{\mu\nu}=\widetilde{h}_{\nu\mu}$ and $\omega_{\mu\nu}=-\omega_{\nu\mu}$, respectively,

$$B_{\mu\nu}=\omega_{\mu\nu}+\widetilde{h}_{\mu\nu}. \quad (24)$$

Finally, taking into account that $\widetilde{T}_{(\alpha\beta)\mu}$ must be traceless we obtain from Eq. (23),

$$A_\alpha=-\tfrac{1}{3}(\partial^\beta\widetilde{h}_{\beta\alpha}-\partial_\alpha\widetilde{h}^\beta_\beta+\partial^\beta\omega_{\beta\alpha}). \quad (25)$$

In this way we have found the general solution for the constraint equation (18) which is

$$T_{(\alpha\beta)\mu}=(\eta_{\mu\alpha}A_\beta-\eta_{\mu\beta}A_\alpha)+(\partial_\alpha B_{\mu\beta}-\partial_\beta B_{\mu\alpha}) -\frac{1}{6}\epsilon_{\alpha\beta\sigma\nu}\int(dy)f^\sigma(x-y)\Theta^{\mu\nu}(y). \quad (26)$$

We have obtained a description of the theory in terms of the potentials $\widetilde{h}_{\mu\nu}$ and $\omega_{\mu\nu}$, together with a Dirac-type string contribution.

Considering for simplicity the free field case, $\Theta_{\mu\nu}=0$, and substituting Eqs. (23) and (25) into Lagrangian (13) we get

$$L=\tfrac{1}{2}\partial^\alpha\widetilde{h}_{\mu\nu}\partial_\alpha\widetilde{h}^{\mu\nu}-\tfrac{2}{3}\partial_\mu\widetilde{h}^{\mu\nu}\partial_\alpha\widetilde{h}^\alpha_\nu-\tfrac{1}{6}\partial_\mu\widetilde{h}^\alpha_\alpha\partial^\mu\widetilde{h}^\alpha_\alpha +\tfrac{1}{3}\partial_\mu\widetilde{h}^\alpha_\alpha\partial_\nu\widetilde{h}^{\mu\nu}-\tfrac{2}{3}\partial_\mu\widetilde{h}^{\mu\nu}\partial^\alpha\omega_{\nu\alpha}+\tfrac{1}{3}\partial_\mu\omega^{\nu\mu}\partial^\alpha\omega_{\nu\alpha}. \quad (27)$$

In fact only the divergence of $\omega_{\nu\alpha}$ appears in the Lagrangian, which implies that $\omega_{\nu\alpha}$ is an auxiliary field, defined up to an arbitrary exact two form. The equations of motion are

$$\partial^\alpha\partial_\alpha(\widetilde{h}_{\mu\nu}-\tfrac{2}{3}\eta_{\mu\nu}\widetilde{h}^\sigma_\sigma)-\tfrac{2}{3}\partial_\alpha(\partial_\mu\widetilde{h}^\alpha_\nu+\partial_\nu\widetilde{h}^\alpha_\mu)+\tfrac{1}{3}(\partial_\nu\partial_\mu\widetilde{h}^\alpha_\alpha +\eta_{\mu\nu}\partial_\alpha\partial_\beta\widetilde{h}^{\alpha\beta})-\tfrac{1}{3}\partial^\alpha(\partial_\mu\omega_{\nu\alpha}+\partial_\nu\omega_{\mu\alpha})=0, \quad (28)$$

$$\partial_\beta(\partial_\mu\widetilde{h}^\beta_\nu-\partial_\nu\widetilde{h}^\beta_\mu)-\partial^\alpha(\partial_\mu\omega_{\nu\alpha}-\partial_\nu\omega_{\mu\alpha})=0. \quad (29)$$

Equation (29) implies that $\partial^\alpha\omega_{\nu\alpha}-\partial_\beta\widetilde{h}^\beta_\nu$ has zero curl. Thus

$$\partial^\alpha\omega_{\nu\alpha}-\partial_\beta\widetilde{h}^\beta_\nu=\partial_\nu\Phi, \quad (30)$$

where Φ is a scalar field. The divergence of Eq. (30) leads to

$$\partial^\nu\partial_\beta\widetilde{h}^\beta_\nu=-\partial^\nu\partial_\nu\Phi. \quad (31)$$

Replacing expression (30) for $\partial^\alpha\omega_{\nu\alpha}$ in Eq. (28), we obtain

$$\partial^\alpha\partial_\alpha\tilde{h}_{\mu\nu}-\partial_\mu\partial_\alpha\tilde{h}^\alpha_\nu-\partial_\nu\partial_\alpha\tilde{h}^\alpha_\mu+\tfrac{1}{3}\partial_\nu\partial_\mu\tilde{h}^\alpha_\alpha+\tfrac{1}{3}\eta_{\mu\nu}(\partial_\alpha\partial_\beta\tilde{h}^{\alpha\beta}-\partial^\beta\partial_\beta\tilde{h}^\alpha_\alpha)-\tfrac{2}{3}\partial_\mu\partial_\nu\Phi=0. \tag{32}$$

Note that the trace of the the above equation does not yield $\partial^2\tilde{h}^\alpha_\alpha-\partial^\alpha\partial^\beta\tilde{h}_{\alpha\beta}=0$, at difference with the case of the linearized sourceless Einstein equations.

