Auxiliary field method in 4- and 3-dimensional Nambu–Jona-Lasinio models

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In order to check the validity of the auxiliary field method in the Nambu–Jona-Lasinio model, the one-loop (= quantum) effects of auxiliary fields on the gap equation are considered with N-component fermion models in four and three dimensions. N is not assumed very large but is regarded as a loop expansion parameter. To overcome infrared divergences caused by the Nambu-Goldstone bosons, an intrinsic fermion mass is assumed. It is shown that the loop expansion can be justified by this intrinsic mass whose lower limit is also given. It is found that due to quantum effects, chiral symmetry breaking (χSB) is restored in $D=4$ and $D=3$ when the four-Fermi coupling is large. However, χSB is enhanced in a small coupling region in $D=3$.

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I. INTRODUCTION

The Nambu–Jona-Lasinio (NJL) model [1] is, needless to say, one of the most famous field theoretical models exhibiting dynamical chiral symmetry breaking (χSB) phenomena. Owing to its simplicity, there have been many studies using the NJL model: some of the recent trends are those that deal with external disturbances such as background gauge fields [2–5] or curved spacetime [6] in order to explore the detailed phase structure. Originally, calculations for dynamical χSB were done in a self-consistent manner to yield the gap equation [1], but later it was revealed that results are more easily obtained in the path integral formulation with the use of an auxiliary [7] or Hubbard-Stratonovich [8] field. The recipe, which we shall call the auxiliary field method [9], becomes exact when N, the number of degrees of freedom of the original dynamical fields, goes to infinity. However, the analysis in lower dimensional bosonic models shows that, even if $N=1$, we can improve the results toward the true value by taking higher orders in the loop expansion [10]. Therefore it is desirable to incorporate the quantum (= loop) effects of auxiliary fields in the gap equation when N remains finite.

However, there is an obstacle to performing the loop expansion in terms of auxiliary fields in the NJL model: because of the massless Nambu-Goldstone bosons that occur in the auxiliary fields, infrared divergences are inevitable in higher loop calculations. Kleinert and Bossche pay attention to this infrared regime and conclude that there is no pion in the NJL model [11], that is, there is no room for the auxiliary fields. Their main contribution is, however, a chiral nonlinear model [12], an effective theory, so more rigorous and careful investigations are needed. Indeed some opposition to this conclusion has occurred [13].

We follow the standard prescription for the effective action formalism by introducing sources coupled to bilinear terms of fermions. In order to control the infrared singularity, we assume an intrinsic mass of fermions, the so called current quark mass. We do not care about the renormalizability so that an ultraviolet cutoff is introduced to define the model. We make no approximation other than the loop expansion.

There have been attempts to consider $O(1/N)$ terms in four-fermion models: some calculate the effective potential for the model with discrete chiral symmetry in order to clarify the renormalizability in $4>D>2$ [14]; others check the Nambu-Goldstone theorem or Gell-Mann–Oakes–Renner or Goldberger-Treiman relations in the four-dimensional NJL model[1] to show that $O(1/N)$ effects weaken the quark condensation $\langle \bar{q}q \rangle$ obtained at the tree order [15]. However, there seems to have been no attempt to study the higher loop contribution, paying attention to the infrared regime, to the vacuum condition, and to the gap equation, in terms of the auxiliary field method. In Sec. II, we present a general path integral formalism to obtain an effective potential in the NJL model. We work with the N-component fermion model in $D=4$ and 3 [16], but as stated above N is merely a loop expansion parameter to be kept finite. Section III deals with the vacuum condition and then the gap equation up to one-loop order of the auxiliary fields. It is concluded that quantum (= loop) effects restore χSB in $D=4$ while the situation in $D=3$ is slightly different; χSB is restored in the strong coupling regime but enhanced in the weak coupling regime. We also find the lower limit of the current quark mass to ensure the loop expansion. The final section is devoted to discussion.

II. MODEL AND BASIC FORMALISM

In this section, we develop a general formalism in order to clarify our goal. The NJL model with an intrinsic mass ε in three as well as four dimensions is given by

[1]It is, however, almost trivial to check these relations under an $O(1/N)$ expansion; since it is well known that those are the consequences of the Ward-Takahashi relations which are persistent in the loop $[=O(1/N)]$ expansion.

