

Domain wall junction in $\mathcal{N}=2$ supersymmetric QED in four dimensions

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An exact solution of the domain wall junction is obtained in $\mathcal{N}=2$ supersymmetric (SUSY) QED with three massive hypermultiplets. The junction preserves two out of eight SUSYs. Both a (magnetic) Fayet-Iliopoulos term and complex masses for hypermultiplets are needed to obtain the junction solution. There are zero modes corresponding to the spontaneously broken translation, SUSY, and $U(1)$. All broken and unbroken SUSY charges are explicitly worked out in the Wess-Zumino gauge in $\mathcal{N}=1$ superfields as well as in components. The relation to models in five dimensions is also clarified.

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I. INTRODUCTION

In recent years, models with extra dimensions have attracted much attention [1,2]. In this brane-world scenario, our world is assumed to be realized on extended topological defects such as domain walls or junctions. On the other hand, supersymmetry (SUSY) provides the most promising idea to build realistic unified theories beyond the standard model [3]. The brane-world scenario in supersymmetric theories can provide an opportunity for realistic model building on walls and/or junctions. Moreover, it can offer a possible explanation of SUSY breaking [4–9], in particular by means of the coexistence of walls [10,11]. SUSY has been useful in obtaining solutions of walls and junctions such as Bogomol’nyi-Prasad-Sommerfield (BPS) states, which preserve a part of the SUSY [12].

Domain walls can conserve half of the SUSY, and are called $\frac{1}{2}$ BPS states. They have been extensively studied in globally supersymmetric theories [13,14]. More recently, an exact BPS wall solution in supergravity theories has been constructed in four dimensions [15] and in five dimensions [16]. We need to consider topological defects such as junctions of walls to consider a fundamental theory in space-time dimensions higher than 5. Domain wall junctions have been studied [17–24] and can preserve a quarter of the original SUSY. An exact analytic solution of the junction has been obtained in $\mathcal{N}=1$ SUSY field theories in four dimensions [20]. The possibility of a junction solution has also been explored in supergravity [23]. The exact solution has been useful in unraveling several unexpected properties of domain wall junctions. The new Nambu-Goldstone fermion modes associated with the junction are found to be non-normalizable [21]. The new central charge associated with the junction was once considered to be a mass of the junction. However, the exact solution showed that the central charge contributes negatively to the energy of the junction [20,21]. Therefore it should more properly be interpreted as a binding energy of the walls which meet at the junction. As another topological defect with codimension 2, an exact solution of vortices on S^2 has also been obtained before [25].

The SUSY theories in dimensions higher than 4 are required to have at least eight supercharges. Theories with eight SUSYs are often called $\mathcal{N}=2$ SUSY theories even in five or six dimensions, since they have twice as many SUSY charges compared to the simple SUSY theories in four dimensions. BPS wall solutions have been constructed in the $\mathcal{N}=2$ SUSY nonlinear sigma models [26–30]. Lump and Q -lump solutions preserving $1/8$ and $1/4$ SUSY, respectively, have also been considered [31,32]. On the other hand, a BPS wall junction has been constructed in linear [20,21] and nonlinear sigma models [24] only in $\mathcal{N}=1$ SUSY models in four dimensions.

The first analytic solution of the BPS junction has been obtained for an $\mathcal{N}=1$ $U(1)\times U(1)$ gauge theory with six charged and one neutral chiral scalar fields with minimal kinetic terms [20], which was constructed as a toy model for the $\mathcal{N}=2$ $SU(2)$ gauge theory with one flavor [33]. Subsequently, it was realized that one can get rid of the vector multiplet by identifying six charged chiral scalar fields pairwise into three chiral scalar fields. One still obtains the same junction solution as a BPS solution [21] in this model with three “charged” and one “neutral” chiral scalar fields with minimal kinetic terms (linear sigma model), without a gauge field at all (Wess-Zumino model). It has also been shown that one can obtain the same solution in an $\mathcal{N}=1$ nonlinear sigma model with only a single “neutral” chiral scalar field, by eliminating the other three “charged” chiral scalar fields appropriately [24]. In all these solutions, one finds that the “neutral” chiral scalar field plays a central role in constructing the junction solution. On the other hand, a neutral scalar field is contained in the $\mathcal{N}=2$ vector multiplet in the case of $\mathcal{N}=2$ SUSY QED. Therefore it is tempting to embed the $\mathcal{N}=1$ gauge theory and its junction solution into the $\mathcal{N}=2$ SUSY QED.

The purpose of this paper is to give an exact analytic solution for the BPS domain wall junction in an $\mathcal{N}=2$ SUSY QED with three massive hypermultiplets. This is the first example of an exact junction solution in $\mathcal{N}=2$ SUSY theories. By explicitly working out eight SUSY transformations, we show that the junction solution preserves two out of eight SUSYs, namely, it is a $\frac{1}{4}$ BPS state. Although the solution has many similarities with the previously obtained $\frac{1}{4}$ BPS junction solution in $\mathcal{N}=1$ SUSY theory, the resulting spectrum of the low-energy effective theory is richer. For in-

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stance, we observe that there are zero modes corresponding to spontaneously broken $U(1)$ global symmetries [29,30]. Similarly to our previous solution in $\mathcal{N}=1$ theory [20], the Nambu-Goldstone modes on the junction background are not normalizable. As pointed out in Ref. [21], it may be possible to obtain a normalizable wave function when it is embedded into supergravity as explored in Ref. [23]. We also show that the same eight SUSY transformations can be derived from a nontrivial dimensional reduction of the $\mathcal{N}=2$ SUSY QED in five dimensions.

The $\mathcal{N}=2$ SUSY theories with vector and hypermultiplets were introduced by Fayet using an automorphism of SUSY algebra [34]. He used both $\mathcal{N}=1$ superfield and component formalisms. The $\mathcal{N}=1$ superfield formalism makes only four SUSYs manifest, but has been useful also in writing down massless nonlinear sigma models [35]. Harmonic superspace formalism can make all eight SUSYs manifest [36–39] and has been used to formulate $\frac{1}{2}$ BPS equations to obtain BPS walls [27,40]. Even in the harmonic superspace formalism, however, it has been useful to use the Wess-Zumino gauge to clarify the physical field content of the theory [27]. The Wess-Zumino gauge in the component formalism allows us to construct all the eight SUSY transformations explicitly. We also find that the action in terms of component fields can be assembled into $\mathcal{N}=1$ superfield formalisms making four out of eight SUSYs manifest in two ways; namely, we can rewrite the same action in terms of two different superfields. One of them makes a set of four SUSYs manifest, and the other makes the set of remaining four SUSYs manifest. Of course we cannot make eight SUSYs manifest in any one of the $\mathcal{N}=1$ superfield formalisms. We shall here employ the $\mathcal{N}=1$ superfield formalism [5,41–44] as well as the component formalism both in the Wess-Zumino gauge.

We find it essential to allow complex mass parameters in order to obtain a junction solution. The $\mathcal{N}=2$ SUSY theories are often derivable by means of a dimensional reduction from five and/or six dimensions [45]. In this spirit, we also show that these $\mathcal{N}=2$ SUSY transformations can be understood in terms of a massive $\mathcal{N}=2$ theory in five dimensions. Since the massive theory in five dimensions can be obtained by a nontrivial dimensional reduction in the manner of Scherk and Schwarz [46] in one spatial direction, the mass parameter should be real. Therefore we find that it is difficult to extend our junction solution in the eight SUSY theory to a junction solution of $\mathcal{N}=2$ SUSY theory in five or six dimensions within the context of our multiflavor QED. If we make a nontrivial dimensional reduction to two spatial directions from six dimensions, we can obtain complex mass parameters. Therefore $\mathcal{N}=2$ SUSY theories in four dimensions can have complex mass parameters which allow the junction solutions.

In Sec. II, our model of $\mathcal{N}=2$ SUSY massive multiflavor QED is introduced; BPS equations are derived as a minimum energy condition and are shown to conserve one out of four SUSYs in the $\mathcal{N}=1$ superfield formalism. In Sec. III, an exact junction solution is obtained as a solution of $\frac{1}{4}$ BPS equations of $\mathcal{N}=1$ superfield formalism. Zero modes are also briefly analyzed. In Sec. IV, the remaining SUSY transformations are found by means of an automorphism of SUSY

algebra. We also show that our model is invariant under the eight SUSY transformations and our BPS junction solution preserves two out of eight SUSYs. Section V is devoted to relating the eight SUSY transformations in four dimensions to $\mathcal{N}=2$ SUSY transformations in five dimensions.

II. $\mathcal{N}=2$ SUSY QED AND BPS EQUATIONS

As one of the simplest models with eight SUSYs, we consider an $\mathcal{N}=2$ SUSY model with local $U(1)$ gauge symmetry in four dimensions with the gauge coupling constant g . If an $\mathcal{N}=1$ SUSY vector superfield V_+ is combined with an $\mathcal{N}=1$ SUSY chiral scalar superfield Φ_+ , an $\mathcal{N}=2$ SUSY vector multiplet is obtained. In order to distinguish the four SUSYs from the remaining four SUSYs which will appear later, we denote the $\mathcal{N}=1$ superfield here by a subscript $+$. Combining $\mathcal{N}=1$ SUSY chiral scalar superfields Q_{+a} with $U(1)$ charge $+1$ and \tilde{Q}_{+a} with $U(1)$ charge -1 gives an $\mathcal{N}=2$ hypermultiplet. The subscript $a=1, \dots, n$ denotes flavor. The $\mathcal{N}=2$ SUSY allows us to introduce the mass m_a of the hypermultiplet for each flavor. Since our gauge symmetry is $U(1)$, the electric $c \in \mathbf{R}$ and magnetic $b \in \mathbf{C}$ Fayet-Iliopoulos (FI) parameters can also be introduced without violating the $\mathcal{N}=2$ SUSY [34]. Assuming a minimal kinetic term for the $\mathcal{N}=2$ vector and hypermultiplets, we thus obtain the $\mathcal{N}=2$ SUSY massive multiflavor QED. Using $\mathcal{N}=1$ superfield formalism, the Lagrangian is given by¹

$$\begin{aligned} \mathcal{L} = & \frac{1}{4g^2} (W_+^\alpha W_\alpha^+ |_{\theta_+^2} + \bar{W}_+^{\dot{\alpha}} \bar{W}_{\dot{\alpha}}^+ |_{\bar{\theta}_+^2}) + \frac{1}{2g^2} \Phi_+^\dagger \Phi_+ \Big|_{\theta_+^2 \bar{\theta}_+^2} \\ & + \sum_{a=1}^n (Q_{+a}^\dagger e^{2V_+} Q_{+a} + \tilde{Q}_{+a}^\dagger e^{-2V_+} \tilde{Q}_{+a}) \Big|_{\theta_+^2 \bar{\theta}_+^2} \\ & - 2cV_+ |_{\theta_+^2 \bar{\theta}_+^2} + \left(\sum_{a=1}^n (\Phi_+ - m_a) Q_{+a} \tilde{Q}_{+a} \Big|_{\theta_+^2} \right. \\ & \left. - b\Phi_+ |_{\theta_+^2} + \text{H.c.} \right), \end{aligned} \quad (2.1)$$

where the $\mathcal{N}=1$ vector multiplet V_+ and the chiral scalar multiplet Φ_+ are multiplied by the gauge coupling g to make the $\mathcal{N}=2$ SUSY more easily visible. The coupling of Φ_+ with the hypermultiplets Q_{+a}, \tilde{Q}_{+a} in the third line of Eq. (2.1) is dictated by the requirement of the $\mathcal{N}=2$ SUSY. If the mass parameters are absent $m_a=0$, the Lagrangian is invariant under the following global $U(n)$ transformations:

$$\begin{aligned} Q_{+a} & \rightarrow Q'_{+a} = Q_{+b} g_{ba}, & \tilde{Q}_{+a} & \rightarrow \tilde{Q}'_{+a} = (g^\dagger)_{ab} \tilde{Q}_{+b}, \\ \Phi_+ & \rightarrow \Phi_+, & V_+ & \rightarrow V_+, & g & \in U(n). \end{aligned} \quad (2.2)$$

The subgroup $U(1)$ of $U(n) = U(1) \times SU(n)$ is gauged. The mass parameters m_a break the remaining global symmetry

¹We use mostly the conventions of Wess and Bagger [47] for the $\mathcal{N}=1$ superfields, spinor, and other notations.

