Quantum order from string-net condensations and the origin of light and massless fermions

Xiao-Gang Wen*
Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA
(Received 24 February 2003; published 3 September 2003)

Recently, it was pointed out that quantum orders and the associated projective symmetry groups can produce and protect massless gauge bosons and massless fermions in local bosonic models. In this paper, we demonstrate that a state with such kinds of quantum order can be viewed as a condensed state of nets of strings. The emergent gauge bosons and fermions in local bosonic models can be regarded as a direct consequence of string-net condensation. The gauge bosons are fluctuations of large closed string nets. The ends of open strings are the charged particles of the corresponding gauge field. For certain types of strings, the ends of open strings can even be fermions. According to the string-net picture, fermions always carry gauge charges. This suggests the existence of a new discrete gauge field that couples to neutrinos and neutrons. We also discuss how chiral symmetry that protects massless Dirac fermions can emerge from the projective symmetry of quantum order.

DOI: 10.1103/PhysRevD.68.065003 PACS number(s): 11.15.−q

I. INTRODUCTION

A. Fundamental questions about light and fermions

We have known about light and fermions for many years. But we still cannot give a satisfactory answer to the following fundamental questions: What are light and fermions? Where do light and fermions come from? Why do light and fermions exist? At the moment, the standard answers to the above fundamental questions appear to be ‘‘light is the particle described by a gauge field’’ and ‘‘fermions are the particles described by anticommuting fields.’’ Here, we would like to argue that there is another possible answer to the above questions: our vacuum is filled with stringlike objects that form a network of arbitrary sizes and those string nets form a quantum condensed state. According to the string-net picture, light (and other gauge bosons) is a vibration of the condensed string nets, and fermions are the ends of strings (or nodes of string nets). String-net condensation provides a unified origin of light and fermions.[1]

Before discussing the above fundamental questions in more detail, we would like to clarify what we mean by ‘‘light exists’’ and ‘‘fermions exist.’’ We know that there is a natural mass scale in physics—the Planck mass. The Planck mass is so large that any observed particle has a mass at least a factor of 10^{16} smaller than the Planck mass. So all observed particles can be treated as massless when compared with the Planck mass. When we ask why some particles exist, we really ask why those particles are massless (or nearly massless when compared with the Planck mass). So the real issue is to understand what makes certain excitations (such as light and fermions) massless. We know that symmetry breaking is a way to get gapless bosonic excitations. We will see that string-net condensation is another way to get gapless excitations. However, string-net condensations can generate massless gauge bosons and massless fermions.

Second, we would like to clarify what we mean by the ‘‘origin of light and fermions.’’ We know that everything has to come from something. So when we ask where light and fermions come from, we have assumed that there are some things simpler and more fundamental than light and fermions. In Sec. II, we define local bosonic models that are simpler than models with gauge fields coupled to fermions. We will regard local bosonic models as more fundamental (the locality principle). We will show that light and fermions can emerge from a local bosonic model if the model contains a condensation of nets of stringlike object in its ground state.

After the above two clarifications, we can state more precisely the meaning of the statement that string-net condensation provides another possible answer to the fundamental questions about light and fermions. When we say gauge bosons and fermions originate from string-net condensation, we really mean that (nearly) *massless* gauge bosons and fermions originate from string-net condensation in a *local bosonic model*.

B. Gapless phonons and symmetry breaking order

Before considering the origin of massless photon and massless fermions, let us consider a simpler massless (or gapless) excitation—the phonon. We can ask three similar questions about the phonon: What is a phonon? Where do phonons come from? Why do phonons exist? We know that those are scientific questions and we know their answers. A phonon is a vibration of a crystal. It comes from a spontaneous translation symmetry breaking. It exists because the translation symmetry breaking phase actually exists in nature. In particular, the gaplessness of the phonon directly originates from and is protected by the spontaneous translation symmetry breaking [1,2]. Many other gapless excitations, such as spin waves, superfluid modes, etc., also come from the condensation of pointlike objects that break certain symmetries.

It is quite interesting to see that our understanding of a gapless excitation—the phonon—is rooted in our understanding of phases of matter. According to Landau's theory [3], phases of matter are different because they have different

*URL: http://dao.mit.edu/~wen

[1]Here, by ‘‘string-net condensation’’ we mean the condensation of nets of stringlike objects of arbitrary sizes.

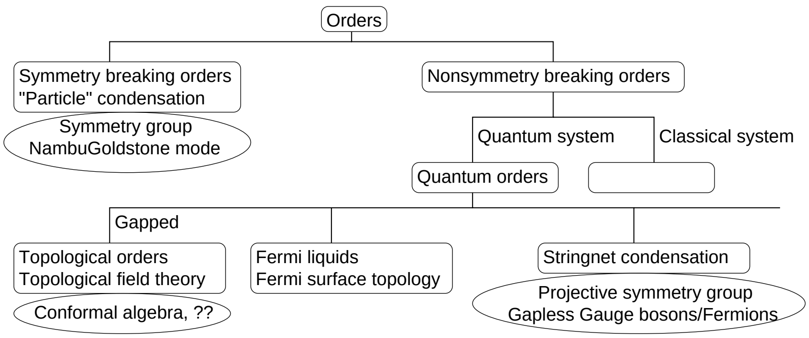

FIG. 1. A classification of different kinds of order in matter. (We view our vacuum as one kind of matter.)

broken symmetries. The symmetry description of phases is very powerful. It allows us to classify all possible crystals. It also provides the origin for gapless phonons and many other gapless excitations. Until a few years ago, it was believed that the condensations of pointlike objects, and the related symmetry breaking and order parameters, can describe all the types of order (or phases) in nature.

C. The existence of light and fermions implies the existence of new kinds of order

Knowing that light is a massless excitation, one may wonder whether light, just like phonons, is also a Nambu-Goldstone mode from a broken symmetry. However, experiments tell us that a $U(1)$ gauge boson, such as light, is really different from a Nambu-Goldstone mode in $3+1$ dimensions. Therefore it is impossible to use Landau's symmetry breaking theory and condensation of pointlike objects to understand the origin and the masslessness of light. Also, Nambu-Goldstone modes are always bosonic; thus it is impossible to use symmetry breaking to understand the origin and the (almost) masslessness of fermions. It seems that there does not exist any order that can give rise to massless light and massless fermions. Because of this, we put light and electrons into a different category from phonons. We regarded them as elementary and introduced them by hand into our theory of nature.

However, if we believe that light and electrons, just like phonons, exist for a reason, then this reason must be a certain order in our vacuum that protects the masslessness of light and electrons. (Here we have assumed that light and electrons are not something that we place in an empty vacuum. Our vacuum is more like an "ocean" which is not empty. Light and electrons are collective excitations that correspond to certain patterns of "water" motion.) Now the question is what kind of order can give rise to light and electrons and protect their masslessness.

If we really believe in the equality between light, electrons, and phonons, then the very existence of light and fermions indicates that our understanding of the states of matter is incomplete. We should deepen and expand our understanding of the states of matter. There should be new states of matter that contain new kinds of order. The new types of order will produce light and electrons and protect their masslessness.

D. Topological order and quantum order

After the discovery of the fractional quantum Hall (FQH) effect [4,5], it became clear that the Landau symmetry breaking theory cannot describe different FQH states, since those states all have the *same* symmetry. It has been proposed that FQH states contain a new kind of order—topological order [6]. Topological order is new because it cannot be described by symmetry breaking, long range correlation, and local order parameters. None of the usual tools that we use to characterize phases applies to topological order. Despite this, topological order is not an empty concept. Topological order can be characterized by a new set of tools, such as the number of degenerate ground states, quasiparticle statistics, and edge states. It was shown that the ground state degeneracy of a topological ordered state is a universal property since the degeneracy is robust against any perturbations [7]. Such a topological degeneracy demonstrates the existence of topological order. It can also be used to perform fault tolerant quantum computations [8].

Recently, the concept of topological order was generalized to quantum order [9,10]. Quantum order is used to describe new kinds of order in gapless quantum states. One way to understand quantum order is to see how it fits into a general classification scheme of types of order (see Fig. 1). First, different types of order can be divided into two classes: symmetry breaking order and nonsymmetry breaking order. The symmetry breaking orders can be described by a local order parameter and can be said to contain a condensation of pointlike objects. All kinds of symmetry breaking order can be understood in terms of Landau's symmetry breaking theory. The nonsymmetry breaking orders cannot be described by symmetry breaking, nor by the related local order parameters and long range correlations. Thus they are a new kind of order. If a quantum system (a state at zero temperature) contains a nonsymmetry breaking order, then the system is said to contain a nontrivial quantum order. We see that a quantum order is simply a nonsymmetry breaking order in a quantum system.

Quantum order can be further divided into many subclasses. If a quantum state is gapped, then the corresponding quantum order will be called topological order. The low energy effective theory of a topological ordered state will be a topological field theory [11]. The second class of quantum order appears in Fermi liquids (or free fermion systems). The different kinds of quantum order in Fermi liquids are classified by the Fermi surface topology [10,12].

E. Quantum order from string-net condensation

In this paper, we will concentrate on the third class of quantum order—the quantum order from condensation of nets of strings, or simply string-net condensation [13,14]. This class of quantum order shares some similarities with the symmetry breaking order of ''particle'' condensation. We know that different types of symmetry breaking order can be classified by symmetry groups. Using group theory, we can classify all the 230 kinds of crystal order in three dimensions. The symmetry also produces and protects gapless Nambu-Goldstone bosons. Similarly, as we will see later in this paper, different string-net condensations (and the corresponding quantum orders) can be classified by a mathematical object called the projective symmetry group (PSG) [9,10]. Using the PSG, we can classify over 100 different 2D spin liquids that all have the same symmetry [9]. Just like the symmetry group, the PSG can also produce and protect gapless excitations. However, unlike the symmetry group, the PSG produces and protects gapless gauge bosons and gapless fermions [9,15,16]. Because of this, we can say that light and massless fermions can have a unified origin. They can come from string-net condensation.

We used to believe that to have light and fermions in our theory, we had to introduce by hand a fundamental $U(1)$ gauge field and anticommuting fermion fields, since at that time we did not know any collective modes that behave like gauge bosons and fermions. Now we know that gauge bosons and fermions appear commonly and naturally in quantum ordered states, as fluctuations of condensed string nets and the ends of open strings. This raises an issue: do light and fermions come from a fundamental $U(1)$ gauge field and anticommuting fields as in the 123 standard model or do they come from a particular quantum order in our vacuum? Clearly, it is more natural to assume that light and fermions come from a quantum order in our vacuum. From the connection between string-net condensation, quantum order, and massless gauge/fermion excitations, it is very tempting to propose the following answers to the fundamental questions about light and (nearly) massless fermions.

What are light and fermions? Light is a fluctuation of condensed string nets of arbitrary sizes. Fermions are ends of open strings.

Where do light and (nearly) massless fermions come from? Light and fermions come from the collective motions of nets of stringlike objects that fill our vacuum.

Why do light and (nearly) massless fermions exist? Light and the fermions exist because our vacuum chooses to have a string-net condensation.

Had our vacuum chosen to have a ''particle'' condensation, there would be only Nambu-Goldstone bosons at low energies. Such a universe would be very boring. String-net condensation and the resulting light and (nearly) massless fermions provide a much more interesting universe, at least interesting enough to support intelligent life to study the origin of light and massless fermions.

The string-net picture of fermions explains why there is always an even number of fermions in our universe. The string-net picture for gauge bosons and fermions also has an experimental prediction: all fermions must carry certain gauge charges [14]. At first sight, this prediction appears to contradict the known experimental fact that neutrons carry no gauge charges. Thus one may think the string-net picture of gauge bosons and fermions has already been falsified by experiments. Here we would like to point out that the string-net picture of gauge bosons and fermions can still be correct if we assume the existence of a new discrete gauge field, such as a Z_2 gauge field, in our universe. In this case, neutrons and neutrinos carry a nonzero charge of the discrete gauge field. Therefore, the string-net picture of gauge bosons and fermions predicts the existence of discrete gauge excitations (such as gauge flux lines) in our universe.

We would like to remark that, despite the similarity, the above string-net picture of gauge bosons and fermions is different from the picture of standard superstring theory. In standard superstring theory, closed strings correspond to gravitons, and open strings correspond to gauge bosons. All the elementary particles correspond to different vibration modes of small strings in superstring theory. Also, the fermions in standard superstring theory come from the fermion fields on the world sheet. In our string-net picture, the vacuum is filled with large nets of strings. The massless gauge bosons correspond to the fluctuations of large closed string nets (i.e., nets of closed strings) and fermions correspond to the ends of open strings in string nets. Anticommuting fields are not needed to produce (nearly) massless fermions. Massless fermions appear as low energy collective modes in a purely bosonic system.

The string-net picture for gauge theories has a long history. The closed-string description of gauge fluctuations is intimately related to the Wilson loop in gauge theory [17–19]. The relation between dynamical gauge theory and a dynamical Wilson-loop theory was suggested in Refs. [20,21]. Reference [22] studied the Hamiltonian of a nonlocal model—lattice gauge theory. It was found that lattice gauge theory contains a string-net structure and the gauge charges can be viewed as the ends of strings. In Refs. [23,24] various duality relations between lattice gauge theories and theories of extended objects were reviewed. In particular, some statistical lattice gauge models were found to be dual to certain statistical membrane models [25]. This duality relation is directly connected to the relation between gauge theory and closed-string-net theory [13] in quantum models.

To have emergent gauge bosons at low energies, the string nets do not have be a fundamental object in the model. The string net can simply be lines of flipped spins in a spin lattice model. Thus deconfined gapless gauge bosons can emerge from a local bosonic model if the Hamiltonian has the right couplings [13,15,27,26].

Emergent fermions from local bosonic models also have a complicated history. References [28–30] discovered that fermions can emerge from purely bosonic gauge theory. The first examples of emergent fermions/anyons from local bosonic models were the fractional quantum Hall states [4,5], where fermionic/anyonic excitations were obtained theoretically from interacting bosons in a magnetic field [31]. In 1987, fermion fields and gauge fields were introduced to express the spin-1/2 Hamiltonian in the slave-boson approach [32,33]. However, writing a bosonic Hamiltonian in terms of fermion fields does not imply the appearance of well defined fermionic quasiparticles. Emergent fermionic excitations can appear only in deconfined phases of the gauge field. References [34–37] constructed several deconfined phases where the fermion fields do describe well defined quasiparticles. However, depending on the property of deconfined phases, those quasiparticles may carry fractional statistics (for the chiral spin states) [34,35,38] or Fermi statistics (for the Z_2 deconfined states) [36,37].

Also in 1987, in a study of resonating-valence-bond states, emergent fermions (the spinons) were proposed in a nearest neighbor dimer model on a square lattice [39–41]. But, according to the deconfinement picture, the results in Refs. [39,40] are valid only when the ground state of the dimer model is in the Z_2 deconfined phase. It appears that the dimer liquid on a square lattice with only nearest neighbor dimers is not a deconfined state [40,41], and thus it is not clear whether or not the nearest neighbor dimer model on a square lattice [40] has fermionic quasiparticles [41]. However, on a triangular lattice, the dimer liquid is indeed a Z_2 deconfined state [42]. Therefore, the results in Refs. [39,40] are valid for the triangular-lattice dimer model and fermionic quasiparticles do emerge in a dimer liquid on a triangular lattice.

All the above models with emergent fermions are (2+1)D models, where the emergent fermions can be understood in terms of binding flux to a charged particle [31]. Recently, it was pointed out in Ref. [14] that the key to emergent fermions is a string structure. Fermions can generally appear as the ends of open strings. The string picture allows construction of a (3+1)D local bosonic model that has emergent fermions [14].

Compared with those previous results, the new features discussed in this paper are as follows. (A) *Massless* gauge bosons and fermions can emerge from *local bosonic models* as a result of string-net condensation. (B) Massless fermions are protected by the string-net condensation (and the associated PSG). (C) String-net condensed states represent a new kind of phase which cannot be described Landau's symmetry breaking theory. Different string-net condensed states are characterized by different PSG's. (D) QED and QCD can emerge from a local bosonic model on a cubic lattice. The effective QED and QCD have $4N$ families of leptons and quarks. Each family has one lepton and two flavors of quarks.

The bottom line is that, within local bosonic models, massless fermions do not just emerge by themselves. Emergent massless fermions, emergent massless gauge bosons, string-net condensations, and PSG's are intimately related. They are just different sides of the same coin—quantum order.

According to the picture of quantum order, elementary particles (such as photons and electrons) may not be elementary after all. They may be collective excitations of a local bosonic system below the Planck scale. Since we cannot do experiments close to the Planck scale, it is hard to determine if photons and electrons are elementary particles or not. In this paper, we would like to show that the string-net picture of light and fermions is at least self-consistent by studying some concrete local boson models that produce massless gauge bosons and massless fermions through string-net condensation. The local boson models studied here are just a few examples among a long list of local boson models [8,26,27,33,34,36–38,40,42–51] that contain emergent fermions and gauge fields.

Here we would like to stress that the string-net picture for the actual gauge bosons and fermions in our universe is only a proposal at the moment. Although string-net condensation can produce and protect massless photons, gluons, quarks, and other charged leptons, we do not know at the moment if string-net condensation can produce neutrinos, which are chiral fermions, and the weak-interaction $SU(2)$ gauge field, which couples chirally to the quarks and the leptons. Also, we do not know if string-net condensation can produce an odd number of families of quarks and leptons. The QED and QCD produced by the known string-net condensations all contain an even number of families so far. The correctness of string-net condensation in our vacuum depends on resolving the above problems. Nature has four fascinating and somewhat strange properties: gauge bosons, Fermi statistics, chiral fermions, and gravity. The string-net condensation picture provides a natural explanation for the first two properties. Two more to go.

On the other hand, if we are concerned about the condensed matter problem of how to use bosons to make artificial light and artificial fermions, then the string-net picture and quantum order do provide an answer. To make artificial light and artificial fermions, we simply let certain string nets condense.

In some recent work, types of quantum order and their connection to emergent gauge bosons and fermions were studied using PSG's, without realizing their connection to string-net condensation [9,15,50]. In this paper, we will show that the quantum ordered states described by PSG's are actually string-net condensed states. The gauge bosons and fermions produced and protected by the PSG's have a very natural string-net interpretation [13,14]. Quantum order, the PSG, and string-net condensation are different parts of the same story. Here we will summarize and expand the previous work and try to present a coherent picture of quantum order, the PSG, and string-net condensation, as well as the associated emergent gauge bosons and fermions.

F. Organization

Section III reviews the work in Ref. [14]. We will study an exactly soluble spin-1/2 model on a square lattice [8,50]. The model was solved using the slave-boson approach [50]. This allowed us to identify the PSG that characterizes the nontrivial quantum order in the ground state [50]. Here, following Ref. [14], we will solve the model from the string-net condensation point of view. Since the ground state of the model can be described by both string-net condensation and the PSG, this allows us to demonstrate the direct connection between string-net condensation and the PSG in Sec. IV. The model is also one of the simplest models that demonstrates the connection between string-net condensation and the emergent gauge field and fermions [8,14].

However, the above exact soluble model does not contain gapless gauge boson and gapless fermions. If we regard the lattice scale as the Planck scale, then gauge bosons and fermions do not "exist" in our model in the sense discussed in Sec. I A. In Sec. V, we will discuss an exact soluble local bosonic model that contains massless Dirac fermions. In Secs. VII and VIII, we will discuss local bosonic models that give rise to massless electrons, quarks, gluons, and photons. Gauge bosons and fermions "exist" in these latter models.

II. LOCAL BOSONIC MODELS

In this paper, we will consider only local bosonic models. Local bosonic models are important since they are really local. We note that fermionic models are in general nonlocal since the fermion operators at different sites do not commute, even when the sites are well separated. Due to their intrinsic locality, local bosonic models are natural candidates for the fundamental theory of nature. In the following we will give a detailed definition of local bosonic models.

To define a physical system, we need to specify (A) a total Hilbert space, (B) a definition of a set of local physical operators, and (C) a Hamiltonian. With this understanding, a local bosonic model is defined to be a model that satisfies the following. (A) The total Hilbert space is a direct product of local Hilbert spaces of finite dimensions. (B) Local physical operators are local bosonic operators. By definition, *local bosonic operators* are operators acting within a local Hilbert space or finite products of those operators for nearby local Hilbert spaces. Those operators are called local bosonic operators since they all commute with each other when far apart. (C) The Hamiltonian is a sum of local physical operators.

A spin-1/2 system on a lattice is an example of local bosonic models. The local Hilbert space is two dimensional and contains $|\uparrow\rangle$ and $|\downarrow\rangle$ states. The local physical operators are $\sigma^a_{\boldsymbol{i}}$, $\sigma^a_{\boldsymbol{i}}\sigma^b_{\boldsymbol{i}+\boldsymbol{x}}$, etc., where σ^a, $a=x,y,z$, are the Pauli matrices.

A free spinless fermion system (in two or higher dimensions) is not a local bosonic model even though it has the same total Hilbert space as the spin-1/2 system. This is because the fermion operators $c_{\boldsymbol{i}}$ on different sites do not commute and are not local bosonic operators. More importantly, the fermion hopping Hamiltonian in two and higher dimensions cannot be written as a sum of local bosonic operators. (Note that in higher dimensions we cannot write all the hopping terms $c^\dagger_{\boldsymbol{i}}c_{\boldsymbol{j}}$ as products of local bosonic operators. However, due to the Jordan-Wigner transformation, a 1D fermion hopping term $c^\dagger_{i+1}c_i$ can be written as a local bosonic operator. Hence, a 1D fermion system can be a local bosonic model if we exclude $c_{\boldsymbol{i}}$ from our definition of local physical operators.)

The bosonic field theory without cutoff is not a local bosonic model. This is because the local Hilbert space does not have a finite dimension. A lattice gauge theory is not a local bosonic model. This is because its total Hilbert space cannot be a direct product of local Hilbert spaces.