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$$\mathcal{L} = -\bar{\psi}(x)(\not\partial + \varepsilon)\psi(x) + \mathcal{L}_{\rm int}, \tag{1}$$

$$\mathcal{L}_{\rm int} \equiv \frac{\lambda}{2N}\begin{cases}\{[\bar{\psi}(x)\psi(x)]^2 + [\bar{\psi}(x)i\gamma_5\psi(x)]^2\}, & D=4,\\ \{[\bar{\psi}(x)\psi(x)]^2 + [\bar{\psi}(x)i\gamma_4\psi(x)]^2 + [\bar{\psi}(x)i\gamma_5\psi(x)]^2\}, & D=3\end{cases},$$

$$\equiv \frac{\lambda}{2N}[\bar{\psi}(x)\mathbf{\Gamma}\psi(x)]^2, \tag{2}$$

with

$$\mathbf{\Gamma} = \Gamma_a \equiv (1, i\mathbf{\Gamma}_5), \quad \mathbf{\Gamma}_5 \equiv \begin{cases}\gamma_5, & D=4,\\ (\gamma_4, \gamma_5), & D=3,\end{cases} \tag{3}$$

where $\not\partial \equiv \gamma_\mu \partial_\mu$, N-component fermion fields have been introduced, and the γ_μ's are 4×4 matrices, even in three dimensions,[2] satisfying

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}, \quad \mu,\nu = 1,2,3,4,5.$$

Explicitly,

$$\gamma_\mu = \begin{pmatrix}\sigma_\mu & 0\\ 0 & -\sigma_\mu\end{pmatrix}, \quad \mu = 1,2,3, \quad \gamma_4 = \begin{pmatrix}0 & \mathbf{1}\\ \mathbf{1} & 0\end{pmatrix},$$

$$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 = \begin{pmatrix}0 & i\mathbf{1}\\ -i\mathbf{1} & 0\end{pmatrix}. \tag{4}$$

The intrinsic mass ε, the so called current quark mass, has been assumed to prevent an infrared divergence in the pion loop integrals. (See the following.)

The quantity we should consider is

$$Z[\mathbf{J}] \equiv \mathrm{Tr}\; T\left(\exp\left[-\int_0^T dt\, H(t)\right]\right), \tag{5}$$

$$H(t) = \int d^{D-1}x\left[\bar{\psi}(x)[\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + \varepsilon - \mathbf{J}(x)\cdot\mathbf{\Gamma}]\times\psi(x) - \mathcal{L}_{\rm int} - \frac{N}{2\lambda}\mathbf{J}^2(x)\right], \tag{6}$$

$$\mathbf{J}(x) = J_a(x) \equiv \begin{cases}(J(x), J_5(x)), & D=4,\\ (J(x), J_4(x), J_5(x)), & D=3,\end{cases} \tag{7}$$

where the $\mathbf{J}(x)$'s are c-number sources and T designates the (imaginary) time-ordered product. (We have introduced the $\mathbf{J}^2$ term just for later notational simplicity [14].) From this we can extract the energy of the ground state by putting $\mathbf{J}\mapsto\mathbf{0}$ as well as $T\mapsto\infty$. The path integral representation reads

$$Z[\mathbf{J}] = \int d[\psi]d[\bar{\psi}]\exp\left(\int d^Dx\left[-\bar{\psi}(\not\partial + \varepsilon - \mathbf{J}\cdot\mathbf{\Gamma})\psi + \frac{\lambda}{2N}(\bar{\psi}\mathbf{\Gamma}\psi)^2 + \frac{N}{2\lambda}\mathbf{J}^2\right]\right). \tag{8}$$

Introducing auxiliary fields in terms of the Gaussian integration

$$1 = \int d[\mathbf{\Sigma}]\exp\left[-\frac{N}{2\lambda}\int d^Dx\left(\mathbf{\Sigma}(x) + \frac{\lambda}{N}\bar{\psi}(x)\mathbf{\Gamma}\psi(x)\right)^2\right], \tag{9}$$

$$\mathbf{\Sigma}(x) = \Sigma_a(x) \equiv \begin{cases}(\sigma(x), \pi(x)), & D=4,\\ (\sigma(x), \pi_1(x), \pi_2(x)), & D=3,\end{cases} \tag{10}$$

to eliminate the four-Fermi interaction, we find

$$Z[\mathbf{J}] = \int d[\psi]d[\bar{\psi}]d[\mathbf{\Sigma}]\exp\left(\int d^Dx\left[-\frac{N}{2\lambda}\mathbf{\Sigma}^2 - \bar{\psi}\{\not\partial + \varepsilon + (\mathbf{\Sigma} - \mathbf{J})\cdot\mathbf{\Gamma}\}\psi + \frac{N}{2\lambda}\mathbf{J}^2\right]\right). \tag{11}$$