$SU(n)$ to $U(1)^{n-1}$. If $b=0$ in addition to $m_a=0$, the $\mathcal{N}=1$ superfield Lagrangian (2.1) appears to have another global $U(1)$ symmetry:

$$\begin{aligned} Q_{+a} &\rightarrow Q'_{+a} = e^{i\alpha} Q_{+a}, & \tilde{Q}_{+a} &\rightarrow \tilde{Q}'_{+a} = e^{i\beta} \tilde{Q}_{+a}, \\ \Phi_+ &\rightarrow \Phi'_+ = e^{-i\beta - i\alpha} \Phi_+, \end{aligned} \quad (2.3)$$

with V_+ invariant. This invariance respects $\mathcal{N}=1$, but is inconsistent with the $\mathcal{N}=2$ SUSY, since the chiral scalar field Φ_+ should have the same transformation as the vector multiplet V_+ to form an $\mathcal{N}=2$ vector multiplet. Summarizing, our model with generic values of m_a has the following $U(1)^n$ symmetries, which are consistent with the $\mathcal{N}=2$ SUSY:

$$\begin{aligned} Q_{+a} &\rightarrow e^{i\alpha_a} Q_{+a}, & \tilde{Q}_{+a} &\rightarrow e^{-i\alpha_a} \tilde{Q}_{+a}, & \Phi_+ &\rightarrow \Phi_+, \\ V_+ &\rightarrow V_+. \end{aligned} \quad (2.4)$$

The diagonal $U(1)$ ($\alpha_1 = \dots = \alpha_n$) is a local gauged symmetry. Other $U(1)^{n-1}$ groups constrained by $\sum_{a=1}^n \alpha_a = 0$ are global symmetries.

To make the physical content of the theory more transparent, we shall use the Wess-Zumino gauge for the $\mathcal{N}=1$ vector superfield V_+ . Then the $\mathcal{N}=1$ vector superfields can be expanded in terms of the Grassmann number θ_+ into component fields

$$\begin{aligned} V_+(x, \theta_+, \bar{\theta}_+) &= -\theta_+ \sigma^m \bar{\theta}_+ v_m(x) + i \theta_+^2 \bar{\theta}_+ \bar{\lambda}(x) \\ &\quad - i \bar{\theta}_+^2 \theta_+ \lambda(x) + \frac{1}{2} \theta_+^2 \bar{\theta}_+^2 X_3(x), \end{aligned} \quad (2.5)$$

where v_m , λ , and X_3 are the gauge field, gaugino, and auxiliary field, respectively. The $\mathcal{N}=1$ chiral scalar superfields can also be expanded into components using $y^m = x^m + i \theta_+ \sigma^m \bar{\theta}_+$ as usual [47]:

$$\begin{aligned} \Phi_+(y, \theta_+) &= \phi(y) + \sqrt{2} \theta_+ [-i \sqrt{2} \psi(y)] \\ &\quad + \theta_+^2 [X_1(y) + i X_2(y)], \end{aligned} \quad (2.6)$$

$$Q_{+a}(y, \theta_+) = q_a(y) + \sqrt{2} \theta_+ \psi_{q_a}(y) + \theta_+^2 F_a(y), \quad (2.7)$$

$$\tilde{Q}_{+a}(y, \theta_+) = \tilde{q}_a(y) + \sqrt{2} \theta_+ \psi_{\tilde{q}_a}(y) + \theta_+^2 \tilde{F}_a(y), \quad (2.8)$$

where the scalar fields are denoted by a lower-case letter corresponding to the superfields, such as positively charged scalar q_a as the first component of the superfield Q_{+a} . Let us note that the subscript $+$ is not carried by component fields, but is carried only by superfields, which are functions of the associated Grassmann number θ_+ .

In terms of component fields, the bosonic part of this Lagrangian becomes

$$\begin{aligned} \mathcal{L}_{\text{boson}} &= -\frac{1}{4g^2} v_{mn} v^{mn} + \frac{1}{2g^2} (X_3)^2 - \frac{1}{2g^2} |\partial_m \phi|^2 + \frac{1}{2g^2} |X_1 + i X_2|^2 + \sum_{a=1}^n [|F_a|^2 + |\tilde{F}_a|^2 - |\mathcal{D}_m q_a|^2 - |\mathcal{D}_m \tilde{q}_a|^2] \\ &\quad + X_3 (q_a^* q_a - \tilde{q}_a^* \tilde{q}_a) - c X_3 + \sum_{a=1}^n [(\phi - m_a) q_a \tilde{F}_a + (\phi - m_a) F_a \tilde{q}_a + (X_1 + i X_2) q_a \tilde{q}_a] - b (X_1 + i X_2) \\ &\quad + \sum_{a=1}^n [(\phi^* - m_a^*) q_a^* \tilde{F}_a^* + (\phi^* - m_a^*) F_a^* \tilde{q}_a^* + (X_1 - i X_2) q_a^* \tilde{q}_a^*] - b^* (X_1 - i X_2), \end{aligned} \quad (2.9)$$

where the field strength v_{mn} and the covariant derivatives \mathcal{D}_m are defined by

$$\begin{aligned} v_{mn} &= \partial_m v_n - \partial_n v_m, & \mathcal{D}_m q_a &\equiv (\partial_m + i v_m) q_a, \\ \mathcal{D}_m \tilde{q}_a &\equiv (\partial_m - i v_m) \tilde{q}_a, \end{aligned} \quad (2.10)$$

respectively. The entire Lagrangian including the fermions will be given in Sec. IV where the full $\mathcal{N}=2$ SUSY will be clarified. We see that the scalar fields q_a with the $U(1)$ charge $+1$ and \tilde{q}_a with charge -1 have a complex mass m_a . Since a complex mass common to all the flavors can be

absorbed by shifting the neutral complex scalar field ϕ , these mass parameters can always be chosen to satisfy

$$\sum_{a=1}^n m_a = 0. \quad (2.11)$$

The real FI parameter c of the D term is usually called the electric FI parameter, and the complex parameter b appearing in the F term is called the magnetic FI parameter [34].

The SUSY auxiliary fields $X_1, X_2, X_3, F_a, \tilde{F}_a$ can be eliminated by solving their algebraic equations of motion:

$$X_3 = -g^2 \left\{ \sum_{a=1}^n (|q_a|^2 - |\tilde{q}_a|^2) - c \right\}, \quad (2.12)$$

$$X_1 + iX_2 = -2g^2 \left(\sum_{a=1}^n q_a^* \tilde{q}_a^* - b^* \right), \quad (2.13)$$

$$F_a = -(\phi^* - m_a^*) \tilde{q}_a^*, \quad (2.14)$$

$$\tilde{F}_a = -(\phi^* - m_a^*) q_a^*. \quad (2.15)$$

Then the Lagrangian is given entirely in terms of physical fields:

$$\begin{aligned} \mathcal{L}_{\text{boson}} = & -\frac{1}{4g^2} v_{mn} v^{mn} - \frac{1}{2g^2} |\partial\phi|^2 - \sum_{a=1}^n (|\mathcal{D}q_a|^2 + |\mathcal{D}\tilde{q}_a|^2) \\ & - 2g^2 \left| \sum_{a=1}^n q_a \tilde{q}_a - b \right|^2 - \sum_{a=1}^n |\phi - m_a|^2 (|q_a|^2 + |\tilde{q}_a|^2) \\ & - \frac{g^2}{2} \left\{ \sum_{a=1}^n (|q_a|^2 - |\tilde{q}_a|^2) - c \right\}^2. \end{aligned} \quad (2.16)$$

A similar model has been considered previously in a different context [28–30].

SUSY vacua are given by vanishing auxiliary fields $X_1 = X_2 = X_3 = 0$, and $F_a = \tilde{F}_a = 0$. In the generic case of dis-

tinct complex mass parameters $m_a \neq m_b$ for $a \neq b$, we find precisely n isolated SUSY vacua (and no other vacua). We denote the modulus and phase of the magnetic FI parameter b by two real parameters $h > 0$ and β as

$$b = h^2 e^{i\beta}. \quad (2.17)$$

The i th vacuum is characterized by nonvanishing values of q_a , \tilde{q}_a , and ϕ :

$$\begin{aligned} \phi = m_a, \quad q_a = & \sqrt{\frac{\sqrt{c^2 + 4h^4 + c}}{2}} e^{i\alpha_a}, \\ \tilde{q}_a = & \sqrt{\frac{\sqrt{c^2 + 4h^4 - c}}{2}} e^{i(\beta - \alpha_a)}, \end{aligned} \quad (2.18)$$

with vanishing values for the remaining hypermultiplets $q_b^* = \tilde{q}_b^* = 0$ ($a \neq b$). At the a th vacuum, the phase α_a is fixed, breaking a $U(1)$ symmetry, which is a linear combination of local gauged $U(1)$ and other global $U(1)^{n-1}$ generators. Because of the Higgs mechanism, the gauge boson should become massive in the vacuum. However, there still remain $U(1)^{n-1}$ global symmetries α_b , $b \neq a$, unbroken as given in Eq. (2.4).

