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Quantum gauge theory on the quantum anti–de Sitter space

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Quantum gauge theory on anti–de Sitter space is studied. The quantum Killing metric and the quadratic quantum Casimir operator are defined. The quantum metrics on the quantum AdS group and the linear transformations leading to them, for both $|q|=1$ and q real, are found explicitly. The quantum Chern-Simons model and its quantum gauge invariance are discussed.

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I. INTRODUCTION

Noncommutative geometry started to appear in this century with progress in the research on operator algebras, the use of algebra in K theory, and the results in index theory. The signal to treat these objects as geometries and look for possible physical relevance came in the 1980s from Connes [1]. In another direction, quantum groups [2] provide natural candidates for noncommutativity. The symmetry is described by noncommutative non-cocommutative * Hopf algebras. Quantum groups were connected with noncommutative differential geometry by Woronowicz [3], who introduced the theory of bicovariant differential calculi. This theory has turned out to be the appropriate language to study gauge theories based on noncommutative spaces. The q-gauge theories were extensively studied in the early 1990s [4] and recently in [5,6]. Actually, there is a map [7] relating the q-deformed gauge fields to the ordinary ones. This map is the analogue of the Seiberg-Witten map [8]. We found this map using the Gerstenhaber product [9] instead of the Groenewold-Moyal star product [10].

In this paper, we study quantum gauge theory on the quantum anti–de Sitter space. We compute the quantum Killing metric, which is an important ingredient in the definition of a quantum invariant Lagrangian. This quantum metric coincides with the Killing metric of the six scalars of the $\mathcal{N}=4$ super Yang-Mills action in the classical limit $q=1$. We discuss the quantum anti–de Sitter space and the quantum orthogonal group $SO_q(6)$. The quantum anti–de Sitter space AdS_5^q is defined as a real form of the complex sphere S_q^5 with a coaction of $SO_q(6)$ which is a real form of Fun$[SO_q(6)]$. We explicitly derive the quantum metrics and the linear transformations leading to them for both $|q|=1$ and q real. Finally, we discuss the quantum invariance of the quantum Chern-Simons action.

Let us first recall that anti–de Sitter space-time is defined as an empty space solution to the Einstein field equations with negative cosmological constant. The metric for five-dimensional anti–de Sitter space (AdS) can be obtained by embedding in a six-dimensional space with two time directions. AdS space-time is homogeneous, has a large isometry group $SO(4,2)$, and leads via a Wigner-Inönü contraction [11] to the Poincaré group. Anti–de Sitter space was first considered by Dirac [12], who discovered the "remarkable representation" which is now known as the singleton. Many other authors [13] have discussed all the representations of the anti–de Sitter group and studied quantum field theory in anti–de Sitter space [14].

The discovery that gauged supergravity theories have ground states corresponding to anti–de Sitter space-time led to a study of the stability of these ground states with respect to fluctuations of the scalar fields [15] as well as a study of supermultiplets in anti–de Sitter space [16]. Currently there is intense activity in the study of the AdS conformal field theory (CFT) correspondence [17]. This conjecture states that type IIB superstring theory with N units of F_5 flux compactified on $\mathrm{AdS}_5\times S^5$ is equivalent to $\mathcal{N}=4$ supersymmetric $SU(N)$ Yang-Mills theory defined on the boundary of AdS_5. Recently, it was proposed [18] that quantum fluctuations in the $\mathrm{AdS}_3\times S^3$ backround have the effect of deforming space-time to a noncommutative manifold. The evidence is based on the quantum group interpretation of the cutoff on single particle chiral primaries. It is thus worthwhile to look for the q-deformed analogue of the AdS/CFT correspondence, find all the representations of the q-deformed superconformal group, and compute the q-deformed correlation functions. We can also generalize the methods used in [19] to the q-deformed case. We postpone the study of q-deformed AdS/CFT correspondence to a future work. This paper is organized as follows. In Sec. II, we recall the general properties for the quantum group $SO_q(6)$ and the bicovariant differential calculus following the general ideas of Woronowicz. We compute the quantum Killing metric using a precise definition of the quantum trace. In Sec. III, we present the quantum anti–de Sitter space and quantum anti–de Sitter group. We construct the quantum anti–de Sitter metric explicitly for both $|q|=1$ and q real. In Sec. IV, we study the quantum gauge invariance of the Chern-Simons term present in the low energy effective action of type IIB superstring theory on AdS_5.

