Nonuniform symmetry breaking in noncommutative $\lambda\Phi^4$ theory

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Spontaneous symmetry breaking in noncommutative cutoff $\lambda\Phi^4$ theory has been analyzed by using the formalism of the effective action for composite operators in the Hartree-Fock approximation. It turns out that there is no phase transition to a constant vacuum expectation of the field and the broken phase corresponds to a nonuniform background. By considering $\langle\phi(x)\rangle = A\cos(\vec{Q}\cdot\vec{x})$ the generated mass gap depends on the angles among the momenta $\vec{k}$ and $\vec{Q}$ and the noncommutativity parameter $\vec{\theta}$. The order of the transition is not easily determinable in our approximation.

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I. INTRODUCTION

The effects of noncommuting coordinates have recently received renewed attention in relation to string theories [1,2] and there is a great deal of effort being put forward to try and understanding the fundamental properties of noncommutative field theories. In particular, the phase structure of $\lambda\Phi^4$ theory has been recently discussed [3–6] (see also [7] and [8] for numerical studies of the theory in three and two Euclidean dimensions), and Gubser and Sondhi [4] showed that there are indications for a first order phase transition to a nonuniform ground state due to noncommutativity.

In this paper we essentially address the problem of spontaneous symmetry breaking within the formalism of the effective action for composite operators introduced by Cornwall, Jackiw, and Tomboulis [9] (CJT), in the Hartree-Fock (HF) approximation. In this approach we have coupled extremum equations, for the field and the full propagator, which shed new light on the transition from the ordered to the disordered phase.

We work in the cutoff field theory mainly for two reasons. First of all, it is not yet clear whether the noncommutative theory is renormalizable [10–15], and moreover the renormalization of the effective potential in the HF approximation is cumbersome also for the commutative case [16,17]. Nevertheless, the proposed approach gives interesting indications on the phase of the theory. In particular we find in the HF approximation that: (a) the transition from $\langle\phi\rangle = 0$ to $\langle\phi\rangle \neq 0$ turns out first order also for the commutative theory, (b) for the noncommutative theory, the minimization of the effective action has no solution for $\langle\phi\rangle = \text{const} \neq 0$ and the broken phase corresponds to a nonuniform background field, and (c) in the nonuniform stripe phase, with $\langle\phi(x)\rangle = A\cos(Q\cdot x)$, [4,18], the generated mass gap depends on the value of $k_\mu\theta_{\mu\nu}Q_\nu$, where k is the momentum, and

$$[x_\mu, x_\nu] = i\theta_{\mu\nu} \tag{1}$$

are the coordinate commutators.

The paper is organized as follows. In Sec. II we briefly review the CJT formalism and apply it in the HF approximation to the commutative $\lambda\Phi^4$ theory; Sec. III is devoted to the noncommutative case with $\phi = \text{const}$; in Sec. IV we study the stripe phase; and Sec. V contains the conclusions.

II. COMMUTATIVE $\lambda\Phi^4$ THEORY

In this section we shall summarize the effective action for composite operator as introduced by CJT (see [9] for details) and study the spontaneous symmetry breaking in the commutative case. The CJT effective action $\Gamma(\phi,G)$ is given by

$$\Gamma(\phi,G) = I(\phi) + \frac{i}{2}\mathrm{Tr}\ln DG^{-1} + \frac{i}{2}\mathrm{Tr}(\Delta^{-1}(\phi)G) + \Gamma_2(\phi,G) - \frac{i}{2}\mathrm{Tr}(1), \tag{2}$$

where $\phi(x)$ is the expectation value of the field on the ground state, $G(x,y)$ is the full connected propagator of the theory, $I(\Phi)$ is the classical effective action,

$$I(\Phi) = \int d^4x L(x), \tag{3}$$

D is the free propagator,

$$iD^{-1}(x-y) = -(\partial^\mu\partial_\mu + m^2)\delta^4(x-y), \tag{4}$$

and

$$i\Delta^{-1}(x-y) = -(\partial^\mu\partial_\mu + m^2)\delta^4(x-y) + \frac{\delta^2 I_{int}(\phi)}{\delta\phi(x)\delta\phi(y)} \tag{5}$$

with the interaction terms $I_{int}(\phi)$ at least cubic in the fields.