The Hamiltonian corresponding to the Lagrangian (2.16) is given by

$$\begin{aligned} \mathcal{H} = & \frac{1}{2g^2} (v_{01}^2 + v_{02}^2 + v_{03}^2 + v_{12}^2 + v_{13}^2 + v_{23}^2) + \frac{1}{2g^2} [|\partial_0\phi|^2 + |\partial_1\phi|^2 + |\partial_2\phi|^2 + |\partial_3\phi|^2] + \sum_{a=1}^n [|\mathcal{D}_0q_a|^2 + |\mathcal{D}_1q_a|^2 + |\mathcal{D}_2q_a|^2 \\ & + |\mathcal{D}_3q_a|^2 + |\mathcal{D}_0\tilde{q}_a|^2 + |\mathcal{D}_1\tilde{q}_a|^2 + |\mathcal{D}_2\tilde{q}_a|^2 + |\mathcal{D}_3\tilde{q}_a|^2] + \sum_{a=1}^n |\phi - m_a|^2 (|q_a|^2 + |\tilde{q}_a|^2) + \frac{g^2}{2} \left\{ \sum_{a=1}^n (|q_a|^2 - |\tilde{q}_a|^2) - c \right\}^2 \\ & + 2g^2 \left| \sum_{a=1}^n q_a \tilde{q}_a - b \right|^2. \end{aligned} \quad (2.19)$$

Since the static domain wall junctions have nontrivial dependence only in two-dimensional spatial coordinates, we shall look for field configurations as a function of x^1 and x^2 coordinates and introduce the complex coordinates $z = x^1 + ix^2$, $\bar{z} = x^1 - ix^2$, $\partial_z = \frac{1}{2}(\partial_1 - i\partial_2)$, and $\partial_{\bar{z}} = \frac{1}{2}(\partial_1 + i\partial_2)$. We also wish to maintain (1+1)-dimensional Lorentz invariance in the x^0, x^3 plane. Therefore we need to require

$$v_0 = v_3 = 0, \quad v_{01} = v_{02} = v_{03} = v_{13} = v_{23} = 0, \quad (2.20)$$

$$\partial_0\phi = 0, \quad \mathcal{D}_0q_a = 0, \quad \mathcal{D}_0\tilde{q}_a = 0, \quad (2.21)$$

$$\partial_3\phi = 0, \quad \mathcal{D}_3q_a = 0, \quad \mathcal{D}_3\tilde{q}_a = 0. \quad (2.22)$$

In order to find the minimum energy configuration for a given boundary condition, we form complete squares [19,20] in the energy density functional \mathcal{E} by introducing an arbitrary phase Ω , $|\Omega| = 1$,

$$\begin{aligned}
\mathcal{E} = & \frac{1}{2g^2} \left[v_{12} + g^2 \left\{ \sum_{a=1}^n (|q_a|^2 - |\tilde{q}_a|^2) - c \right\} \right]^2 + \frac{2}{g^2} \left| \partial_z \phi - g^2 \Omega \left(\sum_{a=1}^n q_a^* \tilde{q}_a^* - b^* \right) \right|^2 + 4 \sum_{a=1}^n \left| \mathcal{D}_z q_a - \frac{1}{2} \Omega (\phi^* - m_a^*) \tilde{q}_a^* \right|^2 \\
& + 4 \sum_{a=1}^n \left| \mathcal{D}_z \tilde{q}_a - \frac{1}{2} \Omega (\phi^* - m_a^*) q_a^* \right|^2 + cv_{12} + \partial_z \left[2\Omega^* \left(\sum_{a=1}^n (\phi - m_a) q_a \tilde{q}_a - b \phi \right) \right] + \partial_{\bar{z}} \left[2\Omega \left(\sum_{a=1}^n (\phi^* - m_a^*) q_a^* \tilde{q}_a^* \right. \right. \\
& \left. \left. - b^* \phi^* \right) \right] + \sum_{a=1}^n \{ \partial_z [q_a^* \mathcal{D}_{\bar{z}} q_a - q_a (\mathcal{D}_z q_a)^*] + \partial_z [\tilde{q}_a^* \mathcal{D}_{\bar{z}} \tilde{q}_a - \tilde{q}_a (\mathcal{D}_z \tilde{q}_a)^*] \} + \sum_{a=1}^n \{ \partial_{\bar{z}} [q_a (\mathcal{D}_z q_a)^* - q_a^* \mathcal{D}_{\bar{z}} q_a] \\
& + \partial_{\bar{z}} [\tilde{q}_a (\mathcal{D}_z \tilde{q}_a)^* - \tilde{q}_a^* \mathcal{D}_{\bar{z}} \tilde{q}_a] \} + \frac{1}{2g^2} \partial_z (\phi^* \partial_z \phi - \phi \partial_z \phi^*) + \frac{1}{2g^2} \partial_{\bar{z}} (\phi \partial_z \phi^* - \phi^* \partial_z \phi). \tag{2.23}
\end{aligned}$$

The last four lines are total derivatives that give surface terms when integrated over the entire x^1, x^2 plane. Since all the remaining terms are complete squares, we find that the integrated energy over the x^1, x^2 plane of the field configuration is always larger than the surface terms, which are completely determined by the boundary condition at spatial infinity. This bound is called the BPS bound and is saturated by requiring the complete squares to vanish:

$$v_{12} = -g^2 \left\{ \sum_{a=1}^n (|q_a|^2 - |\tilde{q}_a|^2) - c \right\}, \tag{2.24}$$

$$\frac{1}{g^2} \frac{\partial \phi}{\partial z} = \Omega \left(\sum_{a=1}^n q_a^* \tilde{q}_a^* - b^* \right), \tag{2.25}$$

$$2\mathcal{D}_z q_a = \Omega (\phi^* - m_a^*) \tilde{q}_a^*, \tag{2.26}$$

$$2\mathcal{D}_z \tilde{q}_a = \Omega (\phi^* - m_a^*) q_a^*. \tag{2.27}$$

These first order differential equations are called the BPS equations. Since the surface terms depend on Ω , the phase factor Ω can be chosen to obtain the best bound. Let us also note that the minimum energy configurations automatically satisfy the equations of motion [19,20].

Since the Lagrangian (2.1) with $\mathcal{N}=1$ superfield exhibits $\mathcal{N}=1$ SUSY manifestly, we can formulate the condition of partial conservation of SUSY. We will see that the above minimum energy conditions (2.24)–(2.27) are precisely the conditions to conserve one out of four SUSYs. We need to consider only the SUSY transformations of fermions, since only bosonic fields can have nonvanishing values. The $\mathcal{N}=1$ SUSY transformation of the gaugino is given by [47]

$$\delta_{\xi_+} \lambda = \sigma^{mn} v_{mn} \xi_+ + iX_3 \xi_+ = \begin{bmatrix} v_{03} - iv_{12} + iX_3 & v_{01} + v_{13} - iv_{23} - iv_{02} \\ v_{01} - v_{13} - iv_{23} + iv_{02} & -v_{03} + iv_{12} + iX_3 \end{bmatrix} \begin{bmatrix} \xi_{+1} \\ \xi_{+2} \end{bmatrix}. \tag{2.28}$$

If we require that a part of the SUSY corresponding to the upper component ξ_{+1} is conserved ($\xi_{+2}=0$), we find [20]

$$v_{12} = X_3, \quad v_{03} = 0, \quad v_{01} = v_{13}, \quad v_{23} = v_{02}. \tag{2.29}$$

Using the algebraic equation of motion for the auxiliary field (2.12), the minimum energy condition (2.24) for the vector multiplet is precisely the same as the condition of partial SUSY conservation condition (2.29). Similarly, the $\mathcal{N}=1$ SUSY transformations of fermions in chiral scalar multiplets are given by [47]

$$\delta_{\xi_+} (-i\sqrt{2}\psi) = i\sqrt{2}\sigma^m \partial_m (\phi - m_a) \bar{\xi}_+ + \sqrt{2}(X_1 + iX_2) \xi_+, \tag{2.30}$$

$$\delta_{\xi_+} \psi_{q_a} = i\sqrt{2}\sigma^m \mathcal{D}_m q_a \bar{\xi}_+ + \sqrt{2}F_a \xi_+, \tag{2.31}$$

$$\delta_{\xi_+} \bar{\psi}_{\tilde{q}_a} = i\sqrt{2}\bar{\sigma}^m \mathcal{D}_m \tilde{q}_a^* \xi_+ + \sqrt{2}\bar{F}_a \bar{\xi}_+. \tag{2.32}$$

For these transformations, we express the derivatives in terms of complex coordinates, assuming x_1, x_2 dependence only:

$$\begin{aligned}
\sigma^m \partial_m &= (\sigma^1 + i\sigma^2) \frac{1}{2} (\partial_1 - i\partial_2) + (\sigma^1 - i\sigma^2) \frac{1}{2} (\partial_1 + i\partial_2) \\
&= 2\sigma^+ \partial_z + 2\sigma^- \partial_{\bar{z}}, \tag{2.33}
\end{aligned}$$

where $\sigma^+ \equiv (\sigma^1 + i\sigma^2)/2$, $\sigma^- \equiv (\sigma^1 - i\sigma^2)/2$. If we require conservation of only one out of four SUSYs specified by

$$-\Omega \sigma^+ \bar{\xi}_+ = i\xi_+ \quad \text{and} \quad \sigma^- \bar{\xi}_+ = 0, \tag{2.34}$$

we obtain [20]

$$\begin{aligned}
\delta_{\xi_+}(-i\sqrt{2}\psi) &= i\sqrt{2}\sigma^m\partial_m(\phi-m_a)\bar{\xi}_+ + \sqrt{2}(X_1+iX_2)\xi_+ \\
&= i2\sqrt{2}(\sigma^+\partial_z + \sigma^-\partial_{\bar{z}})(\phi-m_a)\bar{\xi}_+ \\
&\quad + \sqrt{2}(X_1+iX_2)\xi_+ \\
&= \sqrt{2}\xi_+\Omega^{-1}[2\partial_z(\phi-m_a) + \Omega(X_1+iX_2)],
\end{aligned} \tag{2.35}$$

$$\begin{aligned}
\delta_{\xi_+}\psi_{q_a} &= i\sqrt{2}\sigma^m\mathcal{D}_mq_a\bar{\xi}_+ + \sqrt{2}F_a\xi_+ \\
&= \sqrt{2}\xi_+\Omega^{-1}(2\mathcal{D}_zq_a + \Omega F_a),
\end{aligned} \tag{2.36}$$

$$\begin{aligned}
\delta_{\xi_+}\bar{\psi}_{\tilde{q}_a} &= i\sqrt{2}\bar{\sigma}^m\mathcal{D}_m\tilde{q}_a^*\xi_+ + \sqrt{2}\tilde{F}_a^*\bar{\xi}_+ \\
&= \sqrt{2}\xi_+\Omega^{-1}(2\mathcal{D}_{\bar{z}}\tilde{q}_a + \Omega\tilde{F}_a).
\end{aligned} \tag{2.37}$$

Therefore we find that the condition of conservation of one out of four SUSYs is given by

$$\begin{aligned}
\frac{1}{g^2}\frac{\partial\phi}{\partial z} &= -\Omega(X_1+iX_2), \quad 2\mathcal{D}_zq_a = -\Omega F_a, \\
2\mathcal{D}_{\bar{z}}\tilde{q}_a &= -\Omega\tilde{F}_a.
\end{aligned} \tag{2.38}$$

The algebraic equations of motion for auxiliary fields are given in terms of the superpotential P :

$$X_1+iX_2 = -\left(\frac{\partial P}{\partial\phi}\right)^*, \quad F_a = -\left(\frac{\partial P}{\partial q_a}\right)^*, \quad \tilde{F}_a = -\left(\frac{\partial P}{\partial\tilde{q}_a}\right)^*. \tag{2.39}$$

This superpotential P as a function of scalar fields is given in our case by

$$P = \sum_{a=1}^n (\phi-m_a)q_a\tilde{q}_a - b\phi. \tag{2.40}$$

Using the superpotential (2.40) and the algebraic equations of motion for auxiliary fields (2.39), we see that the minimum energy conditions (2.25)–(2.27) are precisely the same as the conditions for the conservation of one out of four SUSYs (2.38).

III. DOMAIN WALL JUNCTION

In order to obtain an exact solution for the junctions, we shall embed the known solution into a solution of the BPS equations (2.24)–(2.27) for our $\mathcal{N}=2$ SUSY massive multi-flavor QED. We are making one complex structure manifest out of three by using the $\mathcal{N}=1$ superfield formalism. Although the three FI parameters c , $b_1=\text{Re } b$, and $b_2=\text{Im } b$ form an $SU(2)_R$ triplet, the choice of a particular complex structure made the $SU(2)_R$ symmetry not visible. In this circumstance, we find it convenient to choose the FI parameters c and b in Eq. (2.17) as

$$c=0, \quad b=h^2 \in \mathbf{R} \quad (\beta=0). \tag{3.1}$$

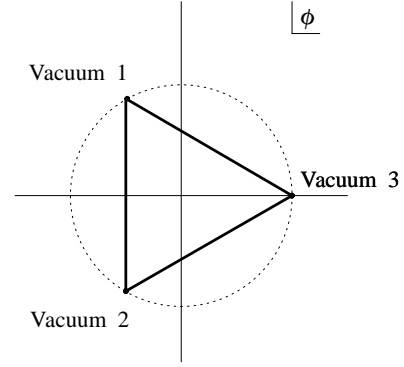


FIG. 1. Vacua in complex ϕ plane.