The fermion integrations yield

$$Z = \int d[\mathbf{\Sigma}]\exp\left[-N\left\{\int d^Dx\,\frac{1}{2\lambda}(\mathbf{\Sigma}^2 - \mathbf{J}^2) - \mathrm{Tr}\ln\{\not\partial + \varepsilon + (\mathbf{\Sigma} - \mathbf{J})\cdot\mathbf{\Gamma}\}\right\}\right]$$

$$\overset{\Sigma\to\Sigma+\mathbf{J}}{=} \int d[\mathbf{\Sigma}]\exp(-NI[\mathbf{\Sigma},\mathbf{J}]) \tag{12}$$

with

$$I[\mathbf{\Sigma},\mathbf{J}] \equiv \int d^Dx\left(\frac{1}{2\lambda}\mathbf{\Sigma}^2 + \frac{1}{\lambda}\mathbf{\Sigma}\cdot\mathbf{J}\right) - \mathrm{Tr}\ln(\not\partial + \varepsilon + \mathbf{\Sigma}\cdot\mathbf{\Gamma}), \tag{13}$$

where Tr designates the spinorial as well as the functional trace. Writing

$$Z[\mathbf{J}] \equiv e^{-NW[\mathbf{J}]}, \tag{14}$$

[2]In three dimensions, we need an additional (N-component) fermion to form the four-component spinor [16,4]

$$\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \quad \bar{\psi} \equiv \psi^\dagger\gamma_3 \equiv (\bar{\psi}_1 \quad -\bar{\psi}_2) \equiv (\psi_1^\dagger\sigma_3 \quad -\psi_2^\dagger\sigma_3),$$

to be able to realize the chiral symmetry (when $\varepsilon = 0$),

$$\psi(x) \to e^{i\alpha\gamma_4}\psi(x), \quad \psi(x) \to e^{i\beta\gamma_5}\psi(x),$$

which is, therefore, a global $U(2)$ symmetry, finally broken down to $U(1)\times U(1)$ by the mass term.

and introducing the "classical" fields

$$\frac{1}{\lambda}\boldsymbol{\phi}\equiv\frac{\delta W}{\delta\mathbf{J}}\overset{\text{Eq. (12)}}{=}\frac{1}{\lambda}\langle\boldsymbol{\Sigma}\rangle\overset{\text{Eq. (8)}}{=}-\frac{1}{N}\langle\bar{\psi}\boldsymbol{\Gamma}\psi\rangle-\frac{1}{\lambda}\mathbf{J},\quad(15)$$

with the expectation values being taken under the expression (12) or under the original one (8), we perform a Legendre transformation with respect to $W[\mathbf{J}]$ to obtain the effective action

$$\Gamma[\boldsymbol{\phi}]=W[\mathbf{J}]-\frac{1}{\lambda}(\mathbf{J}\cdot\boldsymbol{\phi}),\quad(16)$$

where the shorthand notation,

$$(A\cdot B)\equiv\int d^Dx\,A(x)B(x)\quad(17)$$

has been employed. When the $\mathbf{J}$'s are set to be constants, the effective action becomes the effective potential

$$\Gamma[\boldsymbol{\phi}]\overset{\mathbf{J}\mapsto\text{const}}{\Rightarrow}VT\mathcal{V}(\boldsymbol{\phi}),\quad(18)$$

with V being the $(D-1)$-dimensional volume.

We calculate $W[\mathbf{J}]$ with the help of the saddle point method.[3] First, we find the classical solution $\boldsymbol{\Sigma}_0$,

$$0=\left.\frac{\delta I}{\delta\boldsymbol{\Sigma}(x)}\right|_{\boldsymbol{\Sigma}_0}=\frac{1}{\lambda}(\boldsymbol{\Sigma}_0+\mathbf{J})(x)-\text{tr}\,\boldsymbol{\Gamma}S(x,x:\boldsymbol{\Sigma}_0),\quad(19)$$

where $S(x,y:\boldsymbol{\Sigma}_0)$ is a fermion propagator under the background fields

$$[\partial\!\!\!/+\varepsilon+\boldsymbol{\Sigma}_0(x)\cdot\boldsymbol{\Gamma}]S(x,y:\boldsymbol{\Sigma}_0)=\delta(x-y).\quad(20)$$

Second, we expand I around $\boldsymbol{\Sigma}_0$:

$$I=I_0+\frac{1}{2}[\mathbf{I}_0^{(2)}\cdot(\boldsymbol{\Sigma}-\boldsymbol{\Sigma}_0)^2]+\frac{1}{3!}[\mathbf{I}_0^{(3)}\cdot(\boldsymbol{\Sigma}-\boldsymbol{\Sigma}_0)^3]+\cdots,\quad(21)$$

where

$$\mathbf{I}_0^{(n)}\equiv\left.\frac{\delta^nI}{\delta\boldsymbol{\Sigma}^n}\right|_{\boldsymbol{\Sigma}_0},\quad(22)$$

$$(\mathbf{I}_0^{(n)}\cdot\boldsymbol{\Sigma}^n)\equiv\int d^Dx_1\cdots d^Dx_n\frac{\delta^nI}{\delta\Sigma^{a_1}(x_1)\cdots\delta\Sigma^{a_n}(x_n)}\times\Sigma^{a_1}(x_1)\cdots\Sigma^{a_n}(x_n).\quad(23)$$