The term $\Gamma_2(\phi,G)$ is computed as follows. In the classical action $I(\Phi)$ shift the field Φ by $\phi(x)$. The new action $I(\Phi+\phi)$ possesses terms cubic and higher in Φ which define an "interaction" part $I_{int}(\phi,\Phi)$ where the vertices de-

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pend on $\phi(x)$. $\Gamma_2(\phi,G)$ is given by all two particle irreducible vacuum graphs in the theory with vertices determined by $I_{int}(\phi,\Phi)$ and propagator set equal to $G(x,y)$. The usual effective action is recovered by extremizing $\Gamma(\phi,G)$ with respect to G.

We evaluate $\Gamma(\phi,G)$ for the commutative $\lambda\Phi^4$ theory with action

$$I(\phi)=\int d^4x\left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-\frac{1}{2}m^2\phi^2-\frac{\lambda}{4!}\phi^4\right) \tag{6}$$

in the Hartree-Fock approximation which corresponds to retain only the lowest order contribution in coupling constant to $\Gamma_2(\phi,G)$ (see [9]).

The coupled equations for the extrema of $\Gamma(\phi,G)$ are

$$\frac{\delta\Gamma(\phi,G)}{\delta\phi}=0,\quad \frac{\delta\Gamma(\phi,G)}{\delta G}=0. \tag{7}$$

It turns out that in this approximation the propagator can be conveniently parametrized as [9,16,17]

$$G(x,y)=i\int\frac{d^4p}{(2\pi)^4}\frac{e^{-ip(x-y)}}{p^2-M^2(p^2)} \tag{8}$$

and the two previous equations become

$$0=(\partial_\mu\partial^\mu+m^2)\phi(x)+\frac{\lambda}{6}\phi^3(x)+\frac{\lambda}{2}\phi(x)G(x,x), \tag{9}$$

$$M^2=m^2+\frac{\lambda}{2}\phi^2+\frac{\lambda}{2}G(x,x). \tag{10}$$

The extrema of the effective action are for ϕ and M constant. The previous equations contain divergences that are regularized by introducing a cutoff Λ. By requiring that physical quantities are exponentially decoupled from the cutoff we redefine the parameter m^2 to cancel terms proportional to Λ^2, i.e.,

$$\mu^2=m^2+\frac{\lambda}{2}\int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2} \tag{11}$$

and the coupled equations (in the Euclidean space) become

$$M^2=\mu^2+\frac{\lambda}{2}\phi^2+\frac{\lambda}{32\pi^2}M^2\ln M^2, \tag{12}$$

$$0=\phi\left(\frac{\lambda}{3}\phi^2-M^2\right), \tag{13}$$

where all the dimensional quantities have been rescaled in units of the cutoff Λ. The extremum equations have two sets of solutions:

$$\phi=0, \tag{14}$$

$$M^2=\mu^2+\frac{\lambda}{32\pi^2}M^2\ln M^2 \tag{15}$$

and

$$\phi^2=\frac{3M^2}{\lambda}, \tag{16}$$

$$\frac{1}{2}M^2+\mu^2=-\frac{\lambda}{32\pi^2}M^2\ln M^2. \tag{17}$$

Now let us consider the second set of solutions which is the relevant one for the spontaneous symmetry breaking and solve it for various values of μ^2, given as follows.

(a) $\mu^2=0$: In this case, besides the solution $M^2=0$, $\phi=0$ obtained form Eqs. (14), (15) [19], one finds two nonvanishing opposite solutions for ϕ:

$$\phi=\pm\sqrt{\frac{3M_o^2}{\lambda}} \tag{18}$$

with

$$M_o^2=e^{-(16\pi^2)/\lambda}. \tag{19}$$

(b) $\mu^2\neq 0$: To solve Eq. (17) one rescales all quantities in unit of M_o^2, the solution for $\mu^2=0$, thus obtaining

$$\hat{\mu}^2=-\frac{\lambda}{32\pi^2}\hat{M}^2\ln\hat{M}^2, \tag{20}$$

where $\hat{\mu}^2=\mu^2/M_o^2$ and $\hat{M}^2=M^2/M_o^2$. It is easy to verify that for $\hat{\mu}^2>\lambda/(e32\pi^2)$ there is no solution of Eq. (20) and the only solution of Eqs. (14)–(17) is $\phi=0$ with a nonvanishing mass M^2 obtained from Eq. (15). In the region $0<\hat{\mu}^2<\lambda/(e32\pi^2)$ there are two solutions of Eq. (20), $\hat{M}_1^2$ and $\hat{M}_2^2$, and then five extrema corresponding to $\phi=0$ and to