II. THE QUANTUM GROUP $SO_q(6)$ AND BICOVARIANT DIFFERENTIAL CALCULUS

Let $\mathcal{A}$ be the associative unital C algebra generated by $M^n_{\ m}$ $(n,m=1,\ldots,6)$:

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$$M^n{}_m = \begin{pmatrix} M^1{}_1 & M^1{}_2 & M^1{}_3 & M^1{}_4 & M^1{}_5 & M^1{}_6 \\ M^2{}_1 & M^2{}_2 & M^2{}_3 & M^2{}_4 & M^2{}_5 & M^2{}_6 \\ M^3{}_1 & M^3{}_2 & M^3{}_3 & M^3{}_4 & M^3{}_5 & M^3{}_6 \\ M^4{}_1 & M^4{}_2 & M^4{}_3 & M^4{}_4 & M^4{}_5 & M^4{}_6 \\ M^5{}_1 & M^5{}_2 & M^5{}_3 & M^5{}_4 & M^5{}_5 & M^5{}_6 \\ M^6{}_1 & M^6{}_2 & M^6{}_3 & M^6{}_4 & M^6{}_5 & M^6{}_6 \end{pmatrix}. \quad (1)$$

The 6×6 matrix belonging to $SO_q(6)$ preserves the nondegenerate bilinear form C_{nm},

$$C_{nm}M^n{}_k M^m{}_l = C_{kl}, \quad C^{nm}M^k_n M^l_m = C^{kl}, \quad C_{kn}C^{nl} = \delta^l_k, \quad (2)$$

$$C^{nm} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & q^{-2} \\ 0 & 0 & 0 & 0 & q^{-1} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & q & 0 & 0 & 0 & 0 \\ q^2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (3)$$

The noncommutativity of the elements $M^n{}_m$ is controlled by the 36×36 braiding matrix R:

$$R = q\sum_{\substack{i=-3\\ i\neq 0}}^{3} \delta^i_i \otimes \delta^i_i + \sum_{\substack{i,j=-3\\ i\neq j,-j}}^{3} \delta^i_i \otimes \delta^j_j + q^{-1}\sum_{\substack{i=-3\\ i\neq 0}}^{3} \delta^{-i}_{-i}\otimes \delta^i_i$$
$$+ k\left(\sum_{\substack{i,j=-3\\ i>j}}^{3} \delta^i_j \otimes \delta^j_i - \sum_{\substack{i,j=-3\\ i>j}}^{3} q^{\rho_i-\rho_j}\delta^i_j \otimes \delta^{-i}_{-j}\right), \quad (4)$$

where we have used the notation [20]

$$k \equiv q - q^{-1},$$
$$\rho_i = (2,1,0,0,-1,-2). \quad (5)$$

The R matrix satisfies the Yang-Baxter equation

$$R^{ij}{}_{pq}R^{pk}{}_{lr}R^{qr}{}_{mn} = R^{jk}{}_{pq}R^{ip}{}_{rm}R^{rq}{}_{lm} \quad (6)$$

and the relation

$$C_{nm}R^{\pm an}{}_{kc}R^{\pm cm}{}_{lb} = \delta^a_b C_{kl}. \quad (7)$$

The noncommutativity of the elements $M^n{}_m$ is expressed as

$$R^{pq}{}_{nm}M^n{}_k M^m{}_l = M^p{}_n M^q{}_m R^{nm}{}_{kl}. \quad (8)$$

The matrix $\hat{R}$ ($\equiv RP$, where P is a permutation operator, $P{:}A\otimes B = B\otimes A$) enters into local representations of the Birman-Wenzel-Murakami algebra [21]. $\hat{R}$ admits a projector decomposition [22]