Another counterexample of a local bosonic model is a quantum closed-string-net model. A quantum closed-string-net model on a lattice can be defined in the following way. Let us consider only strings that cover nearest neighbor links. A closed-string configuration may have many closed

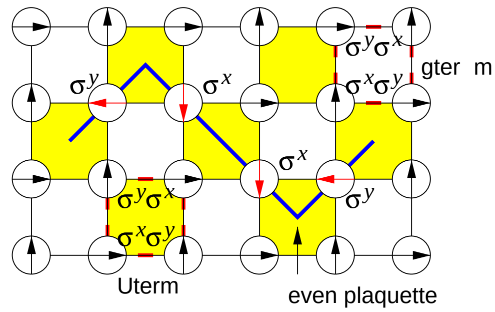

FIG. 2. An open-string excitation on top of the ground state of H_J.

strings with or without overlap. We will call a closed-string configuration a closed string net. For every closed string net, we assign a quantum state. All those quantum states form a basis of the total Hilbert space of the closed-string-net model. Just as in lattice gauge theory, the closed-string-net model is not a local bosonic model since the total Hilbert space cannot be a direct product of local Hilbert spaces. It turns out that closed-string-net models and lattice gauge models are closely related. In fact some closed-string-net models (or statistical membrane models) are equivalent to lattice gauge models [22–25].

III. Z_2 SPIN LIQUID AND STRING-NET CONDENSATION ON A SQUARE LATTICE

A. Hamiltonians with closed-string-net condensation

Let us first consider an arbitrary spin-1/2 model on a square lattice. The first question that we want to ask is what kind of spin interaction can given rise to a low energy gauge theory. If we believe the connection between gauge theory and closed-string-net theory [13,22–25], then one way to obtain a low energy gauge theory is to design a spin interaction that allows strong fluctuations of large closed string nets, but forbids other types of fluctuations (such as local spin flips, open-string-net fluctuations, etc.). (Note that closed string nets are nets of strings formed by intersecting/overlapping closed strings, while open string nets are nets of strings containing at least one open string.) We hope the presence of strong fluctuations of large closed strings will lead to condensation of closed strings of arbitrary sizes, which in turn gives rise to a low energy gauge theory.

Let us start with

$$H_J = -J\sum_{even}\sigma^x_{\boldsymbol{i}} - J\sum_{odd}\sigma^y_{\boldsymbol{i}}, \tag{1}$$

where $\boldsymbol{i}=(i_x,i_y)$ labels the lattice sites, $\sigma^{x,y,z}$ are the Pauli matrices, and Σ_{even} (or Σ_{odd}) is a sum over even sites with $(-)^{\boldsymbol{i}}\equiv(-1)^{i_x+i_y}=1$ [or over odd sites with $(-)^{\boldsymbol{i}}\equiv(-1)^{i_x+i_y}=-1$]. The ground state of H_J, $|0\rangle$, has spins pointing to the x direction on even sites and to the y direction on odd sites (see Fig. 2). Such a state will be defined as a state with no string.

To create a string excitation, we first draw a string that connects nearest neighbor *even* plaquettes (see Fig. 2). We then flip the spins in the string. Such a string state is created by the following string creation operator (or simply string operator):

$$W(C)=\prod_C \sigma_{\boldsymbol{i}}^{a_{\boldsymbol{i}}}, \tag{2}$$

where the product Π_C is over all the sites on the string and $a_{\boldsymbol{i}}=y$ if $\boldsymbol{i}$ is even and $a_{\boldsymbol{i}}=x$ if $\boldsymbol{i}$ is odd. A generic string state has the form

$$|C_1C_2\cdots\rangle=W(C_1)W(C_2)\cdots|0\rangle, \tag{3}$$

where $C_1,C_2,\ldots$ are strings with no overlapping ends. Such a state will be called a string-net state and

$$W(C_{net})=W(C_1)W(C_2)\cdots$$

will be called a string-net operator. The state $|C_1C_2\cdots\rangle$ is an open-string-net state if at least one of C_i is an open string. The corresponding operator $W(C_{net})$ will be called an open-string-net operator. If all C_i are closed loops, then $|C_1C_2\cdots\rangle$ is a closed-string-net state and $W(C_{net})$ a closed-string-net operator. The Hamiltonian has no string-net condensation since its ground state $|0\rangle$ contains no string nets. To obtain a Hamiltonian with closed-string-net condensation, we need to first find a Hamiltonian whose ground state contains a lot of closed string nets of arbitrary sizes and does not contain open string nets.

Let us first write down a Hamiltonian such that closed strings cost no energy and any open strings cost a large amount of energy. One such Hamiltonian has the form

$$H_U=-U\sum_{even}\hat{F}_{\boldsymbol{i}},$$

$$\hat{F}_{\boldsymbol{i}}=\sigma_{\boldsymbol{i}}^x\sigma_{\boldsymbol{i}+\boldsymbol{x}}^y\sigma_{\boldsymbol{i}+\boldsymbol{x}+\boldsymbol{y}}^x\sigma_{\boldsymbol{i}+\boldsymbol{y}}^y. \tag{4}$$

We find that the no-string state $|0\rangle$ is one of the ground states of H_U (assuming $U>0$) with energy $-UN_{site}$. All the closed-string-net states, such as $W(C_{close})|0\rangle$, are also ground states of H_U since $[H_U,W(C_{close})]=0$. An open-string state $W(C_{open})|0\rangle$ is also an eigenstate of H_U but with energy $-UN_{site}+2U$. We see that each end of an open string costs an energy U. We also note that the energy of closed strings does not depend on the length of the closed strings. Thus the closed strings in H_U have no tension. We can introduce a string tension by adding H_J to our Hamiltonian. The string tension will be $2J$ per site (or per segment). We note that any string-net state $|C_1C_2\cdots\rangle$ is an eigenstate of H_U+H_J. Thus, string nets in the model described by H_U+H_J do not fluctuate and hence cannot condense. To make string nets fluctuate, we need a g term:

$$H_g=g\sum_{\boldsymbol{p}} U(C_{\boldsymbol{p}}), \tag{5}$$

where $\boldsymbol{p}$ labels the odd plaquettes and $C_{\boldsymbol{p}}$ is the closed string around the plaquette $\boldsymbol{p}$. In fact,

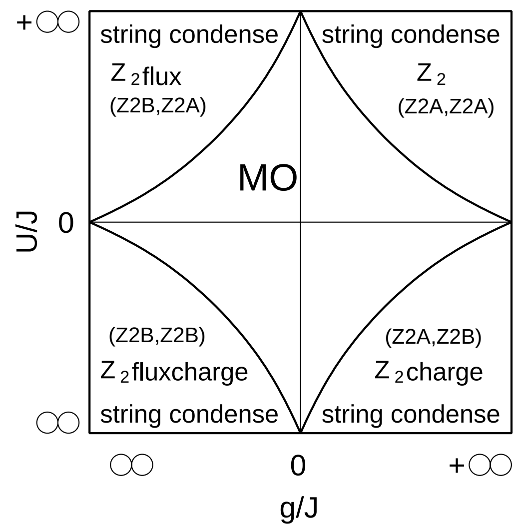

FIG. 3. The proposed phase diagram for the $H=H_U+H_g+H_J$ model. J is assumed to be positive. The four string-net condensed phases are characterized by a pair of PSG's (PSG_{charge},PSG_{vortex}). MO marks a magnetic ordered state.

$$H_g=-g\sum_{odd}\hat{F}_{\boldsymbol{i}}. \tag{6}$$

In this way, we obtain the Hamiltonian of our spin-1/2 model as

$$H=H_U+H_J+H_g. \tag{7}$$

B. String condensation and low energy effective theory

When $J=0$ in Eq. (7), the model is exactly soluble since $[\hat{F}_{\boldsymbol{i}},\hat{F}_{\boldsymbol{j}}]=0$ [8,50]. All the eigenstates of H_U+H_g can be obtained from the common eigenstates of $\hat{F}_{\boldsymbol{i}}$. Since $\hat{F}_{\boldsymbol{i}}^2=1$, the eigenvalues of $\hat{F}_{\boldsymbol{i}}$ are simply ±1. Thus all the eigenstates of H_U+H_g are labeled by ±1 on each plaquette. (Note that this is not true for finite systems where the boundary conditions introduce additional complications [50].) The energies of those eigenstates are the sums of the eigenvalues of $\hat{F}_{\boldsymbol{i}}$ weighted by U and g.

From the results of the exact soluble model, we suggest a phase diagram of our model as sketched in Fig. 3. We will show that the phase diagram contains four different string-net condensed phases and one phase with no string condensation. All the phases have the same symmetry and are distinguished only by their different quantum orders.

Let us first discuss the phase with $U,g>0$. We will assume $J=0$ and $U\gg g$. In this limit, all states containing open strings will have an energy of order U. The low energy states contain only closed strings (or more generally closed string nets) and satisfy

$$\hat{F}_{\boldsymbol{i}}|_{\boldsymbol{i}=even}=1. \tag{8}$$

For infinite systems, the different low energy states are labeled by the eigenvalues of $\hat{F}_{\boldsymbol{i}}$ on odd plaquettes:

$$\hat{F}_{\boldsymbol{i}}|_{\boldsymbol{i}=odd} = \pm 1. \tag{9}$$

In particular, the ground state is given by

$$\hat{F}_{\boldsymbol{i}}|_{\boldsymbol{i}=odd} = 1. \tag{10}$$

All the closed-string-net operators $W(C_{net})$ commute with $H_U + H_g$. Hence the ground state $|\Psi_0\rangle$ of $H_U + H_g$ satisfies

$$\langle \Psi_0 | W(C_{net}) | \Psi_0 \rangle = 1. \tag{11}$$

Thus the $U,g > 0$ ground state has a closed-string-net condensation. The low energy excitations above the ground state can be obtained by flipping $\hat{F}_{\boldsymbol{i}}$ from 1 to -1 on some odd plaquettes.

If we view $\hat{F}_{\boldsymbol{i}}$ on odd plaquettes as the flux in Z_2 gauge theory, we find that the low energy sector of the model is identical to a Z_2 lattice gauge theory, at least for infinite systems. This suggests that the low energy effective theory of our model is a Z_2 lattice gauge theory.

However, one may object to this result by pointing out that the low energy sector of our model is also identical to an Ising model with one spin on each odd plaquette. Thus the low energy effective theory should be the Ising model. We would like to point out that, although the low energy sector of our model is identical to an Ising model for infinite systems, the low energy sector of our model is different from an Ising model for finite systems. For example, on a finite even by even lattice with periodic boundary conditions, the ground state of our model has a fourfold degeneracy [8,50]. The Ising model does not have such a degeneracy. Also, our model contains an excitation that can be identified as a Z_2 charge (see below). Therefore, the low energy effective theory of our model is a Z_2 lattice gauge theory instead of an Ising model. The $\hat{F}_{\boldsymbol{i}} = -1$ excitations on odd plaquettes can be viewed as the Z_2 vortex excitations in the Z_2 lattice gauge theory.

C. Three types of strings and emergent fermions

What is the Z_2 charge excitation? We note that, in the closed-string-net condensed state, the action of the closed-string operator Eq. (2) on the ground state is trivial. This suggests that the action of the open-string operators on the ground state depend only on the ends of strings, since two open strings with the same ends differ only by a closed string. Therefore, an open-string operator creates two particles at its ends when acting on the string condensed state. Since the strings in Eq. (2) connect only even plaquettes, the particle corresponding to the ends of the open strings always live on the even plaquettes. We will call such a string a T1 string. From the commutation relation between $\hat{F}_{\boldsymbol{i}}$ and the open-string operators, we find that the open-string operators flip the signs of $\hat{F}_{\boldsymbol{i}}$ at its ends. Thus each particle created by the open-string operators has an energy $2U$. Now, let us consider the hopping of one such particle around four nearest neighbor even plaquettes (see Fig. 4). We see that the product of the four hopping amplitudes is given by the eigenvalue of $\hat{F}_{\boldsymbol{i}}$ on the odd plaquette in the middle of the four even plaquettes [8,14]. This is exactly the relation between charge and flux. Thus if we identify $\hat{F}_{\boldsymbol{i}}$ on odd plaquettes as a Z_2 flux, then the ends of strings on even plaquettes will correspond to the Z_2 charges. We note that, due to the closed-string condensation, the ends of open strings are not confined and have only short ranged interactions between them. Thus the Z_2 charges behave like quasiparticles with no string attached.

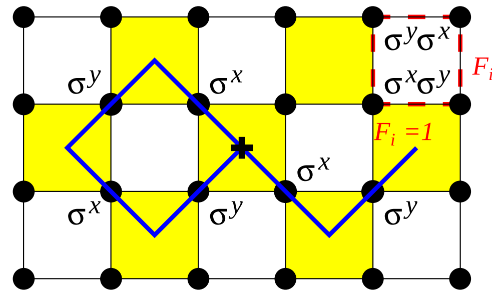

FIG. 4. A hopping of the Z_2 charge around four nearest neighbor even plaquettes.

Just like the Z_2 charges, a pair of Z_2 vortices is also created by an open-string operator. Since the Z_2 vortices correspond to flipped $\hat{F}_{\boldsymbol{i}}$ on *odd* plaquettes, the open-string operator that creates Z_2 vortices is also given by Eq. (2), except now the product is over a string that connects *odd* plaquettes. We will call such a string a T2 string. (The strings connecting *even* plaquettes were called T1 strings.)

We would like to point out that the reference state (i.e., the no-string state) for the T2 string is different from that of the T1 string. The no-T2-string state is given by $|\tilde{0}\rangle$ with spin pointing in the y direction on even sites and the x direction on odd sites. Since the T1 and T2 strings have different reference states, we cannot have a dilute gas of the T1 strings and the T2 strings at the same time. One can easily check that the T2 string operators also commute with $H_U + H_g$. Therefore, the ground state $|\Psi_0\rangle$, in addition to the T1 closed-string condensation, also has a T2 closed-string condensation.

The hopping of a Z_2 vortex is induced by a short T2 open string. Since the T2 open-string operators all commute with each other, the Z_2 vortices behave like bosons. Similarly, the Z_2 charges also behave like bosons. However, T1 open-string operators and T2 open-string operators do not commute. As a result, the ends of T1 strings and the ends of T2 strings have nontrivial mutual statistics. As we have already shown that moving a Z_2 charge around a Z_2 vortex generates a phase π, the Z_2 charges and the Z_2 vortices have semionic mutual statistics.

The T3 strings are defined as bound states of T1 and T2 strings. The T3 string operator has the form $W(C) = \prod_n \sigma_{\boldsymbol{i}_n}^{l_n}$, where C is a string connecting the midpoints of the neighboring links, and $\boldsymbol{i}_n$ are sites on the string. $l_m = z$ if the string does not turn at site $\boldsymbol{i}_m$. $l_m = x$ or y if the string makes a turn at site $\boldsymbol{i}_m$. $l_m = x$ if the turn forms an upper-right or lower-left corner. $l_m = y$ if the turn forms a lower-right or upper-left corner. (See Fig. 6 below for these details.) The ground state also has a condensation of T3 closed strings. The ends of T3 string, as bound states of the Z_2 charges and the Z_2 vortices, are *fermions*. The bound state is formed by a Z_2 charge and a Z_2 vortex on the two plaquettes on the two sides of a link (i.e., $F_{\boldsymbol{i}}=-1$ on the two sides of the link). Thus the fermions live on the links. It is interesting to see that string-net condensation in our model directly leads to a Z_2 gauge structure and three new types of quasiparticles: Z_2 charge, Z_2 vortex, and fermions. Fermions, as the ends of open T3 strings, emerge from our purely bosonic model.

Since the ends of T1 string are Z_2 charges, the T1 strings can be viewed as strings of Z_2 "electric" flux. Similarly, the T2 strings can be viewed as strings of Z_2 "magnetic" flux.

IV. CLASSIFICATION OF DIFFERENT STRING CONDENSATIONS BY PSG'S

A. Four classes of string-net condensations

As we saw in the last section, when $U>0$, $g>0$, and $J=0$, the ground state of our model is given by

$$\hat{F}_{\boldsymbol{i}}|_{\boldsymbol{i}=even}=1, \quad \hat{F}_{\boldsymbol{i}}|_{\boldsymbol{i}=odd}=1. \tag{12}$$

We will call such a phase the Z_2 phase to stress the low energy Z_2 gauge structure. In the Z_2 phase, the T1 string operator $W_1(C_1)$ and the T2 string operator $W_2(C_2)$ have the following expectation values

$$\langle W_1(C_1)\rangle=1, \quad \langle W_2(C_2)\rangle=1. \tag{13}$$

When $U>0$, $g<0$, and $J=0$, the ground state is given by

$$\hat{F}_{\boldsymbol{i}}|_{\boldsymbol{i}=even}=1, \quad \hat{F}_{\boldsymbol{i}}|_{\boldsymbol{i}=odd}=-1. \tag{14}$$

We see that there is π flux through each odd plaquette. We will call such a phase the Z_2 flux phase. The T1 string operator and the T2 string operator have the following expectation values:

$$\langle W_1(C_1)\rangle=(-)^{N_{odd}}, \quad \langle W_2(C_2)\rangle=1, \tag{15}$$

where N_{odd} is the number of odd plaquettes enclosed by the T1 string C_1.

When $U<0$, $g>0$, and $J=0$, the ground state is

$$\hat{F}_{\boldsymbol{i}}|_{\boldsymbol{i}=even}=-1, \quad \hat{F}_{\boldsymbol{i}}|_{\boldsymbol{i}=odd}=1. \tag{16}$$

The ground state has a Z_2 charge on each even plaquette. We will call such a phase the Z_2 charge phase. The T1 string operator and the T2 string operator have the following expectation values:

$$\langle W_1(C_1)\rangle=1, \quad \langle W_2(C_2)\rangle=(-)^{N_{even}}, \tag{17}$$

where N_{even} is the number of even plaquettes enclosed by the T2 string C_2. Note that the Z_2 flux phase and the Z_2 charge phase, different only by a lattice translation, are essentially the same phase.

When $U<0$, $g<0$, and $J=0$, the ground state becomes

$$\hat{F}_{\boldsymbol{i}}|_{\boldsymbol{i}=even}=-1, \quad \hat{F}_{\boldsymbol{i}}|_{\boldsymbol{i}=odd}=-1. \tag{18}$$

There is a Z_2 charge on each even plaquette and π flux through each odd plaquette. We will call such a phase the Z_2 flux charge phase. The T1 string operator and the T2 string operator have the following expectation values:

$$\langle W_1(C_1)\rangle=(-)^{N_{odd}}, \quad \langle W_2(C_2)\rangle=(-)^{N_{even}}. \tag{19}$$

B. PSG's and the ends of condensed strings

From the different $\langle W_1(C_1)\rangle$ and $\langle W_2(C_2)\rangle$, we see that the above four phases have different string-net condensations. However, they all have the same symmetry. This raises an issue. Without symmetry breaking, how do we know the above four phases are really different phases? How do we know that it is impossible to change one string-net condensed state to another without a phase transition?

In the following, we will show that the different string-net condensations can be described by different PSG's (just as different symmetry breaking orders can be described by different symmetry groups of ground states). In Refs. [9,10], different types of quantum order were introduced via their different PSG's. The connection between string-net condensation and the PSG allows us to connect string-net condensation to the quantum order introduced in Refs. [9,10]. In particular, the PSG's are shown to be a universal property of a quantum phase, which can be changed only by phase transitions. Thus the different PSG's for the different string-net condensed states indicate that those different string-net condensed states belong to different quantum phases.

When closed string nets condense, the ends of open strings behave like independent particles. Let us consider two-particle states $|\boldsymbol{p}_1\boldsymbol{p}_2\rangle$ described by the two ends of a T1 string. Note that the ends of the T1 strings, and hence the Z_2 charges, live only on the even plaquettes. Here $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ label the even plaquettes. For our model H_U+H_g, $|\boldsymbol{p}_1\boldsymbol{p}_2\rangle$ is an energy eigenstate and the Z_2 charges do not hop. Here we would like to add a term

$$H_t=t\sum_{\boldsymbol{i}} (\sigma^x_{\boldsymbol{i}}+\sigma^y_{\boldsymbol{i}})+t'\sum_{\boldsymbol{i}} \sigma^z_{\boldsymbol{i}} \tag{20}$$

to the Hamiltonian. The t term $t\Sigma_{\boldsymbol{i}}(\sigma^x_{\boldsymbol{i}}+\sigma^y_{\boldsymbol{i}})$ makes the Z_2 charges hop among the even plaquettes with a hopping amplitude of order t. The dynamics of the two Z_2 charges is described by the following *low energy effective* Hamiltonian in the two-particle Hilbert space:

$$H=H(\boldsymbol{p}_1)+H(\boldsymbol{p}_2), \tag{21}$$

where $H(\boldsymbol{p}_1)$ describes the hopping of the first particle $\boldsymbol{p}_1$ and $H(\boldsymbol{p}_2)$ describes the hopping of the second particle $\boldsymbol{p}_2$. Now we can define the PSG in a string-net condensed state. The PSG is nothing but the symmetry group of the hopping Hamiltonian $H(\boldsymbol{p})$.

We know that in a symmetry breaking phase the low energy effective theory has a lower symmetry than the bare Hamiltnoian at high energies. Thus we can use the symmetry of the low energy effective Hamiltonian to characterize different symmetry breaking phases. Here, using the hopping Hamiltonian and its PSG to characterize different string-net condensations is a similar idea.

Due to the translation symmetry of the underlying model $H_U+H_g+H_t$, we may naively expect the hopping Hamiltonian of the Z_2 charge $H(\boldsymbol{p})$ to also have a translation symmetry

$$H(\boldsymbol{p})=T^{\dagger}_{xy}H(\boldsymbol{p})T_{xy}, \quad T_{xy}|\boldsymbol{p}\rangle=|\boldsymbol{p}+\boldsymbol{x}+\boldsymbol{y}\rangle,$$

$$H(\boldsymbol{p})=T^{\dagger}_{x\bar{y}}H(\boldsymbol{p})T_{x\bar{y}}, \quad T_{x\bar{y}}|\boldsymbol{p}\rangle=|\boldsymbol{p}+\boldsymbol{x}-\boldsymbol{y}\rangle. \tag{22}$$

The above implies that the PSG is the translation symmetry group. It turns out that Eq. (22) is too strong. The underlying spin model can have translation symmetry even when $H(\boldsymbol{p})$ does not satisfy Eq. (22). However, the possible symmetry groups of $H(\boldsymbol{p})$ (the PSG's) are strongly constrained by the translation symmetry of the underlying spin model. In the following, we will explain why the PSG can be different from the symmetry group of the physical spin model, and what conditions the PSG must satisfy in order to be consistent with the translation symmetry of the spin model.