$$\hat{\phi}_{1,2}=\pm\sqrt{\frac{3}{\lambda}\hat{M}_{1,2}^2}. \tag{21}$$

Note that for $\hat{\mu}^2=\lambda/(e32\pi^2)$ the two solutions for $\hat{M}^2$ coincide: $\hat{M}_1^2=\hat{M}_2^2$ and correspondingly there are three different extrema in ϕ. For $\hat{\mu}^2<0$ there is only one solution in $\hat{M}^2$ which corresponds to two nonvanishing extrema in ϕ.

Let us finally translate the previous information on the shape of the effective potential as a function of ϕ for different values of μ^2. For $\hat{\mu}^2>\lambda/(e32\pi^2)$ there is only the extremum at $\phi=0$ and the potential corresponds to plot (a) in Fig. 1. For $\hat{\mu}^2=\lambda/(e32\pi^2)$ two new nonvanishing extrema appear and for $0<\hat{\mu}^2<\lambda/(e32\pi^2)$ there are five extrema and the shape is as in plot (b). Note that when lowering $\hat{\mu}^2$, the maxima of the potential for $\phi\neq 0$ decrease and the corresponding values of ϕ become smaller and approach zero and also the minima decrease but the corresponding values of ϕ increase. For $\hat{\mu}^2=0$ the solution corresponding to the maxima have merged into $\phi=0$ and there are three extrema [see plot (d)]. Then for some critical, finite, and positive value of $\hat{\mu}^2$ the potential must be of the form reported in plot

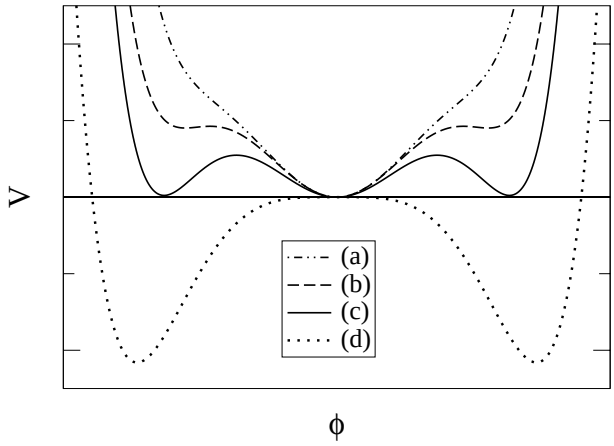

FIG. 1. The effective potential of the scalar commutative theory in the HF approximation for various values of the parameter μ (see text for details).

(c), with three degenerate minima at different values of ϕ. This picture implies a (weak) first order phase transition and suggests that in this case the HF approximation gives a ''coarse grain'' description reliable to establish the occurrence of the transition but probably not its order [20].

III. NONCOMMUTATIVE $\lambda\Phi^4$ THEORY

In this section we shall analyze the extremum equations for the CJT effective action for the noncommutative theory defined by the action

$$I(\phi)=\int d^4x\left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-\frac{1}{2}m^2\phi^2-\frac{\lambda}{4!}\phi^{4*}\right), \quad (22)$$

where the star product is defined by $(i,j=1,\ldots,4)$

$$\begin{aligned}\phi^{4*}(x)&=\phi(x)*\phi(x)*\phi(x)*\phi(x)\\&=\exp\left[\frac{i}{2}\sum_{i<j}\theta_{\mu\nu}\partial^\mu_{x_i}\partial^\nu_{x_j}\right]\\&\quad\times(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4))|_{x_i=x}.\end{aligned} \quad (23)$$

The theory has been discussed in the literature (see, e.g., the review [21]) and the planar approximation $\theta\Lambda^2\to\infty$ has the same behavior of the commutative theory: a phase transition for $\langle\phi\rangle=\phi_0=\text{const}$ and a translational invariant full propagator parametrized as in Eq. (8) with constant M.