Contracting Eq. (14) with $\epsilon_{\kappa\lambda\sigma\alpha}$ we have

$$\tfrac{1}{2}\epsilon_{\kappa\lambda\sigma\alpha}\partial^\kappa\omega^{\sigma\lambda}\equiv D_\alpha=(\partial_\alpha h^\beta_\beta-\partial_\beta h^\beta_\alpha). \tag{33}$$

This equation gives the curl of the antisymmetric component, and also shows that it is a topologically conserved current in the dual description, $\partial^\alpha D_\alpha=0$. This conservation law can also be derived from the equation of motion for $h^{\sigma\kappa}$ when the energy momentum tensor is traceless. It expresses that in this case the scalar curvature vanishes.

III. GAUGE SYMMETRIES

The $T^\mu_{(\alpha\beta)}$ field is invariant under the following local transformations:

$$\delta_\Psi A_\beta=\partial_\beta\Psi,\quad \delta_\Psi B_{\mu\beta}=\eta_{\mu\beta}\delta\Psi, \tag{34}$$

$$\delta_f A_\beta=0,\quad \delta_f B_{\mu\beta}=\partial_\mu f_\beta, \tag{35}$$

which in terms of $\tilde{h}_{\mu\nu}$ and $\omega_{\mu\nu}$ read

$$\delta\omega_{\mu\nu}=-(\partial_\mu f_\nu-\partial_\nu f_\mu), \tag{36}$$

$$\delta\tilde{h}_{\mu\nu}=\eta_{\mu\nu}\Psi+(\partial_\mu f_\nu+\partial_\nu f_\mu). \tag{37}$$

The induced transformation upon the auxiliary field Φ, introduced in Eq. (30), is

$$\delta\Phi=-\Psi-2\partial_\alpha f^\alpha, \tag{38}$$

which shows that it is pure gauge.

The dual theory we have constructed exhibits two kinds of gauge symmetries, one of them similar to that of the Fierz-Pauli spin two theory. Next we show that an adequate gauge fixing for the additional Ψ symmetry reduces our theory to a standard massless spin two form. We can use the freedom in Ψ to set

$$\Phi=-\tilde{h}^\alpha_\alpha. \tag{39}$$

With this choice Eq. (30) becomes

$$\partial^\alpha\omega_{\nu\alpha}-\partial_\beta\tilde{h}^\beta_\nu=-\partial_\nu\tilde{h}^\alpha_\alpha, \tag{40}$$

which leads to

$$\partial^\alpha\partial^\beta\tilde{h}_{\alpha\beta}-\partial^2\tilde{h}^\alpha_\alpha=0. \tag{41}$$

As we have mentioned previously, the above equation is the trace of the sourceless Einstein equations (16). Let us also remark that Eq. (40) is invariant under the remaining gauge transformations generated by the functions f^ν in Eqs. (36) and (37). This is because, according to Eq. (38), the gauge (39) fixes Ψ without constraining f^α. Thus the equations of motion (32) can be rewritten as

$$\partial^\alpha\partial_\alpha\tilde{h}_{\mu\nu}-(\partial_\mu\partial_\alpha\tilde{h}^\alpha_\nu+\partial_\nu\partial_\alpha\tilde{h}^\alpha_\mu)+\partial_\nu\partial_\mu\tilde{h}^\alpha_\alpha=0, \tag{42}$$

where we have explicitly used the trace condition (41). In this way we have recovered the linearized Einstein equations for $\tilde{h}_{\mu\nu}$, thus describing spin two massless excitations. In particular the gauge condition $\Phi=-\tilde{h}^\alpha_\alpha$ leads to $A_\alpha=0$, according to expressions (25) and (30). Therefore, we have shown that the dual theory here obtained, described by Lagrangian (27), is a gauge description where a Ψ orbit is conformed by a set of theories which are gauge equivalent to the linearized Einstein theory. In what follows we will always work in the gauge $\Phi=-\tilde{h}^\alpha_\alpha$.