We note that a string always has two ends. Thus a physical state always has an even number of Z_2 charges. The actions of translation on a two-particle state are given by

$$T^{(2)}_{xy}|\boldsymbol{p}_1,\boldsymbol{p}_2\rangle=e^{\theta_{xy}(\boldsymbol{p}_1,\boldsymbol{p}_2)}|\boldsymbol{p}_1+\boldsymbol{x}+\boldsymbol{y},\boldsymbol{p}_2+\boldsymbol{x}+\boldsymbol{y}\rangle,$$

$$T^{(2)}_{x\bar{y}}|\boldsymbol{p}_1,\boldsymbol{p}_2\rangle=e^{\theta_{x\bar{y}}(\boldsymbol{p}_1,\boldsymbol{p}_2)}|\boldsymbol{p}_1+\boldsymbol{x}-\boldsymbol{y},\boldsymbol{p}_2+\boldsymbol{x}-\boldsymbol{y}\rangle. \tag{23}$$

The phases $e^{\theta_{xy}(\boldsymbol{p}_1,\boldsymbol{p}_2)}$ and $e^{\theta_{x\bar{y}}(\boldsymbol{p}_1,\boldsymbol{p}_2)}$ come from the ambiguity of the location of the string that connects $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$, i.e., the phases can be different if the string connecting the two Z_2 charges has different locations. $T^{(2)}_{xy}$ and $T^{(2)}_{x\bar{y}}$ satisfy the algebra of translations

$$T^{(2)}_{xy}T^{(2)}_{x\bar{y}}=T^{(2)}_{x\bar{y}}T^{(2)}_{xy}. \tag{24}$$

$T^{(2)}_{xy}$ and $T^{(2)}_{x\bar{y}}$ are direct products of translation operators on the single-particle states. Thus, in some sense, the single-particle translations are square roots of two-particle translations.

The most general form of single-particle translations is given by $T_{xy}G_{xy}$ and $T_{x\bar{y}}G_{x\bar{y}}$, where the actions of the operators $T_{xy,x\bar{y}}$ and $G_{xy,x\bar{y}}$ are defined as

$$T_{xy}|\boldsymbol{p}\rangle=|\boldsymbol{p}+\boldsymbol{x}+\boldsymbol{y}\rangle,$$

$$T_{x\bar{y}}|\boldsymbol{p}\rangle=|\boldsymbol{p}+\boldsymbol{x}-\boldsymbol{y}\rangle,$$

$$G_{xy}|\boldsymbol{p}\rangle=e^{i\phi_{xy}(\boldsymbol{p})}|\boldsymbol{p}\rangle,$$

$$G_{x\bar{y}}|\boldsymbol{p}\rangle=e^{i\phi_{x\bar{y}}(\boldsymbol{p})}|\boldsymbol{p}\rangle. \tag{25}$$

In order for the direct products $T^{(2)}_{xy}=T_{xy}G_{xy}\otimes T_{xy}G_{xy}$ and $T^{(2)}_{x\bar{y}}=T_{x\bar{y}}G_{x\bar{y}}\otimes T_{x\bar{y}}G_{x\bar{y}}$ to reproduce the translation algebra Eq. (24), we only require $T_{xy}G_{xy}$ and $T_{x\bar{y}}G_{x\bar{y}}$ to satisfy

$$T_{xy}G_{xy}T_{x\bar{y}}G_{x\bar{y}}=T_{x\bar{y}}G_{x\bar{y}}T_{xy}G_{xy}, \tag{26}$$

or

$$T_{xy}G_{xy}T_{x\bar{y}}G_{x\bar{y}}=-T_{x\bar{y}}G_{x\bar{y}}T_{xy}G_{xy}. \tag{27}$$

The operators $T_{xy}G_{xy}$ and $T_{x\bar{y}}G_{x\bar{y}}$ generate a group. Such a group is the PSG introduced in Ref. [9]. The two different algebras Eq. (26) and Eq. (27) generate two different PSG's; both are consistent with the translation group acting on the two-particle states. We will call the PSG generated by Eq. (26) the $Z2A$ PSG and the PSG generated by Eq. (27) the $Z2B$ PSG.

Let us give a more general definition of a PSG. A PSG is a group. It is an extension of the symmetry group (SG), i.e., a PSG contains a normal subgroup (called an invariant gauge group or IGG) such that

$$\text{PSG/IGG}=\text{SG}. \tag{28}$$

For our case, the SG is the translation group $SG=\{1,T^{(2)}_{xy},T^{(2)}_{x\bar{y}},\dots\}$. For every element in a SG, $a^{(2)}\in\text{SG}$, there are one or several elements in PSG, $a\in\text{PSG}$, such that $a\otimes a=a^{(2)}$. The IGG in our PSG is formed by the transformations G_0 on the single-particle states that satisfy $G_0\otimes G_0=1$. We find that the IGG is generated by

$$G_0|\boldsymbol{p}\rangle=-|\boldsymbol{p}\rangle. \tag{29}$$

G_0, $T_{xy}G_{xy}$, and $T_{x\bar{y}}G_{x\bar{y}}$ generate the $Z2A$ and $Z2B$ PSG's.

Now we see that the underlying translation symmetry does not require the single-particle hopping Hamiltonian $H(\boldsymbol{p})$ to have a translation symmetry. It only requires $H(\boldsymbol{p})$ to be invariant under the $Z2A$ PSG or the $Z2B$ PSG. When $H(\boldsymbol{p})$ is invariant under the $Z2A$ PSG, the hopping Hamiltonian has the usual translation symmetry. When $H(\boldsymbol{p})$ is invariant under the $Z2B$ PSG, the hopping Hamiltonian has a magnetic translation symmetry describing hopping in a magnetic field with π flux through each odd plaquette.

C. PSG's classify different string-net condensations

After understand the possible PSG's for the hopping Hamiltonian of the ends of strings, now we are ready to calculate the actual PSG's. Let us consider two ground states of our model $H_U+H_g+H_t$. One has $\hat{F}_{\boldsymbol{i}}|_{\boldsymbol{i}=odd}=1$ (for $g>0$) and the other has $\hat{F}_{\boldsymbol{i}}|_{\boldsymbol{i}=odd}=-1$ (for $g<0$). Both ground states have the same translation symmetry in the $\boldsymbol{x}+\boldsymbol{y}$ and $\boldsymbol{x}-\boldsymbol{y}$ directions. However, the corresponding single-particle hopping Hamiltonian $H(\boldsymbol{p})$ has different symmetries. For the $\hat{F}_{\boldsymbol{i}}|_{\boldsymbol{i}=odd}=1$ state, there is no flux through odd plaquettes and $H(\boldsymbol{p})$ has the usual translation symmetry. It is invariant under the $Z2A$ PSG. For the $\hat{F}_{\boldsymbol{i}}|_{\boldsymbol{i}=odd}=-1$ state, there is π flux through odd plaquettes and $H(\boldsymbol{p})$ has a magnetic translation symmetry. Its PSG is the $Z2B$ PSG. Thus the $\hat{F}_{\boldsymbol{i}}|_{\boldsymbol{i}=odd}=1$ state and the $\hat{F}_{\boldsymbol{i}}|_{\boldsymbol{i}=odd}=-1$ state have different orders even though they have the same symmetry. The different quantum orders in the two states can be characterized by their different PSG's.

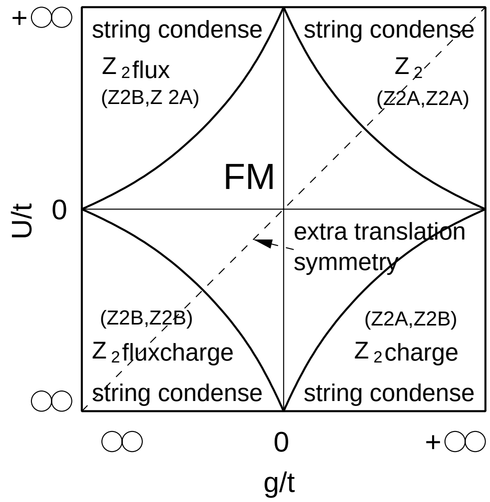

FIG. 5. The proposed phase diagram for the $H=H_U+H_g+H_t$ model. $t=t'$ is assumed to be positive. The four string-net condensed phases are characterized by a pair of PSG's $(\mathrm{PSG}_{charge},\mathrm{PSG}_{vortex})$. FM marks a ferromagnetic phase.

The above discussion also applies to the Z_2 vortex and T2 strings. Thus the quantum orders in our model are described by a pair of PSG's $(\mathrm{PSG}_{charge},\mathrm{PSG}_{vortex})$, one for the Z_2 charge and one for the Z_2 vortex. The PSG pairs $(\mathrm{PSG}_{charge},\mathrm{PSG}_{vortex})$ allow us to distinguish four different string-net condensed states of the model $H=H_U+H_g+H_t$. (See Fig. 5.)

Now let us assume $U=g$ in our model:

$$H_U+H_g+H_t=H_t-V\sum_{\boldsymbol{i}}\hat{F}_{\boldsymbol{i}}. \tag{30}$$

The new physical spin model has a larger translation symmetry generated by $\Delta\boldsymbol{i}=\boldsymbol{x}$ and $\Delta\boldsymbol{i}=\boldsymbol{y}$ (see Fig. 5). Due to the enlarged symmetry group, the quantum order in the new system should be characterized by a new PSG. In the following, we will calculate the new PSG.

The single-particle states are given by $|\boldsymbol{p}\rangle$. When $\boldsymbol{p}$ is even, $|\boldsymbol{p}\rangle$ corresponds to a Z_2 charge and when $\boldsymbol{p}$ is odd, $|\boldsymbol{p}\rangle$ corresponds to a Z_2 vortex. We see that a translation by $\boldsymbol{x}$ (or $\boldsymbol{y}$) will change a Z_2 charge to a Z_2 vortex or a Z_2 vortex to a Z_2 charge. Therefore the effective single-particle hopping Hamiltonian $H(\boldsymbol{p})$ contains hops only between even plaquettes or odd plaquettes. The single-particle Hamiltonian $H(\boldsymbol{p})$ is invariant under the following two transformations G_0 and G_0':

$$G_0|\boldsymbol{p}\rangle=-|\boldsymbol{p}\rangle, \quad G_0'|\boldsymbol{p}\rangle=(-)^{\boldsymbol{p}}|\boldsymbol{p}\rangle. \tag{31}$$

We note that $G_0\otimes G_0=G_0'\otimes G_0'=1$. Therefore both G_0 and G_0' correspond to the identity element of the symmetry group of two-particle states. (G_0,G_0') generate the IGG of the new PSG. The new IGG is $Z_2\times Z_2$.

The translations of single-particle states by $\boldsymbol{x}$ and by $\boldsymbol{y}$ are generated by T_xG_x and T_yG_y. The translations by $\boldsymbol{x}+\boldsymbol{y}$ and by $\boldsymbol{x}-\boldsymbol{y}$ are given by

$$T_{xy}G_{xy}=T_yG_yT_xG_x,$$

$$T_{x\bar{y}}G_{x\bar{y}}=(T_yG_y)^{-1}T_xG_x. \tag{32}$$

Since $T_{xy}G_{xy}$ and $T_{x\bar{y}}G_{x\bar{y}}$ are the translations of the Z_2 charge and the Z_2 vortex discussed above, we find

$$(T_{x\bar{y}}G_{x\bar{y}})^{-1}(T_{xy}G_{xy})^{-1}T_{x\bar{y}}G_{x\bar{y}}T_{xy}G_{xy}=\eta, \tag{33}$$

where $\eta=1$ for the $(Z2A,Z2A)$ state with $\hat{F}_{\boldsymbol{i}}=1$ and $\eta=-1$ for the $(Z2B,Z2B)$ state with $\hat{F}_{\boldsymbol{i}}=-1$. T_xG_x and T_yG_y must also satisfy

$$(T_yG_y)^{-1}(T_xG_x)^{-1}T_yG_yT_xG_x\in \mathrm{IGG} \tag{34}$$

since in the two-particle states

$$(T_y^{(2)})^{-1}(T_x^{(2)})^{-1}T_y^{(2)}T_x^{(2)}=1. \tag{35}$$

Therefore, $(T_yG_y)^{-1}(T_xG_x)^{-1}T_yG_yT_xG_x$ may take the possible values 1, -1, $(-)^{\boldsymbol{p}}$, and $-(-)^{\boldsymbol{p}}$. Only the choices $\eta^{\boldsymbol{p}}$ and $-\eta^{\boldsymbol{p}}$ are consistent with Eq. (33) and we have

$$(T_yG_y)^{-1}(T_xG_x)^{-1}T_yG_yT_xG_x=\eta'\,\eta^{\boldsymbol{p}}. \tag{36}$$

We wish to point out that the different choices of $\eta'=\pm1$ do not lead to different PSG's. This is because if T_xG_x is a symmetry of the $H(\boldsymbol{p})$, then $T_xG_x(-)^{\boldsymbol{p}}$ is also a symmetry of the $H(\boldsymbol{p})$. However, the change $G_x\to G_x(-)^{\boldsymbol{p}}$ will change the sign of η'. Thus $\eta'=1$ and $\eta'=-1$ will lead to the same PSG. But the different signs of η will lead to different PSG's.

(G_0,G_0') and (T_xG_x,T_yG_y) generate the new PSG. The single-particle Hamiltonian $H(\boldsymbol{p})$ is invariant under such a PSG. $\eta=1$ and $\eta=-1$ correspond to two different PSG's that characterize two different quantum orders. The ground state for $V>0$ and $|V|\gg t$ [see Eq. (30)] is described by the $\eta=1$ PSG. The ground state for $V<0$ and $|V|\gg t$ is described by the $\eta=-1$ PSG. The two ground states have different quantum orders and different string-net condensations.

D. Different PSG's from the ends of different condensed strings

In this section we still assume $U=g$ and consider only the translationally invariant model Eq. (30). In the above we discussed the PSG for the ends of one type of condensed string in different states. In this section, we will concentrate on only one ground state. We know that the ground state of our spin-1/2 model contains condensations of several types of string. We wish to calculate the different PSG's for the different condensed strings.

The PSG's for the condensed T1 and T2 strings were obtained above. Here we will discuss the PSG for the T3 string. Since the ends of the T3 strings live on the links, the corresponding single-particle hopping Hamiltonian $H_f(\boldsymbol{l})$ describes fermion hopping between links. Clearly, the symmetry group (the PSG) of $H_f(\boldsymbol{l})$ can be different from that of $H(\boldsymbol{p})$.

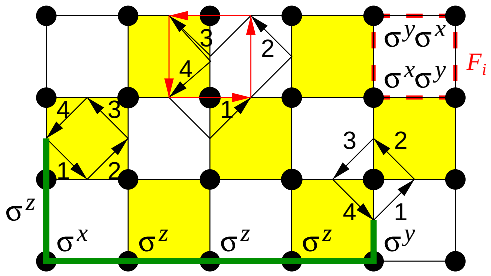

FIG. 6. Fermion hopping around a plaquette, around a square, and around a site.

Let us consider fermion hopping around some small loops. The four hops of a fermion around a site $\boldsymbol{i}$ (see Fig. 6) are generated by $\sigma^y_{\boldsymbol{i}}$, $\sigma^x_{\boldsymbol{i}}$, $\sigma^y_{\boldsymbol{i}}$, and $\sigma^x_{\boldsymbol{i}}$. The total amplitude of a fermion hopping around a site is $\sigma^y_{\boldsymbol{i}}\sigma^x_{\boldsymbol{i}}\sigma^y_{\boldsymbol{i}}\sigma^x_{\boldsymbol{i}}=-1$. The fermion hopping around a site always sees π flux. The four hops of a fermion around a plaquette $\boldsymbol{p}$ (see Fig. 6) are generated by $\sigma^x_{\boldsymbol{i}_0}$, $\sigma^y_{\boldsymbol{i}_0+\boldsymbol{x}}$, $\sigma^x_{\boldsymbol{i}_0+\boldsymbol{x}+\boldsymbol{y}}$, and $\sigma^y_{\boldsymbol{i}_0+\boldsymbol{y}}$, where $\boldsymbol{i}_0$ is the lower left corner of the plaquette $\boldsymbol{p}$. The total amplitude of a fermion hopping around a plaquette is given by $\sigma^y_{\boldsymbol{i}_0+\boldsymbol{y}}\sigma^x_{\boldsymbol{i}_0+\boldsymbol{x}+\boldsymbol{y}}\sigma^y_{\boldsymbol{i}_0+\boldsymbol{x}}\sigma^x_{\boldsymbol{i}_0}=\hat{F}_{\boldsymbol{i}_0}$. When $V>0$, the ground state has $\hat{F}_{\boldsymbol{i}}=1$. However, since site $\boldsymbol{i}_0$ is next to the end of the T3 string, we have $\hat{F}_{\boldsymbol{i}_0}=-\hat{F}_{\boldsymbol{i}}=-1$. In this case, a fermion hopping around a plaquette sees π flux. For a $V<0$ ground state, we find that a fermion hopping around a plaquette sees no flux.

Let us define the fermion hopping $\boldsymbol{l}\to\boldsymbol{l}+\boldsymbol{x}$ as a combination of two hops $\boldsymbol{l}\to\boldsymbol{l}+\boldsymbol{x}/2-\boldsymbol{y}/2\to\boldsymbol{l}+\boldsymbol{x}$ and the fermion hopping $\boldsymbol{l}\to\boldsymbol{l}+\boldsymbol{y}$ as a combination of $\boldsymbol{l}\to\boldsymbol{l}+\boldsymbol{x}/2+\boldsymbol{y}/2\to\boldsymbol{l}+\boldsymbol{y}$ (see Fig. 6). Under such a definition, a fermion hopping around a square $\boldsymbol{l}\to\boldsymbol{l}+\boldsymbol{x}\to\boldsymbol{l}+\boldsymbol{x}+\boldsymbol{y}\to\boldsymbol{l}+\boldsymbol{y}\to\boldsymbol{l}$ corresponds to a fermion hopping around a site and a fermion hopping around a plaquette as discussed above (see Fig. 6). Therefore, the total amplitude for a fermion hopping around a square is given by the sign of V: sgn(V). We find that the translation symmetries (T_xG_x,T_yG_y) of the fermion hopping $H_f(\boldsymbol{l})$ satisfy

$$(T_yG_y)^{-1}(T_xG_x)^{-1}T_yG_yT_xG_x=\text{sgn}(V), \tag{37}$$

which is different from the translation algebra for $H(\boldsymbol{p})$ [Eq. (36)]. $H_f(\boldsymbol{l})$ is also invariant under G_0:

$$G_0|\boldsymbol{l}\rangle=-|\boldsymbol{l}\rangle. \tag{38}$$

(G_0,T_xG_x,T_yG_y) generate the symmetry group—the fermion PSG—of $H_f(\boldsymbol{l})$. We will call the fermion PSG Eq. (37) for sgn(V)=1 the $Z2A$ PSG and the fermion PSG for sgn(V)=−1 the $Z2B$ PSG. We see that the quantum orders in the ground state can also be characterized using the fermion PSG. The quantum order in the $V>0$ ground state is characterized by the $Z2A$ PSG and the quantum order in the $V<0$ ground state is characterized by the $Z2B$ PSG.

In Ref. [50], the spin-1/2 model Eq. (30) (with $t=t'=0$) was viewed as a hard-core boson model. The model was solved using the slave-boson approach by splitting the boson into two fermions. Then it was shown that the fermion hopping Hamiltonians for the $V>0$ and $V<0$ states have different symmetries, or are invariant under different PSG's. According to the arguments in Ref. [9], the different PSG's imply different quantum orders in the $V>0$ and $V<0$ ground states. The PSG's obtained in Ref. [50] for the $V>0$ and $V<0$ phases agrees exactly with the fermion PSG's that we obtained above. This example shows that the PSG's introduced in Refs. [10,50] are the symmetry groups of the hopping Hamiltonian of the ends of condensed strings. The PSG description and the string-net condensation description of quantum order are intimately related.

Here we would like to point out that the PSG's introduced in Refs. [9,10] are all fermion PSG's. They are only one of many different kinds of PSG's that can be used to characterize quantum order. In general, a quantum ordered state may contain condensations of several types of string. The ends of each type of condensed string will have their own PSG.

V. MASSLESS FERMION AND PSG IN STRING-NET CONDENSED STATE

In Refs. [9,16], it was pointed out that the PSG can protect the masslessness of the emergent fermions, just as symmetry can protect the masslessness of Nambu-Goldstone bosons. In this section, we are going to study an exactly soluble spin-$\frac{1}{2}\frac{1}{2}$ model with string-net condensation and emergent massless fermions. Through this soluble model, we demonstrate how the PSG that characterizes the string-net condensation can protect the masslessness of the fermions. The exactly soluble model that we are going to study is motivated by Kitaev's exact soluble spin-1/2 model on a honeycomb lattice [52].