Let us now check whether this behavior survives to the genuine noncommutative effects, i.e., for finite $\theta\Lambda^2$. With a translational invariant propagator

$$G(x,y)=\int\frac{d^4p}{(2\pi)^4}e^{-ip(x-y)}G(p), \quad (24)$$

the CJT effective action in momentum space reads

$$\begin{aligned}\Gamma(\phi,G)=&\frac{1}{2}\int\frac{d^4p}{(2\pi)^4}(p^2-m^2)\phi(p)\phi(-p)-\frac{\lambda}{4!}\left[\prod_{a=1}^{4}\int\frac{d^4p_a}{(2\pi)^4}\phi(p_a)\right]\delta^4\left(\sum_a p_a\right)\\&\times\exp\left(\frac{i}{2}p_1\wedge p_2\right)\exp\left(\frac{i}{2}p_3\wedge p_4\right)+\frac{i}{2}\delta^4(0)\int\frac{d^4p}{(2\pi)^4}\ln D(p)G^{-1}(p)+\frac{1}{2}\delta^4(0)\\&\times\int\frac{d^4p}{(2\pi)^4}(p^2-m^2)G(p)-\frac{\lambda}{6}\int\frac{d^4p}{(2\pi)^4}\int\frac{d^4q}{(2\pi)^4}\phi(p)\phi(-p)G(q)\left[1+\frac{1}{2}\exp(iq\wedge p)\right]-\frac{\lambda}{12}\delta^4(0)\\&\times\int\frac{d^4p}{(2\pi)^4}\int\frac{d^4q}{(2\pi)^4}G(p)G(q)\left[1+\frac{1}{2}\exp(iq\wedge p)\right],\end{aligned} \quad (25)$$

where $q\wedge p\equiv q_\mu\theta^{\mu\nu}p_\nu$. In the noncommutative case let us parametrize

$$G(q)=\frac{i}{q^2-M^2(q)}, \quad (26)$$

where M^2 is a function of the four-momentum.

From Eq. (25) we get two coupled extremum equations for $M^2(q)$ and $\phi(q)$:

$$\delta^4(0)\left[M^2(q)-m^2-\frac{\lambda}{3}\int\frac{d^4p}{(2\pi)^4}\frac{i}{[p^2-M^2(p)]}\left(1+\frac{1}{2}\exp(iq\wedge p)\right)\right]=\frac{\lambda}{3}\int\frac{d^4p}{(2\pi)^4}\phi(p)\phi(-p)\left(1+\frac{1}{2}\exp(iq\wedge p)\right), \quad (27)$$

$$\left[q^2-m^2-\frac{\lambda}{3}\int\frac{d^4p}{(2\pi)^4}\frac{i}{[p^2-M^2(p)]}\left(1+\frac{1}{2}\exp(iq\wedge p)\right)\right]\phi(-q)$$
$$=\frac{\lambda}{12}\int\frac{d^4p}{(2\pi)^4}\int\frac{d^4k}{(2\pi)^4}\phi(p)\phi(k-p)\phi(-q-k)\exp\left(\frac{i}{2}p\wedge k\right)\left[\exp\left(\frac{i}{2}k\wedge q\right)+\exp\left(-\frac{i}{2}k\wedge q\right)\right]. \tag{28}$$

With the help of Eq. (27), one can get rid of the constant m^2 in Eq. (28) which becomes

$$[q^2-M^2(q)]\phi(-q)+[\delta^4(0)]^{-1}\phi(-q)\frac{\lambda}{3}$$
$$\times\int\frac{d^4p}{(2\pi)^4}\phi(p)\phi(-p)\left(1+\frac{1}{2}\exp(iq\wedge p)\right)$$
$$=\frac{\lambda}{12}\int\frac{d^4p}{(2\pi)^4}\int\frac{d^4k}{(2\pi)^4}\phi(p)\phi(k-p)\phi(-q-k)$$
$$\times\exp\left(\frac{i}{2}p\wedge k\right)\left[\exp\left(\frac{i}{2}k\wedge q\right)+\exp\left(-\frac{i}{2}k\wedge q\right)\right]. \tag{29}$$