Then the a th vacuum values of fields in Eq. (2.18) become

$$\phi = m_a, \quad q_b = h e^{i\alpha_a} \delta_{ba}, \quad \tilde{q}_b = h e^{-i\alpha_a} \delta_{ba}. \tag{3.2}$$

Since the known junction solution was obtained [20] with the Z_3 symmetry for SUSY vacua and with a relation between the vacuum values of the charged chiral scalar fields q_a, \tilde{q}_a and the neutral scalar fields ϕ , we should require $n=3$ flavors with Z_3 symmetry, and a relation between the mass scales of m_b and the Fayet-Iliopoulos term h . Altogether we assume the following particular values for the parameters of our model:

$$m_b = \frac{2gh}{\sqrt{3}} e^{i(2\pi/3)b}, \quad b=1,2,3. \tag{3.3}$$

The resulting vacua are illustrated in the complex ϕ plane in Fig. 1.

Combined with the algebraic equation of motion for the auxiliary field X_3 Eq. (2.12) for a vector superfield, we can satisfy the BPS equations (2.29) trivially by choosing

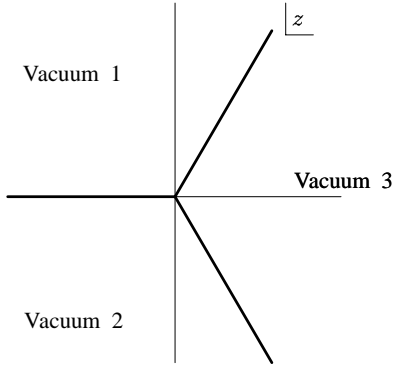
$$v_0(x^1, x^2) = v_3(x^1, x^2) = 0, \quad |q_b(x^1, x^2)| = |\tilde{q}_b(x^1, x^2)|. \tag{3.4}$$

Suggested by this condition, we assume the following relation between values of the hypermultiplets [20]:

$$q_b(x^1, x^2) e^{-i\alpha_b} = \tilde{q}_b(x^1, x^2) e^{i\alpha_b} \in \mathbf{R}, \tag{3.5}$$

in accordance with the vacuum values (3.2) which should be reached at infinity. This assumption will be justified *a posteriori* after finding solutions. We are interested in a BPS junction configuration separating three vacuum domains $a=1,2,3$ with Z_3 symmetry, where the third vacuum is placed at infinity along the positive real axis as illustrated in Fig. 2. This configuration corresponds to the choice of the phase factor $\Omega = -1$ [20,21]. Now the remaining BPS equations for chiral scalar multiplets (hypermultiplets and the chiral scalar ϕ in the $\mathcal{N}=2$ vector multiplet) read

$$2\frac{\partial q_b}{\partial z} = \left(\frac{2gh}{\sqrt{3}} e^{i(2\pi/3)b} - \phi \right)^* q_b, \tag{3.6}$$


 FIG. 2. The Z_3 junction in real space $z=x^1+ix^2$.

$$\frac{1}{g^2} \frac{\partial \phi}{\partial z} = h^2 - \sum_{b=1}^3 |q_b|^2. \quad (3.7)$$

We define a dimensionless complex coordinate \hat{z} and real dimensionless fields \hat{q}_b and $\hat{\phi}$ by rescaling with the normalization factor associated with the vacuum values as

$$z = \frac{\sqrt{3}}{2} \frac{1}{gh} \hat{z}, \quad q_b = h e^{i\alpha_b} \hat{q}_b, \quad \phi = \frac{2}{\sqrt{3}} gh \hat{\phi}. \quad (3.8)$$

Then the BPS equations become

$$2 \frac{\partial \hat{q}_b}{\partial \hat{z}} = (e^{i(2\pi/3)b} - \hat{\phi})^* \hat{q}_b, \quad \hat{q}_b \in \mathbf{R}, \quad (3.9)$$

$$2 \frac{\partial \hat{\phi}}{\partial \hat{z}} = \frac{3}{2} \left(1 - \sum_b \hat{q}_b^2 \right). \quad (3.10)$$

We can now recognize the familiar form of the BPS equation allowing the junction as an exact solution [20,24]. Let us define the following auxiliary quantities f_b :

$$f_b = \exp\left(\frac{1}{2} (e^{-i(2\pi/3)b} \hat{z}^* + e^{i(2\pi/3)b} \hat{z})\right), \quad b=1,2,3. \quad (3.11)$$

The following identities can be derived for these auxiliary quantities [20,24]:

$$\begin{aligned} 2 \frac{\partial}{\partial \hat{z}} \left(\frac{f_b}{f_1+f_2+f_3} \right) &= \left(e^{-i(2\pi/3)b} - \frac{\sum_c e^{-i(2\pi/3)c} f_c}{f_1+f_2+f_3} \right) \\ &\times \frac{f_b}{f_1+f_2+f_3}, \end{aligned} \quad (3.12)$$

$$2 \frac{\partial}{\partial \hat{z}} \left(\frac{\sum_b e^{i(2\pi/3)b} f_b}{f_1+f_2+f_3} \right) = \frac{3}{2} \left(1 - \frac{\sum_b f_b^2}{(f_1+f_2+f_3)^2} \right). \quad (3.13)$$

Therefore we find the solutions for the BPS equations

$$\hat{q}_b = \frac{f_b}{f_1+f_2+f_3}, \quad \hat{\phi} = \frac{\sum_b e^{i(2\pi/3)b} f_b}{f_1+f_2+f_3}. \quad (3.14)$$

These solutions can be rewritten in terms of our original variables q_b , ϕ , and z ,

$$q_b = h e^{i\alpha_b} \frac{f_b}{f_1+f_2+f_3}, \quad (3.15)$$

$$\tilde{q}_b = h e^{-i\alpha_b} \frac{f_b}{f_1+f_2+f_3}, \quad (3.16)$$

$$\phi = \frac{2gh}{\sqrt{3}} \frac{\sum_b e^{i(2\pi/3)b} f_b}{f_1+f_2+f_3}, \quad (3.17)$$

where

$$f_b = \exp\left(\frac{2gh}{\sqrt{3}} \frac{1}{2} (e^{-i(2\pi/3)b} z + e^{i(2\pi/3)b} z^*)\right), \quad b=1,2,3. \quad (3.18)$$

By rotating the field configuration by $2\pi/3$, we find that the solution (3.18) is precisely the same field configuration as in the previous junction solution in the $\mathcal{N}=1$ SUSY theory [20] provided the dimensionful parameter h is related to the parameter Λ through $h = \sqrt{2}\Lambda$. The field ϕ of the junction configuration takes values inside the triangle connecting the three vacua as illustrated in Fig. 1. The straight line segments between vacua on the ϕ plane correspond to spatial infinity in real space, $z \rightarrow \infty$. As an illustration of the asymptotic behavior of the junction configuration, the values of the fields q_a , $a=1,2,3$, choosing $\alpha_a=0$ are shown along the real axis in Fig. 3. The energy density computed analytically in our previous solution of the $\mathcal{N}=1$ SUSY model [20] can easily be converted into our case of $\mathcal{N}=2$ SUSY QED. The energy density of the junction solution in our $\mathcal{N}=2$ SUSY model is shown in Fig. 4.

Let us discuss zero modes in this junction solution. First we notice that we have two massless bosons corresponding to spontaneously broken translation in two directions x^1, x^2 . As for the two global $U(1)$ symmetries, both of them are broken in the junction solution, and there are two corresponding Nambu-Goldstone bosons. For each domain wall, only two hypermultiplets have nontrivial field configurations. Therefore only one of the two $U(1)$ global symmetries is spontaneously broken and only one Nambu-Goldstone boson associated with the $U(1)$ phase rotation appears [29,30]. Since the broken $U(1)$ symmetry on each wall is a different combination of the two $U(1)$ global symmetries, the associated Nambu-Goldstone boson on each wall is also a different linear combination of these two. This situation is very similar to the property of the Nambu-Goldstone fermions on the junction that was observed previously [20,21]. As for fermions, six out of eight SUSYs are spontaneously broken. Therefore we have six Nambu-Goldstone fermions. On each

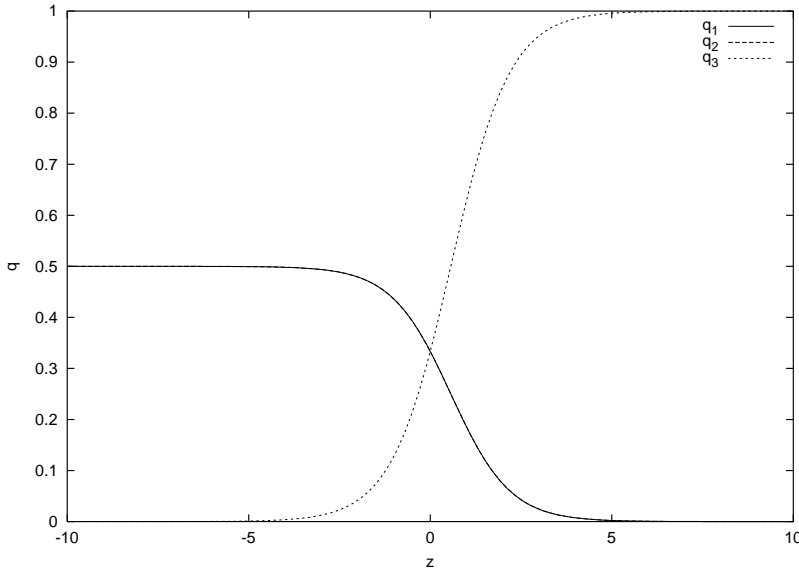


FIG. 3. Along $z \rightarrow +\infty$ one obtains the vacuum 3, where only q_3 takes nonzero values. Along $z \rightarrow -\infty$ one obtains the middle point of the wall between the first and second vacua, where $|q_1| = |q_2|$.

wall, only four SUSYs are broken. Therefore there are four Nambu-Goldstone fermions on each wall. These four Nambu-Goldstone fermions on each wall are different linear combinations of the six Nambu-Goldstone fermions in the junction as a whole, since different linear combinations of the SUSY charges are broken on each wall. This situation is analogous to the previously obtained junction solution in $\mathcal{N} = 1$ theory [20,21]. It has been pointed out that the wall provides a model to “localize” gauge bosons [29]. In this context, one should note that one out of two Nambu-Goldstone bosons associated with the two $U(1)$ ’s mixes with the $U(1)$ gauge boson. We postpone a more detailed analysis of walls and junctions involving the gauge field for subsequent publications.

Let us discuss possible junction configurations in other models. We have obtained an analytic solution for the junction provided the mass parameters m_i are tuned to the gauge coupling g and the Fayet-Iliopoulos parameter $b = h^2$ as in Eq. (3.3). However, it is almost clear from continuity that junction configurations should exist even if the mass parameters are perturbed infinitesimally away from the tuned values. On the other hand, a junction configuration is not al-

lowed if the masses are on a straight line in the complex mass plane, for instance, if all masses are aligned on the real axis. Since the tension is given by the absolute value of the difference of the superpotentials, the possible junction configuration cannot satisfy the condition of mechanical stability [19] if the masses are on a straight line. We conjecture that there should exist a “line of marginal stability” somewhere between the Z_3 symmetric mass parameters and the mass parameters on a straight line. The junction should become unstable across the line of marginal stability. An interesting example of the line of marginal stability for domain walls has been explicitly demonstrated for a similar Wess-Zumino model [49]. Lines of marginal stability for monopoles and dyons are well studied in $\mathcal{N} = 2$ gauge theories [50].