From Eqs. (14) and (23), the duality relation among the potential fields is

$$\epsilon^{\kappa\lambda\sigma\beta}(\partial_\alpha h^\alpha_\beta-\partial_\beta h^\alpha_\alpha)-\epsilon^{\kappa\lambda\alpha\beta}\partial_\alpha h^\sigma_\beta =(\partial^\kappa\tilde{h}^{\sigma\lambda}-\partial^\lambda\tilde{h}^{\sigma\kappa})+(\partial^\kappa\omega^{\sigma\lambda}-\partial^\lambda\omega^{\sigma\kappa}). \tag{43}$$

At this stage we can completely determine the fields of the dual theory. A standard gauge choice in the linearized spin two theory via the functions f^ν gives $\tilde{h}_{\alpha\beta}$, which in turn fixes the divergence of $\omega_{\mu\nu}$ through Eq. (40), and the curl of $\omega_{\mu\nu}$ through Eq. (33), thus yielding $\omega_{\mu\nu}$.

We can now obtain the relationship between the corresponding Riemann tensors. Recalling its definition

$$R^\lambda{}_{\mu\nu\kappa}=\tfrac{1}{2}[\partial_\mu(\partial_\kappa h^\lambda_\nu-\partial_\nu h^\lambda_\kappa)-\partial^\lambda(\partial_\kappa h_{\mu\nu}-\partial_\nu h_{\mu\kappa})], \tag{44}$$

and using Eqs. (43) and (33) we have

$$\tilde{R}_\mu{}^{\sigma\lambda\kappa}=\tfrac{1}{2}\epsilon^{\kappa\lambda}{}_{\alpha\beta}R_\mu{}^{\sigma\alpha\beta}\ , \tag{45}$$

which exhibits the local transformation between the field strengths arising from the nonlocal relation among the potentials.

Finally, it is interesting to explore the relation among the gauge transformations in both theories. The gauge freedom due to ε^μ in the original theory is mapped into gauge transformations of the antisymmetric tensor $\omega_{\sigma\kappa}$,

$$\delta\omega_{\sigma\kappa}=-\epsilon_{\mu\nu\sigma\kappa}\partial^\mu\varepsilon^\nu, \tag{46}$$

while the new gauge freedom of $\tilde{h}^{\mu\nu}$ due to f^μ is independent from them. Thus the dual Lagrangian is invariant under Eq. (46) because it depends only on the divergence of $\omega_{\sigma\kappa}$, which does not change under this gauge transformation.

IV. EXAMPLES

In this section we discuss two examples which illustrate the construction and effects of the proposed dual transformations. The first one refers to the behavior of the polarization components of a gravitational wave. The second one discusses the field produced by a point mass and shows how it is mapped into the potentials ω_{0i} and $\tilde{h}_{0i}$, that have a form analogous to the electromagnetic potential of a magnetic monopole.

A. Gravitational waves

In this case the gauge can be fixed using the transverse traceless gauge TT. Working in the momentum space the gravitational field is

$$h_{\mu\nu}(k)=h^{+}(k)e^{+}_{\mu\nu}(k)+h^{\times}(k)e^{\times}_{\mu\nu}(k), \tag{47}$$

where $k^{\mu}=(k^{0},\vec{k})$, with $k_{0}=|\vec{k}|$, and the two possible helicities have polarization tensors

$$e^{+}_{\mu\nu}(k)=m_{\mu}m_{\nu}-n_{\mu}n_{\nu},\quad e^{\times}_{\mu\nu}(k)=m_{\mu}n_{\nu}+n_{\mu}m_{\nu}. \tag{48}$$

The spacelike quadrivectors $m^{\mu}=(0,\hat{m})$, $n^{\mu}=(0,\hat{n})$ are such that

$$\hat{m}\cdot\hat{m}=\hat{n}\cdot\hat{n}=1,\quad \hat{m}\cdot\hat{n}=0,\quad \hat{m}\cdot\vec{k}=\hat{n}\cdot\vec{k}=0. \tag{49}$$

That is to say, $\hat{m}$, $\hat{n}$, and $\hat{k}=\vec{k}/|\vec{k}|$ form an orthonormal triad with $\hat{n}=\hat{m}\times\hat{k}$. Thus, the properties that define the TT gauge are $\partial^{\mu}h_{\mu\nu}=h^{\nu}_{\nu}=h^{0\nu}=0$. In this gauge we have $D^{\lambda}=0$ so that Eq. (33) leads to

$$\epsilon_{\lambda\sigma\kappa\alpha}\partial^{\lambda}\omega^{\sigma\kappa}=0. \tag{50}$$

Given the field of a gravitational wave as in Eq. (47), we will now find the dual fields. To begin with we also choose the TT gauge for the field $\tilde{h}_{\alpha\beta}$, which implies

$$\partial^{\mu}\omega_{\mu\nu}=0, \tag{51}$$

according to Eq. (40). Thus we obtain $\omega_{\mu\nu}=0$. In the chosen gauge we have the following expression for $\tilde{h}_{\mu\nu}(k)$:

$$\tilde{h}_{\mu\nu}(k)=\tilde{h}_{+}e^{+}_{\mu\nu}(k)+\tilde{h}_{\times}e^{\times}_{\mu\nu}(k). \tag{52}$$