Then, analogously to what has been done for the commutative case in Eq. (11), we cancel the terms proportional to Λ^2 in Eq. (27) by defining

$$\mu^2=m^2+\frac{\lambda}{3}\int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2}, \tag{30}$$

and we can directly check whether a constant background

$$\phi(q)=\phi_o\delta^4(q) \tag{31}$$

is a solution of the extremum equations. Indeed, by substituting Eqs. (30) and (31) in Eqs. (27) and (29) we get

$$M^2(q)=\mu^2+\frac{\lambda}{2}\phi_o^2+\frac{\lambda}{3}\int\frac{d^4p}{(2\pi)^4}\frac{iM^2(p)}{p^2[p^2-M^2(p)]}$$
$$+\frac{\lambda}{6}\int\frac{d^4p}{(2\pi)^4}\frac{i\exp(iq\wedge p)}{[p^2-M^2(p)]} \tag{32}$$

$$0=\phi_o\left(\frac{\lambda}{3}\phi_o^2-M^2(q)\right)\delta^4(q). \tag{33}$$

We note that in Eq. (33) we can replace $M^2(q)$ with $M^2(0)$ because of the delta function $\delta^4(q)$. As usual there is the solution $\phi_o=0$ and $M^2(q)$ given by Eq. (32) (where the term proportional to the field ϕ_0 has been discarded).

This case has been studied in [4] with the interesting result that for $q_\mu\theta^{\mu\nu}\theta_{\nu\rho}q^\rho\to 0$ the function $M^2(q)$ has a singular behavior

$$M^2(q)\to\frac{c}{(q^2/\Lambda^2)(\theta\Lambda^2)}, \tag{34}$$

where c is a constant and $\theta^{\mu\nu}$ is taken of maximal rank and eigenvalues $\pm\theta$. For $\theta\neq 0$ the singularity appears in the infrared limit $q^2\to 0$. This is a genuine effect of the noncommutative structure of the theory and does not change if one considers the same equation for $\phi_o\neq 0$ because it is due to the phase factor in the integral in Eq. (32).

Let us consider Eqs. (32) and (33) in the case of constant finite and nonvanishing background $\phi_o\neq 0$. As noted above, due to the noncommutative terms, M^2 constant is not a solution of Eq. (32): $M^2(q)$ must depend on q and moreover $M^2(q)$ for small q is singular as in Eq. (34). Then the condition $(\lambda/3)\phi_o^2-M^2(0)=0$ from Eq. (33) does not admit a finite constant solution ϕ_o. Therefore a finite constant solution $\phi_o\neq 0$ is ruled out by the analysis of the combined equations.

It is interesting to note that an indication of the impossibility of finding a constant field solution of our extremum equations could have been obtained directly from Eqs. (27) and (28). In fact after substituting in these equations the constant field solution, Eq. (31), the terms proportional to Λ^2 that appear in the two extremum equations cannot be simultaneously cancelled by a single counterterm, namely, m^2 as fixed in Eq. (30), and therefore all solutions of the coupled equations are plagued with integrals that grow as Λ^2.

In our previous analysis the problem of cancelling the terms proportional to Λ^2 has been hidden by the replacement performed to get Eq. (29) which apparently made the case of constant background field free of divergences, although in the end we could not find any suitable solution because of the singular behavior of $M^2(q)$ at $q=0$ shown in Eq. (34). By looking at Eq. (27), it is easy to realize that this q dependent singular behavior is directly related to the incomplete cancellation of the terms proportional to Λ^2 for finite $\theta\Lambda^2$. Indeed this pathology is not present in the planar limit.

In conclusion, we have to reject the constant solution $\phi_o\neq 0$ and look for spontaneous symmetry breaking only in a nonuniform phase.

IV. THE STRIPE PHASE

As pointed out in [18] the phase transition to a nonuniform state is related to a periodic correlation function $\langle\phi(x)\phi(0)\rangle$ which oscillates in sign for large x. For this reason we consider a time independent stripe pattern

$$\langle\phi(\vec{x})\rangle=A\cos(\vec{Q}\cdot\vec{x}) \tag{35}$$

and calculate the CJT effective action in the Hartree-Fock approximation in the static limit [9]. Let us then assume that $\theta^{\mu\nu}$ has no time component $\theta^{0i}=0$ and $\theta^{ij}=\varepsilon^{ijk}\theta_k$.

It is impossible to study the transition to the stripe phase with the most general class of propagators G and we shall limit ourselves to a Raileigh-Ritz variational approach where, however a meaningful ansatz for G requires at least some physical indications on its asymptotic behaviors. Indeed the nonuniform background given in Eq. (35) has a new typical scale $|\vec{x}|\sim 1/|\vec{Q}|$. For small $|\vec{Q}|$ (in cutoff units), the effect of the nonuniform background will be relevant only for large distances and the background will be a slowly varying function of $\vec{x}$.