It is also likely that there exist junctions of n -walls, although it is difficult to work out explicit solutions except in particular nonlinear sigma models [24].

Similarly to our previous solution [21], the Nambu-Goldstone bosons and fermions on our junction solution are not normalizable. Therefore the usual wisdom of a low-energy effective Lagrangian approach cannot be applied easily to our junction solution. However, it has been observed that a graviton zero mode is localized at the junction, if the bulk space-time is warped [51]. This is due to a suppression factor produced by the bulk AdS(-like) space-time. It may also be possible to exploit this mechanism to obtain normalizable zero modes localized at the junction. The general properties of possible junction solutions have been studied in the presence of gravity [23], although no explicit solution has been obtained. This is an interesting subject for the future.

IV. EIGHT SUSY TRANSFORMATIONS

In this and the following sections, we return to the general case of n flavors. There have been a number of studies to formulate $\mathcal{N} = 2$ SUSY field theories in five dimensions in terms of the $\mathcal{N} = 1$ superfield formalism [5,41–44]. Inspired by these studies [5,41,42], we will redefine the auxiliary

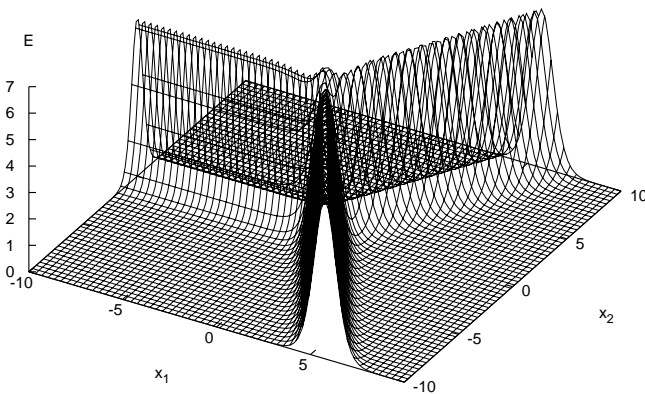


FIG. 4. Energy density of the Z_3 junction configuration of $\mathcal{N} = 2$ SUSY QED.

fields of chiral scalar fields for a hypermultiplet (2.7) and (2.8) in order to identify all eight supersymmetry transformations

$$F_a = F'_a - (\phi^* - m_a^*) \tilde{q}_a^*, \quad \bar{F}_a = -\bar{F}'_a - q_a^* (\phi^* - m_a^*). \quad (4.1)$$

Instead of Eqs. (2.5)–(2.8), the component expansions of the superfields in powers of the Grassmann number θ_+ now read

$$V_+ = -\theta_+ \sigma^m \bar{\theta}_+ v_m + i \theta_+^2 \bar{\theta}_+ \bar{\lambda} - i \bar{\theta}_+^2 \theta_+ \lambda + \frac{1}{2} \theta_+^2 \bar{\theta}_+^2 X_3, \quad (4.2)$$

$$\Phi_+ = \phi + \sqrt{2} \theta_+ (-i \sqrt{2} \psi) + \theta_+^2 (X_1 + i X_2), \quad (4.3)$$

$$Q_{+a} = q_a + \sqrt{2} \theta_+ \psi_{q_a} + \theta_+^2 [F'_a - (\phi^* - m_a^*) \tilde{q}_a^*], \quad (4.4)$$

$$\tilde{Q}_{+a} = \tilde{q}_a + \sqrt{2} \theta_+ \psi_{\tilde{q}_a} + \theta_+^2 [-\bar{F}'_a - q_a^* (\phi^* - m_a^*)]. \quad (4.5)$$

In terms of these component fields, the full Lagrangian (2.1) is given by

$$\begin{aligned} \mathcal{L} = & -\frac{1}{4g^2} v_{mn} v^{mn} - \frac{1}{g^2} i \lambda \sigma^m \partial_m \bar{\lambda} + \frac{1}{2g^2} (X_3)^2 + \frac{1}{2g^2} |X_1 + i X_2|^2 - \frac{1}{2g^2} |\partial_m \phi|^2 - \frac{1}{g^2} i \bar{\psi} \bar{\sigma}^m \partial_m \psi - c X_3 - b (X_1 + i X_2) \\ & - b^* (X_1 - i X_2) + \sum_{a=1}^n \{ [F'_a - (\phi^* - m_a^*) \tilde{q}_a^*] [F'_a - (\phi^* - m_a^*) \tilde{q}_a^*] - |\mathcal{D}_m q_a|^2 - i \bar{\psi}_{q_a} \bar{\sigma}^m \mathcal{D}_m \psi_{q_a} - i \sqrt{2} (\bar{\psi}_{q_a} \bar{\lambda} q_a - \psi_{q_a} \lambda q_a^*) \\ & + X_3 |q_a|^2 \} + \sum_{a=1}^n \{ [-\bar{F}'_a - q_a^* (\phi^* - m_a^*)] [-\bar{F}'_a - q_a^* (\phi^* - m_a^*)] - |\mathcal{D}_m \tilde{q}_a|^2 - i \psi_{\tilde{q}_a} \sigma^m \mathcal{D}_m \bar{\psi}_{\tilde{q}_a} + i \sqrt{2} (\bar{\psi}_{\tilde{q}_a} \bar{\lambda} \tilde{q}_a - \psi_{\tilde{q}_a} \lambda \tilde{q}_a^*) \\ & - X_3 |\tilde{q}_a|^2 \} + \sum_{a=1}^n \{ (\phi - m_a) \{ \tilde{q}_a [F'_a - (\phi^* - m_a^*) \tilde{q}_a^*] + [-\bar{F}'_a - q_a^* (\phi^* - m_a^*)] q_a \} + (X_1 + i X_2) \tilde{q}_a q_a - \psi_{\tilde{q}_a} (\phi - m_a) \psi_{q_a} \\ & + i \sqrt{2} \psi_{\tilde{q}_a} \psi_{q_a} + i \sqrt{2} \psi_{q_a} \psi_{\tilde{q}_a} \} + \sum_{a=1}^n \{ (\phi^* - m_a^*) \{ q_a^* [-\bar{F}'_a - q_a^* (\phi^* - m_a^*)] + [F'_a - (\phi^* - m_a^*) \tilde{q}_a^*] \} + (X_1 - i X_2) q_a^* \tilde{q}_a^* \\ & - \bar{\psi}_{q_a} (\phi^* - m_a^*) \bar{\psi}_{\tilde{q}_a} - i \sqrt{2} \bar{\psi}_{q_a} \bar{\psi}_{\tilde{q}_a} - i \sqrt{2} \bar{\psi}_{\tilde{q}_a} \bar{\psi}_{q_a} \}. \end{aligned} \quad (4.6)$$

This is the full Lagrangian including fermion terms compared to the bosonic one in Eq. (2.9).

To obtain the $\mathcal{N}=1$ SUSY transformations in the Wess-Zumino gauge, one has to combine an ordinary SUSY transformation with an accompanying gauge transformation to preserve the Wess-Zumino gauge. Let us consider the four SUSY transformations δ_{ξ_+} given by an infinitesimal Grassmann number ξ_+ in the direction of θ_+ in Eqs. (4.2)–(4.5). The $\mathcal{N}=2$ vector multiplet represented by superfields (4.2) and (4.3) transforms under the infinitesimal SUSY δ_{ξ_+} along the left-handed spinor θ_+ in the Wess-Zumino gauge as [47]

$$\delta_{\xi_+} v^m = i \bar{\xi}_+ \bar{\sigma}^m \lambda + i \xi_+ \sigma^m \bar{\lambda}, \quad (4.7)$$

$$\delta_{\xi_+} \lambda = \sigma^{mn} v_{mn} \xi_+ + i X_3 \xi_+, \quad (4.8)$$

$$\delta_{\xi_+} X_3 = \bar{\xi}_+ \bar{\sigma}^m \partial_m \lambda - \xi_+ \sigma^m \partial_m \bar{\lambda}, \quad (4.9)$$

$$\delta_{\xi_+} \phi = \sqrt{2} \xi_+ (-i \sqrt{2} \psi), \quad (4.10)$$

$$\delta_{\xi_+} (-i \sqrt{2} \psi) = i \sqrt{2} \sigma^m \partial_m \phi \bar{\xi}_+ + \sqrt{2} (X_1 + i X_2) \xi_+, \quad (4.11)$$

$$\delta_{\xi_+} (X_1 + i X_2) = i \sqrt{2} \bar{\xi}_+ \bar{\sigma}^m \partial_m (-i \sqrt{2} \psi). \quad (4.12)$$

Similarly, we obtain the supersymmetry transformation rules for hypermultiplets in the Wess-Zumino gauge as

$$\delta_{\xi_+} q_a = \sqrt{2} \xi_+ \psi_{q_a}, \quad (4.13)$$

$$\begin{aligned} \delta_{\xi_+} \psi_{q_a} = & i \sqrt{2} \sigma^m \mathcal{D}_m q_a \bar{\xi}_+ + \sqrt{2} [F'_a \\ & - (\phi^* - m_a^*) \tilde{q}_a^*] \xi_+, \end{aligned} \quad (4.14)$$

$$\delta_{\xi_+} [F'_a - (\phi^* - m_a^*) \tilde{q}_a^*] = i \sqrt{2} \bar{\xi}_+ \bar{\sigma}^m \mathcal{D}_m \psi_{q_a} + 2i \bar{\xi}_+ \bar{\lambda} q_a, \quad (4.15)$$

$$\delta_{\xi_+} \tilde{q}_a = \sqrt{2} \xi_+ \psi_{\tilde{q}_a}, \quad (4.16)$$

$$\delta_{\xi_+} \psi_{\tilde{q}_a} = i\sqrt{2}\sigma^m \mathcal{D}_m \tilde{q}_a \bar{\xi}_+ + \sqrt{2}[-\tilde{F}'_a - (\phi^* - m_a^*)q_a^*] \xi_+, \quad (4.17)$$

$$\delta_{\xi_+} [-\tilde{F}'_a - (\phi^* - m_a^*)q_a^*] = i\sqrt{2}\bar{\xi}_+ \bar{\sigma}^m \mathcal{D}_m \psi_{\tilde{q}_a} - 2i\bar{\xi}_+ \lambda \tilde{q}_a. \quad (4.18)$$

Let us now consider the remaining four SUSY transformations δ_{ξ_-} also along the left-handed spinor θ_- , which are not manifest by the above $\mathcal{N}=1$ superfield formalism (2.1) or the associated component formalism (4.6) in the Wess-Zumino gauge. Please note that the Grassmann numbers ξ_-, θ_- are left-handed chiral spinors, similarly to ξ_+, θ_+ , and are not to be confused with the right-handed spinors. To work out δ_{ξ_-} , we shall follow the classical method of Fayet [34]. The two sets of left-handed chiral spinor Grassmann numbers θ_+ and θ_- should form a doublet under an internal $SU(2)_R$ transformation \mathfrak{M} whose representation matrix is denoted by a 2×2 matrix M :

$$\mathfrak{M} \begin{pmatrix} \theta_+ \\ \theta_- \end{pmatrix} \mathfrak{M}^{-1} = M \begin{pmatrix} \theta_+ \\ \theta_- \end{pmatrix}. \quad (4.19)$$