According to Eq. (43) the duality relations between both theories are

$$\tilde{D}_{\kappa\sigma\lambda}=-\tfrac{1}{2}\epsilon_{\kappa\lambda}{}^{\alpha\beta}D_{\alpha\sigma\beta}, \tag{53}$$

where $D^{\lambda\sigma\kappa}\equiv\partial^{\lambda}h^{\sigma\kappa}-\partial^{\kappa}h^{\sigma\lambda}$. The properties ($\epsilon^{0123}=+1$)

$$\epsilon_{\alpha\beta}{}^{\mu\nu}k_{\mu}m_{\nu}=(k_{\alpha}n_{\beta}-k_{\beta}n_{\alpha}),$$

$$\epsilon_{\alpha\beta}{}^{\mu\nu}k_{\mu}n_{\nu}=-(k_{\alpha}m_{\beta}-k_{\beta}m_{\alpha}), \tag{54}$$

lead to

$$\epsilon_{\kappa\lambda}{}^{\nu\mu}(k_{\nu}e^{+}_{\sigma\mu}-k_{\mu}e^{+}_{\sigma\nu})=+2[k_{\kappa}e^{\times}_{\sigma\lambda}-k_{\lambda}e^{\times}_{\sigma\kappa}],$$

$$\epsilon_{\kappa\lambda}{}^{\nu\mu}(k_{\nu}e^{\times}_{\sigma\mu}-k_{\mu}e^{\times}_{\sigma\nu})=-2[k_{\kappa}e^{+}_{\sigma\lambda}-k_{\lambda}e^{+}_{\sigma\kappa}]. \tag{55}$$

The next step is to substitute Eqs. (47) and (52) in relation (53). The elements $e^{+}_{\alpha\beta}$ and $e^{\times}_{\alpha\beta}$ of the tensor basis in the LHS of Eq. (53) are mixed by the epsilon symbol according to Eqs. (55), which interchanges the labels $+$ and $\times$ of the basis tensors. Comparing with the corresponding terms of the RHS we obtain the relations

$$\tilde{h}_{+}=h_{\times},\quad \tilde{h}_{\times}=h_{+}. \tag{56}$$

Summarizing, the net result of the dualization procedure is to interchange the helicity states.

B. Point mass

In the de Donder gauge $\partial^{\nu}h_{\mu\nu}=1/2\partial_{\mu}h^{\alpha}{}_{\alpha}$, the linearized gravitational field produced by a point mass M is

$$h_{\mu\nu}=-\frac{2M}{r}\delta_{\mu\nu}, \tag{57}$$

where the metric is $\eta_{\mu\nu}=\mathrm{diag}(+,-,-,-)$. Note that $h_{\mu\nu}$ is proportional to $\delta_{\mu\nu}$ and not to $\eta_{\mu\nu}$. The trace of the gravitational field is

$$h=\eta^{\mu\nu}h_{\mu\nu}=\frac{4M}{r}, \tag{58}$$

so that we can rewrite $h_{\mu\nu}=-1/2h\,\delta_{\mu\nu}$.

The curl of $\omega^{\sigma\lambda}$ is fixed by the original gravitational field $h_{\mu\nu}$ through Eq. (33) which yields

$$-\epsilon_{\kappa\lambda\sigma\alpha}\partial^{\kappa}\omega^{\sigma\lambda}=D_{\alpha}=-4\frac{M}{r^{3}}x_{\alpha}. \tag{59}$$

To solve this equation we must specify the divergence of $\omega^{\sigma\lambda}$, which we do by choosing

$$\partial^{\mu}\tilde{h}_{\mu\nu}=\partial_{\nu}\tilde{h}^{\alpha}_{\alpha}, \tag{60}$$

as the gauge in which the dual field $\tilde{h}_{\mu\nu}$ is described. In this way Eq. (40) yields

$$\partial^{\mu}\omega_{\mu\nu}=0. \tag{61}$$

We solve Eq. (59) as usual. The zero divergence condition (61) yields $\omega^{H}_{\mu\nu}=0$ for the regular solutions of the corresponding homogeneous equation. As we see from Eq. (59), it is not possible to give a global particular solution for $\omega^{\sigma\lambda}$, because the divergence of the RHS is always zero, while the divergence of the RHS gives $16\pi M\delta(\vec{r})$. The problem is similar to finding the electromagnetic potential for a magnetic monopole. Using a Dirac string of the form $f^{\mu}(x)=n^{\mu}f(x)$, with the constant vector $n^{\mu}=(0,\hat{n})$, we obtain that the only nonzero components of $\omega^{\sigma\lambda}$ are $\omega^{0i}=-\omega^{i0}=H^{i}$ with

$$\vec{H}=(H^{1},H^{2},H^{3})=2M\frac{\hat{n}\times\vec{r}}{r(r+\hat{n}\cdot\vec{r})}, \tag{62}$$

which are singular on the negative $\hat{n}$ axis. The reader can verify that we have $\boldsymbol{\nabla}\cdot\vec{H}=0$ outside the singularity line.