Then for momenta $|\vec{p}|\gg|\vec{Q}|$, the breaking of the translational and rotational invariance is expected to be negligible and a good ansatz for the tridimensional propagator in momentum space is

$$G(\vec{p})=\frac{1}{2\sqrt{p^2+M_o^2}}, \tag{36}$$

where, analogously to the constant background case, M_o^2 is a constant.

In the region $|\vec{p}|<|\vec{Q}|$, the previous ansatz is of course not reliable, and to obtain further information on the behavior of G let us preliminarily assume that the breaking of the translational invariance appears in the field expectation value only, i.e., in Eq. (35), while we consider a general translational invariant form of the propagator with

$$G(\vec{p})=\frac{1}{2\sqrt{p^2+M^2(\vec{p})}}. \tag{37}$$

Then we analyze the extremum equations obtained by minimizing, with respect to $M^2(\vec{p})$, A and Q^2, the quantity $E(\phi,G)$ defined as

$$-\delta(0)E(\phi,G)=\Gamma(\phi,G), \tag{38}$$

with $\Gamma(\phi,G)$ computed in the static limit [9]. The three coupled equation for $M^2(\vec{p})$, A, and Q^2, respectively, turn out as

$$M^2(\vec{p})=m^2+A^2\frac{\lambda}{6}\left(1+\frac{1}{2}\cos(\vec{p}\wedge\vec{Q})\right)+\frac{\lambda}{3}\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\sqrt{k^2+M^2(\vec{k})}}\left(1+\frac{1}{2}\cos(\vec{p}\wedge\vec{k})\right), \tag{39}$$

$$A\left[Q^2+m^2+A^2\frac{\lambda}{8}+\frac{\lambda}{3}\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\sqrt{k^2+M^2(\vec{k})}}\times\left(1+\frac{1}{2}\cos(\vec{Q}\wedge\vec{k})\right)\right]=0 \tag{40}$$

$$Q^2-\frac{\lambda}{12}\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\sqrt{k^2+M^2(\vec{k})}}(\vec{k}\wedge\vec{Q})\sin(\vec{k}\wedge\vec{Q})=0. \tag{41}$$

The cancellation of the terms proportional to Λ^2 is now obtained by defining

$$\mu^2=m^2+\frac{\lambda}{3}\int\frac{d^3k}{(2\pi)^3}\frac{1}{2|\vec{k}|}. \tag{42}$$

Let us first discuss Eq. (41) and look for a small Q^2 solution. Due to the strong oscillating factor, for small Q the integration region is dominated by large k and then we can replace G with its asymptotic behavior in Eq. (37) or, in other words, $M^2(\vec{k})\sim M_o^2$.

By choosing the configuration $\vec{\theta}=(0,0,\theta)$ and $\vec{Q}=(Q/\sqrt{2},Q/\sqrt{2},0)$, the small Q^2 self-consistent solution turns out as

$$\frac{Q^2}{\Lambda^2}=\left(\frac{\lambda}{24\pi^2}\right)^{1/2}\frac{1}{\theta\Lambda^2}, \tag{43}$$

where we consider from now on large but finite values of $\theta\Lambda^2$.

The next step is to consider the gap equation (39). As previously discussed for large $p^2\gg Q^2$ one expects $M^2(\vec{p})\sim M_o^2$. For $p^2\ll Q^2$ the self-consistent behavior of $M^2(\vec{p})$ is

$$M^2(\vec{p})|_{p\to 0}\sim\alpha+\frac{A^2\lambda}{12}\left[1+\frac{1}{2}\cos(\vec{p}\cdot(\vec{Q}\times\vec{\theta}))\right]+\frac{\lambda}{6\pi^2}\frac{1}{|\vec{p}\times\vec{\theta}|^2}, \tag{44}$$

where α is a constant and $\times$ indicates the usual vector product. The last term corresponds to the small p contribution analogous to the singular behavior in Eq. (34).