$$\hat{R} = qP_S - q^{-1}P_A + q^{-5}P_T, \quad (9)$$

where P_S, P_A, P_T are the projection operators onto the three eigenspaces of $\hat{R}$ with dimensions, respectively, 20, 15, 1: they project the tensor product $x\otimes x$ of the fundamental corepresentation x of $SO_q(6)$ into the corresponding irreducible corepresentations:

$$P_S = \frac{1}{q+q^{-1}}[\hat{R} + q^{-1}I - (q^{-1}+q^{-5})P_T],$$
$$P_A = \frac{1}{q+q^{-1}}[-\hat{R} + qI - (q+q^{-5})P_T],$$
$$P_{T}{}^{ab}_{cd} = (C_{ef}C^{ef})^{-1}C^{ab}C_{cd}. \quad (10)$$

The algebra Fun$[SO_q(6)]$ is a Hopf algebra with comultiplication Δ, counit ϵ, and antipode S which are as follows. The comultiplication (also called the coproduct) is

$$\Delta(M^n{}_m) = M^n{}_k \otimes M^k{}_m. \quad (11)$$

This coproduct Δ on Fun$[SO_q(6)]$ is directly related, for $q=1$ (the nondeformed case), to the pullback induced by left multiplication of the group on itself. The counit ϵ is given by

$$\varepsilon(M^n{}_m) = \delta^n_m, \quad (12)$$

and the antipode S (coinverse) is

$$S(M^n{}_k)M^k{}_m = M^n{}_k S(M^k{}_m) = \delta^n{}_m, \quad (13)$$

$$S(M^n{}_m) = C^{nk}M^l{}_k C_{lm}. \quad (14)$$

Now we consider the bicovariant bimodule Γ over $SO_q(6)$. Let θ^a be a left-invariant basis of ${}_{inv}\Gamma$, the linear subspace of all left-invariant elements of Γ, i.e., $\Delta_L(\theta^a) = I\otimes\theta^a$. In the $q=1$ case the left coaction Δ_L coincides with the pullback for one-forms. There exists an adjoint representation $M_b{}^a$ of the quantum group, defined by the right action on the left-invariant θ^a:

$$\Delta_R(\theta^a) = \theta^b \otimes M_b{}^a, \quad M_b{}^a \in \mathcal{A}, \quad (15)$$

where $\mathcal{A}$ is an associative unital C algebra.

The right coaction Ad$_R$ is given by

$$\mathrm{Ad}_R(M_i{}^j) = M_l{}^k \otimes S(M_i{}^l)M_k{}^j. \quad (16)$$

The exterior derivative d is defined as

$$dM_m{}^n = \frac{1}{\mathcal{N}}[X, M_m{}^n]_- = (\chi^{ab} * M_m{}^n)\theta_{ab}$$
$$= \chi^{ab}(M_m{}^k)M_k{}^n\theta_{ab}, \quad (17)$$

where $X = C_{ab}\theta^{ab} = q^{-2}\theta^{16} + q^{-1}\theta^{25} + \theta^{34} + \theta^{43} + q\theta^{52} + q^2\theta^{61}$ is the singlet representation of θ^{ab} and is both left and right coinvariant, $N\in C$ is the normalization constant, which we take purely imaginary, $N^* = -N$, and χ^{ab} are the quantum analogues of right-invariant vectors given by

$$\chi^{ab}=\frac{1}{\mathcal{N}}(C^{cd}f^{\ ab}_{Adcd}-C^{ab}\epsilon). \tag{18}$$