Now we determine the dual field $\tilde{h}_{\mu\nu}$. To this end we use the duality relations (43) between the fields $h_{\mu\nu}$ and $\tilde{h}_{\mu\nu}$. Going back to the notation $T^{(\kappa\lambda)\sigma}$ for the LHS of Eq. (43) and using Eq. (57) we can show that the only non-zero contribution is

$$T_{(ij)0} = -\epsilon_{ij0k}\,\partial^k h. \tag{63}$$

In this way, the explicit expressions for Eqs. (43) are

$$0 = \partial_i \tilde{h}^{00} - \partial^0 \tilde{h}^{0i}, \tag{64}$$

$$0 = \partial_i (\tilde{h}^{j0} + \omega^{j0}) - \partial^0 \tilde{h}^{ij}, \tag{65}$$

$$0 = (\partial^i \tilde{h}^{kj} - \partial^j \tilde{h}^{ki}), \tag{66}$$

$$\epsilon_{ijk0}\partial^k h = (\partial^i \tilde{h}^{0j} - \partial^j \tilde{h}^{0i}) + (\partial^i \omega^{0j} - \partial^j \omega^{0i}). \tag{67}$$

Equation (66), together with the symmetry of $\tilde{h}^{kj}$, implies that $\tilde{h}^{kj} = \partial^k \partial^j U$ for a scalar U. At this point we note that to implement condition (61) we have made use of the gauge transformations of $\omega_{\mu\nu}$, which depend on the curl of f_μ. We still have the gauge freedom given by $f_\mu = \partial_\mu \Delta$, which only involves a scalar function Δ. The corresponding transformation of $\tilde{h}_{\mu\nu}$ is $\delta \tilde{h}_{\mu\nu} = 2\partial_\mu \partial_\nu \Delta$. We can make use of this gauge freedom to set $\tilde{h}^{jk} = 0$. Equation (65) then gives place to $\tilde{h}^{j0} = -\omega^{j0}$, where we are discarding constant solutions that do not go to zero at infinity. In consequence, the second term in Eq. (64) vanishes and we have $\tilde{h}^{00} = 0$. Thus, from the first three equations we get

$$\tilde{h}^{00} = 0, \quad \tilde{h}^{ij} = 0, \quad \tilde{h}^{j0} = -\omega^{j0}. \tag{68}$$

The remaining equation (67) is

$$\partial^i \omega^{0j} - \partial^j \omega^{0i} = -\tfrac{1}{2}\epsilon_{0ijk}\,\partial^k h, \tag{69}$$

which the reader can verify is only a rewriting of Eq. (59), which ω^{0i} indeed satisfies.

Summarizing, we see that the duality introduced here maps the field of a point source into ω^{0i} and $\tilde{h}^{0i}$, which have the form of the electromagnetic potential of a magnetic monopole with its corresponding Dirac string singularity. The potential $\tilde{h}$ breaks the rotational symmetry of the problem, but this is a gauge artifact, and as Eq. (45) shows the gauge invariant quantities are symmetric under spatial rotations.

V. SUMMARY AND FINAL COMMENTS

Using a parent Lagrangian approach we have constructed dual theories for linearized gravity. The starting point is the zero mass case of the parent Lagrangian for massive spin-two theories developed in Ref. [2]. The equation of motion for the original field $h_{\mu\nu}$ leads to a constraint implying that the dual field $T_{(\mu\nu)\rho}$ can be written as the field strength of a potential. The presence of sources required the introduction of Dirac-type line singularities in order to have a global solution for the potentials. The general solution for the constraint leads to a dual description in terms of an auxiliary field $\omega_{\mu\nu}$, which enters only through its divergence, together with $\tilde{h}_{\mu\nu}$. The resulting theory has the standard gauge symmetry of linearized gravity plus an additional Ψ symmetry, according to Eqs. (34). By an adequate gauge fixing of the latter symmetry one recovers the Einstein equations for $\tilde{h}_{\mu\nu}$ together with the standard symmetries. They still affect the field $\omega_{\mu\nu}$ [see Eqs. (36) and (37)], which becomes determined through the gauge fixing of the gravitational fields $h_{\mu\nu}$ and $\tilde{h}_{\mu\nu}$. In fact, such gauge fixing determines the curl and divergence of $\omega_{\mu\nu}$, respectively, as can be seen from Eqs. (33) and (40). The relation between the dual theories is established at the level of the nonlocal equation (43) involving the corresponding potentials $h_{\mu\nu}$, $\tilde{h}_{\mu\nu}$ and $\omega_{\mu\nu}$. We show that this equation translates into the somewhat expected local relation between the corresponding linearized Riemann tensors (45), thus providing further evidence for the auxiliary character of the field $\omega_{\mu\nu}$.