Third, we put $(\boldsymbol{\Sigma}-\boldsymbol{\Sigma}_0)\mapsto\boldsymbol{\Sigma}/\sqrt{N}$ and then perform the Gaussian integration with respect to $\boldsymbol{\Sigma}$ to obtain

$$W[\mathbf{J}]=I_0+\frac{1}{2N}\text{Tr}\ln\mathbf{I}_0^{(2)}+O\left(\frac{1}{N^2}\right),\quad(24)$$

where

$$I_0=\frac{1}{2\lambda}(\boldsymbol{\Sigma}_0\cdot\boldsymbol{\Sigma}_0)+\frac{1}{\lambda}(\boldsymbol{\Sigma}_0\cdot\mathbf{J})-\text{Tr}\ln(\partial\!\!\!/+\varepsilon+\boldsymbol{\Sigma}_0\cdot\boldsymbol{\Gamma}),\quad(25)$$

$$(\mathbf{I}_0^{(2)})_{ab}=\left.\frac{\delta^2I}{\delta\Sigma^a(x)\delta\Sigma^b(y)}\right|_{\boldsymbol{\Sigma}_0}=\frac{1}{\lambda}\delta(x-y)\delta_{ab}+\text{tr}\,\Gamma_aS(x,y:\boldsymbol{\Sigma}_0)\Gamma_bS(y,x:\boldsymbol{\Sigma}_0).\quad(26)$$

Here the trace is taken only for the spinor space. $\mathbf{I}_0^{(2)}$ is a matrix in the $\boldsymbol{\Sigma}$ space (2×2 in $D=4$, 3×3 in $D=3$). I_0 is the "tree" part while $\text{Tr}\ln\mathbf{I}_0^{(2)}$ is the "one-loop" part of the auxiliary fields. Using Eq. (15) we find

$$\boldsymbol{\phi}(x)=\boldsymbol{\Sigma}_0(x)+\frac{\lambda}{2N}\frac{\delta}{\delta\mathbf{J}(x)}(\text{Tr}\ln\mathbf{I}_0^{(2)})\equiv\boldsymbol{\Sigma}_0(x)+\frac{\boldsymbol{\Sigma}_1(x)}{N}.\quad(27)$$

Note that the difference between $\boldsymbol{\phi}$ and $\boldsymbol{\Sigma}_0$ is $O(1/N)$. Inserting Eq. (27) into the effective action (16) with the use of Eqs. (24), (25), and (26), we obtain

$$\Gamma[\boldsymbol{\phi}]=\frac{1}{2\lambda}(\boldsymbol{\phi}\cdot\boldsymbol{\phi})-\text{Tr}\ln(\partial\!\!\!/+\varepsilon+\boldsymbol{\phi}\cdot\boldsymbol{\Gamma})+\frac{1}{2N}\text{Tr}\ln\left(\frac{\mathbf{I}}{\lambda}\delta(x-y)+\text{tr}\,\boldsymbol{\Gamma}S(x,y:\boldsymbol{\phi})\boldsymbol{\Gamma}S(y,x:\boldsymbol{\phi})\right)+O\left(\frac{1}{N^2}\right).\quad(28)$$

By setting the $\mathbf{J}$'s constant, the effective potential (18) reads

$$\mathcal{V}(\boldsymbol{\phi})=\frac{1}{2\lambda}\boldsymbol{\phi}^2-\frac{1}{VT}\text{Tr}\ln(\partial\!\!\!/+\varepsilon+\boldsymbol{\phi}\cdot\boldsymbol{\Gamma})+\frac{1}{2NVT}\text{Tr}\ln\times\left(\frac{\mathbf{I}}{\lambda}\delta(x-y)+\text{tr}\boldsymbol{\Gamma}S(x,y:\boldsymbol{\phi})\boldsymbol{\Gamma}S(y,x:\boldsymbol{\phi})\right)+O\left(\frac{1}{N^2}\right).\quad(29)$$

In Eq. (29) the first two terms are the tree part and the third is the one-loop part of the auxiliary fields, whose functional trace should also be taken for the $\boldsymbol{\Sigma}$ space. The vacuum is chosen by

$$\left.\frac{\partial\mathcal{V}}{\partial\boldsymbol{\phi}}\right|_{\mathbf{J}=\mathbf{0}}=0.\quad(30)$$

Armed with these results, we now proceed to a detailed calculation.

[3]The result becomes exact when N goes to infinity, which, however, is not the case in this analysis: N is a mere expansion parameter that is finally put to unity.