Finally one can qualitatively analyze the phase transition by looking at the equation for A, Eq. (40), that can be written in the form

$$A^2 = -\frac{8}{\lambda}\left\{ Q^2+\mu^2+\frac{\lambda}{3}\int \frac{d^3k}{(2\pi)^3}\frac{1}{2\sqrt{k^2+M^2(\vec{k})}} \times\left(1-\frac{\sqrt{k^2+M^2(\vec{k})}}{|\vec{k}|}+\frac{1}{2}\cos(\vec{k}\cdot(\vec{Q}\times\vec{\theta}))\right)\right\}. \tag{45}$$

Indeed, if the contribution of the integrals in Eq. (45) is negligible, a solution $A^2\neq 0$ for $Q^2\neq 0$ is allowed only for sufficiently negative μ^2. Moreover in this approximation the energy gap between the stripe phase and the $A=0$ phase is about $\Delta E\sim -\lambda A^2/64$, showing that the broken phase is energetically favored.

On the basis of this preliminary analysis, to establish the occurrence of the spontaneous symmetry breaking one can apply the following settings of self-consistent approximations for the numerical evaluation of the energy defined in Eq. (38).

(1) Slowly changing background, i.e.,

$$\frac{Q^2}{\Lambda^2}=\left(\frac{\lambda}{24\pi^2}\right)^{1/2}\frac{1}{\theta\Lambda^2}\ll 1 \tag{46}$$

with small λ and large, finite $\theta\Lambda^2$.

(2) $G(x,y)$ translational invariant with $G(\vec{p})$ as in Eq. (37) with $M^2(\vec{p})$ given by

$$M^2(\vec{p})=M_o^2+\frac{A^2\lambda}{12}\left[1+\frac{1}{2}\cos(\vec{p}\cdot(\vec{Q}\times\vec{\theta}))\right] +\frac{\lambda}{6\pi^2}\frac{1}{|\vec{p}\times\vec{\theta}|^2}\quad \text{for } p^2\leqslant Q^2, \tag{47}$$

$$M^2(\vec{p})=M_o^2\quad \text{for } p^2\geqslant Q^2. \tag{48}$$

This choice is motivated by the self-consistent asymptotic solution of Eq. (39) [which is displayed in Eq. (44)] and, above all, by the previous observation that the transition is mainly driven by $Q^2\neq 0$ and depends weakly on the other details of our ansatz for $M^2(\vec{p})$ in the small p region.

By means of Eqs. (25) and (46)–(48), we computed, in cutoff units,

$$\Delta E(A,M_o^2,\theta)=E(A,M_o^2,\theta)-E(0,0,0) \tag{49}$$

for λ and $\theta\Lambda^2$ fixed and studied the occurrence of the phase transition by changing the mass parameter μ^2.

For $\mu^2=0$ there is no spontaneous symmetry breaking. Figure 2 shows ΔE for $\mu^2=0$, $\lambda=10^{-2}$, and $\theta\Lambda^2=100$ for different values of M_0^2 in the range between 10^{-2} and 10^{-6}.

For values of μ^2 below a negative threshold, we observe spontaneous symmetry breaking. In Fig. 3 different plots of ΔE are reported for $\mu^2=-5\times 10^{-4}$ and again M_0^2 in the range between 10^{-2} and 10^{-6}. λ and $\theta\Lambda^2$ are the same as in the previous case and the absolute minimum is for $A\sim 0.55$.

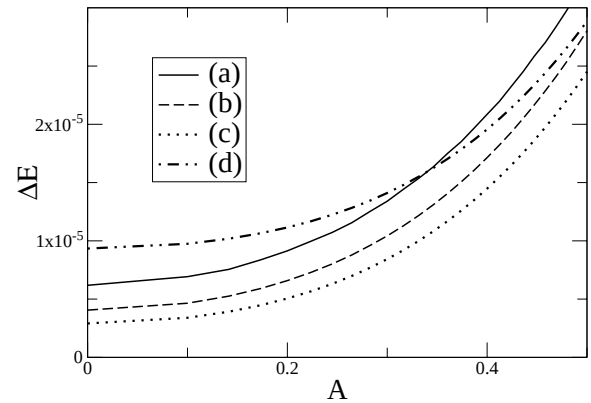

FIG. 2. ΔE vs A with $\lambda=10^{-2}$, $\theta=100$, $\mu^2=0$, and $M_o^2=10^{-6}$ (a), $M_o^2=10^{-3}$ (b), $M_o^2=8\times 10^{-3}$ (c), $M_o^2=4\times 10^{-2}$ (d).