It is enough to consider a discrete transformation, for instance, an $SU(2)_R$ rotation around the second axis by π :

$$\mathfrak{M}_0 = \exp(i\pi I_2), \quad M_0 = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (4.20)$$

Following Fayet, we demand that v_m , ϕ , ψ_{q_a} , and $\psi_{\tilde{q}_a}$ be $SU(2)_R$ singlets, and that

$$\begin{pmatrix} \lambda \\ \psi \end{pmatrix}, \quad \begin{pmatrix} q_a \\ \tilde{q}_a^* \end{pmatrix}, \quad \begin{pmatrix} -\tilde{q}_a \\ q_a^* \end{pmatrix} \quad (4.21)$$

be $SU(2)_R$ doublets. The equations of motion (2.12) and (2.13) show that

$$\begin{pmatrix} X_1 \\ -X_2 \\ X_3 \end{pmatrix} \quad (4.22)$$

transforms as an $SU(2)_R$ triplet. The equations of motion for auxiliary fields in Eqs. (2.14), (2.15) for the hypermultiplets become

$$F_a'^* = F_a^* + (\phi - m_a)\tilde{q}_a = 0, \quad \tilde{F}'_a = -\tilde{F}_a - q_a^*(\phi^* - m_a^*) = 0, \quad (4.23)$$

which suggest that the auxiliary fields in hypermultiplets generate $SU(2)_R$ doublets

$$\begin{pmatrix} (m_a^* - \phi^*)q_a - \tilde{F}'_a^* \\ (m_a^* - \phi^*)\tilde{q}_a^* + F'_a \end{pmatrix} \sim \begin{pmatrix} q_a \\ \tilde{q}_a^* \end{pmatrix},$$

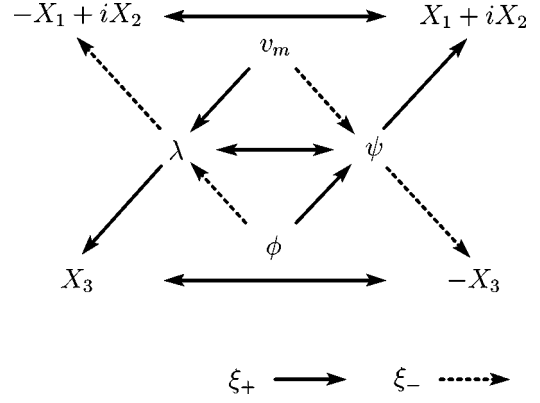


FIG. 5. The transformation laws for vector multiplets.

$$\begin{pmatrix} -\tilde{q}_a(m_a^* - \phi^*) - F_a'^* \\ q_a^*(m_a^* - \phi^*) - \tilde{F}'_a \end{pmatrix} \sim \begin{pmatrix} -\tilde{q}_a \\ q_a^* \end{pmatrix}. \quad (4.24)$$

By applying the discrete $SU(2)_R$ internal transformation (4.20) with the assignment (4.21)–(4.24) to the first four SUSYs (4.7)–(4.18) represented by an infinitesimal ξ_+ , we can find another four SUSY δ_{ξ_-} corresponding to an infinitesimal Grassmann number ξ_- . We obtain δ_{ξ_-} transformations of the $\mathcal{N}=2$ vector multiplet in the Wess-Zumino gauge:

$$\delta_{\xi_-} v^m = i\xi_- \sigma^m \bar{\psi} + i\bar{\xi}_- \bar{\sigma}^m \psi \quad (4.25)$$

$$\delta_{\xi_-} \psi = \sigma^{mn} v_{mn} \xi_- - iX_3 \xi_-, \quad (4.26)$$

$$\delta_{\xi_-} (-X_3) = -\xi_- \sigma^m \partial_m \bar{\psi} + \bar{\xi}_- \bar{\sigma}^m \partial_m \psi \quad (4.27)$$

$$\delta_{\xi_-} \phi = \sqrt{2}\xi_- (\sqrt{2}i\lambda), \quad (4.28)$$

$$\delta_{\xi_-} (\sqrt{2}i\lambda) = i\sqrt{2}\sigma^m \partial_m \phi \bar{\xi}_- + \sqrt{2}(-X_1 + iX_2)\xi_-, \quad (4.29)$$

$$\delta_{\xi_-} (-X_1 + iX_2) = i\sqrt{2}\bar{\xi}_- \bar{\sigma}^m \partial_m (\sqrt{2}i\lambda). \quad (4.30)$$

The transformation laws for the $\mathcal{N}=2$ vector multiplet (4.7)–(4.12) with respect to the Grassmann numbers θ_+ and (4.25)–(4.30) with respect to the Grassmann numbers θ_- are illustrated in Fig. 5. For a hypermultiplet, the δ_{ξ_-} transformations in the Wess-Zumino gauge are given by

$$\delta_{\xi_-} \tilde{q}_a^* = \sqrt{2}\xi_- \psi_{q_a}, \quad (4.31)$$

$$\delta_{\xi_-} \psi_{q_a} = i\sqrt{2}\sigma^m \mathcal{D}_m \tilde{q}_a^* \bar{\xi}_- + \sqrt{2}[\tilde{F}'_a^* + (\phi^* - m_a^*)q_a] \xi_-, \quad (4.32)$$

$$\delta_{\xi_-} [\tilde{F}'_a^* + (\phi^* - m_a^*)q_a] = i\sqrt{2}\bar{\xi}_- \bar{\sigma}^m \mathcal{D}_m \psi_{q_a} + 2i\bar{\xi}_- \bar{\psi} \tilde{q}_a^*, \quad (4.33)$$

$$\delta_{\xi_-} q_a^* = \sqrt{2}\xi_- (-\psi_{\tilde{q}_a}), \quad (4.34)$$

$$\begin{aligned} \delta_{\xi_-}(-\psi_{q_a}^-) &= i\sqrt{2}\sigma^m \mathcal{D}_m q_a^* \bar{\xi}_- \\ &+ \sqrt{2}[-F_a'^* + (\phi^* - m_a^*)\tilde{q}_a]\xi_-, \end{aligned} \quad (4.35)$$

$$\begin{aligned} \delta_{\xi_-}[-F_a'^* + (\phi^* - m_a^*)\tilde{q}_a] \\ = i\sqrt{2}\bar{\xi}_- \bar{\sigma}^m \mathcal{D}_m(-\psi_{q_a}^-) - 2i\bar{\xi}_- \bar{\psi} q_a^*. \end{aligned} \quad (4.36)$$

The transformation laws for the $\mathcal{N}=2$ hypermultiplet (4.13)–(4.18) with respect to the Grassmann numbers θ_+ and (4.34), (4.33) with respect to the Grassmann numbers θ_- are illustrated in Fig. 6.

To summarize the transformation properties (4.25)–(4.33) under δ_{ξ_-} , it is convenient to define the following superfields using another set of Grassmann numbers θ_- :

$$V_- = -\theta_- \sigma^m \bar{\theta}_- v_m + i\theta_-^2 \bar{\theta}_- \bar{\psi} - i\bar{\theta}_-^2 \theta_- \psi + \frac{1}{2}\theta_-^2 \bar{\theta}_-^2 (-X_3), \quad (4.37)$$

$$\Phi_- = \phi + \sqrt{2}\theta_-(\sqrt{2}i\lambda) + \theta_-^2(-X_1 + iX_2), \quad (4.38)$$

$$Q_{-a} = \tilde{q}_a^* + \sqrt{2}\theta_- \psi_{q_a} + \theta_-^2[\bar{F}_a'^* + (\phi^* - m_a^*)q_a], \quad (4.39)$$

$$\bar{Q}_{-a} = q_a^* + \sqrt{2}\theta_-(\psi_{q_a}^-) + \theta_-^2[-F_a'^* + \tilde{q}_a(\phi^* - m_a^*)]. \quad (4.40)$$

We can now rewrite the Lagrangian (4.6) to make the second set of four SUSY transformations δ_{ξ_-} (4.25)–(4.33) manifest:

$$\begin{aligned} \mathcal{L} = & -\frac{1}{4g^2}v_{mn}v^{mn} - \frac{1}{g^2}i\bar{\psi}\bar{\sigma}^m\partial_m\psi + \frac{1}{2g^2}(-X_3)^2 + \frac{1}{2g^2}|X_1 - iX_2|^2 - \frac{1}{2g^2}|\partial_m\phi|^2 - \frac{1}{g^2}i\lambda\sigma^m\partial_m\bar{\lambda} + c(-X_3) + b(-X_1 - iX_2) \\ & + b^*(-X_1 + iX_2) + \sum_{a=1}^n \{[-\bar{F}_a' - q_a^*(\phi - m_a)][-\bar{F}_a'^* - (\phi^* - m_a^*)q_a] - |\mathcal{D}_m\tilde{q}_a|^2 - i\bar{\psi}_{q_a}\bar{\sigma}^m\mathcal{D}_m\psi_{q_a} - i\sqrt{2}(\bar{\psi}_{q_a}\bar{\psi}\tilde{q}_a^* \\ & - \psi_{q_a}\psi\tilde{q}_a) + (-X_3)|\tilde{q}_a|^2\} + \sum_{a=1}^n \{[F_a'^* - \tilde{q}_a(\phi^* - m_a^*)][F_a' - (\phi - m_a)\tilde{q}_a^*] - |\mathcal{D}_m q_a|^2 - i\psi_{q_a}\sigma^m\mathcal{D}_m\bar{\psi}_{q_a} \\ & + i\sqrt{2}(-\bar{\psi}_{q_a}\bar{\psi}q_a^* + \psi_{q_a}\psi q_a) - (-X_3)|q_a|^2\} + \sum_{a=1}^n ((\phi - m_a)\{q_a^*[-\bar{F}_a'^* - (\phi^* - m_a^*)q_a] + [F_a'^* - \tilde{q}_a(\phi^* - m_a^*)]\tilde{q}_a^*\} \\ & + (X_1 - iX_2)q_a^*\tilde{q}_a^* - \psi_{q_a}^-(\phi - m_a)\psi_{q_a} - i\sqrt{2}\psi_{q_a}^-\lambda\tilde{q}_a^* + i\sqrt{2}\psi_{q_a}\lambda q_a^*) + \sum_{a=1}^n ((\phi^* - m_a^*)\{\tilde{q}_a[F_a' - (\phi - m_a)\tilde{q}_a^*] \\ & + [-\bar{F}_a' - q_a^*(\phi - m_a)]q_a\} + (X_1 + iX_2)\tilde{q}_a q_a - \bar{\psi}_{q_a}(\phi^* - m_a^*)\bar{\psi}_{q_a} + i\sqrt{2}\bar{\psi}_{q_a}\bar{\lambda}\tilde{q}_a - i\sqrt{2}\bar{\psi}_{q_a}\bar{\lambda}q_a). \end{aligned} \quad (4.41)$$

We can finally assemble the above component form of the Lagrangian in an $\mathcal{N}=1$ superfield formalism using the second set of Grassmann numbers θ_- in Eqs. (4.37)–(4.40), in contrast to those superfields with θ_+ in Eqs. (4.2)–(4.5):

$$\begin{aligned} \mathcal{L} = & \frac{1}{4g^2}(W_-^\alpha W_{-\alpha}|\theta_-^2 + \bar{W}_{-\alpha} \bar{W}_{-\alpha}|\bar{\theta}_-^2) + \frac{1}{2g^2}\Phi_-^\dagger\Phi_- \Big|_{\theta_-^2\bar{\theta}_-^2} \\ & + \sum_{a=1}^n (Q_{-a}^\dagger e^{2V} Q_{-a} + \bar{Q}_{-a}^\dagger e^{-2V} \bar{Q}_{-a})|_{\theta_-^2\bar{\theta}_-^2} \\ & + 2cV_-|_{\theta_-^2\bar{\theta}_-^2} + \left(\sum_{a=1}^n (\Phi_- - m_a)Q_{-a}\bar{Q}_{-a}|_{\theta_-^2\bar{\theta}_-^2} \right. \\ & \left. + b^*\Phi_-|_{\theta_-^2} + \text{H.c.} \right). \end{aligned} \quad (4.42)$$

Let us note that the Lagrangian is not invariant under the automorphism $SU(2)_R$ if FI parameters are present. Obviously, the Lagrangian is invariant under the eight SUSY transformations when it is invariant under the first set of four SUSYs and the discrete $SU(2)_R$ transformation \mathfrak{M}_0 . However, it is important to realize that the Lagrangian represented by Eqs. (4.6) and (4.42) is invariant under all the eight SUSY transformations (4.7)–(4.18) and (4.25)–(4.33) irrespective of the values of the FI parameters c and b . This is because the difference between the Lagrangian transformed by \mathfrak{M}_0 and the original one is given by the FI terms which are transformed to total derivatives by SUSY transformations δ_{ξ_+} and δ_{ξ_-} . Therefore the action is invariant as usual for the SUSY theories.