III. VACUUM AND THE GAP EQUATION

We write the "classical" fields at $\mathbf{J}=\mathbf{0}$ as

$$\boldsymbol{\phi}=(m,\boldsymbol{\Sigma}_\pi), \quad \boldsymbol{\Sigma}_\pi\equiv\begin{cases}\pi, & D=4,\\ (\pi_1,\pi_2), & D=3,\end{cases} \tag{31}$$

and

$$\tilde{\boldsymbol{\phi}}\equiv(m+\varepsilon,\boldsymbol{\Sigma}_\pi) \tag{32}$$

to study the vacuum condition (30) on the tree part of the effective potential (29),

$$\mathcal{V}_0=\frac{1}{2\lambda}(m^2+\boldsymbol{\Sigma}_\pi^2)-\frac{1}{VT}\mathrm{Tr}\ln(\partial\!\!\!/+\tilde{\boldsymbol{\phi}}\cdot\boldsymbol{\Gamma})$$
$$=\frac{1}{2\lambda}(m^2+\boldsymbol{\Sigma}_\pi^2)-\frac{\mathrm{tr}\,\mathbf{I}}{2}\int\frac{d^Dp}{(2\pi)^D}\ln(p^2+\tilde{\boldsymbol{\phi}}^2), \tag{33}$$

giving

$$\frac{\partial\mathcal{V}_0}{\partial m}=0=\frac{m}{\lambda}-4(m+\varepsilon)\int\frac{d^Dp}{(2\pi)^D}\frac{1}{p^2+\tilde{\boldsymbol{\phi}}^2}, \tag{34}$$

$$\frac{\partial\mathcal{V}_0}{\partial\boldsymbol{\Sigma}_\pi}=0=\frac{\boldsymbol{\Sigma}_\pi}{\lambda}-4\boldsymbol{\Sigma}_\pi\int\frac{d^Dp}{(2\pi)^D}\frac{1}{p^2+\tilde{\boldsymbol{\phi}}^2}. \tag{35}$$

(Recall that tr $\mathbf{I}=4$ for both $D=4$ and $D=3$.) Therefore the solution is

$$\boldsymbol{\Sigma}_\pi=\mathbf{0}, \tag{36}$$

$$\frac{m}{\lambda}=4(m+\varepsilon)\int\frac{d^Dp}{(2\pi)^D}\frac{1}{p^2+(m+\varepsilon)^2}, \tag{37}$$

where the second relation, called the gap equation at the tree order, reads

$$\frac{1}{\lambda_D}=\frac{\sqrt{x}}{\sqrt{x}-\epsilon}g_D^{(0)}(x), \tag{38}$$

where

$$\epsilon\equiv\frac{\varepsilon}{\Lambda}, \quad x\equiv\frac{(m+\varepsilon)^2}{\Lambda^2} \quad (\epsilon^2\leqslant x\leqslant 1), \tag{39}$$

$$g_4^{(0)}(x)\equiv 1-x\ln\left(\frac{1+x}{x}\right), \tag{40}$$

$$g_3^{(0)}(x)\equiv 2\left[1-\sqrt{x}\tan^{-1}\frac{1}{\sqrt{x}}\right], \tag{41}$$

and

$$\lambda_D\equiv\frac{\lambda\Lambda^{D-2}}{4^{D/2-1}\pi^{D/2}\Gamma(D/2)}, \tag{42}$$

with $\Gamma(D/2)$ being the gamma function $\Gamma(2)=1$, $\Gamma(3/2)=\sqrt{\pi}/2$. In view of Eq. (38),

$$\lambda \overset{\sqrt{x}\mapsto\epsilon}{=} O(\sqrt{x}-\epsilon), \tag{43}$$

which implies the trivial fact that the mass is ε even in the free theory $\lambda=0$.

The one-loop part $\mathcal{V}_1$ of the effective potential (29) reads

$$\mathcal{V}_1=\frac{1}{2N}\mathrm{tr}\int\frac{d^Dp}{(2\pi)^D}\ln\left(\frac{\mathbf{I}}{\lambda}+\boldsymbol{\Pi}(p)\right), \tag{44}$$

where tr should be taken for the spinorial space as well as the $\boldsymbol{\Sigma}$. In Eq. (44) the argument of the logarithm is nothing but a two-point function of the auxiliary fields; therefore, we call $\boldsymbol{\Pi}$ the vacuum polarization matrix,