The χ^{ab} functionals close on the quantum Lie algebra

$$[\chi^{ab},\chi^{cd}](M_n{}^m)=\mathbf{C}^{abcd}_{ef}\chi^{ef}(M_n{}^m), \tag{19}$$

where $\mathbf{C}^{abcd}_{ef}$ are the q-structure constants. To construct a quantum- and gauge-invariant Lagrangian, we need a well defined quantum trace. We require that this trace is invariant under the right adjoint coaction:

$$\mathrm{Tr}(M_i{}^j)=\mathrm{Tr}[\mathrm{Ad}_R(M_i{}^j)]. \tag{20}$$

Clearly, this equation is satisfied if one defines the quantum trace in the right adjoint representation as

$$\mathrm{Tr}[\mathrm{Ad}_R(M_i{}^j)]=-C_{jk}M_i{}^jC^{ik}. \tag{21}$$

The quantum trace allows us to introduce the quantum Killing metric as in the usual nondeformed case ($q=1$):

$$g^{ab,cd}=\mathrm{Tr}[\chi^{ab}(M_i{}^k)\chi^{cd}(M_k{}^j)]. \tag{22}$$

This quantum metric is an important ingredient in the definition of a quantum gauge invariant Lagrangian. We can also define the symmetric, antisymmetric, and trace parts of this Killing metric as

$$g^{Sab,cd}=\mathrm{Tr}[\chi^{Sab}(M_k{}^n)\chi^{Scd}(M_m{}^k)], \tag{23}$$

$$g^{Aab,cd}=\mathrm{Tr}[\chi^{Aab}(M_k{}^n)\chi^{Acd}(M_m{}^k)], \tag{24}$$

$$g^{Tab,cd}=\mathrm{Tr}[\chi^{Tab}(M_k{}^n)\chi^{Tcd}(M_m{}^k)], \tag{25}$$

where

$$\chi^{Sab}=P^{ab}_{Scd}\chi^{cd},$$

$$\chi^{Aab}=P^{ab}_{Acd}\chi^{cd},$$

$$\chi^{Tab}=P^{ab}_{Tcd}\chi^{cd}, \tag{26}$$

and where the projectors P_S,P_A,P_T are defined in Eq. (10). The quadratic quantum Casimir operator is defined as

$$C=g_{ab,cd}\chi^{ab}\chi^{cd}, \tag{27}$$

where $g_{ab,cd}$ is the inverse of the quantum Killing metric $g^{ab,cd}$.

III. QUANTUM ANTI–de SITTER SPACE

Let us recall that the classical anti–de Sitter space AdS_5 is a five-dimensional manifold with constant curvature and signature $(+,-,-,-,-)$. It can be embedded as a hyperboloid into a six-dimensional flat space with signature $(+,+,-,-,-,-)$ by

$$z_0^2+z_5^2-z_1^2-z_2^2-z_3^2-z_4^2=R^2, \tag{28}$$

where R will be called the ‘‘radius’’ of the AdS_5 space.

To define the quantum anti–de Sitter space we follow the method of Ref. [24] used for AdS_4^q. The quantum anti–de Sitter space is defined as a real form of the complex quantum sphere S_q^5.

Let us introduce the quantum Euclidean space $\mathrm{Fun}(E_q^6)$ generated by the elements x^i with commutation relations

$$(P_A)^{ij}_{kl}x_ix_j=0. \tag{29}$$

Now we define the coordinates of the quantum sphere S_q^5 generated by $t^i=x^i/r$ where r is central:

$$(P_A)^{ij}_{kl}t^kt^l=0, \tag{30}$$

$$t\cdot t\equiv C_{kl}t^kt^l=1. \tag{31}$$

For $|q|=1$, we consider the reality structure

$$\overline{t^i}=-(-1)^{E_i}t^jC_{ji} \tag{32}$$

where $E_i=(1,0,0,0,0,-1)$ for $i=1,2,\ldots,6$ are the eigenvalues of energy in the vector representation.