A. Exactly soluble spin-$\frac{1}{2}\frac{1}{2}$ model

The exactly soluble model is a local bosonic model on a square lattice. To construct the model, we start with four Majorana fermions $\lambda^a_{\boldsymbol{i}}$, $a=x,\bar{x},y,\bar{y}$, and one complex fermion ψ. $\lambda^a_{\boldsymbol{i}}$ satisfy

$$\{\lambda^a_{\boldsymbol{i}},\lambda^b_{\boldsymbol{j}}\}=2\delta_{ab}\delta_{\boldsymbol{ij}}. \tag{39}$$

We note that

$$\hat{U}_{\boldsymbol{i},\boldsymbol{i}+\boldsymbol{x}}=-i\lambda^x_{\boldsymbol{i}}\lambda^{\bar{x}}_{\boldsymbol{i}+\boldsymbol{x}},\quad \hat{U}_{\boldsymbol{i},\boldsymbol{i}+\boldsymbol{y}}=-i\lambda^y_{\boldsymbol{i}}\lambda^{\bar{y}}_{\boldsymbol{i}+\boldsymbol{y}},\quad \hat{U}_{\boldsymbol{ij}}=\hat{U}_{\boldsymbol{ji}} \tag{40}$$

form a commuting set of operators. Using such a commuting set of operators, we can construct the following exactly soluble interacting fermion model:

$$H=g\sum_{\boldsymbol{i}}\hat{F}_{\boldsymbol{i}}+t\sum_{\boldsymbol{i}}(i\hat{U}_{\boldsymbol{i},\boldsymbol{i}+\boldsymbol{x}}\psi^\dagger_{\boldsymbol{i}}\psi_{\boldsymbol{i}+\boldsymbol{x}}+i\hat{U}_{\boldsymbol{i},\boldsymbol{i}+\boldsymbol{y}}\psi^\dagger_{\boldsymbol{i}}\psi_{\boldsymbol{i}+\boldsymbol{y}}+\text{H.c.}),$$

$$\hat{F}_{\boldsymbol{i}}=\hat{U}_{\boldsymbol{i},\boldsymbol{i}_1}\hat{U}_{\boldsymbol{i}_1,\boldsymbol{i}_2}\hat{U}_{\boldsymbol{i}_2,\boldsymbol{i}_3}\hat{U}_{\boldsymbol{i}_3,\boldsymbol{i}}, \tag{41}$$

where $\boldsymbol{i}_1=\boldsymbol{i}+\boldsymbol{x}$, $\boldsymbol{i}_2=\boldsymbol{i}+\boldsymbol{x}+\boldsymbol{y}$, $\boldsymbol{i}_3=\boldsymbol{i}+\boldsymbol{y}$, and t is real. We will call $\hat{F}_{\boldsymbol{i}}$ a Z_2 flux operator. To obtain the Hilbert space within which the Hamiltonian H acts, we group $\lambda^{x,\bar{x},y,\bar{y}}$ into two complex fermion operators

$$2\psi_{1,\boldsymbol{i}}=\lambda^x_{\boldsymbol{i}}+i\lambda^{\bar{x}}_{\boldsymbol{i}}, \quad 2\psi_{2,\boldsymbol{i}}=\lambda^y_{\boldsymbol{i}}+i\lambda^{\bar{y}}_{\boldsymbol{i}} \tag{42}$$

on each site. The complex fermion operators $\psi_{1,2}$ and ψ generate an eight-dimensional Hilbert space on each site.

Since $\hat{U}_{\boldsymbol{ij}}$ commute with each other, we can find the common eigenstates of the $\hat{U}_{\boldsymbol{ij}}$ operators, $|\{s_{\boldsymbol{ij}}\},n\rangle$, where $s_{\boldsymbol{ij}}$ is the eigenvalue of $\hat{U}_{\boldsymbol{ij}}$, and n labels different degenerate common eigenstates. Since $(\hat{U}_{\boldsymbol{ij}})^2=1$ and $\hat{U}_{\boldsymbol{ij}}=\hat{U}_{\boldsymbol{ji}}$, $s_{\boldsymbol{ij}}$ satisfies $s_{\boldsymbol{ij}}=\pm 1$ and $s_{\boldsymbol{ij}}=s_{\boldsymbol{ji}}$. Within the subspace with a fixed set of $s_{\boldsymbol{ij}}$, $\{|\{s_{\boldsymbol{ij}}\},n\rangle | n=1,2,\dots\}$, the Hamiltonian has the form

$$H=g\sum_{\boldsymbol{i}} f_{\boldsymbol{i}}+t\sum_{\boldsymbol{i}} (is_{\boldsymbol{i},\boldsymbol{i}+\boldsymbol{x}}\psi^\dagger_{\boldsymbol{i}}\psi_{\boldsymbol{i}+\boldsymbol{x}}+is_{\boldsymbol{i},\boldsymbol{i}+\boldsymbol{y}}\psi^\dagger_{\boldsymbol{i}}\psi_{\boldsymbol{i}+\boldsymbol{y}}+\text{H.c.}),$$

$$f_{\boldsymbol{i}}=s_{\boldsymbol{i},\boldsymbol{i}_1}s_{\boldsymbol{i}_1,\boldsymbol{i}_2}s_{\boldsymbol{i}_2,\boldsymbol{i}_3}s_{\boldsymbol{i}_3,\boldsymbol{i}}, \tag{43}$$

which is a free fermion Hamiltonian. Thus we can find all the many-body eigenstates of $\psi_{\boldsymbol{i}}$, $|\{s_{\boldsymbol{ij}}\},\Psi_n\rangle$, and their energies $E(\{s_{\boldsymbol{ij}}\},n)$ in each subspace. In this way we solve the interacting fermion model exactly.

We note that the Hamiltonian H can change the fermion number on each site only by an even number. Thus H acts within a subspace that has an even number of fermions on each site. We will call this subspace the physical Hilbert space. The physical Hilbert space has only four states per site. When defined on the physical space, H becomes a local bosonic system which actually describes a spin-$(\frac{1}{2}\times\frac{1}{2})$ system (with no spin rotation symmetry). We will call such a system a spin-$\frac{1}{2}\frac{1}{2}$ system. To obtain an expression for H within the physical Hilbert space, we introduce two Majorana fermions $\eta_{1,\boldsymbol{i}}$ and $\eta_{2,\boldsymbol{i}}$ to represent $\psi_{\boldsymbol{i}}$: $2\psi_{\boldsymbol{i}}=\eta_{1,\boldsymbol{i}}+i\eta_{2,\boldsymbol{i}}$. We note that $\lambda^a\eta_1$, $a=x,\bar{x},y,\bar{y}$, act within the four-dimensional physical Hilbert space on each site, and thus are 4×4 matrices. Also, $\{-i\lambda^a\eta_1,-i\lambda^b\eta_1\}=2\delta_{ab}$; thus the four 4×4 matrices $\lambda^a\eta_1$ satisfy the algebra of Dirac matrices. Therefore we can express $\lambda^a\eta_1$ in terms of Dirac matrices γ^a:

$$\lambda^a\eta_1=i\gamma^a,$$

$$\gamma^x=\sigma^x\otimes\sigma^x, \quad \gamma^{\bar{x}}=\sigma^y\otimes\sigma^x,$$

$$\gamma^y=\sigma^z\otimes\sigma^x, \quad \gamma^{\bar{y}}=\sigma^0\otimes\sigma^y. \tag{44}$$

We can also define γ^5 as

$$\gamma^5\equiv\gamma^x\gamma^{\bar{x}}\gamma^y\gamma^{\bar{y}}=-\sigma^0\otimes\sigma^z$$
$$=\lambda^x\lambda^{\bar{x}}\lambda^y\lambda^{\bar{y}}=i\eta_1\eta_2, \tag{45}$$

where we have used $1-2\psi^\dagger\psi=-i\eta_1\eta_2$ and $(-i\lambda^x\lambda^{\bar{x}})(-i\lambda^y\lambda^{\bar{y}})(-i\eta_1\eta_2)=1$ for states with even numbers of fermions. With the above definitions of γ^a and γ^5, we find that

$$\lambda^a\eta_2=\gamma^a\gamma^5 \tag{46}$$

and

$$\lambda^a\psi=\frac{i}{2}(\gamma^a+\gamma^a\gamma^5)\equiv i\gamma^{-,a},$$

$$\lambda^a\psi^\dagger=\frac{i}{2}(\gamma^a-\gamma^a\gamma^5)\equiv i\gamma^{+,a},$$

$$\gamma^{-,a}=(\gamma^{+,a})^\dagger. \tag{47}$$

We also have

$$\lambda^a\lambda^b=\gamma^a\gamma^b\equiv\gamma^{ab}. \tag{48}$$

The above relations allows us to write H in terms of 4×4 Dirac matrices. For example,

$$\hat{F}_{\boldsymbol{i}}=-\gamma^{yx}_{\boldsymbol{i}}\gamma^{\bar{x}y}_{\boldsymbol{i}+\boldsymbol{x}}\gamma^{\bar{y}\bar{x}}_{\boldsymbol{i}+\boldsymbol{x}+\boldsymbol{y}}\gamma^{x\bar{y}}_{\boldsymbol{i}+\boldsymbol{y}} \tag{49}$$

and

$$\hat{U}_{\boldsymbol{i},\boldsymbol{i}+\boldsymbol{x}}\psi^\dagger_{\boldsymbol{i}}\psi_{\boldsymbol{i}+\boldsymbol{x}}=-i\gamma^{+,x}_{\boldsymbol{i}}\gamma^{-,\bar{x}}_{\boldsymbol{i}+\boldsymbol{x}},$$

$$\hat{U}_{\boldsymbol{i},\boldsymbol{i}+\boldsymbol{y}}\psi^\dagger_{\boldsymbol{i}}\psi_{\boldsymbol{i}+\boldsymbol{y}}=-i\gamma^{+,y}_{\boldsymbol{i}}\gamma^{-,\bar{y}}_{\boldsymbol{i}+\boldsymbol{y}}. \tag{50}$$

The physical states in the physical Hilbert space are invariant under local Z_2 gauge transformations generated by

$$G=\prod_{\boldsymbol{i}} G^{n_{\boldsymbol{i}}}_{\boldsymbol{i}},$$

$$n_{\boldsymbol{i}}=\psi^\dagger_{1,\boldsymbol{i}}\psi_{1,\boldsymbol{i}}+\psi^\dagger_{2,\boldsymbol{i}}\psi_{2,\boldsymbol{i}}+\psi^\dagger_{\boldsymbol{i}}\psi_{\boldsymbol{i}}, \tag{51}$$

where $G_{\boldsymbol{i}}$ is an arbitrary function with only two values ± 1 and $n_{\boldsymbol{i}}$ is the number of fermions on site $\boldsymbol{i}$. We note that the Z_2 gauge transformations change $\psi_{I\boldsymbol{i}}\to G_{\boldsymbol{i}}\psi_{I\boldsymbol{i}}$. The projection into the physical Hilbert space with even numbers of fermions per site makes our theory a Z_2 gauge theory.

Since the Hamiltonian H in Eq. (41) is Z_2 gauge invariant, $[G,H]=0$, the eigenstate of H within the physical Hilbert space can be obtained from $|\{s_{\boldsymbol{ij}}\},\Psi_n\rangle$ by projecting into the physical Hilbert space: $\mathcal{P}|\{s_{\boldsymbol{ij}}\},\Psi_n\rangle$. The projected state $\mathcal{P}|\{s_{\boldsymbol{ij}}\},\Psi_n\rangle$ (or the physical state), if nonzero, is an eigenstate of the spin-$\frac{1}{2}\frac{1}{2}$ model with energy $E(\{s_{\boldsymbol{ij}}\},n)$. The Z_2 gauge invariance implies that

$$\mathcal{P}|\{s_{\boldsymbol{ij}}\},\Psi_n\rangle=\mathcal{P}|\{\tilde{s}_{\boldsymbol{ij}}\},\Psi_n\rangle,$$

$$E(\{s_{\boldsymbol{ij}}\},n)=E(\{\tilde{s}_{\boldsymbol{ij}}\},n), \tag{52}$$

if $s_{\boldsymbol{ij}}$ and $\tilde{s}_{\boldsymbol{ij}}$ are Z_2 gauge equivalent:

$$\tilde{s}_{\boldsymbol{ij}}=G(\boldsymbol{i})s_{\boldsymbol{ij}}G(\boldsymbol{j}). \tag{53}$$

Let us count the states to show that the projected states $\mathcal{P}|\{s_{\boldsymbol{ij}}\},\Psi_n\rangle$ generate all states in the physical Hilbert space. Let us consider a periodic lattice with $N_{site}=L_xL_y$ sites. First there are $2^{2N_{site}}$ choices of $s_{\boldsymbol{ij}}$. We note that there are $2^{N_{site}}$ different Z_2 gauge transformations. But the constant gauge transformation $G(\boldsymbol{i})=-1$ does not change $s_{\boldsymbol{ij}}$. Thus there are $2^{N_{site}}/2$ different $s_{\boldsymbol{ij}}$'s in each Z_2 gauge equivalent class. Therefore, there are $2\times 2^{N_{site}}$ different Z_2 gauge equivalent classes of $s_{\boldsymbol{ij}}$'s. We also note that

$$
\begin{aligned}
\prod_{\boldsymbol{i}} s_{\boldsymbol{i},\boldsymbol{i}+\boldsymbol{x}}s_{\boldsymbol{i},\boldsymbol{i}+\boldsymbol{y}}
&=(-)^{L_x+L_y}\prod_{\boldsymbol{i}}(-i\lambda_{\boldsymbol{i}}^{x}\lambda_{\boldsymbol{i}}^{\bar{x}})(-i\lambda_{\boldsymbol{i}}^{y}\lambda_{\boldsymbol{i}}^{\bar{y}})\\
&=(-)^{L_x+L_y+\sum_{\boldsymbol{i}}(\psi_{1,\boldsymbol{i}}^{\dagger}\psi_{1,\boldsymbol{i}}+\psi_{2,\boldsymbol{i}}^{\dagger}\psi_{2,\boldsymbol{i}})}.
\end{aligned}
\tag{54}
$$

Thus, among the $2\times 2^{N_{site}}$ different classes of $s_{\boldsymbol{ij}}$'s, $2^{N_{site}}$ of them satisfy $\Pi_{\boldsymbol{i}}s_{\boldsymbol{i},\boldsymbol{i}+\boldsymbol{x}}s_{\boldsymbol{i},\boldsymbol{i}+\boldsymbol{y}}=(-)^{L_x+L_y}$ and have even numbers of $\psi_{1,\boldsymbol{i}}$ and $\psi_{2,\boldsymbol{i}}$ fermions. The other $2^{N_{site}}$ of them satisfy $\Pi_{\boldsymbol{i}}s_{\boldsymbol{i},\boldsymbol{i}+\boldsymbol{x}}s_{\boldsymbol{i},\boldsymbol{i}+\boldsymbol{y}}=-(-)^{L_x+L_y}$ and have odd numbers of $\psi_{1,\boldsymbol{i}}$ and $\psi_{2,\boldsymbol{i}}$ fermions.

For each fixed $s_{\boldsymbol{ij}}$, there are $2^{N_{site}}$ many-body states of the $\psi_{\boldsymbol{i}}$ fermions, i.e., n in $|\{s_{\boldsymbol{ij}}\},\Psi_n\rangle$ runs from 1 to $2^{N_{site}}$. Among those $2^{N_{site}}$ many-body states, $2^{N_{site}}/2$ have even numbers of $\psi_{\boldsymbol{i}}$ fermions and $2^{N_{site}}/2$ have odd numbers of $\psi_{\boldsymbol{i}}$ fermions. In order for the projection $\mathcal{P}|\{s_{\boldsymbol{ij}}\},\Psi_n\rangle$ to be nonzero, the total number of fermions must be even. A physical state has even numbers of $(\psi_{1,\boldsymbol{i}},\psi_{2,\boldsymbol{i}})$ fermions and even numbers of $\psi_{\boldsymbol{i}}$ fermions, or it has odd numbers of $(\psi_{1,\boldsymbol{i}},\psi_{2,\boldsymbol{i}})$ fermions and odd numbers of $\psi_{\boldsymbol{i}}$ fermions. Thus there are $2^{N_{site}}\times 2^{N_{site}}/2+2^{N_{site}}\times 2^{N_{site}}/2=4^{N_{site}}$ distinct physical states that can be produced by the projection. Thus the projection produces all the states in the physical Hilbert space.

B. Physical properties of the spin-$\frac{1}{2}\frac{1}{2}$ model

Let us define a closed-string operator to be

$$W(C_{close})=\hat{U}_{\boldsymbol{i}_1\boldsymbol{i}_2}\hat{U}_{\boldsymbol{i}_2\boldsymbol{i}_3}\cdots\hat{U}_{\boldsymbol{i}_n\boldsymbol{i}_1}, \tag{55}$$

where C_{close} is a closed oriented string $C_{close}=\boldsymbol{i}_1\rightarrow\boldsymbol{i}_2\rightarrow\cdots\rightarrow\boldsymbol{i}_n\rightarrow\boldsymbol{i}_1$ formed by nearest neighbor links. Since C_{close} can intersect with itself, C_{close} can also be viewed as a closed string net. We will also call $W(C_{close})$ a closed-string-net operator.

The closed-string-net operators act within the physical Hilbert space and commute with the Hamiltonian Eq. (41). Thus there is a string-net condensation since $\langle W(C_{close})\rangle=\pm 1$ in the ground state of Eq. (41). The above strings correspond to the T3 string discussed in Sec. III C. Unlike the spin-1/2 model, we do not have condensed T1 and T2 closed strings in the spin-$\frac{1}{2}\frac{1}{2}$ model.

We can also define open-string operators that act within the physical Hilbert space:

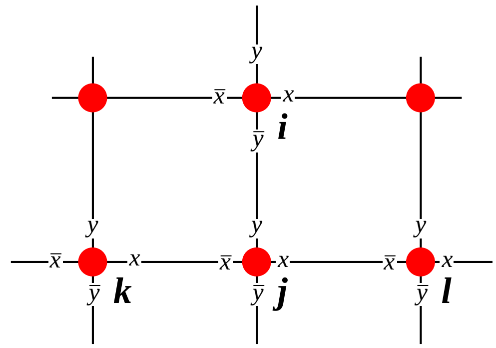

FIG. 7. A particle can hop between different sites $\boldsymbol{i},\boldsymbol{j},\boldsymbol{k},\boldsymbol{l}$.

$$
\begin{aligned}
W(C_{open})&=\lambda_{\boldsymbol{i}_1}^{a}\hat{U}_{\boldsymbol{i}_1\boldsymbol{i}_2}\hat{U}_{\boldsymbol{i}_2\boldsymbol{i}_3}\cdots\hat{U}_{\boldsymbol{i}_{n-1}\boldsymbol{i}_n}\lambda_{\boldsymbol{i}_n}^{b},\\
\tilde{W}(C_{open})&=\psi_{\boldsymbol{i}_1}^{\dagger}\hat{U}_{\boldsymbol{i}_1\boldsymbol{i}_2}\hat{U}_{\boldsymbol{i}_2\boldsymbol{i}_3}\cdots\hat{U}_{\boldsymbol{i}_{n-1}\boldsymbol{i}_n}\psi_{\boldsymbol{i}_n},
\end{aligned}
\tag{56}
$$

where C_{open} is an open oriented string $C_{open}=\boldsymbol{i}_1\rightarrow\boldsymbol{i}_2\rightarrow\cdots\rightarrow\boldsymbol{i}_n$ formed by nearest neighbor links. $W(C)$ corresponds to the open T3 string defined in Sec. III C. Just as in the spin-1/2 model Eq. (30), the ends of such strings correspond to gapped fermions (if $|g|\gg|t|$). The ends of $\tilde{W}$ strings differ from the ends of W strings only by a local bosonic operator. Thus the ends of $\tilde{W}$ strings are also fermions.

To really prove that the ends of $\tilde{W}$ strings are fermions, we need to show that the hopping of the ends of $\tilde{W}$ strings satisfies the fermion hopping algebra introduced in Ref. [14]:

$$
\begin{aligned}
&t_{jl}t_{kj}t_{ji}=-t_{ji}t_{kj}t_{jl},\\
&[t_{ij},t_{kl}]=0\quad\text{if}\quad i,j,k,l\quad\text{are all different},
\end{aligned}
\tag{57}
$$

where t_{ji} describes the hopping from site i to site j. It was shown that the particles are fermions if their hopping satisfies the algebra Eq. (57). We note that the ends of the $\tilde{W}$ strings live on the sites. The labels $i,j,\ldots$ in the above equation correspond to lattice sites $\boldsymbol{i},\boldsymbol{j},\ldots$. The hops between sites $\boldsymbol{i},\boldsymbol{j},\boldsymbol{k},\boldsymbol{l}$ in Fig. 7 are given by

$$t_{\boldsymbol{i}+\boldsymbol{a},\boldsymbol{i}}=\psi_{\boldsymbol{i}+\boldsymbol{a}}^{\dagger}\hat{U}_{\boldsymbol{i}+\boldsymbol{a},\boldsymbol{i}}\psi_{\boldsymbol{i}},\qquad \boldsymbol{a}=\pm\boldsymbol{x},\pm\boldsymbol{y}. \tag{58}$$

Note that the hops between nearest neighbors are taken from the Hamiltonian Eq. (41). Since $\hat{U}_{\boldsymbol{ij}}$ commute with each other, the algebra of the above hopping operators is just that of fermion hopping operators. In particular, the above hopping operators satisfy the fermion hopping algebra Eq. (57). Hence, the ends of the $\tilde{W}$ strings are fermions.

For each fixed configuration $s_{\boldsymbol{ij}}$, there are $2^{N_{site}}/2$ different states (with even or odd numbers of total ψ fermions). Their energies are given by the fermion hopping Hamiltonian Eq. (43). Let $E_0(\{s_{\boldsymbol{ij}}\})$ be the ground state energy of Eq. (43). The ground state and the ground state energy of our spin-$\frac{1}{2}\frac{1}{2}$ model Eq. (61) are obtained by choosing a configuration $s_{\boldsymbol{ij}}$ that minimizes $E_0(\{s_{\boldsymbol{ij}}\})$. We note that $E_0(\{s_{\boldsymbol{ij}}\})$ is invariant under the Z_2 gauge transformation Eq. (53).

When $g \gg |t|$, the ground state of Eq. (41) has $\hat{F}_{\boldsymbol{i}} = -1$, which minimizes the dominan $g\Sigma_{\boldsymbol{i}}\hat{F}_{\boldsymbol{i}}$ term. The ground state configuration is given by

$$s_{\boldsymbol{i},\boldsymbol{i}+\boldsymbol{x}} = (-)^{i_y}, \quad s_{\boldsymbol{i},\boldsymbol{i}+\boldsymbol{y}} = 1. \tag{59}$$

The $\psi_{\boldsymbol{i}}$ fermion hopping Hamiltonian Eq. (43) for the above configuration describes fermion hopping with π flux per plaquette. The fermion spectrum has the form

$$E_{\boldsymbol{k}} = \pm 2\sqrt{t^2 \sin^2(k_x) + t^2 \sin^2(k_y)}. \tag{60}$$

The low energy excitations of such a hopping Hamiltonian are described by two two-component massless Dirac fermions in (2+1)D. We see that the ends of the $\widetilde{W}$ strings are massless Dirac fermions.