Two examples have been considered which illustrate the construction of the dual theory together with the physical significance of the dual gravitational field $\tilde{h}_{\mu\nu}$. In the case of a gravitational wave, duality just interchanges the polarizations. When considering the field produced by a point mass, the dual configuration is a Dirac-type string. This last example shows that the duality transformation interchanges the role of gravitoelectric and gravitomagnetic fields, defined as proportional to the gradient of the Newtonian potential and the curl of the h^{0i} field, respectively. Such a possibility was conjectured on the basis of the formal similarity between Maxwell equations for the electromagnetism and Einstein equations in the context of the parametrized post-Newtonian expansion for gravitation [12]. This duality relation also has a geometrical motivation because, in the same way as the original Newtonian potential for a point mass is the weak field approximation for the Schwarzschild metric, the dual field $\tilde{h}^{\mu\nu}$ we found is the weak field approximation for the massless Taub-NUT metric [13], which corresponds to spaces where gravitomagnetic charges can be defined [14]. Following the analogy with the original work of Dirac on magnetic monopoles, the possibility of a mass quantization due to the existence of a gravitomagnetic charge has also been considered [12,15].

Finally we have explored an alternative possibility to obtain a dual theory for massless spin-two fields. The idea is to take the zero mass case of the massive $T_{(\mu\nu)\rho}$ theory previously developed, which is dual to massive Fierz-Pauli. Duality in this construction is realized in terms of constraints that enforce a reduced phase space with the correct spin content. It is by no means an obvious matter how these constraints and their classification into first and second class subsets (which determines the count of degrees of freedom) will be modified by the zero mass condition. Hence it is difficult to know in advance which will be the spin content of the resulting theory. We have studied this case in the Appendix, concluding that the resulting massless theory describes spin-zero excitations. This result corrects our previous preliminary calculation of the number of degrees of freedom of the massless theory reported in Refs. [1] and [2], which erroneously stated that this number was two. This phenomenon provides a clear manifestation of the van Dam–Veltman–Zakharov [16] zero mass discontinuity, which leads to massless theories having different spin content with respect to the original massive cases. We can understand our result in terms of irreducible representations of $SO(3)$. Since we basically start from four antisymmetric two forms embedded in $T_{(\mu\nu)\rho}$, we are dealing with the product $(2,0)\times(1,0)$ which decomposes into $(3,0)+(2,1)$. Previous results [9,17] indicate that the representation (2,1) carries zero degrees of freedom, while the representation (3,0) corresponds to the Kalb-Ramond field.

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APPENDIX: THE MASSLESS THEORY FOR $T_{(\mu\nu)\sigma}$

There is still another possibility we can explore when constructing massless theories. In an earlier paper we discussed dual Lagrangians for massive spin two fields [2]. In this approach the fields were not in irreducible representations of the Poincaré group at the Lagrangian level, but the Euler-Lagrange equations lead to constraints that reduced the configuration space to the adequate representation. We can take $m=0$ in these theories with the hope of obtaining an alternative massless spin-two formulation. Moreover, it is not obvious *a priori* how the constraints will be modified by this choice and hence, which will be the spin content of the resulting theory. In this appendix we explore these matters and show how the van Dam–Veltman–Zakharov [16] zero mass discontinuity, which leads to massless theories having different spin content with respect to the original massive cases, is realized.

First we give a brief review of the well-known case of the Kalb-Ramond field, from the perspective of the Dirac method, in order to provide a unified description of the massive and massless cases, which clearly shows the difference in the final counting of the true degrees of freedom. Subsequently we present a more detailed account of the zero mass case corresponding to the massive spin-two theory for $T_{(\alpha\beta)\mu}$ previously developed [2].