To follow the behavior of ΔE as a function of μ^2, in Fig. 4 we give ΔE for fixed $M_o^2=10^{-2}$ and different values of μ^2. The absolute minimum goes slowly to zero around $\mu^2\sim -2\times 10^{-4}$. The qualitative picture does not strongly depend on the particular values taken for λ and $\theta\Lambda^2$ as long as we stay in the low Q^2 region where the whole approach is self-consistent.

V. CONCLUSIONS

The variational Raileigh-Ritz approximation to the CJT effective action shows that for large $\theta\Lambda^2$, i.e., small Q^2 the transition to a broken stripe phase occurs. The mass generated by the gap equation depends on the mutual direction among $\vec{\theta}$, $\vec{Q}$, and the momentum vector $\vec{p}$. This phenomenon occurs also in noncommutative electrodynamics [22,23] where for the electromagnetic waves the modified dispersion relation

$$\frac{\omega}{c}=|\vec{k}|(1-\vec{b}_T\cdot\vec{\theta}_T) \tag{50}$$

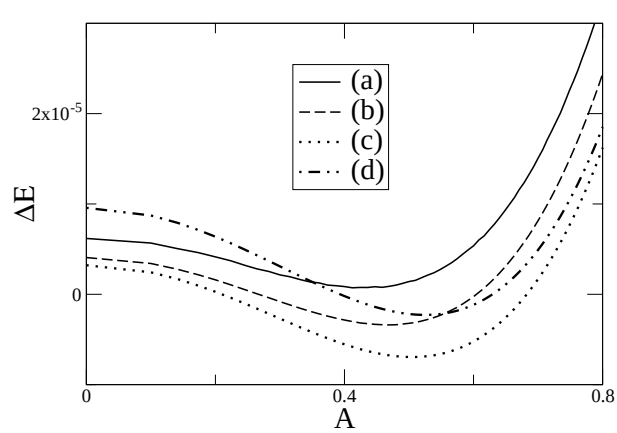

FIG. 3. ΔE vs A with $\lambda=10^{-2}$, $\theta=100$, $\mu^2=-5\times 10^{-4}$, and $M_o^2=10^{-6}$ (a), $M_o^2=10^{-3}$ (b), $M_o^2=1.2\times 10^{-2}$ (c), $M_o^2=4\times 10^{-2}$ (d).

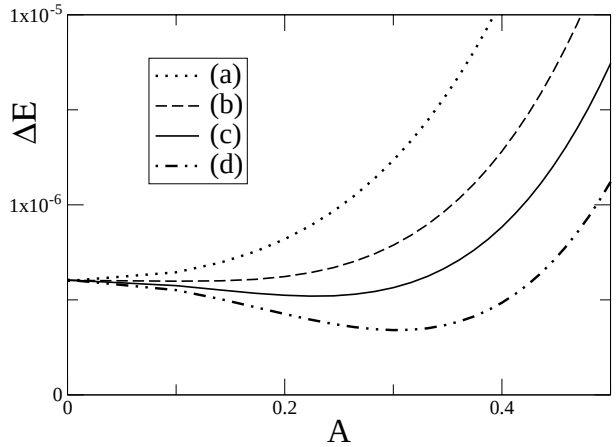

FIG. 4. ΔE vs A with $\lambda=10^{-2}$, $\theta=100$, $M_0^2=10^{-2}$, and $\mu^2=-10^{-4}$ (a), $\mu^2=-2\times10^{-4}$ (b), $\mu^2=-2.5\times10^{-4}$ (c), $\mu^2=-3\times10^{-4}$ (d).

depends on the angles among the wave vector $\vec{k}$ and the transverse components (with respect to $\vec{k}$) of the background magnetic field $\vec{b}$ and of the vector $\vec{\theta}$.

In our approximated numerical analysis we checked that the occurrence of spontaneous symmetry breaking is weakly related to the ansatz made on $M^2(\vec{p})$ in the small p region. The transition essentially depends on $Q^2\neq0$ and this is the *a posteriori* main motivation to believe that also a translational invariant approximation for the propagator can reproduce the qualitative features of the transition also of the noncommutative theory. On the other hand, due to our coarse grained ansatz on the propagator, we are not able to make a precise statement on the order of the phase transition which, in turn, depends on the dynamical details of the theory.

Finally we considered a simplest periodic structure for the background field in Eq. (35) because no qualitative changes are expected for more complicated superpositions as suggested in [4].

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