Our model also contains Z_2 gauge excitations. The Z_2 vortices are created by flipping $\hat{F}_{\boldsymbol{i}} = -1$ to $\hat{F}_{\boldsymbol{i}} = 1$ in some plaquettes. The Z_2 vortex behaves like a π flux to the gapless fermions. Thus the gapless fermions carry a unit Z_2 charge. The low energy effective theory of our model consists of massless Dirac fermions coupled to a Z_2 gauge field.

C. Projective symmetry and massless fermions

We know that symmetry breaking can produce and protect gapless Nambu-Goldstone modes. In Refs. [9,16], it was proposed that, in addition to symmetry breaking, quantum order can also produce and protect gapless excitations. The gapless excitations produced and protected by quantum order can be gapless gauge bosons and/or gapless fermions. In this paper we show that the quantum orders discussed in Refs. [9,16] are due to string-net condensations. Therefore, more precisely it is string-net condensations that produce and protect gapless gauge bosons and/or gapless fermions. The string-net condensations and gapless excitations are connected in the following way. Let us consider a Hamiltonian that has a symmetry described by a symmetry group SG. We assume the ground state has a string-net condensation. Then, the hopping Hamiltonian for the ends of the condensed string will be invariant under a larger group—the projective symmetry group PSG, as discussed in Sec. IV B. The PSG is an extension of the symmetry group SG, i.e., the PSG contain a normal subgroup IGG such that PSG/IGG = SG. The relation between the PSG and gapless gauge bosons is simple. Let $\mathcal{G}$ be the maximum continuous subgroup of the IGG. Then the gapless gauge bosons are described by a gauge theory with $\mathcal{G}$ as the gauge group [9,15]. Sometimes the ends of strings are fermions. However, the relation between gapless fermions and the PSG is more complicated. Through a case by case study of some PSG's [9,16], we find that certain PSG's indeed guarantee the existence of gapless fermions.

In this section, we are going to study a large family of exactly soluble local bosonic models which depends on many continuous parameters. The ground states of the local bosonic models have a string-net condensation and do not break any symmetry. We will show that the projective symmetry of the ends of condensed strings protects a massless fermion. As a result, our exactly soluble model always has massless fermion excitations regardless of the value of the continuous parameters (as long as they are within a certain range). This puts the results of Refs. [9,16], which were based on mean-field theory, on a firmer ground.

The exactly soluble local bosonic model is the spin-$\frac{1}{2}\frac{1}{2}$ model

$$H_{\frac{1}{2}\frac{1}{2}} = -g\sum_{\boldsymbol{i}} \gamma_{\boldsymbol{i}}^{yx}\gamma_{\boldsymbol{i}+\boldsymbol{x}}^{\overline{xy}}\gamma_{\boldsymbol{i}+\boldsymbol{x}+\boldsymbol{y}}^{\overline{yx}}\gamma_{\boldsymbol{i}+\boldsymbol{y}}^{x\bar{y}} + \sum_{\boldsymbol{i}} (t\gamma_{\boldsymbol{i}}^{+,x}\gamma_{\boldsymbol{i}+\boldsymbol{x}}^{-,\bar{x}} + t\gamma_{\boldsymbol{i}}^{+,y}\gamma_{\boldsymbol{i}+\boldsymbol{y}}^{-,\bar{y}} + \text{H.c.}), \tag{61}$$

where γ^{ab} and $\gamma^{\pm,a}$ are given in Eq. (48) and Eq. (47). We will discuss a more general Hamiltonian later.

The Hamiltonian is not invariant under $x \to -x$ parity P_x. But it has $x \to -x$ parity symmetry if P_x is followed by a spin rotation $\gamma^x \leftrightarrow \gamma^{\bar{x}}$. That is, $\gamma_{P_x}P_x H(\gamma_{P_x}P_x)^{-1} = H$ with

$$\gamma_{P_x} = \gamma^5 \frac{\gamma^x - \gamma^{\bar{x}}}{\sqrt{2}}. \tag{62}$$

Similarly, for $y \to -y$ parity P_y, we have $\gamma_{P_y}P_y H(\gamma_{P_y}P_y)^{-1} = H$ with

$$\gamma_{P_y} = \gamma^5 \frac{\gamma^y - \gamma^{\bar{y}}}{\sqrt{2}}. \tag{63}$$

In the fermion representation γ_{P_x} and γ_{P_y} generate the following transformations:

$$\gamma_{P_x}: \quad \lambda_{\boldsymbol{i}}^x \leftrightarrow \lambda_{\boldsymbol{i}}^{\bar{x}}, \quad \psi_{\boldsymbol{i}} \leftrightarrow \psi_{\boldsymbol{i}}^\dagger,$$
$$\gamma_{P_y}: \quad \lambda_{\boldsymbol{i}}^y \leftrightarrow \lambda_{\boldsymbol{i}}^{\bar{y}}, \quad \psi_{\boldsymbol{i}} \leftrightarrow \psi_{\boldsymbol{i}}^\dagger. \tag{64}$$

Now let us study how the symmetries $T_{x,y}$ and $\gamma_{P_{x,y}}P_{x,y}$ are realized in the hopping Hamiltonian Eq. (43) for the ends of condensed strings. As discussed in Sec. IV B, the hopping Hamiltonian may not be invariant under the symmetry transformations $T_{x,y}$ and $\gamma_{P_{x,y}}P_{x,y}$ directly. The hopping Hamiltonian has only a projective symmetry generated by a symmetry transformation followed by a Z_2 gauge transformation $G(\boldsymbol{i})$. Since the π-flux configuration does not break any symmetries, we expect the hopping Hamiltonian for the π-flux configuration to be invariant under $G_x T_x$, $G_y T_y$, $G_{P_x}\gamma_{P_x}P_x$, and $G_{P_y}\gamma_{P_y}P_y$, where $G_{x,y}$ and $G_{P_{x,y}}$ are the corresponding gauge transformations. The actions of $T_{x,y}$ and $\gamma_{P_{x,y}}P_{x,y}$ on the ψ fermion are given by

$$T_x: \quad \psi_{(i_x,i_y)} \to \psi_{(i_x+1,i_y)},$$
$$T_y: \quad \psi_{(i_x,i_y)} \to \psi_{(i_x,i_y+1)},$$
$$\gamma_{P_x}P_x: \quad \psi_{(i_x,i_y)} \leftrightarrow \psi^\dagger_{(-i_x,i_y)},$$
$$\gamma_{P_y}P_y: \quad \psi_{(i_x,i_y)} \leftrightarrow \psi^\dagger_{(i_x,-i_y)}. \tag{65}$$

For the π-flux configuration Eq. (59), we need to choose the following $G_{x,y}$ and $G_{P_{x,y}}$ in order for the combined transformation $G_{x,y}T_{x,y}$ and $G_{P_{x,y}}\gamma_{P_{x,y}}P_{x,y}$ to be the symmetries of the hopping Hamiltonian Eq. (43):

$$G_x=1, \quad G_y=(-)^{i_x},$$

$$G_{P_x}=(-)^{i_x}, \quad G_{P_y}=(-)^{i_y}. \tag{66}$$

The hopping Hamiltonian is also invariant under a global Z_2 gauge transformation:

$$G_0: \quad \psi_{\boldsymbol{i}}\to-\psi_{\boldsymbol{i}}. \tag{67}$$

The transformations $\{G_0,G_{x,y}T_{x,y},G_{P_{x,y}}\gamma_{P_{x,y}}P_{x,y}\}$ generate the PSG of the hopping Hamiltonian.

To show that the above PSG protects the masslessness of the fermions, we consider a more general Hamiltonian by adding

$$\delta H_{\frac{1}{2}\frac{1}{2}}=\sum_{C_{\boldsymbol{ij}}}[t(C_{\boldsymbol{ij}})\tilde{W}(C_{\boldsymbol{ij}})+\text{H.c.}] \tag{68}$$

to $H_{\frac{1}{2}\frac{1}{2}}$, where $C_{\boldsymbol{ij}}$ is an open string connecting site $\boldsymbol{i}$ and site $\boldsymbol{j}$ and $\tilde{W}(C_{\boldsymbol{ij}})$ is given in Eq. (56). The new Hamiltonian is still exactly soluble. We will choose $t(C_{\boldsymbol{ij}})$ such that the new Hamiltonian has translation symmetries and $P_{x,y}$ parity symmetries. In the following, we would like to show that the new Hamiltonian with these symmetries always has massless Dirac fermion excitations [assuming that $t(C_{\boldsymbol{ij}})$ is not too big compared to g].

When $t(C_{\boldsymbol{ij}})$ is not too large, the ground state is still described by the π-flux configuration. The new hopping Hamiltonian for the π-flux configuration has the more general form

$$H=\sum_{\langle\boldsymbol{ij}\rangle}(\chi_{\boldsymbol{ij}}\psi_{\boldsymbol{i}}^{\dagger}\psi_{\boldsymbol{j}}+\text{H.c.}). \tag{69}$$

The symmetry of the physical spin-$\frac{1}{2}\frac{1}{2}$ Hamiltonian requires the above hopping Hamiltonian to be invariant under the PSG discussed above. Such an invariance will guarantee the existence of massless fermions.

The invariance under G_xT_x and G_yT_y requires that

$$\chi_{\boldsymbol{i},\boldsymbol{i}+\boldsymbol{m}}=(-)^{i_ym_x}\chi_{\boldsymbol{m}}. \tag{70}$$

In momentum space,

$$\chi(\boldsymbol{k}_1,\boldsymbol{k}_2)\equiv N_{site}^{-1}\sum_{\boldsymbol{ij}}e^{-i\boldsymbol{k}_1\cdot\boldsymbol{i}+i\boldsymbol{k}_2\cdot\boldsymbol{j}}\chi_{\boldsymbol{ij}}$$
$$=\epsilon_0(\boldsymbol{k}_2)\delta_{\boldsymbol{k}_1-\boldsymbol{k}_2}+\epsilon_1(\boldsymbol{k}_2)\delta_{\boldsymbol{k}_1-\boldsymbol{k}_2+\boldsymbol{Q}_y}, \tag{71}$$

where

$$\epsilon_0(\boldsymbol{k})=\sum_{m_x=even}e^{i\boldsymbol{k}\cdot\boldsymbol{m}}\chi_{\boldsymbol{m}},$$

$$\epsilon_1(\boldsymbol{k})=\sum_{m_x=odd}e^{i\boldsymbol{k}\cdot\boldsymbol{m}}\chi_{\boldsymbol{m}}. \tag{72}$$

We note that $\epsilon_0(\boldsymbol{k})$ and $\epsilon_1(\boldsymbol{k})$ are periodic functions in the Brillouin zone. They also satisfy

$$\epsilon_0(\boldsymbol{k})=\epsilon_0(\boldsymbol{k}+\boldsymbol{Q}_x), \quad \epsilon_1(\boldsymbol{k})=-\epsilon_1(\boldsymbol{k}+\boldsymbol{Q}_x), \tag{73}$$

where $\boldsymbol{Q}_x=\pi\boldsymbol{x}$ and $\boldsymbol{Q}_y=\pi\boldsymbol{y}$. In momentum space, we can rewrite H as

$$H=\sum_{\boldsymbol{k}}\Psi_{\boldsymbol{k}}^{\dagger}\Gamma(\boldsymbol{k})\Psi_{\boldsymbol{k}}, \tag{74}$$

where $\Psi_{\boldsymbol{k}}^T=(\psi_{\boldsymbol{k}},\psi_{\boldsymbol{k}+\boldsymbol{Q}_y})$. The sum $\Sigma_{\boldsymbol{k}}$ is over the reduced Brillouin zone $-\pi<k_x<\pi$ and $-\pi/2<k_y<\pi/2$. $\Gamma(\boldsymbol{k})$ has the form

$$\Gamma(\boldsymbol{k})=\begin{pmatrix}\epsilon_0(\boldsymbol{k}) & \epsilon_1(\boldsymbol{k}+\boldsymbol{Q}_y)\\ \epsilon_1(\boldsymbol{k}) & \epsilon_0(\boldsymbol{k}+\boldsymbol{Q}_y)\end{pmatrix}. \tag{75}$$

Note that the transformation γ_{P_x}: $\psi\leftrightarrow\psi^{\dagger}$ changes $\Sigma\chi_{\boldsymbol{ij}}\psi_{\boldsymbol{i}}^{\dagger}\psi_{\boldsymbol{j}}$ to $\Sigma\tilde{\chi}_{\boldsymbol{ij}}\psi_{\boldsymbol{i}}^{\dagger}\psi_{\boldsymbol{j}}$ with $\tilde{\chi}_{\boldsymbol{ij}}=-\chi_{\boldsymbol{ji}}$. Thus the invariance under $G_{P_x}\gamma_{P_x}P_x$ requires that

$$-\chi_{P_x\boldsymbol{j},P_x\boldsymbol{i}}=G_{P_x}(\boldsymbol{i})\chi_{\boldsymbol{ij}}G_{P_x}(\boldsymbol{j}) \tag{76}$$

or

$$\chi_{-P_x\boldsymbol{m}}=-(-)^{m_xm_y+m_x}\chi_{\boldsymbol{m}}. \tag{77}$$

In momentum space, the above becomes

$$\epsilon_0(P_y\boldsymbol{k})=-\epsilon_0(\boldsymbol{k}),$$

$$\epsilon_1(P_y\boldsymbol{k})=\epsilon_1(\boldsymbol{k}+\boldsymbol{Q}_y). \tag{78}$$

Similarly, the invariance under $G_{P_y}\gamma_{P_y}P_y$ requires that

$$-\chi_{P_y\boldsymbol{j},P_y\boldsymbol{i}}=G_{P_y}(\boldsymbol{i})\chi_{\boldsymbol{ij}}G_{P_y}(\boldsymbol{j}) \tag{79}$$

or

$$\chi_{-P_y\boldsymbol{m}}=-(-)^{m_xm_y+m_y}\chi_{\boldsymbol{m}}. \tag{80}$$

In momentum space

$$\epsilon_0(P_x\boldsymbol{k})=-\epsilon_0(\boldsymbol{k}+\boldsymbol{Q}_y),$$

$$\epsilon_1(P_x\boldsymbol{k})=-\epsilon_1(\boldsymbol{k}). \tag{81}$$

We see that the translation $T_{x,y}$ and $x\to-x$ parity $\gamma_{P_x}P_x$ symmetries of the spin-$\frac{1}{2}\frac{1}{2}$ Hamiltonian require that $\epsilon_0(\boldsymbol{k})=-\epsilon_0(P_y\boldsymbol{k})$ and hence $\epsilon_0(\boldsymbol{k})|_{k_y=0}=0$. Similarly, the translation $T_{x,y}$ and $y\to-y$ parity $\gamma_{P_y}P_y$ symmetries require that $\epsilon_1(\boldsymbol{k})|_{k_x=0}=0$. Thus the $T_{x,y}$ and $\gamma_{P_{x,y}}P_{x,y}$ symmetries require that $\Gamma(\boldsymbol{k})|_{\boldsymbol{k}=0}=0$. Using Eq. (73), we find that $\Gamma(0)=0$ implies that $\Gamma(\boldsymbol{Q}_x)=0$. The spin-$\frac{1}{2}\frac{1}{2}$ Hamiltonian Eq. (61) has (at least) two two-component massless Dirac fermions if it has two translation $T_{x,y}$ and two parity $\gamma_{P_{x,y}}P_{x,y}$ symmetries. We see that string-net condensation and the associated projective symmetry produce and protect massless Dirac fermions.

VI. MASSLESS FERMIONS AND STRING-NET CONDENSATION ON A CUBIC LATTICE

The above calculation and the 2D model can be generalized to a 3D cubic lattice. We introduce six Majorana fermions $\lambda_{\boldsymbol{i}}^a$, where $a=x,\bar{x},y,\bar{y},z,\bar{z}$. One set of commuting operators on a square lattice has the form

$$\hat{U}_{\boldsymbol{i},\boldsymbol{i}+\boldsymbol{x}}=-i\lambda_{\boldsymbol{i}}^x\lambda_{\boldsymbol{i}+\boldsymbol{x}}^{\bar{x}},$$

$$\hat{U}_{\boldsymbol{i},\boldsymbol{i}+\boldsymbol{y}}=-i\lambda_{\boldsymbol{i}}^y\lambda_{\boldsymbol{i}+\boldsymbol{y}}^{\bar{y}},$$

$$\hat{U}_{\boldsymbol{i},\boldsymbol{i}+\boldsymbol{z}}=-i\lambda_{\boldsymbol{i}}^z\lambda_{\boldsymbol{i}+\boldsymbol{z}}^{\bar{z}},$$

$$\hat{U}_{\boldsymbol{i},\boldsymbol{j}}^\dagger=\hat{U}_{\boldsymbol{j},\boldsymbol{i}}. \tag{82}$$

Using $\hat{U}_{\boldsymbol{i},\boldsymbol{j}}$ and a complex fermion $\psi_{\boldsymbol{i}}$, we can construct an exactly soluble interacting Hamiltonian on a cubic lattice:

$$H_{\frac{1}{2}\frac{1}{2}\frac{1}{2}}=g\sum_{\boldsymbol{p}}\hat{F}_{\boldsymbol{p}}+t\sum_{\boldsymbol{i}}\sum_{\boldsymbol{a}=\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}}(i\hat{U}_{\boldsymbol{i},\boldsymbol{i}+\boldsymbol{a}}\psi_{\boldsymbol{i}}^\dagger\psi_{\boldsymbol{i}+\boldsymbol{a}}+\text{H.c.}),$$

$$\hat{F}_{\boldsymbol{p}}=\hat{U}_{\boldsymbol{i}_1,\boldsymbol{i}_2}\hat{U}_{\boldsymbol{i}_2,\boldsymbol{i}_3}\hat{U}_{\boldsymbol{i}_3,\boldsymbol{i}_4}\hat{U}_{\boldsymbol{i}_4,\boldsymbol{i}_1}, \tag{83}$$

where $\Sigma_{\boldsymbol{p}}$ sums over all the square faces of the cubic lattice. $\boldsymbol{i}_1$, $\boldsymbol{i}_2$, $\boldsymbol{i}_3$, and $\boldsymbol{i}_4$ label the four corners of the square $\boldsymbol{p}$. The Hilbert space of the system is generated by the complex fermion operators $\psi_{\boldsymbol{i}}$ and

$$2\psi_{1,\boldsymbol{i}}=\lambda_{\boldsymbol{i}}^x+i\lambda_{\boldsymbol{i}}^{\bar{x}},$$

$$2\psi_{2,\boldsymbol{i}}=\lambda_{\boldsymbol{i}}^y+i\lambda_{\boldsymbol{i}}^{\bar{y}},$$

$$2\psi_{3,\boldsymbol{i}}=\lambda_{\boldsymbol{i}}^z+i\lambda_{\boldsymbol{i}}^{\bar{z}}, \tag{84}$$

and there are 16 states per site.

The physical Hilbert space is defined as a subspace with even numbers of fermions per site. The physical Hilbert space has eight states per site. When restricted to the physical Hilbert space, $H_{\frac{1}{2}\frac{1}{2}\frac{1}{2}}$ defines our spin-$\frac{1}{2}\frac{1}{2}\frac{1}{2}$ system, which is a *local bosonic system*.

When $g\gg|t|$, our spin-$\frac{1}{2}\frac{1}{2}\frac{1}{2}$ model has two four-component massless Dirac fermions as its low lying excitations. The model also has Z_2 gauge excitations and the massless Dirac fermions carry unit Z_2 gauge charge. Again, the model has a string-net condensation in its ground state. Both the Z_2 gauge excitation and the massless fermion are produced and protected by the string-net condensation and the associated PSG.

VII. ARTIFICIAL LIGHT AND ARTIFICIAL MASSLESS ELECTRONS ON A CUBIC LATTICE

In this section, we are going to combine the above 3D model and the rotor model discussed in Ref. [26] and Ref. [13] to obtain a quasiexactly soluble local bosonic model that contains massless Dirac fermions coupled to massless $U(1)$ gauge bosons.

A. 3D rotor model and artificial light

A rotor is described by an angular variable $\hat{\theta}$. The angular momentum of $\hat{\theta}$, denoted as S^z, is quantized as integers. The 3D rotor model under consideration has one rotor on every link of a cubic lattice. We use $\boldsymbol{ij}$ to label the nearest neighbor links. $\boldsymbol{ij}$ and $\boldsymbol{ji}$ label the same links. For convenience, we will define $\hat{\theta}_{\boldsymbol{ij}}=-\hat{\theta}_{\boldsymbol{ji}}$ and $S_{\boldsymbol{ij}}^z=-S_{\boldsymbol{ji}}^z$. The 3D rotor Hamiltonian has the form

$$H_{rotor}=U\sum_{\boldsymbol{i}}\left(\sum_{\boldsymbol{a}}S_{\boldsymbol{i},\boldsymbol{i}+\boldsymbol{a}}^z\right)^2+\frac{1}{2}J\sum_{\boldsymbol{i},\boldsymbol{a}}(S_{\boldsymbol{i},\boldsymbol{i}+\boldsymbol{a}}^z)^2$$

$$+g_1\sum_{\boldsymbol{p}}\cos(\hat{\theta}_{\boldsymbol{i}_1\boldsymbol{i}_2}+\hat{\theta}_{\boldsymbol{i}_2\boldsymbol{i}_3}+\hat{\theta}_{\boldsymbol{i}_3\boldsymbol{i}_4}+\hat{\theta}_{\boldsymbol{i}_4\boldsymbol{i}_1}). \tag{85}$$

Here $\boldsymbol{i}=(i_x,i_y,i_z)$ label the sites of the cubic lattice, and $\boldsymbol{a}=\pm\boldsymbol{x},\pm\boldsymbol{y},\pm\boldsymbol{z}$. The $\Sigma_{\boldsymbol{p}}$ sum over all the square faces of the cubic lattice. $\boldsymbol{i}_1$, $\boldsymbol{i}_2$, $\boldsymbol{i}_3$, and $\boldsymbol{i}_4$ label the four corners of the square $\boldsymbol{p}$.