The massive Kalb-Ramond theory for $H_{\mu\nu\lambda}$ considered in Ref. [17] can be more conveniently described in terms of the field

$$b_\alpha=\tfrac{1}{6}\epsilon_{\alpha\mu\nu\lambda}H^{\mu\nu\lambda}, \tag{A1}$$

with Lagrangian

$$L=\frac{1}{2}(\partial_\alpha h^\alpha)(\partial_\beta h^\beta)-\frac{m^2}{2}h^\alpha h_\alpha. \tag{A2}$$

The momenta are

$$\Pi_0=\frac{\partial L}{\partial(\partial^0 h^0)}=(\partial_\alpha h^\alpha),\quad \Pi_i=\frac{\partial L}{\partial(\partial^0 h^i)}=0, \tag{A3}$$

leading, respectively, to

$$\dot h_0=(\Pi_0-\partial_i h^i), \tag{A4}$$

together with the primary constraint $\Pi_i=0$. The Hamiltonian $H=\Pi_0\dot h^0-L+\lambda^i\Pi_i$ is

$$H=\frac{1}{2}\Pi_0^2+h^i\partial_i\Pi_0+\frac{m^2}{2}(h^0h_0+h^ih_i)+\lambda^i\Pi_i. \tag{A5}$$

The secondary constraints are

$$\dot\Pi_k=0\Rightarrow 0=\Theta_k=\partial_k\Pi_0+cm^2h_k. \tag{A6}$$

The tertiary constraints

$$0=\dot\Theta_k=m^2(\partial_k h_0-\lambda_k) \tag{A7}$$

finalize the Dirac procedure in both cases. When $m\neq 0$ they determine the Lagrange multiplier λ_k. When $m=0$ the consistency is automatically satisfied. Summarizing, we have the constraints

$$\Pi_i=0,\quad \Theta_k=\partial_k\Pi_0+cm^2h_k=0, \tag{A8}$$

whose classification in terms of first or second class strongly depends upon the theory being massive or massless. In the case $m\neq 0$ the six constraints are second class, yielding $\frac{1}{2}(2\times4-6)=1$ true degrees of freedom, thus reproducing the standard scalar field. However, the situation changes drastically in the case $m=0$. Here, the secondary constraint $\partial_k\Pi_0=0$ reduces to only one, $\Pi_0=cte$, and the remaining four constraints are first class. This leaves $\frac{1}{2}(2\times4-2\times4)=0$ true degrees of freedom. This is in accordance with the results of Refs. [17] and [9].

Next we study the zero mass case of the dual massive spin-two theory previously developed. Taking $m=0$ in Eq. (61) of Ref. [2], the resulting Lagrangian is

$$\mathcal{L}=\tfrac{4}{9}F_{(\alpha\beta\gamma)\nu}F^{(\alpha\beta\gamma)\nu}+\tfrac{2}{3}F_{(\alpha\beta\gamma)\nu}F^{(\alpha\beta\nu)\gamma}-F_{(\alpha\beta\mu)}{}^{\mu}F^{(\alpha\beta\nu)}{}_{\nu}, \tag{A9}$$

with

$$F^{(\alpha\beta\gamma)\nu}=\partial^\alpha T^{(\beta\gamma)\nu}+\partial^\beta T^{(\gamma\alpha)\nu}+\partial^\gamma T^{(\alpha\beta)\nu}. \tag{A10}$$

Lagrangian (A9) dynamically fixes $T^{(\alpha\beta)}{}_\beta=0$, and therefore it is not necessary to impose this constraint through a Lagrange multiplier. The corresponding equations of motion are

$$-4\partial_\alpha F^{(\alpha\nu\rho)\sigma}-2\partial_\alpha F^{(\alpha\nu\sigma)\rho}+2\partial_\alpha F^{(\alpha\rho\sigma)\nu}-2\partial_\alpha F^{(\nu\rho\sigma)\alpha}+3\partial^\sigma F^{(\nu\rho\mu)}{}_\mu=0. \tag{A11}$$

In order to analyze the constraints following the Dirac procedure we start by rewriting Lagrangian (A9) in terms of spatial and temporal components

$$\mathcal{L}=F_{(0ij)0}F^{(0ij)0}-2F^{(0ij)0}F^{k}_{(ijk)}-2F_{(0ij)}{}^{j}F^{(0ik)}{}_{k}+\tfrac{4}{3}F_{(0ij)k}F^{(0ij)k}+\tfrac{4}{3}F_{(0ij)k}F^{(0ik)j}+\tfrac{4}{3}F_{(0ij)k}F^{(ijk)0}+\tfrac{1}{3}F_{(ijk)l}F^{(ijk)l}+\tfrac{4}{9}F_{(ijk)0}F^{(ijk)0}. \quad (A12)$$