$$\boldsymbol{\Pi}_{ab}(p)\equiv\int\frac{d^Dl}{(2\pi)^D}\mathrm{tr}\left(\frac{1}{i(l\!\!/+p\!\!/ /2)+\tilde{\boldsymbol{\phi}}\cdot\boldsymbol{\Gamma}}\Gamma_a\frac{1}{i(l\!\!/+p\!\!/ /2)+\tilde{\boldsymbol{\phi}}\cdot\boldsymbol{\Gamma}}\Gamma_b\right)$$
$$=4\int\frac{d^Dl}{(2\pi)^D}\frac{1}{[(l+p/2)^2+\tilde{\boldsymbol{\phi}}^2][(l-p/2)^2+\tilde{\boldsymbol{\phi}}^2]}$$
$$\times\begin{cases}\begin{pmatrix}-\left(l^2-\frac{p^2}{4}\right)+(m+\varepsilon)^2-\pi^2 & 2(m+\varepsilon)\pi\\ 2(m+\varepsilon)\pi & -\left(l^2-\frac{p^2}{4}\right)-(m+\varepsilon)^2+\pi^2\end{pmatrix}, & D=4,\\[2ex] \begin{pmatrix}-\left(l^2-\frac{p^2}{4}\right)+(m+\varepsilon)^2-\Sigma_\pi^2 & 2(m+\varepsilon)\pi_1 & 2(m+\varepsilon)\pi_2\\ 2(m+\varepsilon)\pi_1 & -\left(l^2-\frac{p^2}{4}\right)-(m+\varepsilon)^2+\pi_1^2-\pi_2^2 & 2\pi_1\pi_2\\ 2(m+\varepsilon)\pi_2 & 2\pi_1\pi_2 & -\left(l^2-\frac{p^2}{4}\right)-(m+\varepsilon)^2-\pi_1^2+\pi_2^2\end{pmatrix}, & D=3.\end{cases} \tag{45}$$

By noting that

$$\mathrm{tr}\left[\left(\frac{\mathbf{I}}{\lambda}+\mathbf{\Pi}\right)^{-1}\cdot\frac{\partial\mathbf{\Pi}}{\partial(\Sigma_\pi)^a}\right]=(\Sigma_\pi)^a(\cdots) \tag{46}$$

and the tree relation (35), the vacuum condition for $\mathbf{\Sigma}_\pi$ up to one loop is

$$\left.\frac{\partial\mathcal{V}}{\partial\mathbf{\Sigma}_\pi}\right|_{\mathbf{J}=\mathbf{0}}=\mathbf{\Sigma}_\pi\left(\frac{1}{\lambda}+\cdots\right) \tag{47}$$

with $\mathcal{V}=\mathcal{V}_0+\mathcal{V}_1$, allowing us to choose $\mathbf{\Sigma}_\pi=\mathbf{0}$ as the vacuum. Also, in the one-loop part of the gap equation,

$$\left.\frac{\partial\mathcal{V}_1}{\partial m}\right|_{\mathbf{J}=\mathbf{0},\mathbf{\Sigma}_\pi=\mathbf{0}} \tag{48}$$

we can utilize the tree result (37).

Therefore in the gap equation we put $\mathbf{\Sigma}_\pi=\mathbf{0}$ in the expression (45) to find the diagonal matrix

$$\mathbf{\Pi}|_{\mathbf{\Sigma}_\pi=\mathbf{0}}=\begin{cases}\begin{pmatrix}\Pi_1 & 0\\ 0 & \Pi_2\end{pmatrix}, & D=4,\\ \begin{pmatrix}\Pi_1 & 0 & 0\\ 0 & \Pi_2 & 0\\ 0 & 0 & \Pi_2\end{pmatrix}, & D=3,\end{cases} \tag{49}$$

with

$$\left.\begin{matrix}\Pi_1\\ \Pi_2\end{matrix}\right\}\equiv 4\int\frac{d^Dl}{(2\pi)^D}\times\frac{1}{\{(l+p/2)^2+(m+\varepsilon)^2\}\{(l-p/2)^2+(m+\varepsilon)^2\}}\times\begin{cases}-\left(l^2-\dfrac{p^2}{4}\right)+(m+\varepsilon)^2\\ -\left(l^2-\dfrac{p^2}{4}\right)-(m+\varepsilon)^2\end{cases}. \tag{50}$$

Writing

$$l^2-\frac{p^2}{4}=\frac{(l+p/2)^2+(l-p/2)^2}{2}-\frac{p^2}{2},$$

then

$$\left.\begin{matrix}\Pi_1\\ \Pi_2\end{matrix}\right\}=-2\left[\int\frac{d^Dl}{(2\pi)^D}\frac{1}{(l+p/2)^2+(m+\varepsilon)^2}+\int\frac{d^Dl}{(2\pi)^D}\frac{1}{(l-p/2)^2+(m+\varepsilon)^2}\right]+8\int\frac{d^Dl}{(2\pi)^D}\times\frac{\begin{cases}(m+\varepsilon)^2+p^2/4\\ p^2/4\end{cases}}{\{(l+p/2)^2+(m+\varepsilon)^2\}\{(l-p/2)^2+(m+\varepsilon)^2\}}. \tag{51}$$