When $J=g_1=0$ and $U>0$, the state with all $S_{\boldsymbol{ij}}^z=0$ is the ground state. Such a state will be regarded as a state with no strings. We can create a string or a string net from the no-string state using the following string (or string-net) operator:

$$W_{U(1)}(C)=\prod_C e^{i\hat{\theta}_{\boldsymbol{ij}}}, \tag{86}$$

where C is a string (or a string net) formed by the nearest neighbor links, and Π_C is the product over all the nearest neighbor links $\boldsymbol{ij}$ on the string (or string net). Since the closed-string-net operator $W_{U(1)}(C_{close})$ commutes with H_{rotor} when $J=g_1=0$, $W_{U(1)}(C_{close})$ generates a large set of degenerate ground states. The degenerate ground states are described by closed string nets.

There is another way to generate the degenerate ground states. We note that all the degenerate ground states satisfy $\Sigma_{\boldsymbol{a}}S_{\boldsymbol{i},\boldsymbol{i}+\boldsymbol{a}}^z=0$. Let $|\{\theta_{\boldsymbol{ij}}\}\rangle$ be the common eigenstate of $\hat{\theta}_{\boldsymbol{ij}}$: $\hat{\theta}_{\boldsymbol{ij}}|\{\theta_{\boldsymbol{ij}}\}\rangle=\theta_{\boldsymbol{ij}}|\{\theta_{\boldsymbol{ij}}\}\rangle$. Then the projection onto the $\Sigma_{\boldsymbol{a}}S_{\boldsymbol{i},\boldsymbol{i}+\boldsymbol{a}}^z=0$ subspace $\mathcal{P}|\{\theta_{\boldsymbol{ij}}\}\rangle$ gives us a degenerate ground state. We note that

$$\exp\left(i\sum_{\boldsymbol{i}}\phi_{\boldsymbol{i}}\sum_{\boldsymbol{a}}S_{\boldsymbol{i},\boldsymbol{i}+\boldsymbol{a}}^z\right) \tag{87}$$

generates a $U(1)$ gauge transformation $|\{\theta_{\boldsymbol{ij}}\}\rangle\rightarrow|\{\tilde{\theta}_{\boldsymbol{ij}}\}\rangle$, where

$$\tilde{\theta}_{\boldsymbol{ij}}=\theta_{\boldsymbol{ij}}+\phi_{\boldsymbol{i}}-\phi_{\boldsymbol{j}}. \tag{88}$$

Thus two $U(1)$ gauge equivalent configurations θ_{ij} and $\tilde{\theta}_{ij}$ give rise to the same projected state

$$\mathcal{P}|\{\theta_{ij}\}\rangle = \mathcal{P}|\{\tilde{\theta}_{ij}\}\rangle. \tag{89}$$

We find that the degenerate ground states are described by $U(1)$ gauge equivalent classes of θ_{ij}. The degenerate ground states also have a $U(1)$ gauge structure.

When $J=0$ but $g_1\neq 0$, the degeneracy in the ground states is lifted. One can show that, in this case, $\mathcal{P}|\{\theta_{ij}\}\rangle$ is an energy eigenstate with energy $g_1\Sigma_{p}\cos(\theta_{i_1i_2}+\theta_{i_2i_3}+\theta_{i_3i_4}+\theta_{i_4i_1})$. Clearly, two $U(1)$ gauge equivalent configurations θ_{ij} and $\tilde{\theta}_{ij}$ have the same energy. A nonzero g_1 makes the closed string nets fluctuate, and vanishing J means that the strings in the string nets have no tension. Thus the $J=0$ ground state has strong fluctuations of large closed string nets, and the ground state has a closed-string-net condensation [13].

When $J\neq 0$, $\mathcal{P}|\{\theta_{ij}\}\rangle$ is no longer an eigenstate. The fluctuations of θ_{ij} describe a dynamical $U(1)$ gauge theory with θ_{ij} as the gauge potential [13,26].

The ends of open strings carry a unit charge of the $U(1)$ gauge field. Since the $U(1)$ gauge field is compact, our model also has monopole excitations with magnetic charge 1/2 (i.e., the monopole generates 2π flux). Both charges and monopoles are bosons. However, according to Ref. [30], a bound state of a unit charge and a monopole of magnetic charge 1/2 is a fermion. Thus the 3D rotor model also has emergent massive fermions.

B. (Quasi)exactly soluble QED on a cubic lattice

To obtain massless Dirac fermions and $U(1)$ gauge bosons from a local bosonic model, we mix the spin-$\frac{1}{2}\frac{1}{2}\frac{1}{2}$ model and the rotor model to get

$$H_{QED}=U\sum_{i}\left(\psi_i^\dagger\psi_i+\sum_{a}S^z_{i,i+a}\right)^2+\frac{J}{2}\sum_{i,a}(S^z_{i,i+a})^2 + g_1\sum_{p}\cos(\hat{\Phi}_p)+g\sum_{p}\hat{F}_p + t\sum_{i}\sum_{a=x,y,z}(ie^{i\hat{\theta}_{ij}}\hat{U}_{i,i+a}\psi_i^\dagger\psi_{i+a}+\text{H.c.}), \tag{90}$$

where $\hat{\Phi}_p=\hat{\theta}_{i_1i_2}+\hat{\theta}_{i_2i_3}+\hat{\theta}_{i_3i_4}+\hat{\theta}_{i_4i_1}$. If we restrict ourselves within the physical Hilbert space with even numbers of fermions per site, the above model is a local bosonic model.

Let us first set $J=0$. In this case, the above model can be solved exactly. First let us also set $U=0$. In this case $\hat{\theta}_{ij}$ and $\hat{U}_{ij}$ commute with H_{QED} and commute with each other. Let $|\{\theta_{ij},s_{ij}\},n\rangle$ be the common eigenstates of $\hat{\theta}_{ij}$ and $\hat{U}_{ij}$, where $n=1,2,\ldots,2^{N_{site}}$ labels different degenerate common eigenstates. Within the subspace expanded by $|\{\theta_{ij},s_{ij}\},n\rangle$, $n=1,2,\ldots,2^{N_{site}}$, H_{QED} reduces to

$$H_{hop}=g_1\sum_{p}\cos(\Phi_p)+g\sum_{p}f_p + t\sum_{i}\sum_{a=x,y,z}(ie^{i\theta_{ij}}s_{i,i+a}\psi_i^\dagger\psi_{i+a}+\text{H.c.}), \tag{91}$$

which is a free fermion hopping model. Let $|\{\theta_{ij},s_{ij}\},\Psi_n\rangle$ be the many-fermion eigenstate of the above fermion hopping model and let $E(\{\theta_{ij},s_{ij}\},n)$ be its energy. Then $|\{\theta_{ij)},s_{ij}\},\Psi_n\rangle$ is also an eigenstate of $H_{QED}|_{J=0,U=0}$ with energy $E(\{\theta_{ij},s_{ij}\},n)$.

We note that

$$\hat{N}_i=\psi_i^\dagger\psi_i+\sum_{a}S^z_{i,i+a} \tag{92}$$

commute with each other and commute with H_{QED}. Thus the eigenstates of $H_{QED}|_{J=0}$ can be obtained from the eigenstates of $H_{QED}|_{J=0,U=0}$ by projecting onto the subspace with $\hat{N}_i=N_i$:

$$\mathcal{P}_{\{N_i\}}|\{\theta_{ij},s_{ij}\},\Psi_n\rangle. \tag{93}$$

The above state is an eigenstate of $H_{QED}|_{J=0}$ with energy

$$U\sum_{i}N_i+E(\{\theta_{ij},s_{ij}\},n). \tag{94}$$

Equations (93) and (94) are our exact solution of $H_{QED}|_{J=0}$. (We have implicitly assumed that $\mathcal{P}_{\{N_i\}}$ also performs the projection onto the physical Hilbert space of even numbers of fermions per site.)

When U is positive and large, the low energy excitations appear only in the sector $N_i=0$. Those low energy eigenstates are given by $\mathcal{P}|\{\theta_{ij},s_{ij}\},\Psi_n\rangle$ where $\mathcal{P}$ is the projection onto the $N_i=0$ subspace and the even-fermion subspace. Their energy is $E(\{\theta_{ij},s_{ij}\},n)$.

Let us further assume that $-g_1\gg|t|$ and $g\gg|t|$. In this limit, the ground state has $f_p=-1$ and $\Phi_p=0$. We can choose

$$\theta_{i,i+a}=0,\quad a=x,y,z,$$
$$s_{i,i+x}=1,$$
$$s_{i,i+y}=(-)^{i_x},$$
$$s_{i,i+z}=(-)^{i_x+i_y} \tag{95}$$

to describe such a configuration. For this configuration, Eq. (91) describes a staggered fermion Hamiltonian [24,53,54]. The ground state wave function $\mathcal{P}|\{\theta_{ij},s_{ij}\},\Psi_0\rangle$ is an eigenstate of the $U(1)$ closed-string-net operator $W_{U(1)}(C_{close})$ with eigenvalue 1. It is also an eigenstate of the Z_2 closed-string-net operator $W(C_{close})$ with eigenvalue $(-)^{N_p}$ where N_p is the number of square plaquettes enclosed by C_{close}. We see that there is a condensation of closed $U(1)$ and Z_2 string nets in the $J=0$ ground state. In such a string-net condensed state, there are gapless fermionic excitations, which are described by fermion hopping in the π-flux phase.

In momentum space, the fermion hopping Hamiltonian Eq. (91) for the π-flux configuration has the form

$$H_{hop}=\sum_{\boldsymbol{k}}{}' \; \Psi^\dagger_{a,\boldsymbol{k}}\Gamma(\boldsymbol{k})\Psi_{a,\boldsymbol{k}}+\text{const}, \tag{96}$$

where

$$\Psi^T_{a,\boldsymbol{k}}=(\psi_{a,\boldsymbol{k}},\psi_{a,\boldsymbol{k}+\boldsymbol{Q}_x},\psi_{a,\boldsymbol{k}+\boldsymbol{Q}_y},\psi_{a,\boldsymbol{k}+\boldsymbol{Q}_x+\boldsymbol{Q}_y}),$$

$$\Gamma(\boldsymbol{k})=2t[\sin(k_x)\Gamma_1+\sin(k_y)\Gamma_2+\sin(k_z)\Gamma_3],$$

and $\Gamma_1=\tau^3\otimes\tau^0$, $\Gamma_2=\tau^1\otimes\tau^3$, and $\Gamma_3=\tau^1\otimes\tau^1$. Here $\tau^{1,2,3}$ are the Pauli matrices and τ^0 is the 2×2 identity matrix. The momentum summation $\Sigma'_{\boldsymbol{k}}$ is over the range $k_x \in(-\pi/2,\pi/2)$, $k_y\in(-\pi/2,\pi/2)$, and $k_z\in(-\pi,\pi)$. Since $\{\Gamma_i,\Gamma_j\}=2\delta_{ij}$, $i,j=1,2,3$, we find that the fermions have the dispersion

$$E(\boldsymbol{k})=\pm 2t\sqrt{\sin^2(k_x)+\sin^2(k_y)+\sin^2(k_z)}. \tag{97}$$

We see that the dispersion has two nodes at $\boldsymbol{k}=0$ and $\boldsymbol{k}=(0,0,\pi)$. Thus, Eq. (91) will give rise to two massless four-component Dirac fermions in the continuum limit.

After including the $U(1)$ gauge fluctuations described by $\theta_{\boldsymbol{ij}}$ and the Z_2 gauge fluctuations described by $s_{\boldsymbol{ij}}$, the massless Dirac fermions interact with the $U(1)$ and the Z_2 gauge fields as fermions with unit charge. Therefore the total low energy effective theory of our model is a QED with two families of Dirac fermions of unit charge (plus an extra Z_2 gauge field). We will call these fermions artificial electrons. The continuum effective theory has the form

$$\mathcal{L}=\bar{\psi}_I D_0\gamma^0\psi_I+v_f\bar{\psi}_I D_i\gamma^i\psi_I+\frac{C}{Jl_0}\boldsymbol{E}^2-l_0g_1\boldsymbol{B}^2+\cdots, \tag{98}$$

where l_0 is the lattice constant, $I=1,2$, $D_0=\partial_t+ia_0$, $D_i=\partial_i+ia_i|_{i=1,2,3}$, $v_f=2l_0t$, $\gamma^\mu|_{\mu=0,1,2,3}$ are 4×4 Dirac matrices, and $\bar{\psi}_I=\psi_I^\dagger\gamma^0$.

We wish to point out the constant C is Eq. (98) is of order 1. Thus the coefficient of the $\boldsymbol{E}^2$ term $C/Jl_0\to\infty$ when $J=0$. For a finite J, the $U(1)$ gauge field will have a nontrivial dynamics. We also point out that, without fine-tuning, the speed of artificial light, $c_a\sim l_0\sqrt{Jg_1}$, and the speed of artificial electrons, v_f, do not have to be the same in our model. Thus Lorentz symmetry is not guaranteed.

We would like to remark that, for finite J, the $U(1)$ closed-string operators no longer condense. A necessary (but not sufficient) condition for closed strings to condense is that the ground state expectation value of the closed-string operator satisfy the perimeter law

$$\langle W_{U(1)}(C_{close})\rangle=Ae^{-L_C/\xi}, \tag{99}$$

where L_C is the length of the closed string and (A,ξ) are constants for large closed strings. We note that the closed-string operators are the Wilson-loop operators of the $U(1)$ gauge field. If the (3+1)D $U(1)$ gauge theory is in the Coulomb phase where the artificial light is gapless, it was found that [18]

$$\langle W(C_{close})\rangle=A(C)e^{-L_C/\xi}, \tag{100}$$

where $A(C)$ depends on the shape of the closed string C_{close} even in the large string limit. Thus the closed strings in our model do not exactly condense. The $U(1)$ Coulomb phase is, in some sense, similar to the algebraic long range order phase of the (1+1)D interacting boson model, where the bosons do not exactly condense but the boson operator has an algebraic long range correlation.

C. Emergent chiral symmetry from the PSG

Equation (98) describes the low energy dynamics of the ends of open strings (the fermion ψ) and the "condensed" closed string nets [the $U(1)$ gauge field]. The fermions and gauge boson are massless and interact with each other. Here we would like to address an important question: after integrating out high energy fermions and gauge fluctuations, do the fermions and gauge bosons remain massless? In general, interactions between massless excitations will generate a mass term for them, unless the masslessness is protected by symmetry or something else. We know that, due to the $U(1)$ gauge invariance, the radiative corrections cannot generate counterterms that break the $U(1)$ gauge invariance. Thus radiative corrections cannot generate mass for the $U(1)$ gauge boson. For the fermions, if the theory has a chiral symmetry $\psi_I\to e^{i\theta\gamma^5}\psi_I$, $\gamma^5=\gamma^0\gamma^1\gamma^2\gamma^3$, then the radiative corrections cannot generate counterterms that break the chiral symmetry and thus cannot generate mass for the fermions. Although the low energy effective theory Eq. (98) appears to have chiral symmetry, in fact it does not. This is because Eq. (98) is derived from a lattice model. It contains many other higher order terms summarized by the ellipsis in Eq. (98). Those higher order terms do not have chiral symmetry. To see this, we note that the action of γ^5 on $\Psi_{a,\boldsymbol{k}}$ is realized by a 4×4 matrix $\gamma^5\propto\Gamma_1\Gamma_2\Gamma_3\propto\tau^3\otimes\tau^2$. We also note that the periodic boundary conditions of $\Psi_{a,\boldsymbol{k}}$ in the reduced Brillouin zone are given by

$$\Psi_{a,\boldsymbol{k}+\boldsymbol{Q}_x}=\tau^1\otimes\tau^0\Psi_{a,\boldsymbol{k}}, \quad \Psi_{a,\boldsymbol{k}+\boldsymbol{Q}_y}=\tau^0\otimes\tau^1\Psi_{a,\boldsymbol{k}}. \tag{101}$$

We find that the action of γ^5 is incompatible with the periodic boundary conditions since γ^5 does not commute with $\tau^1\otimes\tau^0$ and $\tau^0\otimes\tau^1$. Therefore the chiral symmetry generated by γ^5 cannot be realized on a lattice. Due to the lack of chiral symmetry, it appears that the radiative corrections can generate a mass term

$$\delta\mathcal{L}=\bar{\psi}_{I,a}m\psi_{I,a}, \tag{102}$$

which is allowed by the symmetry.

The lack of chiral symmetry on the lattice makes it very difficult to study massless fermions/quarks in lattice gauge theory. In the last few years, this problem was solved using the Ginsparg-Wilson relation [55–58]. In the following, we would like to show that there is another way to solve the massless-fermion/chiral-symmetry problem. We will show that our model has an emergent chiral symmetry that appears only at low energies. The low energy chiral symmetry comes from the nontrivial quantum order and the associated PSG in the string-net condensed ground state [9,15,16]. The Dirac operator in our model satisfies the linear relation

$$WDW^{\dagger}=D, \quad W\in \text{PSG}, \tag{103}$$

in contrast to the nonlinear Ginsparg-Wilson relation

$$D\gamma^5+\gamma^5 D=aD\gamma^5 D. \tag{104}$$

Because of the low energy chiral symmetry, the two families of Dirac fermions remain massless even after we include the radiative corrections from the interaction with the $U(1)$ gauge bosons.

To see how the string-net condensation and the related PSG protect the massless fermions, we follow closely the discussion in Sec. V C. The Hamiltonian Eq. (90) is a mixture of the rotor model and the spin-$\frac{1}{2}\frac{1}{2}\frac{1}{2}$ model. The symmetry properties of the rotor part are simple. Here, we will concentrate on the spin-$\frac{1}{2}\frac{1}{2}\frac{1}{2}$ part. Equation (90) is not invariant under the six parity transformations $P_{x,y,z}$ and $P_{xy,yz,zx}$ that generate $x\leftrightarrow -x$, $y\leftrightarrow -y$, $z\leftrightarrow -z$, $x\leftrightarrow y$, $y\leftrightarrow z$, and $z\leftrightarrow x$. But it is invariant under the parity $P_{x,y,z}$ and $P_{xy,yz,zx}$ followed by the spin rotations $\gamma_{P_{x,y,z}}$ and $\gamma_{P_{xy,yz,zx}}$, respectively. In the fermion representation $\gamma_{P_{x,y,z}}$ and $\gamma_{P_{xy,yz,zx}}$ generate the following transformations:

$$\begin{aligned}
\gamma_{P_x}&: \quad \lambda_{\boldsymbol{i}}^{x}\leftrightarrow\lambda_{\boldsymbol{i}}^{\bar{x}}, \quad \psi_{\boldsymbol{i}}\leftrightarrow\psi_{\boldsymbol{i}}^{\dagger},\\
\gamma_{P_y}&: \quad \lambda_{\boldsymbol{i}}^{y}\leftrightarrow\lambda_{\boldsymbol{i}}^{\bar{y}}, \quad \psi_{\boldsymbol{i}}\leftrightarrow\psi_{\boldsymbol{i}}^{\dagger},\\
\gamma_{P_z}&: \quad \lambda_{\boldsymbol{i}}^{z}\leftrightarrow\lambda_{\boldsymbol{i}}^{\bar{z}}, \quad \psi_{\boldsymbol{i}}\leftrightarrow\psi_{\boldsymbol{i}}^{\dagger},\\
\gamma_{P_{xy}}&: \quad \lambda_{\boldsymbol{i}}^{x}\leftrightarrow\lambda_{\boldsymbol{i}}^{y}, \quad \lambda_{\boldsymbol{i}}^{\bar{x}}\leftrightarrow\lambda_{\boldsymbol{i}}^{\bar{y}},\\
\gamma_{P_{yz}}&: \quad \lambda_{\boldsymbol{i}}^{y}\leftrightarrow\lambda_{\boldsymbol{i}}^{z}, \quad \lambda_{\boldsymbol{i}}^{\bar{y}}\leftrightarrow\lambda_{\boldsymbol{i}}^{\bar{z}},\\
\gamma_{P_{zx}}&: \quad \lambda_{\boldsymbol{i}}^{z}\leftrightarrow\lambda_{\boldsymbol{i}}^{x}, \quad \lambda_{\boldsymbol{i}}^{\bar{z}}\leftrightarrow\lambda_{\boldsymbol{i}}^{\bar{x}}.
\end{aligned} \tag{105}$$

The symmetries $T_{x,y}$, $\gamma_{P_{x,y,z}}P_{x,y,z}$, and $\gamma_{P_{xy,yz,zx}}P_{xy,yz,zx}$ are realized in the hopping Hamiltonian Eq. (91) through the PSG. The hopping Hamiltonian is invariant only under symmetry transformations followed by proper Z_2 gauge transformations $G(\boldsymbol{i})$. Since the π-flux configuration $s_{\boldsymbol{ij}}$ of the spin-$\frac{1}{2}\frac{1}{2}\frac{1}{2}$ sector and the zero-flux configuration $\theta_{\boldsymbol{ij}}$ of the rotor sector do not break any symmetries, we expect the hopping Hamiltonian Eq. (91) to be invariant under $G_{x,y,z}T_{x,y,z}$, $G_{P_{x,y,z}}\gamma_{P_{x,y,z}}P_{x,y,z}$, and $G_{P_{xy,yz,zx}}\gamma_{P_{xy,yz,zx}}P_{xy,yz,zx}$. The actions of $T_{x,y}$ and $\gamma_{P_{xy,yz,zx}}P_{xy,yz,zx}$ on the ψ fermion are standard coordinate transformations. The action of $\gamma_{P_{x,y,z}}P_{x,y,z}$ on the ψ fermion is given by

$$\begin{aligned}
\gamma_{P_x}P_x&: \quad \psi_{(i_x,i_y,i_z)}\leftrightarrow\psi^{\dagger}_{(-i_x,i_y,i_z)},\\
\gamma_{P_y}P_y&: \quad \psi_{(i_x,i_y,i_z)}\leftrightarrow\psi^{\dagger}_{(i_x,-i_y,i_z)},\\
\gamma_{P_y}P_y&: \quad \psi_{(i_x,i_y,i_z)}\leftrightarrow\psi^{\dagger}_{(i_x,i_y,-i_z)}.
\end{aligned} \tag{106}$$

For the π-flux configuration Eq. (95), we need to choose the following $G_{x,y,z}$, $G_{P_{x,y,z}}$, and $G_{P_{xy,yz,zx}}$ in order for the combined transformation $G_{x,y}T_{x,y}$, $G_{P_{x,y}}\gamma_{P_{x,y}}P_{x,y}$, and $G_{P_{xy,yz,zx}}\gamma_{P_{xy,yz,zx}}P_{xy,yz,zx}$ to be the symmetries of the hopping Hamiltonian Eq. (91):

$$\begin{aligned}
&G_x=(-)^{i_y+i_z}, \quad G_y=(-)^{i_z}, \quad G_z=1,\\
&G_{P_x}=(-)^{i_x}, \quad G_{P_y}=(-)^{i_y}, \quad G_{P_z}=(-)^{i_z},\\
&G_{P_{xy}}=(-)^{i_x i_y}, \quad G_{P_{yz}}=(-)^{i_y i_z},\\
&G_{P_{zx}}=(-)^{i_x i_y+i_y i_z+i_z i_x}.
\end{aligned} \tag{107}$$

The hopping Hamiltonian is also invariant under a global Z_2 gauge transformation

$$G_0: \quad \psi_{\boldsymbol{i}}\rightarrow -\psi_{\boldsymbol{i}}. \tag{108}$$

The transformations $\{G_{x,y,z}T_{x,y,z}, \ G_{P_{x,y,z}}\gamma_{P_{x,y,z}}P_{x,y,z}, \ G_{P_{xy,yz,zx}}\gamma_{P_{xy,yz,zx}}P_{xy,yz,zx}, G_0\}$ generate a PSG (a part of the full PSG) of the hopping Hamiltonian.