The primary constraints arising from the definition of the momenta are

$$\Omega^{i}=\Pi^{(ik)}{}_{k}=0, \quad (A13)$$

$$\Gamma^{i}=\Pi^{(i0)0}=0, \quad (A14)$$

$$\Gamma^{ij}=\Pi^{(0i)j}=0, \quad (A15)$$

$$\Lambda=\epsilon_{ijk}(\Pi^{(ij)k}-4\partial^{i}T^{(jk)0})=0. \quad (A16)$$

The Hamiltonian is

$$H=\tfrac{1}{4}\Pi^{(ij)0}\Pi_{(ij)0}+\tfrac{1}{8}\Pi_{(ij)k}\Pi^{(ij)k}+\Pi^{(ij)0}F_{(ijk)}{}^{k}-\tfrac{2}{3}F^{(ijk)0}F_{(ijk)0}+2T_{(j0)0}\partial_{i}\Pi^{(ij)0}-2T_{(0j)k}\partial_{i}\Pi^{(ij)k} \quad (A17)$$

$$+\lambda_{i}\Pi^{(i0)0}+\lambda_{ij}\Pi^{(0i)j}+\lambda(\Pi-4F^{0})+\mu_{i}\Pi^{(ik)}{}_{k}, \quad (A18)$$

with

$$\Pi=-\tfrac{1}{2}\epsilon_{ijk}\Pi^{(ij)k}, \quad F^{0}=-\tfrac{1}{2}\epsilon_{ijk}\partial^{i}T^{(jk)0}. \quad (A19)$$

We see that $T_{(0j)k}$ and $T_{(0j)0}$ act as Lagrange multipliers, stating that $\partial_i\Pi^{(ij)k}=0$ and $\partial_i\Pi^{(ij)0}=0$. Therefore the degrees of freedom must be in $T_{(ij)\mu}$.

The time evolution of the primary constraints yields an additional set of secondary constraints

$$\Sigma^{i0}=\partial_{j}\Pi^{(ji)0}, \quad (A20)$$

$$\Sigma^{ij}=\partial_{k}\Pi^{(ki)j}, \quad (A21)$$

$$\Sigma=\epsilon_{ijk}(\partial^{i}\Pi^{(jk)0}+\tfrac{4}{3}\partial_{r}F^{(ijk)r}). \quad (A22)$$

There are no tertiary constraints.

Our set of constraints contains the first class subset

$$\Omega^{i}=\Pi^{(ik)}{}_{k}=0, \quad \rightarrow \quad 3, \quad (A23)$$

$$\Gamma^{i}=\Pi^{(i0)0}=0, \quad \rightarrow \quad 3, \quad (A24)$$

$$\Gamma^{ij}=\Pi^{(0i)j}=0, \quad \rightarrow \quad 9, \quad (A25)$$

$$\Sigma^{i0}=\partial_{j}\Pi^{(ji)0}=0, \quad (\partial_i\partial_j\Pi^{(ji)0}=0) \quad \rightarrow \quad 3-1=2, \quad (A26)$$

$$\Sigma^{ij}=\partial_{k}\Pi^{(ki)j}=0, \quad (\partial_k\Pi^{(ki)}{}_{i}=0, \quad \partial_i\partial_k\Pi^{(ki)j}=0) \quad \rightarrow \quad 9-3-1=5. \quad (A27)$$

In parentheses we have indicated the identities that must be subtracted when counting the number of independent constraints, which is shown to the right of each equation. Their total number is 22. The second class subset is

$$\Lambda=-\tfrac{1}{2}\epsilon_{ijk}(\Pi^{(ij)k}-\tfrac{4}{3}F^{(ijk)0})=-\tfrac{1}{2}\epsilon_{ijk}(\Pi^{(ij)k}-4\partial^{k}T^{(ij)0}), \quad (A28)$$

$$\Sigma=\epsilon_{ijk}(\partial^{i}\Pi^{(jk)0}+\tfrac{4}{3}\partial_{r}F^{(ijk)r})=\epsilon_{ijk}\partial^{i}(\Pi^{(jk)0}+4\partial_{r}T^{(jk)r}). \quad (A29)$$

In this way the standard count of the independent degrees of freedom N gives

$$N=\tfrac{1}{2}(2\times24-2\times22-2)=1, \quad (A30)$$

showing that the massless dual theory describes a spin zero excitation.

The above count is most clearly seen in a plane wave configuration with $k^{\mu}=(k,0\,0\,k)$. In this case the constraints become

$$\Pi^{(i0)0}=0, \quad \Pi^{(0i)j}=0, \quad i=1,2,3, \quad (A31)$$

$$\Pi^{(31)0}=\Pi^{(32)0}=0, \quad (A32)$$

$$\Pi^{(12)2}=\Pi^{(21)1}=0, \quad (A33)$$

$$\Pi^{(31)2}=\Pi^{(32)1}=\Pi^{(31)1}=\Pi^{(32)2}=0, \quad (A34)$$

$$\Pi^{(31)3}=\Pi^{(32)3}=0, \quad (A35)$$

with

$$T^{(12)0}=-\frac{1}{4k}\Pi^{(12)3}, \quad \Pi^{(12)0}=4k\,T^{(12)3}. \quad (A36)$$

The canonical pair that remains is $(T^{(12)3},\ \Pi^{(12)3})$, which means that there is only one degree of freedom, corresponding to a spin zero field.

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