We introduce proper units and define

$$y^i\equiv t^iR, \tag{33}$$

$$y\cdot y=y^iy^jC_{ij}=R^2 \tag{34}$$

for a constant $R\in R_{>0}$. We now introduce new real variables z^i by

$$y^1=\frac{z^0+iz^5}{\sqrt{2}},\quad y^6=\frac{z^0-iz^5}{\sqrt{2}},$$

$$y^2=i\frac{z^1+iz^2}{\sqrt{2}},\quad y^5=i\frac{z^1-iz^2}{\sqrt{2}},$$

$$y^3=i\frac{z^3+iz^4}{\sqrt{2}},\quad y^4=i\frac{z^3-iz^4}{\sqrt{2}}. \tag{35}$$

Plugging these into Eq. (34) gives, for $q=1$, the classical AdS space given by Eq. (28).

Correspondingly, on $\mathrm{Fun}[SO_q(2,4)]$, we can consider the reality structure

$$\overline{M}^i{}_j=(-)^{E_i+E_j}C^{jm}M^l{}_mC_{li}. \tag{36}$$

Let us now follow the method of Ref. [25] used for AdS_4^q. For $|q|=1$ we consider the conjugation [22] defined as $M^\times=M$. The unique associated quantum space conjugation is $(x^a)^\times=x^a$. By this conjugation on the quantum orthogonal we cannot get the desired quantum AdS space. We introduce another operation on the quantum orthogonal group as

$$M^\dagger=DMD^{-1}, \tag{37}$$

where the matrix D is given by

$$D=\begin{pmatrix}1&&&&&\\&1&&&&\\&&-1&&&\\&&&-1&&\\&&&&1&\\&&&&&1\end{pmatrix}. \qquad (38)$$

We can easily prove that the D matrix is a special element of the quantum orthogonal group [23]. The quantum AdS group is obtained by the combined operation $M^*\equiv M^{\times\dagger}=DMD^{-1}$. The induced conjugation on the quantum space is $x^*\equiv x^{\times\dagger}=Dx$. We can check that the conjugation really gives the quantum AdS group and quantum AdS space. We should find a linear transformation $x\to x'=Ux$, $M\to M'=UMU^{-1}$ such that the new coordinates x' and M' are real and the new metric $C'=(U^{-1})^tCU^{-1}$ is diagonal in the $q\to 1$ limit, $C'|_{q=1}=\mathrm{diag}(1,-1,-1,-1,-1,1)$. We find the following U matrix:

$$U=\frac{1}{\sqrt{2}}\begin{pmatrix}1&0&0&0&0&1\\-1&0&0&0&0&1\\0&0&1&-1&0&0\\0&0&i&i&0&0\\0&-1&0&0&1&0\\0&1&0&0&1&0\end{pmatrix} \qquad (39)$$

and the corresponding quantum metric is

$$C'=\begin{pmatrix}\frac{1}{2}q^2+\frac{1}{2q^2}&-\frac{1}{2}q^2+\frac{1}{2q^2}&0&0&0&0\\ \frac{1}{2}q^2-\frac{1}{2q^2}&-\frac{1}{2}q^2-\frac{1}{2q^2}&0&0&0&0\\0&0&-1&0&0&0\\0&0&0&-1&0&0\\0&0&0&0&-\frac{1}{2}q-\frac{1}{2q}&\frac{1}{2}q-\frac{1}{2q}\\0&0&0&0&-\frac{1}{2}q+\frac{1}{2q}&\frac{1}{2}q+\frac{1}{2q}\end{pmatrix}. \qquad (40)$$

For q real we consider the second conjugation given in [22] and realized via the metric, i.e., $M^{\star}=C^tMC^t$. The condition on the braiding R matrix is $\overline{R}=R$. To get the quantum AdS group and the quantum AdS space we have to consider another operation on the quantum orthogonal space as

$$M^{\ddagger}=AMA^{-1} \qquad (41)$$

where the matrix A is given by

$$A=\begin{pmatrix}1&&&&&\\&-1&&&&\\&&-1&&&\\&&&-1&&\\&&&&-1&\\&&&&&1\end{pmatrix}. \qquad (42)$$