To study the robustness of massless fermions, we consider a more general Hamiltonian by adding

$$\delta H=\sum_{C_{\boldsymbol{ij}}}\left[t(C_{\boldsymbol{ij}})\tilde{W}_{U(1)}(C_{\boldsymbol{ij}})+\text{H.c.}\right] \tag{109}$$

to H_{QED}, where $C_{\boldsymbol{ij}}$ is an open string connecting site $\boldsymbol{i}$ and site $\boldsymbol{j}$ and $\tilde{W}_{U(1)}(C_{\boldsymbol{ij}})$ an open-string operator

$$\tilde{W}_{U(1)}(C_{open})=\psi^{\dagger}_{\boldsymbol{i}_1}e^{i\theta_{\boldsymbol{i}_1\boldsymbol{i}_2}}\hat{U}_{\boldsymbol{i}_1\boldsymbol{i}_2}\cdots e^{i\theta_{\boldsymbol{i}_{n-1}\boldsymbol{i}_n}}\hat{U}_{\boldsymbol{i}_{n-1}\boldsymbol{i}_n}\psi_{\boldsymbol{i}_n}. \tag{110}$$

The new Hamiltonian is still exactly soluble, when $J=0$. We will choose $t(C_{\boldsymbol{ij}})$ such that the new Hamiltonian has translation symmetries and $P_{x,y,z}$ parity symmetries. We find that the resulting projective symmetry imposes enough constraint on the hopping Hamiltonian for the ends of condensed strings such that the Hamiltonian always has massless Dirac fermions [assuming that $t(C_{\boldsymbol{ij}})$ is not too big compared to g and g_1]. Although the PSG transformations $G_{P_{xy,yz,zx}}\gamma_{P_{xy,yz,zx}}P_{xy,yz,zx}$ are not needed for the existence of the massless fermions, we will still include them in the following discussion.

For small $t(C_{\boldsymbol{ij}})$, the ground state is still described by the π-flux configuration. The new hopping Hamiltonian for the π-flux configuration has the more general form

$$H=\sum_{\langle \boldsymbol{ij}\rangle}\left(\chi_{\boldsymbol{ij}}\psi^{\dagger}_{\boldsymbol{i}}\psi_{\boldsymbol{j}}+\text{H.c.}\right). \tag{111}$$

The symmetry of the generalized H_{QED} requires that the above hopping Hamiltonian be invariant under the PSG generated by $\{G_0, G_{x,y,z}T_{x,y,z}, G_{P_{x,y,z}}\gamma_{P_{x,y,z}}P_{x,y,z}, G_{P_{xy,yz,zx}}\gamma_{P_{xy,yz,zx}}P_{xy,yz,zx}\}$.

The invariance under $G_{x,y,z}T_{x,y,z}$ requires that

$$\chi_{\boldsymbol{i},\boldsymbol{i}+\boldsymbol{m}} = (-)^{i_y m_z}(-)^{i_x(m_y+m_z)}\chi_{\boldsymbol{m}}. \tag{112}$$

In momentum space,

$$\chi(\boldsymbol{k}_1,\boldsymbol{k}_2) \equiv N_{site}^{-1}\sum_{\boldsymbol{ij}} e^{-i\boldsymbol{k}_1\cdot\boldsymbol{i}+i\boldsymbol{k}_2\cdot\boldsymbol{j}}\chi_{\boldsymbol{ij}}$$

$$= \sum_{\alpha,\beta=0,1} \epsilon_{\alpha\beta}(\boldsymbol{k}_2)\,\delta_{\boldsymbol{k}_1-\boldsymbol{k}_2+\alpha\boldsymbol{Q}_x+\beta\boldsymbol{Q}_y}, \tag{113}$$

where

$$\epsilon_{00}(\boldsymbol{k}) = \sum_{m_y+m_z=even, m_z=even} e^{i\boldsymbol{k}\cdot\boldsymbol{m}}\chi_{\boldsymbol{m}},$$

$$\epsilon_{10}(\boldsymbol{k}) = \sum_{m_y+m_z=odd, m_z=even} e^{i\boldsymbol{k}\cdot\boldsymbol{m}}\chi_{\boldsymbol{m}},$$

$$\epsilon_{01}(\boldsymbol{k}) = \sum_{m_y+m_z=even, m_z=odd} e^{i\boldsymbol{k}\cdot\boldsymbol{m}}\chi_{\boldsymbol{m}},$$

$$\epsilon_{11}(\boldsymbol{k}) = \sum_{m_y+m_z=odd, m_z=odd} e^{i\boldsymbol{k}\cdot\boldsymbol{m}}\chi_{\boldsymbol{m}}. \tag{114}$$

We note that $\epsilon_{\alpha\beta}(\boldsymbol{k})$ are periodic functions in the lattice Brillouin zone $-\pi < k_{x,y,z} < \pi$. They also satisfy

$$\epsilon_{\alpha\beta}(\boldsymbol{k}) = (-)^{\alpha}\epsilon_{\alpha\beta}(\boldsymbol{k}+\boldsymbol{Q}_y+\boldsymbol{Q}_z),$$

$$\epsilon_{\alpha\beta}(\boldsymbol{k}) = (-)^{\beta}\epsilon_{\alpha\beta}(\boldsymbol{k}+\boldsymbol{Q}_z). \tag{115}$$

The $\Gamma(\boldsymbol{k})$ in Eq. (96) now has the form

$$\Gamma(\boldsymbol{k}) = \begin{pmatrix} \epsilon_{00}(\boldsymbol{k}) & \epsilon_{10}(\boldsymbol{k}+\boldsymbol{Q}_x) & \epsilon_{01}(\boldsymbol{k}+\boldsymbol{Q}_y) & \epsilon_{11}(\boldsymbol{k}+\boldsymbol{Q}_x+\boldsymbol{Q}_y) \\ \epsilon_{10}(\boldsymbol{k}) & \epsilon_{00}(\boldsymbol{k}+\boldsymbol{Q}_x) & \epsilon_{11}(\boldsymbol{k}+\boldsymbol{Q}_y) & \epsilon_{01}(\boldsymbol{k}+\boldsymbol{Q}_x+\boldsymbol{Q}_y) \\ \epsilon_{01}(\boldsymbol{k}) & \epsilon_{11}(\boldsymbol{k}+\boldsymbol{Q}_x) & \epsilon_{00}(\boldsymbol{k}+\boldsymbol{Q}_y) & \epsilon_{10}(\boldsymbol{k}+\boldsymbol{Q}_x+\boldsymbol{Q}_y) \\ \epsilon_{11}(\boldsymbol{k}) & \epsilon_{01}(\boldsymbol{k}+\boldsymbol{Q}_x) & \epsilon_{10}(\boldsymbol{k}+\boldsymbol{Q}_y) & \epsilon_{00}(\boldsymbol{k}+\boldsymbol{Q}_x+\boldsymbol{Q}_y) \end{pmatrix}. \tag{116}$$

Just as discussed in Sec. V C, the invariance under $G_{P_x}\gamma_{P_x}P_x$ requires that

$$-\chi_{P_x\boldsymbol{j},P_x\boldsymbol{i}} = G_{P_x}(\boldsymbol{i})\chi_{\boldsymbol{ij}}G_{P_x}(\boldsymbol{j}) \tag{117}$$

or

$$\chi_{-P_x\boldsymbol{m}} = -(-)^{m_x m_y+m_y m_z+m_z m_x}(-)^{m_x}\chi_{\boldsymbol{m}}. \tag{118}$$

In momentum space, the above becomes

$$\epsilon_{00}(-P_x\boldsymbol{k}) = -\epsilon_{00}(\boldsymbol{k}+\boldsymbol{Q}_x),$$

$$\epsilon_{10}(-P_x\boldsymbol{k}) = -\epsilon_{10}(\boldsymbol{k}),$$

$$\epsilon_{01}(-P_x\boldsymbol{k}) = \epsilon_{01}(\boldsymbol{k}+\boldsymbol{Q}_x),$$

$$\epsilon_{11}(-P_x\boldsymbol{k}) = -\epsilon_{11}(\boldsymbol{k}), \tag{119}$$

where $\boldsymbol{Q}_z = \pi\boldsymbol{z}$. Similarly, the invariance under $G_{P_y}\gamma_{P_y}P_y$ requires that

$$\chi_{-P_y\boldsymbol{m}} = -(-)^{m_x m_y+m_y m_z+m_z m_x}(-)^{m_y}\chi_{\boldsymbol{m}}. \tag{120}$$

In momentum space

$$\epsilon_{00}(-P_y\boldsymbol{k}) = -\epsilon_{00}(\boldsymbol{k}),$$

$$\epsilon_{10}(-P_y\boldsymbol{k}) = \epsilon_{10}(\boldsymbol{k}+\boldsymbol{Q}_x),$$

$$\epsilon_{01}(-P_y\boldsymbol{k}) = -\epsilon_{01}(\boldsymbol{k}),$$

$$\epsilon_{11}(-P_y\boldsymbol{k}) = -\epsilon_{11}(\boldsymbol{k}+\boldsymbol{Q}_x). \tag{121}$$

The invariance under $G_{P_z}\gamma_{P_z}P_z$ requires that

$$\chi_{-P_z\boldsymbol{m}} = -(-)^{m_x m_y+m_y m_z+m_z m_x}(-)^{m_z}\chi_{\boldsymbol{m}}. \tag{122}$$

In momentum space

$$\epsilon_{00}(-P_z\boldsymbol{k}) = -\epsilon_{00}(\boldsymbol{k}),$$

$$\epsilon_{10}(-P_z\boldsymbol{k}) = -\epsilon_{10}(\boldsymbol{k}+\boldsymbol{Q}_x),$$

$$\epsilon_{01}(-P_z\boldsymbol{k}) = -\epsilon_{01}(\boldsymbol{k}),$$

$$\epsilon_{11}(-P_z\boldsymbol{k}) = \epsilon_{11}(\boldsymbol{k}+\boldsymbol{Q}_x). \tag{123}$$

The invariance under $G_{P_{xy}}\gamma_{P_{xy}}P_{xy}$ requires that

$$\chi_{P_{xy}\boldsymbol{i},P_{xy}\boldsymbol{j}} = G_{P_{xy}}(\boldsymbol{i})\chi_{\boldsymbol{ij}}G_{P_{xy}}(\boldsymbol{j}) \tag{124}$$

or

$$\chi_{P_{xy}\boldsymbol{m}}=(-)^{m_x m_y}\chi_{\boldsymbol{m}}. \tag{125}$$

In momentum space

$$\begin{aligned}
\epsilon_{00}(P_{xy}\boldsymbol{k})&=\epsilon_{00}(\boldsymbol{k}),\\
\epsilon_{10}(P_{xy}\boldsymbol{k})&=\epsilon_{10}(\boldsymbol{k}+\boldsymbol{Q}_x),\\
\epsilon_{01}(P_{xy}\boldsymbol{k})&=\epsilon_{01}(\boldsymbol{k}+\boldsymbol{Q}_x),\\
\epsilon_{11}(P_{xy}\boldsymbol{k})&=\epsilon_{11}(\boldsymbol{k}).
\end{aligned} \tag{126}$$

The invariance under $G_{P_{yz}}\gamma_{P_{yz}}P_{yz}$ requires that

$$\chi_{P_{yz}\boldsymbol{m}}=(-)^{m_y m_z}\chi_{\boldsymbol{m}} \tag{127}$$

or

$$\begin{aligned}
\epsilon_{00}(P_{yz}\boldsymbol{k})&=\epsilon_{00}(\boldsymbol{k}),\\
\epsilon_{10}(P_{yz}\boldsymbol{k})&=-\epsilon_{10}(\boldsymbol{k}),\\
\epsilon_{01}(P_{yz}\boldsymbol{k})&=-\epsilon_{01}(\boldsymbol{k}),\\
\epsilon_{11}(P_{yz}\boldsymbol{k})&=\epsilon_{11}(\boldsymbol{k}).
\end{aligned} \tag{128}$$

The invariance under $G_{P_{zx}}\gamma_{P_{zx}}P_{zx}$ requires that

$$\chi_{P_{zx}\boldsymbol{m}}=(-)^{m_x m_y+m_y m_z+m_z m_x}\chi_{\boldsymbol{m}} \tag{129}$$

or

$$\begin{aligned}
\epsilon_{00}(P_{zx}\boldsymbol{k})&=\epsilon_{00}(\boldsymbol{k}),\\
\epsilon_{10}(P_{zx}\boldsymbol{k})&=\epsilon_{10}(\boldsymbol{k}+\boldsymbol{Q}_x),\\
\epsilon_{01}(P_{zx}\boldsymbol{k})&=-\epsilon_{01}(\boldsymbol{k}),\\
\epsilon_{11}(P_{zx}\boldsymbol{k})&=\epsilon_{11}(\boldsymbol{k}+\boldsymbol{Q}_x).
\end{aligned} \tag{130}$$

We see that Eq. (119) requires that $\epsilon_{10}(\boldsymbol{k})|_{k_y=k_z=0}=0$ and $\epsilon_{11}(\boldsymbol{k})|_{k_y=k_z=0}=0$. Equation (123) requires that $\epsilon_{00}(\boldsymbol{k})|_{k_x=k_y=0}=0$ and $\epsilon_{01}(\boldsymbol{k})|_{k_x=k_y=0}=0$. Thus $\epsilon_{\alpha\beta}(0)=0$. When combined with Eq. (115), Eq. (119), and Eq. (123), we find

$$\epsilon_{\alpha\beta}(\alpha_x\boldsymbol{Q}_x+\alpha_y\boldsymbol{Q}_y+\alpha_z\boldsymbol{Q}_z)=0,\quad \alpha_x,\alpha_y,\alpha_z=0,1. \tag{131}$$

Therefore $\Gamma(\boldsymbol{k})=0$ when $\boldsymbol{k}=0,\boldsymbol{Q}_z$. The two translation $T_{x,y}$ and the three parity $\gamma_{P_{x,y,z}}P_{x,y,z}$ symmetries of H_{QED} guarantee the existence of at least two four-component massless Dirac fermions, or, more precisely, no symmetric local perturbations in the local bosonic model H_{QED} can generate mass terms for the two massless Dirac fermions in the unperturbed Hamiltonian.

Since the mass term in the continuum effective field theory is not allowed by the underlying lattice PSG, we say that our model has an emergent chiral symmetry. The masslessness of the Dirac fermion is protected by the quantum order and the associated PSG.

VIII. QED AND QCD FROM A BOSONIC MODEL ON A CUBIC LATTICE

In this section, we are going to generalize the results in Ref. [59] and Ref. [15] and use a bosonic model on a cubic lattice to generate QED and QCD with $2N_f$ families of massless quarks and leptons. To describe the local Hilbert space on site $\boldsymbol{i}$ in our bosonic model, it is convenient to introduce fermions $\lambda_{\boldsymbol{i}}^a$ and $\psi_{\boldsymbol{i}}^{n\alpha}$, where $a=1,\dots,N_f$, $n=1,\dots,2N_f$, and $\alpha=1,2,3$. $\lambda_{\boldsymbol{i}}^a$ is in the fundamental representation of an $SU(N_f)$ group. $\psi_{\boldsymbol{i}}^{n\alpha}$ is in the fundamental representation of an $SU(3)$ color group and an $SU(2N_f)$ group. The Hilbert space of fermions is bigger than the Hilbert space of our boson model. Only the physical subspace of the fermion Hilbert space becomes the Hilbert space of our boson model. The physical state on each site is formed by color singlet states that satisfy

$$\left(\lambda_{\boldsymbol{i}}^{a\dagger}\lambda_{\boldsymbol{i}}^{a}\delta^{\alpha\beta}+\psi_{\boldsymbol{i}}^{n\alpha\dagger}\psi_{\boldsymbol{i}}^{n\beta}-\delta^{\alpha\beta}\frac{3}{2}N_f\right)|\Phi_{phys}\rangle=0, \tag{132}$$

where N_f is assumed to be even. Once restricted within the physical Hilbert space, the fermion model becomes our local bosonic model.

In the fermion representation, the local physical operators in our bosonic model are given by

$$\begin{aligned}
S_{\boldsymbol{i}}^{mn}&=\psi_{\boldsymbol{i}}^{m\alpha\dagger}\psi_{\boldsymbol{i}}^{n\alpha}-\frac{1}{2N_f}\delta^{mn}\psi_{\boldsymbol{i}}^{l\alpha\dagger}\psi_{\boldsymbol{i}}^{l\alpha},\\
M_{\boldsymbol{i}}^{ab}&=\lambda_{\boldsymbol{i}}^{a\dagger}\lambda_{\boldsymbol{i}}^{b}-\frac{1}{N_f}\delta^{ab}\lambda_{\boldsymbol{i}}^{c\dagger}\lambda_{\boldsymbol{i}}^{c},\\
\Gamma_{\boldsymbol{i}}^{a,lmn}&=\lambda_{\boldsymbol{i}}^{a\dagger}\psi_{\boldsymbol{i}}^{l\alpha}\psi_{\boldsymbol{i}}^{m\beta}\psi_{\boldsymbol{i}}^{n\gamma}\epsilon_{\alpha\beta\gamma}.
\end{aligned} \tag{133}$$

We note that by definition $M_{\boldsymbol{i}}^{aa}=S_{\boldsymbol{i}}^{nn}=0$. The Hamiltonian of our boson model is given by

$$\begin{aligned}
H=\frac{J_1}{N_f}\sum_{\langle\boldsymbol{ij}\rangle}S_{\boldsymbol{i}}^{mn}S_{\boldsymbol{j}}^{nm}+\frac{J_2}{N_f}\sum_{\langle\boldsymbol{ij}\rangle}M_{\boldsymbol{i}}^{ab}M_{\boldsymbol{j}}^{ba}\\
+\frac{J_3}{N_f^3}\sum_{\langle\boldsymbol{ij}\rangle}[\Gamma_{\boldsymbol{i}}^{a,lmn}\Gamma_{\boldsymbol{j}}^{a,lmn\dagger}+\text{H.c.}].
\end{aligned} \tag{134}$$

Let us assume, for the time being, that $J_3=0$. In terms of fermions, the above Hamiltonian can be rewritten as

$$\begin{aligned}
H=-\frac{J_1}{N_f}\sum_{\langle\boldsymbol{ij}\rangle}\psi_{\boldsymbol{j}}^{n\beta}\psi_{\boldsymbol{i}}^{n\alpha\dagger}\psi_{\boldsymbol{i}}^{m\alpha}\psi_{\boldsymbol{j}}^{m\beta\dagger}-\frac{J_2}{N_f}\sum_{\langle\boldsymbol{ij}\rangle}\lambda_{\boldsymbol{j}}^{a}\lambda_{\boldsymbol{i}}^{a\dagger}\lambda_{\boldsymbol{i}}^{b}\lambda_{\boldsymbol{j}}^{b\dagger}\\
+\text{const.}
\end{aligned} \tag{135}$$

Using the path integral, we can rewrite the above model as

$$Z=\int \mathcal{D}(\psi^\dagger)\mathcal{D}(\psi)\mathcal{D}(a_0)\mathcal{D}(u)\mathcal{D}(\chi)e^{i\int dtL},$$

$$\begin{aligned}L=&\psi_{\boldsymbol{i}}^{n\dagger}i[\partial_t+ia_0(\boldsymbol{i})]\psi_{\boldsymbol{i}}^n-\sum_{\langle \boldsymbol{ij}\rangle}(\psi_{\boldsymbol{i}}^{n\dagger}u_{\boldsymbol{ij}}\psi_{\boldsymbol{j}}^n+\text{H.c.})\\&+\lambda_{\boldsymbol{i}}^{a\dagger}i[\partial_t+i\,\text{Tr}\,a_0(\boldsymbol{i})]\lambda_{\boldsymbol{i}}^a\\&-\sum_{\langle \boldsymbol{ij}\rangle}(\lambda_{\boldsymbol{i}}^{a\dagger}\chi_{\boldsymbol{ij}}\lambda_{\boldsymbol{j}}^a+\text{H.c.})-\frac{N_f}{J_1}\sum_{\langle \boldsymbol{ij}\rangle}\text{Tr}(u_{\boldsymbol{ij}}u_{\boldsymbol{ij}}^\dagger)\\&-\frac{N_f}{J_2}\sum_{\langle \boldsymbol{ij}\rangle}\chi_{\boldsymbol{ij}}\chi_{\boldsymbol{ij}}^\dagger,\end{aligned}\tag{136}$$

where $(\psi_{\boldsymbol{i}}^n)^T=(\psi_{\boldsymbol{i}}^{n,1},\psi_{\boldsymbol{i}}^{n,2},\psi_{\boldsymbol{i}}^{n,3})$, and $a_0(\boldsymbol{i})$ and $u_{\boldsymbol{ij}}$ are 3×3 complex matrices that satisfy

$$u_{\boldsymbol{ij}}^\dagger=u_{\boldsymbol{ji}},\quad a_0(\boldsymbol{i})=a_0^\dagger(\boldsymbol{i}).\tag{137}$$

When $J_3\neq0$, the Lagrangian may contain terms that mix $\chi_{\boldsymbol{ij}}$ and $u_{\boldsymbol{ij}}$:

$$\begin{aligned}L=&\psi_{\boldsymbol{i}}^{n\dagger}i[\partial_t+ia_0(\boldsymbol{i})]\psi_{\boldsymbol{i}}^n-\sum_{\langle \boldsymbol{ij}\rangle}(\psi_{\boldsymbol{i}}^{n\dagger}u_{\boldsymbol{ij}}\psi_{\boldsymbol{j}}^n+\text{H.c.})\\&+\lambda_{\boldsymbol{i}}^{a\dagger}i[\partial_t+i\,\text{Tr}\,a_0(\boldsymbol{i})]\lambda_{\boldsymbol{i}}^a-\sum_{\langle \boldsymbol{ij}\rangle}(\lambda_{\boldsymbol{i}}^{a\dagger}\chi_{\boldsymbol{ij}}\lambda_{\boldsymbol{j}}^a+\text{H.c.})\\&-\frac{N_f}{J_1}\sum_{\langle \boldsymbol{ij}\rangle}\text{Tr}(u_{\boldsymbol{ij}}u_{\boldsymbol{ij}}^\dagger)-\frac{N_f}{J_2}\sum_{\langle \boldsymbol{ij}\rangle}\chi_{\boldsymbol{ij}}\chi_{\boldsymbol{ij}}^\dagger\\&+CN_f\sum_{\langle \boldsymbol{ij}\rangle}[\chi_{\boldsymbol{ij}}\det(u_{\boldsymbol{ji}})+\text{H.c.}],\end{aligned}\tag{138}$$

where C is an $O(1)$ constant. We note that the above Lagrangian describes a $U(1)\times SU(3)$ lattice gauge theory coupled to fermions.