Theories with eight SUSYs like our model have been shown to possess three complex structures [26,48]. Our formulation in terms of two sets of Grassmann numbers θ_+, θ_- does not make this property manifest. In fact, our choice of

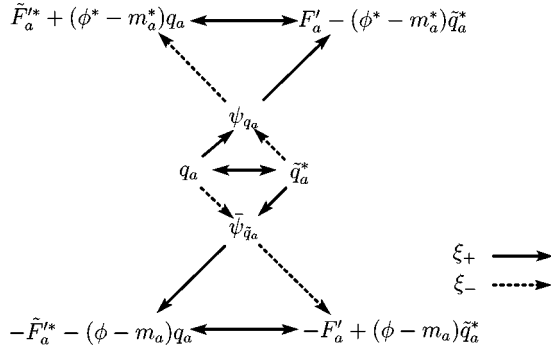


FIG. 6. The transformation laws for hypermultiplets.

$c=0$, $b \in \mathbf{R}$ breaks $SU(2)_R$ symmetry and a particular complex structure has been selected. However, this particular choice of complex structure will turn out to be useful for the analysis of our model and solution. For instance, we will show that we can choose one of the two conserved SUSY directions from θ_+ and the other from θ_- .

Since all the eight SUSY transformations are clarified, we are now in a position to determine precisely how many SUSY charges out of these eight are conserved by our solution (3.15)–(3.17) of the domain wall junction. We have already found in Eq. (2.34) that one out of four SUSY δ_{ξ_-} is conserved. We need to examine another four SUSY δ_{ξ_-} . Let us first consider the fermions $\psi_{\tilde{q}_a}$ and ψ_{q_a} in the $\mathcal{N}=1$ chiral scalar superfields \tilde{Q}_{-a} in Eq. (4.39) and Q_{-a} in Eq. (4.40) for the hypermultiplets

$$\begin{aligned} \delta_{\xi_-} \psi_{q_a} &= i\sqrt{2}\sigma^m \partial_m \tilde{q}_a^* \bar{\xi}_- + \sqrt{2}(\phi^* - m_a^*)q_a \xi_- \\ &= i2\sqrt{2} \left(\sigma^+ \partial_z \tilde{q}_a^* \bar{\xi}_- + \sigma^- \partial_{\bar{z}} \tilde{q}_a^* \bar{\xi}_- \right. \\ &\quad \left. - i\frac{1}{2}(\phi^* - m_a^*)q_a \xi_- \right) \\ &= i2\sqrt{2}(\sigma^+ \bar{\xi}_- + i\xi_-) \partial_z \tilde{q}_a^* + i2\sqrt{2}\sigma^- \bar{\xi}_- \partial_{\bar{z}} \tilde{q}_a^*, \end{aligned} \quad (4.43)$$

$$\begin{aligned} \delta_{\xi_-} (-\psi_{\tilde{q}_a}) &= i\sqrt{2}\sigma^m \partial_m q_a^* \bar{\xi}_- + \sqrt{2}(\phi^* - m_a^*)\tilde{q}_a \xi_- \\ &= i2\sqrt{2} \left(\sigma^+ \partial_z q_a^* \bar{\xi}_- + \sigma^- \partial_{\bar{z}} q_a^* \bar{\xi}_- \right. \\ &\quad \left. - i\frac{1}{2}(\phi^* - m_a^*)\tilde{q}_a \xi_- \right) \\ &= i2\sqrt{2}\sigma^- \bar{\xi}_- \partial_z q_a^* + i2\sqrt{2}(\sigma^+ \bar{\xi}_- + i\xi_-) \partial_{\bar{z}} q_a^*. \end{aligned} \quad (4.44)$$

Therefore the conserved SUSY direction is given by

$$\sigma^+ \bar{\xi}_- = -i\xi_- \quad \text{and} \quad \sigma^- \bar{\xi}_- = 0. \quad (4.45)$$

We have to study the SUSY transformation of the remaining fermions λ and $\bar{\psi}$ in the $\mathcal{N}=1$ chiral scalar superfields Φ_- and V_- using the BPS equations (2.29) and (2.38) with $\Omega = -1$ and algebraic equations of motion for the auxiliary fields (2.39). Moreover, we note that $X_2 = X_3 = 0$ in our solution. Then we find that the same SUSY directions are conserved for λ :

$$\begin{aligned} \delta_{\xi_-} (i\sqrt{2}\lambda) &= i\sqrt{2}\sigma^m \partial_m \phi \bar{\xi}_- + \sqrt{2}(-X_1 + iX_2)\xi_-, \\ &= i2\sqrt{2}(\sigma^+ \partial_z + \sigma^- \partial_{\bar{z}})\phi \bar{\xi}_- \\ &\quad + \sqrt{2}(-X_1 + iX_2)\xi_-, \\ &= i2\sqrt{2}(\sigma^+ \bar{\xi}_- + i\xi_-) \partial_z \phi + i2\sqrt{2}\sigma^- \bar{\xi}_- \partial_{\bar{z}} \phi = 0, \end{aligned} \quad (4.46)$$

with the same infinitesimal SUSY transformation parameters (4.45). The right-hand side of the ξ_- transformation of ψ in Eq. (4.26) involves only v_m and X_3 , which vanish in our solution. Therefore they conserve all the SUSY transformations trivially.

Summarizing our results for all the eight SUSYs, we see that there are two conserved directions in the Grassmann parameter:

$$\sigma^+ \bar{\xi}_+ = i\xi_+ \quad \text{and} \quad \sigma^- \bar{\xi}_+ = 0, \quad (4.47)$$

$$\sigma^+ \bar{\xi}_- = -i\xi_- \quad \text{and} \quad \sigma^- \bar{\xi}_- = 0. \quad (4.48)$$

That is, we have determined the two conserved directions

$$\xi_{+1} = i(\xi_{+1})^*, \quad \xi_{+2} = 0, \quad \xi_{-1} = -i(\xi_{-1})^*, \quad \xi_{-2} = 0, \quad (4.49)$$

where we set

$$\xi_+ = \begin{pmatrix} \xi_{+1} \\ \xi_{+2} \end{pmatrix}, \quad \bar{\xi}_+ = \begin{pmatrix} \bar{\xi}_+^1 \\ \bar{\xi}_+^2 \end{pmatrix} = \begin{pmatrix} (\xi_{+2})^* \\ -(\xi_{+1})^* \end{pmatrix}, \quad (4.50)$$

$$\xi_- = \begin{pmatrix} \xi_{-1} \\ \xi_{-2} \end{pmatrix}, \quad \bar{\xi}_- = \begin{pmatrix} \bar{\xi}_-^1 \\ \bar{\xi}_-^2 \end{pmatrix} = \begin{pmatrix} (\xi_{-2})^* \\ -(\xi_{-1})^* \end{pmatrix}. \quad (4.51)$$

We have now established that our domain wall solution preserves two out of eight SUSYs.

V. DIMENSIONAL REDUCTION FROM FIVE DIMENSIONS

The $\mathcal{N}=1$ superfield formalism is of course most useful to study $\mathcal{N}=1$ SUSY theories in four dimensions. It can also be used to describe the $\mathcal{N}=2$ SUSY theories in four dimensions making only four out of eight SUSYs manifest. In order to describe $\mathcal{N}=2$ SUSY theories in five dimensions, however, we need to sacrifice the five-dimensional Lorentz invariance [5,41–44]. In terms of components, we can always express the $\mathcal{N}=2$ SUSY theories with all the necessary

auxiliary fields to close the algebra off shell in the Wess-Zumino gauge.

The highest dimension allowed by the $\mathcal{N}=2$ SUSY is 6. The hypermultiplet in six dimensions cannot have masses. The five-dimensional theories can be obtained by a dimensional reduction from six dimensions. If we perform a nontrivial dimensional reduction, allowing the momenta in the sixth dimension, we obtain a massive hypermultiplet [45]. Therefore the hypermultiplets in five dimensions can have only real mass parameters. The $\mathcal{N}=2$ SUSY theories in four dimensions can be obtained by dimensionally reducing the five-dimensional theories. The real scalar field Σ in five dimensions comes originally from the sixth component of the gauge field, and the combination $\Sigma + iv_5$ becomes a complex scalar in four dimensions, when we consider the reduction to four dimensions. To obtain the four-dimensional theory with complex mass parameters, we should perform a nontrivial dimensional reduction in the x^5 direction. In this process, the mass terms arise as momenta in the fifth dimension. Conversely, we can recover the five-dimensional theory by restoring the fifth dimension from the imaginary part of the complex mass parameter $m_a \equiv m_{aR} + im_{aI}$ as

$$\partial_5 q_a = -im_{aI} q_a, \quad \partial_5 \psi_{q_a} = -im_{aI} \psi_{q_a}. \quad (5.1)$$

The imaginary part of the complex scalar field ϕ can also be identified as the fifth component of the vector field v_5 as

$$\phi = \Sigma + iv_5. \quad (5.2)$$

We can recover the covariant derivative along the fifth direction as

$$i(v_5 - m_{aI})q_a = (\partial_5 + iv_5)q_a = \mathcal{D}_5 q_a. \quad (5.3)$$

Therefore the mass terms associated with the quark fields are reduced to covariant derivatives in the fifth dimension,

$$\begin{aligned} (\phi - m_a)q_a &= [(\Sigma - m_{aR}) + i(v_5 - m_{aI})]q_a \\ &= [(\Sigma - m_{aR}) + \mathcal{D}_5]q_a. \end{aligned} \quad (5.4)$$