We obtain the AdS quantum group by using the conjugation $M^{\star\ddagger}=AC^tMC^tA^{-1}$. The induced conjugation on the quantum space is $x^{\star\ddagger}=C^tAx$. To prove that this combination really gives the quantum AdS group and quantum AdS space we should find a linear transformation $x\to x'=Vx, M\to M'=VMV^{-1}$ such that the new coordinates x' and M' are real and the new metric $C'=(V^{-1})^tCV^{-1}$ is diagonal in the $q\to 1$ limit, $C'|_{q=1}=\mathrm{diag}(1,-1,-1,-1,-1,1)$. We find the following V matrix:

$$V=\frac{1}{\sqrt{2}}\begin{pmatrix}1&0&0&0&0&q^2\\0&-i&0&0&-iq&0\\0&0&1&-1&0&0\\0&0&i&i&0&0\\0&q^{-1}&0&0&-1&0\\iq^{-2}&0&0&0&0&-i\end{pmatrix}, \qquad (43)$$

and the quantum metric C' is given by

$$C' = \begin{pmatrix} \frac{1}{2}+\frac{1}{2q^4} & 0 & 0 & 0 & 0 & -\frac{1}{2}iq^2+\frac{1}{2}\frac{i}{q^2} \\ 0 & -\frac{1}{2}-\frac{1}{2q^2} & 0 & 0 & \frac{1}{2}iq-\frac{1}{2}\frac{i}{q} & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -\frac{1}{2}iq+\frac{1}{2}\frac{i}{q} & 0 & 0 & -\frac{1}{2}-\frac{1}{2}q^2 & 0 \\ \frac{1}{2}iq^2-\frac{1}{2}\frac{i}{q^2} & 0 & 0 & 0 & 0 & \frac{1}{2}q^4+\frac{1}{2} \end{pmatrix}. \tag{44}$$

IV. QUANTUM GAUGE THEORY ON QUANTUM AdS_5^q SPACE

Let us first recall that the infinitesimal quantum gauge transformations are defined as [5]

$$\delta_\alpha A = -d\alpha_{ab}\chi^{ab} + A_{ab}\cdot\alpha_{cd}(\chi^{ab}\otimes\chi^{cd})\mathrm{Ad}_R,$$
$$\delta_\alpha F = F_{ab}\cdot\alpha_{cd}(\chi^{ab}\otimes\chi^{cd})\mathrm{Ad}_R,$$
$$\delta_\alpha B = B_{ab}\cdot\alpha_{cd}(\chi^{ab}\otimes\chi^{cd})\mathrm{Ad}_R. \tag{45}$$

The low energy effective action of type IIB superstring theory on AdS_5 contains the Chern-Simons term for the $SL(2,Z)$ doublet two-form fields B_{NS} and B_{RR} [28]. The quantum analogue of the Chern-Simons term is given by

$$S_{CS} = \frac{-iN}{2\pi}\int_{\mathrm{AdS}_5^q}\mathrm{Tr}(B_{RR}*dB_{NS}), \tag{46}$$

where d is the exterior derivative, $B_{RR}^{*}dB_{NS} = B_{RRab}\cdot dB_{NScd}(\chi^{ab}*\chi^{cd})$, and where $(\cdot)$ denotes the product of an element of $\Omega^{(2)}(\mathrm{AdS}_5^q)$ with an element of $\Omega^{(3)}(\mathrm{AdS}_5^q)$, giving an element of $\Omega^{(5)}(\mathrm{AdS}_5^q)$.

Anti–de Sitter space is nothing but the quantum sphere S_q^5 with a suitable reality structure. As expected, the integral on AdS_5^q can be obtained from the integral over the Euclidean sphere S_q^5 by analytic continuation in q [24]. The integral over S_q^5 is written in terms of the Haar measure on the quantum group $O_q(6,R)$ [26]. We recall that there is a unique invariant integral over the quantum Euclidean sphere [27].