The field $a_0(\boldsymbol{i})$ in the Lagrangian is introduced to enforce the constraint

$$\psi_{\boldsymbol{i}}^{n\alpha\dagger}\psi_{\boldsymbol{i}}^{n\beta}-\psi_{\boldsymbol{i}}^{n\beta}\psi_{\boldsymbol{i}}^{n\alpha\dagger}+\lambda_{\boldsymbol{i}}^{a\dagger}\lambda_{\boldsymbol{i}}^a\delta^{\alpha\beta}-\lambda_{\boldsymbol{i}}^a\lambda_{\boldsymbol{i}}^{a\dagger}\delta^{\alpha\beta}=0.\tag{139}$$

As in standard gauge theory, the above constraint really means a constraint on physical states, i.e., all physical states must satisfy

$$\left(\lambda_{\boldsymbol{i}}^{a\dagger}\lambda_{\boldsymbol{i}}^a\delta^{\alpha\beta}+\psi_{\boldsymbol{i}}^{n\alpha\dagger}\psi_{\boldsymbol{i}}^{n\beta}-\delta^{\alpha\beta}\frac{3}{2}N_f\right)|\Phi_{phys}\rangle=0.\tag{140}$$

The above is the constraint needed to obtain the Hilbert space of our bosonic model.

Here we would like to stress that writing a bosonic model in terms of a gauge theory does not imply the existence of physical gauge bosons at low energy. Using projective construction, we can write any model in terms of a gauge theory of any gauge group [33,60]. The existence of low energy gauge fluctuations is a property of the ground state. It has nothing to do with how we write the Hamiltonian in terms of this or that gauge theory.

Certainly, if the ground state is known to have certain gauge fluctuations, then writing the Hamiltonian in terms of a particular gauge theory that happens to have the same gauge group will help us to derive the low energy effective theory. Even when we do not know the low energy gauge fluctuations in the ground state, we can still try to write the Hamiltonian in a form that contains a certain gauge theory and try to derive the low energy effective gauge theory. Most of the time, we find that the gauge fluctuations in the low energy effective theory are so strong that the gauge theory is in the confining phase. This indicates that we have chosen the wrong form of the Hamiltonian. However, if we are lucky enough to choose the right form of the Hamiltonian with the right gauge group, then the gauge fluctuations in the low energy effective theory will be weak and the gauge fields a_0, $\chi_{\boldsymbol{ij}}$, and $u_{\boldsymbol{ij}}$ will be almost like classical fields. In this case, we can say that the ground state of the Hamiltonian contains low energy gauge fluctuations described by a_0, $\chi_{\boldsymbol{ij}}$, and $u_{\boldsymbol{ij}}$. In the following, we will show that the $U(1)\times SU(3)$ fermion model Eq. (136) is the right form for us to write the Hamiltonian Eq. (134) of our bosonic model.

After integrating out the fermions, we obtain the following effective theory for $a_0(\boldsymbol{i})$, $\chi_{\boldsymbol{ij}}$, and $u_{\boldsymbol{ij}}$:

$$Z=\int \mathcal{D}(a_0)\mathcal{D}(u)e^{i\int dtN_f\tilde{L}_{eff}(u,a_0)},\tag{141}$$

where $\tilde{L}_{eff}$ does not depend on N_f. We see that, in the large N_f limit, $\chi_{\boldsymbol{ij}}$, $u_{\boldsymbol{ij}}$, and a_0 indeed become classical fields with weak fluctuations.

In the semiclassical limit, the ground state of the system is given by the ansatz $(\overline{\chi_{\boldsymbol{ij}}},\overline{u_{\boldsymbol{ij}}},\overline{a_0}(\boldsymbol{i}))$ that minimizes the energy $-\tilde{L}_{eff}$. We will assume that such an ansatz has π flux on every plaquette and takes the form

$$\begin{aligned}&\overline{\chi_{\boldsymbol{i},\boldsymbol{i}+\hat{\boldsymbol{x}}}}=-i\chi, &&\overline{\chi_{\boldsymbol{i},\boldsymbol{i}+\hat{\boldsymbol{y}}}}=-i(-)^{i_x}\chi,\\ & &&\overline{\chi_{\boldsymbol{i},\boldsymbol{i}+\hat{\boldsymbol{z}}}}=-i(-)^{i_x+i_y}\chi,\\&\overline{u_{\boldsymbol{i},\boldsymbol{i}+\hat{\boldsymbol{x}}}}=-iu, &&\overline{u_{\boldsymbol{i},\boldsymbol{i}+\hat{\boldsymbol{y}}}}=-i(-)^{i_x}u,\\&\overline{u_{\boldsymbol{i},\boldsymbol{i}+\hat{\boldsymbol{z}}}}=-i(-)^{i_x+i_y}u, &&a_0(\boldsymbol{i})=0.\end{aligned}\tag{142}$$

(If the π-flux ansatz does not minimize the energy, we can always modify the Hamiltonian of our bosonic model to make the π-flux ansatz have the minimal energy.) Despite the $\boldsymbol{i}$ dependence, the above ansatz actually describes translation, rotation, parity, and charge conjugation symmetric states. This is because the symmetry transformed ansatz, although not equal to the original ansatz, is gauge equivalent to the original ansatz.

The mean-field Hamiltonian for fermions has the form

$$H=\sum_{\langle \boldsymbol{ij}\rangle}(\psi_{\boldsymbol{i}}^{n\dagger}\overline{u_{\boldsymbol{ij}}}\psi_{\boldsymbol{j}}^n+\lambda_{\boldsymbol{i}}^{a\dagger}\overline{\chi_{\boldsymbol{ij}}}\lambda_{\boldsymbol{j}}^a+\text{H.c.}).\tag{143}$$

The fermion dispersion has two nodes at $\boldsymbol{k}=0$ and $\boldsymbol{k}=(0,0,\pi)$. Thus there are $2N_f\times7$ massless four-component Dirac fermions in the continuum limit. They correspond to quarks and leptons of $2N_f$ different families. Each family contains six quarks (two flavors times three colors) that carry $SU(3)$ colors and charge 1/3 for the $U(1)$ gauge field, and one lepton that carries no $SU(3)$ colors and charge 1 for the $U(1)$ gauge field.

Including the collective fluctuations of the ansatz, the $U(1)\times SU(3)=U(3)$ fermion theory has the form

$$L=\sum_{\boldsymbol{i}} \psi_{\boldsymbol{i}}^{n\dagger} i[\partial_t+ia_0(\boldsymbol{i})]\psi_{\boldsymbol{j}}^{n}+\sum_{\boldsymbol{ij}} \psi_{\boldsymbol{i}}^{n\dagger}\overline{u}_{\boldsymbol{ij}}e^{ia_{\boldsymbol{ij}}}\psi_{\boldsymbol{j}}^{n}+\sum_{\boldsymbol{i}} \lambda_{\boldsymbol{i}}^{a\dagger} \times i[\partial_t+i\,\mathrm{Tr}\,a_0(\boldsymbol{i})]\lambda_{\boldsymbol{j}}^{a}+\sum_{\boldsymbol{ij}} \lambda_{\boldsymbol{i}}^{a\dagger}\overline{\chi}_{\boldsymbol{ij}}\det(e^{ia_{\boldsymbol{ij}}})\lambda_{\boldsymbol{j}}^{a}, \tag{144}$$

where $a_{\boldsymbol{ij}}$ are 3×3 Hermitian matrices, describing $U(1)$ and $SU(3)$ gauge fields. In the continuum limit, the above becomes

$$\mathcal{L}=\overline{\psi}_{I,n}D_0\gamma^0\psi_{I,n}+v_f\overline{\psi}_{I,n}D_i\gamma^i\psi_{I,n}+\overline{\lambda}_{I,a}D_0'\gamma^0\lambda_{I,a}+v_f'\overline{\lambda}_{I,a}D_i'\gamma^i\lambda_{I,a} \tag{145}$$

with $v_f\sim l_0J_1$, $v_f'\sim l_0J_2$, $D_\mu=\partial_\mu+ia_\mu$, $D_\mu'=\partial_\mu+i\,\mathrm{Tr}\,a_\mu$, $I=1,2$, and γ^μ are 4×4 Dirac matrices [24,53,54]. $\lambda_{I,a}$ and $\psi_{I,n}$ are Dirac fermion fields. $\psi_{I,n}$ forms a fundamental representation of color $SU(3)$.

If we integrate out a_0 and $a_{\boldsymbol{ij}}$ in Eq. (144) first, we will recover the bosonic model Eq. (134). If we integrate out the high energy fermions first, the $U(1)\times SU(3)$ gauge field a_μ will acquire a dynamics. We obtain the following low energy effective theory in the continuum limit:

$$\mathcal{L}=\overline{\psi}_{I,n}D_0\gamma^0\psi_{I,n}+v_f\overline{\psi}_{I,n}D_i\gamma^i\psi_{I,n}+\overline{\lambda}_{I,a}D_0'\gamma^0\lambda_{I,a}+v_f'\overline{\lambda}_{I,a}D_i'\gamma^i\lambda_{I,a}+\frac{1}{\alpha_S}[\mathrm{Tr}\,F_{0i}F^{0i}+c_a^2\mathrm{Tr}\,F_{ij}F^{ij}]+\cdots, \tag{146}$$

where the velocity of the $U(3)$ gauge bosons is $c_a\sim l_0J_{1,2}$, and the ellipsis represents higher derivative terms and the coupling constant α_S is of order $1/N_f$.

In the large N_f limit, fluctuations of the gauge fields are weak. The model Eq. (146) describes a $U(1)\times SU(3)$ gauge theory coupled weakly to $2N_f$ families of massless fermions. Therefore, our bosonic model can generate massless artificial quarks and artificial leptons that couple to artificial light and artificial gluons. As discussed in Ref. [15], the PSG of the ansatz Eq. (142) protects the masslessness of the artificial quarks and the artificial leptons. Our model has an emergent chiral symmetry.

IX. CONCLUSION

In this paper, we studied a new class of ordered states—string-net condensed states—in local bosonic models. The new kind of order does not break any symmetry and cannot be described by Landau's symmetry breaking theory. We show that different string-net condensation can be characterized (and, hopefully, classified) by the projective symmetry in the hopping Hamiltonian for the ends of condensed strings. Similar to symmetry breaking states (or "particle" condensed states), string-net condensed states can also produce and protect gapless excitations. However, unlike symmetry breaking states, which can only produce and protect gapless scaler bosons (or Nambu-Goldstone modes), string-net condensed states can produce and protect gapless gauge bosons and gapless fermions. It is amazing to see that gapless fermions can even appear in local bosonic models.

Motivated by the above results, we propose the following locality principle: *The fundamental theory for our universe is a local bosonic model.* Using several local bosonic models as examples, we try to argue that the locality principle is not obviously wrong, if we assume that there is a string-net condensation in our vacuum. The string-net condensation can naturally produce and protect massless photons (as well as gluons) and (nearly) massless electrons/quarks. However, to really prove the string-net condensation in our vacuum, we need to show that string-net condensation can generate chiral fermions. Also, the above locality principle has not taken quantum gravity into account. It may need to be generalized to include quantum gravity. In any case, we can say that we have a plausible understanding of where light and fermions come from. The existence of light and fermions is no longer mysterious once we realize that they can come from local bosonic models via string-net condensations.

The string-net condensation and the associated PSG also provide a new solution to the chiral symmetry and fermion mass problems in lattice QED and lattice QCD. We show that the symmetry of the lattice bosonic model leads to the PSG of the hopping Hamiltonian for the ends of condensed strings. If the ends of condensed strings are fermions, then the PSG can sometimes protect the masslessness of the fermions, even though the chiral symmetry in the continuum limit cannot be generalized to the lattice. Thus the PSG can lead to an emergent chiral symmetry that protects massless Dirac fermions.

In this paper, we have been stressing that string-net condensation and the associated PSG can protect the masslessness of fermions. However, most fermions in nature do have masses, although very small compared to the Planck mass. One may wonder where those small masses come from. Here we would like to point out that the PSG argument for masslessness works only for radiative corrections. In other words, the fermions protected by string-net condensation and PSG cannot gain any mass from additive radiative corrections caused by high energy fluctuations. However, if the model has infrared divergence, then infrared divergence can give the would-be-massless fermions some mass. The acquired mass should have the scale of the infrared divergence. The (3+1)D QED model studied in this paper does not have any infrared divergence. Thus, the artificial electrons in the model are exactly massless. But in the bosonic model discussed in Sec. VIII the $SU(3)$ gauge coupling α_S runs as

$$\frac{d\alpha_S^{-1}}{d\ln(M^2)}=\frac{11-(2/3)(2N_f)}{4\pi}, \tag{147}$$

where M is the cutoff scale. Thus, when $N_f \leqslant 8$, α_S has a logarithmic infrared divergence. In general, for models with $U(1)$ and $SU(3)$ gauge interactions and the right content of fermions, the $SU(3)$ gauge interactions can have a weak logarithmic infrared divergence in (3+1)D [61,62]. This weak divergence could generate mass of order $e^{-C/\alpha_S(M_P)}M_P$, where M_P is the Planck mass or the grand unified theory (GUT) scale (the cutoff scale of the lattice theory), $C=O(1)$, and $\alpha_S(M_P)$ is the dimensionless gauge coupling constant at the Planck scale. A $C/\alpha_S(M_P)\sim 40$ can produce the desired separation between the Planck mass/GUT scale and the masses of the observed fermions. It is interesting to see that, in order to use the string-net condensation picture to explain the origin of gauge bosons and nearly massless fermions, it is important to have a four-dimensional space-time. When space-time has five or more dimensions, the gauge-fermion interactions do not have any infrared divergence. In this case, if a string-net condensation produces massless fermions, those fermions will remain massless down to zero energy. In (2+1)D, the gauge interaction between massless fermions is so strong that one cannot have fermionic quasiparticles at low energies [63–65]. It is amazing to see that 3+1 is the only space-time dimension where the gauge bosons and fermions produced by string-net condensation have weak enough interactions so that they can be identified at low energies and, at the same time, have strong enough interactions to have a rich nontrivial structure at low energies.

ACKNOWLEDGMENT

This research is supported by NSF Grant No. DMR-01-23156 and by NSF-MRSEC Grant No. DMR-02-13282.

[1] Y. Nambu, Phys. Rev. Lett. **4**, 380 (1960).
[2] J. Goldstone, Nuovo Cimento **19**, 154 (1961).
[3] L.D. Landau, Phys. Z. Sowjetunion **11**, 26 (1937).
[4] D.C. Tsui, H.L. Stormer, and A.C. Gossard, Phys. Rev. Lett. **48**, 1559 (1982).
[5] R.B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).
[6] X.-G. Wen, Adv. Phys. **44**, 405 (1995).
[7] X.-G. Wen and Q. Niu, Phys. Rev. B **41**, 9377 (1990).
[8] A.Y. Kitaev, Ann. Phys. (N.Y.) **303**, 2 (2003).
[9] X.-G. Wen, Phys. Rev. B **65**, 165113 (2002).
[10] X.-G. Wen, Phys. Lett. A **300**, 175 (2002).
[11] E. Witten, Commun. Math. Phys. **121**, 351 (1989).
[12] I.M. Lifshitz, Sov. Phys. JETP **11**, 1130 (1960).
[13] X.-G. Wen, Phys. Rev. B (to be published), cond-mat/0210040.
[14] M. Levin and X.-G. Wen, Phys. Rev. B **67**, 245316 (2003).
[15] X.-G. Wen, Phys. Rev. Lett. **88**, 011602 (2002).
[16] X.-G. Wen and A. Zee, Phys. Rev. B **66**, 235110 (2002).
[17] F. Wegner, J. Math. Phys. **12**, 2259 (1971).
[18] J.B. Kogut, Rev. Mod. Phys. **51**, 659 (1979).
[19] K.G. Wilson, Phys. Rev. D **10**, 2445 (1974).
[20] A.M. Polyakov, Phys. Lett. **82B**, 247 (1979).
[21] A.M. Polyakov, Nucl. Phys. B (Proc. Suppl.) **68**, 1 (1998).
[22] J. Kogut and L. Susskind, Phys. Rev. D **11**, 395 (1975).
[23] R. Savit, Rev. Mod. Phys. **52**, 453 (1980).
[24] C. Itzykson and J.-M. Drouffe, *Statistical Field Theory* (Cambridge University Press, Cambridge, England, 1989).
[25] T. Banks, R. Myerson, and J.B. Kogut, Nucl. Phys. **B129**, 493 (1977).
[26] O.I. Motrunich and T. Senthil, Phys. Rev. Lett. **89**, 277004 (2002).
[27] D. Forster, H.B. Nielsen, and M. Ninomiya, Phys. Lett. **94B**, 135 (1980).
[28] R. Jackiw and C. Rebbi, Phys. Rev. Lett. **36**, 1116 (1976).
[29] P. Hasenfratz and G. 't Hooft, Phys. Rev. Lett. **36**, 1119 (1976).
[30] A.S. Goldhaber, Phys. Rev. Lett. **36**, 1122 (1976).
[31] D. Arovas, J.R. Schrieffer, and F. Wilczek, Phys. Rev. Lett. **53**, 722 (1984).
[32] G. Baskaran, Z. Zou, and P.W. Anderson, Solid State Commun. **63**, 973 (1987).
[33] G. Baskaran and P.W. Anderson, Phys. Rev. B **37**, 580 (1988).
[34] X.-G. Wen, F. Wilczek, and A. Zee, Phys. Rev. B **39**, 11413 (1989).
[35] D. Kheshchenko and P. Wiegmann, Mod. Phys. Lett. B **3**, 1383 (1989).
[36] N. Read and S. Sachdev, Phys. Rev. Lett. **66**, 1773 (1991).
[37] X.-G. Wen, Phys. Rev. B **44**, 2664 (1991).
[38] V. Kalmeyer and R.B. Laughlin, Phys. Rev. Lett. **59**, 2095 (1987).
[39] S.A. Kivelson, D.S. Rokhsar, and J.P. Sethna, Phys. Rev. B **35**, 8865 (1987).
[40] D.S. Rokhsar and S.A. Kivelson, Phys. Rev. Lett. **61**, 2376 (1988).
[41] N. Read and B. Chakraborty, Phys. Rev. B **40**, 7133 (1989).
[42] R. Moessner and S.L. Sondhi, Phys. Rev. Lett. **86**, 1881 (2001).
[43] A. D'Adda, P.D. Vecchia, and M. Lüscher, Nucl. Phys. **B146**, 63 (1978).
[44] E. Witten, Nucl. Phys. **B149**, 285 (1979).
[45] I. Affleck and J.B. Marston, Phys. Rev. B **37**, 3774 (1988).
[46] G. Kotliar and J. Liu, Phys. Rev. B **38**, 5142 (1988).
[47] S. Sachdev and K. Park, Ann. Phys. (N.Y.) **298**, 58 (2002).
[48] L. Balents, M.P.A. Fisher, and S.M. Girvin, Phys. Rev. B **65**, 224412 (2002).
[49] L.B. Ioffe, M.V. Feigel'man, A. Ioselevich, D. Ivanov, M. Troyer, and G. Blatter, Nature (London) **415**, 503 (2002).
[50] X.-G. Wen, Phys. Rev. Lett. **90**, 016803 (2003).
[51] O.I. Motrunich, Phys. Rev. B **67**, 115108 (2003).
[52] A. Y. Kitaev (unpublished).
[53] T. Banks, L. Susskind, and J. Kogut, Phys. Rev. D **13**, 1043 (1976).
[54] A. Zee, in *M. A. B. Bég Memorial Volume* (World Scientific, Singapore, 1992).
[55] P. Ginsparg and K. Wilson, Phys. Rev. D **25**, 2649 (1982).
[56] H. Neuberger, Phys. Lett. B **417**, 141 (1997).
[57] P. Hasenfratz, Nucl. Phys. **B525**, 401 (1998).

[58] M. Lüscher, Phys. Lett. B **428**, 342 (1998).
[59] N. Read and S. Sachdev, Nucl. Phys. **B316**, 609 (1989).
[60] X.-G. Wen, Phys. Rev. B **60**, 8827 (1999).
[61] F. Wilczek, Rev. Mod. Phys. **71**, S85 (1999).
[62] C.J. Hogan, Rev. Mod. Phys. **72**, 1149 (2000).
[63] T. Appelquist and D. Nash, Phys. Rev. Lett. **64**, 721 (1990).
[64] W. Rantner and X.-G. Wen, Phys. Rev. Lett. **86**, 3871 (2001).
[65] M. Franz and Z. Tesanovic, Phys. Rev. Lett. **87**, 257003 (2001).