Shifting the momentum (although we are in a cutoff world), we obtain

$$\left.\begin{matrix}\Pi_1\\ \Pi_2\end{matrix}\right\}=-4\int\frac{d^Dl}{(2\pi)^D}\frac{1}{l^2+(m+\varepsilon)^2}+8\int_{-1/2}^{1/2}dt\int\frac{d^Dl}{(2\pi)^D}\frac{\begin{cases}(m+\varepsilon)^2+p^2/4\\ p^2/4\end{cases}}{[l^2+p^2(1/4-t^2)+(m+\varepsilon)^2]^2}. \tag{52}$$

Using the tree result for the gap equation (37), we obtain

$$\frac{1}{\lambda}+\left\{\begin{matrix}\Pi_1\\ \Pi_2\end{matrix}\right.=\frac{1}{\lambda}-4\int\frac{d^Dl}{(2\pi)^D}\frac{1}{l^2+(m+\varepsilon)^2}+8\int_{-1/2}^{1/2}dt\times\int\frac{d^Dl}{(2\pi)^D}\frac{\begin{cases}(m+\varepsilon)^2+p^2/4\\ p^2/4\end{cases}}{[l^2+p^2(1/4-t^2)+(m+\varepsilon)^2]^2}=\frac{1}{\lambda}\left(1-\lambda_Dg_D^{(0)}(x)+4\lambda_Dq_D(x,s)\begin{cases}x+s\\ s\end{cases}\right), \tag{53}$$

where

$$q_4(x,s)\equiv\frac{1}{2}\left[\ln\frac{1+x}{x}+\frac{1+2x+2s}{2\sqrt{s(1+x+s)}}\ln\frac{\sqrt{1+x+s}+\sqrt{s}}{\sqrt{1+x+s}-\sqrt{s}}-\sqrt{(x+s)/s}\ln\frac{\sqrt{x+s}+\sqrt{s}}{\sqrt{x+s}-\sqrt{s}}\right], \tag{54}$$

$$q_3(x,s)\equiv\int_0^1dt\frac{1}{\sqrt{x+s(1-t^2)}}\tan^{-1}\frac{1}{\sqrt{x+s(1-t^2)}}-\frac{1}{2\sqrt{s(1+x+s)}}\ln\left(\frac{\sqrt{1+x+s}+\sqrt{s}}{\sqrt{1+x+s}-\sqrt{s}}\right) \tag{55}$$

with

$$s\equiv\frac{p^2}{4\Lambda^2},\quad 0\leqslant s\leqslant\frac{1}{4}, \tag{56}$$

and λ_D given by Eq. (42).

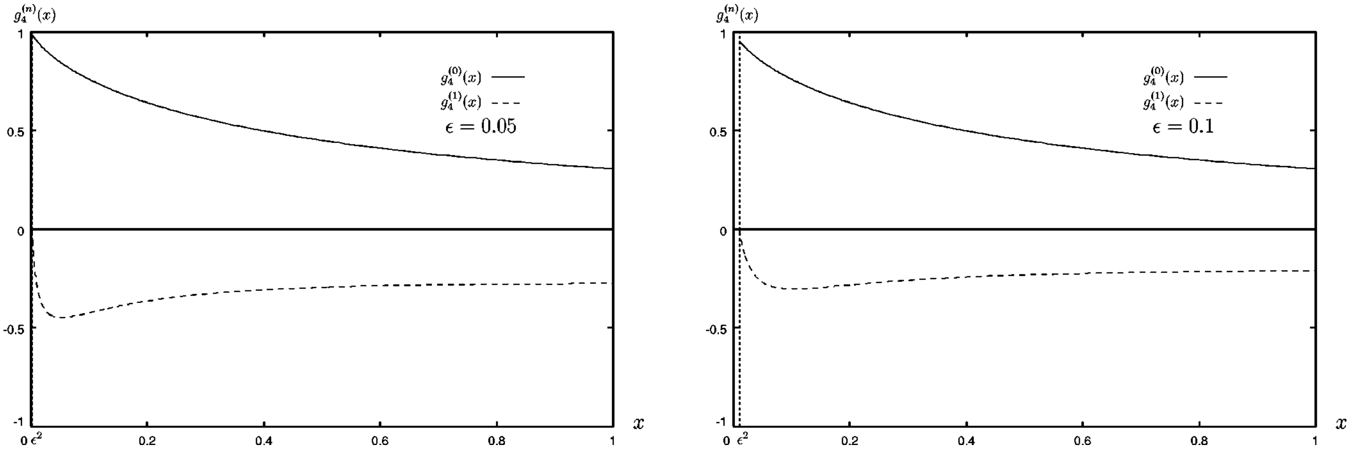

FIG. 1. $g_4^{(0)}(x)$ (solid line) and $g_4^{(1)}(x)$ (dotted line) are plotted with ϵ set at 0.05 (left) and 0.1 (right). $x \equiv (m+\varepsilon)^2/\Lambda^2$ and $\epsilon \equiv \varepsilon/\Lambda$. Note that $g_4^{(0)}(x)$ is positive but $g_4^{(1)}(x)$ is negative everywhere.