As in the undeformed case the Chern-Simons term is not manifestly quantum gauge invariant. However, it can be written as follows [29]. If AdS_5^q is the boundary of a quantum six-manifold X over which the two B fields extend, and we write B as $H=dB$, then we can write the above action in a manifestly gauge invariant way as

$$S_{CS} = \frac{-iN}{2\pi}\int_X \mathrm{Tr}(H_{RR}*H_{NS}). \tag{47}$$

Indeed, we can easily prove that this action is quantum gauge invariant:

$$\begin{aligned}(H_{RR}*H_{NS})'(M_i{}^j) &= (H'_{RR}*H'_{NS})(M_i{}^j)\\ &= (H_{RR}\otimes T)\mathrm{Ad}_R(M_i{}^k)(H_{NS}\otimes T)\mathrm{Ad}_R(M_k{}^j)\\ &= (H_{RR}*H_{NS}\otimes T)\mathrm{Ad}_R(M_i{}^j),\end{aligned} \tag{48}$$

where T is a finite gauge transformation [5]. The quantum trace is invariant and is given by

$$\begin{aligned}\mathrm{Tr}[H'_{RR}*H'_{NS}(M_i{}^j)] &= \mathrm{Tr}(H_{RR}*H_{NS}\otimes T)\mathrm{Ad}_R(M_i{}^j)\\ &= (H_{RR}*H_{NS}\otimes T)\mathrm{Tr}[\mathrm{Ad}_R(M_i{}^j)]\\ &= (H_{RR}*H_{NS}\otimes T)[\mathrm{Tr}(M_i{}^j)\otimes 1_{\mathcal{A}}]\\ &= \mathrm{Tr}(H_{RR}*H_{NS}\otimes 1_X),\end{aligned} \tag{49}$$

where we have used $T(1_{\mathcal{A}})=1_X$.

The infinitesimal quantum gauge transformation reads

$$\begin{aligned}\mathrm{Tr}(H_{RR}*H_{NS}\wedge\alpha)(M_i{}^j) &= (H_{RRab}\cdot H_{NScd}\cdot\alpha_{ef})(\chi^{ab}\otimes\chi^{cd}\\ &\quad\otimes\chi^{ef})\mathrm{Tr}[\mathrm{Ad}_R(M_i{}^j)]\\ &= (H_{RRab}\cdot H_{NScd}\cdot\alpha_{ef})(\chi^{ab}\otimes\chi^{cd}\\ &\quad\otimes\chi^{ef})\mathrm{Tr}[\mathrm{Ad}_R(M_i{}^j)\otimes 1_{\mathcal{A}}]=0,\end{aligned} \tag{50}$$

where we have used $\chi^{ef}(1_{\mathcal{A}})=0$. In quantizing the quantum Chern-Simons term, one must introduce, in addition to B, certain extra fields: anticommuting ghosts and antighosts c and $\bar{c}$ and a scalar auxiliary field b (sometimes called the Nielsen-Lautrup auxiliary field), all in the adjoint representation of the group. The q-deformed Chern-Simons action is then separately invariant under the q-deformed BRST and anti-BRST transformations [5]. The quantization of the

q-deformed Chern-Simons model can be done following the path integral approach developed in [30] using braided integration [31].

V. CONCLUDING REMARKS

Exploration of the connection between quantum groups and AdS/CFT correspondence is an interesting problem, which will certainly shed light on still open questions in quantum gravity and quantum gauge theory. In this paper we began by explicitly constructing the metrics of the quantum anti–de Sitter space for both $|q|=1$ and q real. We introduced an appropriate quantum trace in the adjoint representation of the quantum group, leading us to define a quantum Killing metric and a quadratic quantum Casimir operator. Finally, we studied the quantum gauge invariance of the quantum Chern-Simons action defined on the quantum anti–de Sitter space.

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