In the spirit of the $\mathcal{N}=1$ superfield formalism, we can express the $\mathcal{N}=2$ vector and hypermultiplets in five dimensions by two kinds of superfields, similarly to our results (4.2)–(4.5) and (4.37)–(4.40) for four-dimensional $\mathcal{N}=2$ theories. The θ_+ superfields are given by

$$\begin{aligned} V_+(x, \theta_+, \bar{\theta}_+) &= -\theta_+ \sigma^m \bar{\theta}_+ v_m + i\theta_+^2 \bar{\theta}_+ \bar{\lambda} - i\bar{\theta}_+^2 \theta_+ \lambda \\ &+ \frac{1}{2} \theta_+^2 \bar{\theta}_+^2 (X_3 - \partial_5 \Sigma), \end{aligned} \quad (5.5)$$

$$\begin{aligned} \Phi_+(y, \theta_+) &= (\Sigma + iv_5) + \sqrt{2} \theta_+ (-i\sqrt{2} \psi) \\ &+ \theta_+^2 (X_1 + iX_2), \end{aligned} \quad (5.6)$$

$$\begin{aligned} Q_{+a}(y, \theta_+) &= q_a + \sqrt{2} \theta_+ \psi_{q_a} + \theta_+^2 [F'_a + \mathcal{D}_5 \tilde{q}_a^* \\ &- (\Sigma - m_{aR}) \tilde{q}_a^*], \end{aligned} \quad (5.7)$$

$$\begin{aligned} \bar{Q}_{+a}(y, \theta_+) &= \tilde{q}_a + \sqrt{2} \theta_+ \psi_{q_a}^- + \theta_+^2 [-\bar{F}'_a - \mathcal{D}_5 q_a^* \\ &- q_a^* (\Sigma - m_{aR})]. \end{aligned} \quad (5.8)$$

The θ_- superfields are given by

$$\begin{aligned} V_-(x, \theta_-, \bar{\theta}_-) &= -\theta_- \sigma^m \bar{\theta}_- v_m + i\theta_-^2 \bar{\theta}_- \bar{\psi} - i\bar{\theta}_-^2 \theta_- \psi \\ &+ \frac{1}{2} \theta_-^2 \bar{\theta}_-^2 (-X_3 - \partial_5 \Sigma), \end{aligned} \quad (5.9)$$

$$\begin{aligned} \Phi_-(y, \theta_-) &= (\Sigma + iv_5) + \sqrt{2} \theta_- (\sqrt{2} i \lambda) \\ &+ \theta_-^2 (-X_1 + iX_2), \end{aligned} \quad (5.10)$$

$$\begin{aligned} Q_{-a}(y, \theta_-) &= \tilde{q}_a^* + \sqrt{2} \theta_- \psi_{q_a} + \theta_-^2 [\bar{F}'_a^* - \mathcal{D}_5 q_a \\ &+ (\Sigma - m_{aR}) q_a], \end{aligned} \quad (5.11)$$

$$\begin{aligned} \bar{Q}_{-a}(y, \theta_-) &= q_a^* + \sqrt{2} \theta_- (-\psi_{q_a}^-) + \theta_-^2 [-F'_a^* + \mathcal{D}_5 \tilde{q}_a \\ &+ \tilde{q}_a (\Sigma - m_{aR})]. \end{aligned} \quad (5.12)$$

Please note that all the fields depend on the coordinate x^5 in five dimensions, in spite of having almost the same appearance as the four-dimensional superfields.

Since the mass term in five dimensions can be obtained as a nontrivial dimensional reduction from six dimensions, we can have only real mass parameters in the Lagrangian m_{aR} [42,45]. Therefore we find the Lagrangian in terms of the θ_+ $\mathcal{N}=1$ superfields in Eqs. (5.5)–(5.8) as

$$\begin{aligned} \mathcal{L} &= \frac{1}{4g^2} (W_+^\alpha W_+^\dagger |_{\theta_+^2} + \bar{W}_+^\alpha \bar{W}_+^\dagger |_{\bar{\theta}_+^2}) + \frac{1}{g^2} \left(\partial_5 V - \frac{\Phi_+^\dagger + \Phi_+}{2} \right) \Big|_{\theta_+^2 \bar{\theta}_+^2} + \sum_{a=1}^n (Q_{+a}^\dagger e^{2V} + Q_{+a} + \bar{Q}_{+a}^\dagger e^{-2V} + \bar{Q}_{+a}) |_{\theta_+^2 \bar{\theta}_+^2} \\ &- 2cV_+ |_{\theta_+^2 \bar{\theta}_+^2} + \left(\sum_{a=1}^n \bar{Q}_{+a} (\partial_5 + \Phi_+ - m_a) Q_{+a} |_{\theta_+^2} - b\Phi_+ |_{\theta_+^2} + \text{H.c.} \right). \end{aligned} \quad (5.13)$$

The same Lagrangian is given in terms of the θ_- $\mathcal{N}=1$ superfields in Eqs. (5.9)–(5.12) as

$$\begin{aligned} \mathcal{L} = & \frac{1}{4g^2} (W_\alpha^- W_\alpha^- |_{\theta_-^2} + \bar{W}_{\dot{\alpha}}^- \bar{W}_{\dot{\alpha}}^- |_{\bar{\theta}_-^2}) + \frac{1}{g^2} \left(\partial_5 V - \frac{\Phi_-^\dagger + \Phi_-}{2} \right) \Big|_{\theta_-^2 \bar{\theta}_-^2} + \sum_{a=1}^n (Q_{-a}^\dagger e^{2V} Q_{-a} + \tilde{Q}_{-a}^\dagger e^{-2V} \tilde{Q}_{-a}) |_{\theta_-^2 \bar{\theta}_-^2} + 2c V_- |_{\theta_-^2 \bar{\theta}_-^2} \\ & + \left(\sum_{a=1}^n \tilde{Q}_{-a} (\partial_5 + \Phi_- - m_a) Q_{-a} |_{\theta_-^2} + b^* \Phi_- |_{\theta_-^2} + \text{H.c.} \right). \end{aligned} \quad (5.14)$$

In terms of components, we can express the Lagrangian more symmetrically with respect to the $SU(2)_R$ symmetry:

$$\mathcal{L} = \mathcal{L}_{\text{boson}} + \mathcal{L}_{\text{fermion}}, \quad (5.15)$$

$$\begin{aligned} \mathcal{L}_{\text{boson}} = & -\frac{1}{4g^2} F_{MN} F^{MN} - \frac{1}{2g^2} (\partial_M \Sigma)^2 + \frac{1}{2g^2} \{ (X_1)^2 + (X_2)^2 + (X_3)^2 \} + c(-X_3) + b(-X_1 - iX_2) + b^*(-X_1 + iX_2) \\ & + \sum_{a=1}^n [-|\mathcal{D}_M q_a|^2 - |\mathcal{D}_M \tilde{q}_a|^2 + |F'|^2 + |\bar{F}'|^2 - (\Sigma - m_{aR})^2 (|q_a|^2 + |\tilde{q}_a|^2) + X_3 (|q_a|^2 - |\tilde{q}_a|^2) \\ & + (X_1 + iX_2) \tilde{q}_a q_a + (X_1 - iX_2) q_a^* \tilde{q}_a^*], \end{aligned} \quad (5.16)$$

$$\begin{aligned} \mathcal{L}_{\text{fermion}} = & -\frac{1}{g^2} \lambda \bar{\sigma}^m \partial_m \bar{\lambda} - \frac{1}{g^2} \bar{\psi} \bar{\sigma}^m \partial_m \psi - \frac{1}{g^2} \psi \partial_5 \lambda - \frac{1}{g^2} \bar{\psi} \partial_5 \bar{\lambda} + \sum_{a=1}^n [-i \bar{\psi}_{q_a} \bar{\sigma}^m \mathcal{D}_m \psi_{q_a} - i \psi_{\tilde{q}_a} \sigma^m \mathcal{D}_m \bar{\psi}_{\tilde{q}_a} \\ & - \psi_{q_a} \mathcal{D}_5 \psi_{q_a} - \bar{\psi}_{\tilde{q}_a} \mathcal{D}_5 \bar{\psi}_{\tilde{q}_a} - \psi_{q_a} (\Sigma - m_{aR}) \psi_{q_a} - \bar{\psi}_{\tilde{q}_a} (\Sigma - m_{aR}) \bar{\psi}_{\tilde{q}_a} + i\sqrt{2} \{ (\psi_{q_a} \psi - \bar{\psi}_{\tilde{q}_a} \bar{\lambda}) q_a - (\psi_{q_a} \lambda + \bar{\psi}_{\tilde{q}_a} \bar{\psi}) \tilde{q}_a^* \\ & + (\psi_{q_a} \lambda - \bar{\psi}_{\tilde{q}_a} \bar{\psi}) q_a^* + (\psi_{q_a} \psi + \bar{\psi}_{\tilde{q}_a} \bar{\lambda}) \tilde{q}_a \}] \\ = & -\frac{1}{2g^2} \bar{\lambda}_i \gamma^M \partial_M \lambda^i + \sum_{a=1}^n [-\bar{\psi}_a \gamma^M \partial_M \psi_a - \bar{\psi}_a (\Sigma - m_{aR}) \psi_a - i\sqrt{2} \bar{\psi}_a \lambda^i \epsilon_{ij} q_a^j + i\sqrt{2} \bar{\lambda}_i \psi_a \epsilon^{ij} q_{ja}^*], \end{aligned} \quad (5.17)$$

where the capitalized indices M, N, \dots run over 0,1,2,3,5. The gamma matrices in five dimensions are given by 4×4 matrices as

$$\gamma^M = \left(\begin{pmatrix} 0 & \sigma^m \\ \bar{\sigma}^m & 0 \end{pmatrix}, \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \right), \quad (5.18)$$

where $\sigma^m = (1, \vec{\sigma})$ and $\bar{\sigma}^m = (1, -\vec{\sigma})$ [47]. To achieve the $\mathcal{N} = 2$ SUSY off-shell formalism, it is convenient to make $SU(2)_R$ manifest. In the case of a vector multiplet, the spinors in five dimensions are most conveniently organized in terms of the symplectic [$SU(2)$] Majorana spinors λ^i , $i = 1, 2$, transforming as doublets under the $SU(2)_R$ symmetry. The $SU(2)$ Majorana spinor is defined by

$$\lambda^i = \epsilon^{ij} C \bar{\lambda}_j^T, \quad (5.19)$$

where the charge conjugation matrix C in five dimensions satisfies $C \gamma^M C^{-1} = (\gamma^M)^T$, $C^T = -C$, and $CC^\dagger = 1$. An explicit form may be given by $C = \text{diag}(i\sigma^2, i\sigma^2)$. The spinors in the $\mathcal{N} = 2$ vector multiplet can be assembled into a four-component $SU(2)$ Majorana spinor λ^i as

$$\lambda^1 = \begin{pmatrix} \lambda_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} \psi_\alpha \\ -\bar{\lambda}^{\dot{\alpha}} \end{pmatrix}, \quad (5.20)$$

$$\bar{\lambda}_1 = (\psi^\alpha \quad \bar{\lambda}_{\dot{\alpha}}), \quad \bar{\lambda}_2 = (-\lambda^\alpha \quad \bar{\psi}_{\dot{\alpha}}). \quad (5.21)$$

The spinors in the hypermultiplets are singlets under the $SU(2)_R$ symmetry and are assembled into a four-component spinor ψ_a for each flavor:

$$\psi = \begin{pmatrix} \psi_{q_a} \\ \bar{\psi}_{\tilde{q}_a} \end{pmatrix}, \quad \bar{\psi} = (\psi_{\tilde{q}_a} \quad -\bar{\psi}_{q_a}). \quad (5.22)$$

The scalars and auxiliary fields in the hypermultiplet transform as doublets under $SU(2)_R$ as given in Eqs. (4.21) and (4.24).

The above five-dimensional Lagrangian (5.15)–(5.17) makes it clear that we can have only real mass parameters for a hypermultiplet that is obtained as a momentum in one extra dimension (the sixth dimension) through a nontrivial (Scherk-Schwarz) [46] dimensional reduction from six dimensions [45]. On the other hand, we need to have at least three discrete vacua in the complex field plane. This situation can be realized in our model through the complex masses of hypermultiplets. This is the reason why we cannot generalize our junction solution to five or six dimensions.

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