It should be noticed that in Eq. (44) the $\boldsymbol{\Gamma}_5$ part of $1/\lambda + \Pi_2$, the (inverse) propagator of pions, vanishes when $\epsilon \mapsto 0$ and $s \mapsto 0$; that is, pions are the massless Nambu-Goldstone particle(s). The quantity ϵ, therefore, plays the role of an infrared cutoff.

The one-loop part of the gap equation is derived from a restricted (one-loop) effective potential obtained by putting $\boldsymbol{\Sigma}_\pi = \mathbf{0}$ in Eq. (44),

$$\mathcal{V}_1|_{\boldsymbol{\Sigma}_\pi=\mathbf{0}} = \frac{1}{2N}\frac{\Lambda^D}{\pi^{D/2}\Gamma(D/2)}\int_0^{1/4} ds\, s^{(D-2)/2} Q_D(x,s) + x\text{-independent terms}, \tag{57}$$

where

$$Q_4(x,s) \equiv \ln[1-\lambda_4 g_4^{(0)}(x) + 4\lambda_4(x+s)q_4(x,s)] + \ln[1-\lambda_4 g_4^{(0)}(x) + 4\lambda_4 s q_4(x,s)], \tag{58}$$

$$Q_3(x,s) \equiv \ln[1-\lambda_3 g_3^{(0)}(x) + 4\lambda_3(x+s)q_3(x,s)] + 2\ln[1-\lambda_3 g_3^{(0)}(x) + 4\lambda_3 s q_3(x,s)], \tag{59}$$

to give

$$\left.\frac{\partial \mathcal{V}_1}{\partial x}\right|_{\Sigma_\pi=\mathbf{0},\mathbf{J}=\mathbf{0}} = \frac{1}{2N}\frac{\Lambda^D}{\pi^{D/2}\Gamma(D/2)}\int_0^{1/4} ds\, s^{(D-2)/2}\frac{\partial Q_D(x,s)}{\partial x} \equiv -\frac{1}{2N}\frac{\Lambda^D}{4^{D/2-1}\pi^{D/2}\Gamma(D/2)} g_D^{(1)}(x), \tag{60}$$

where

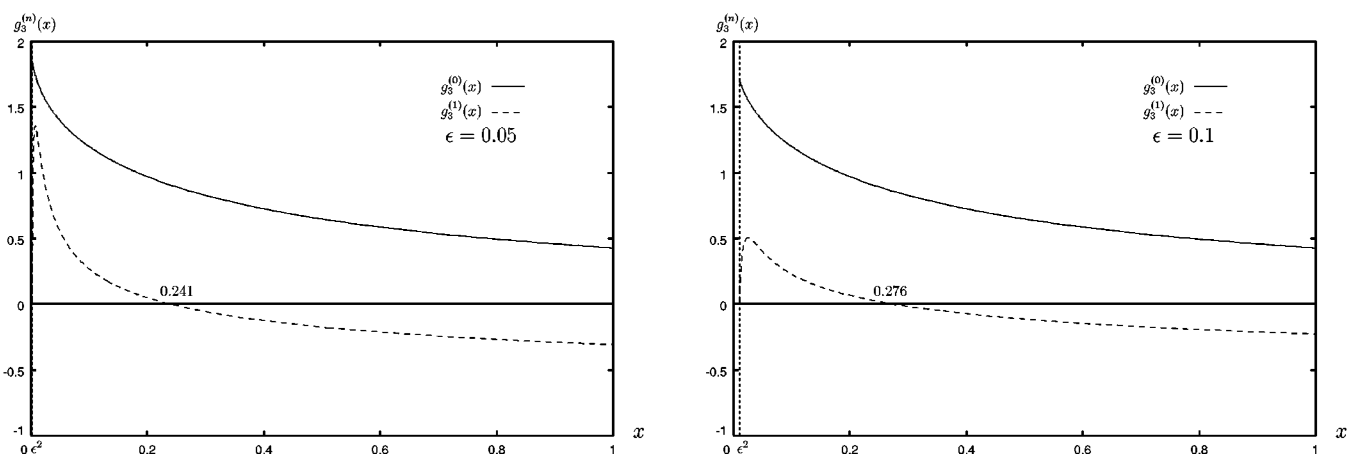

FIG. 2. $g_3^{(0)}(x)$ (solid line) and $g_3^{(1)}(x)$ (dotted line) are plotted with ϵ set at 0.05 (left) and 0.1 (right). Note that $g_3^{(1)}(x)$ has a zero at $x = 0.241$ ($\epsilon = 0.05$) or at $x = 0.276$ ($\epsilon = 0.1$).

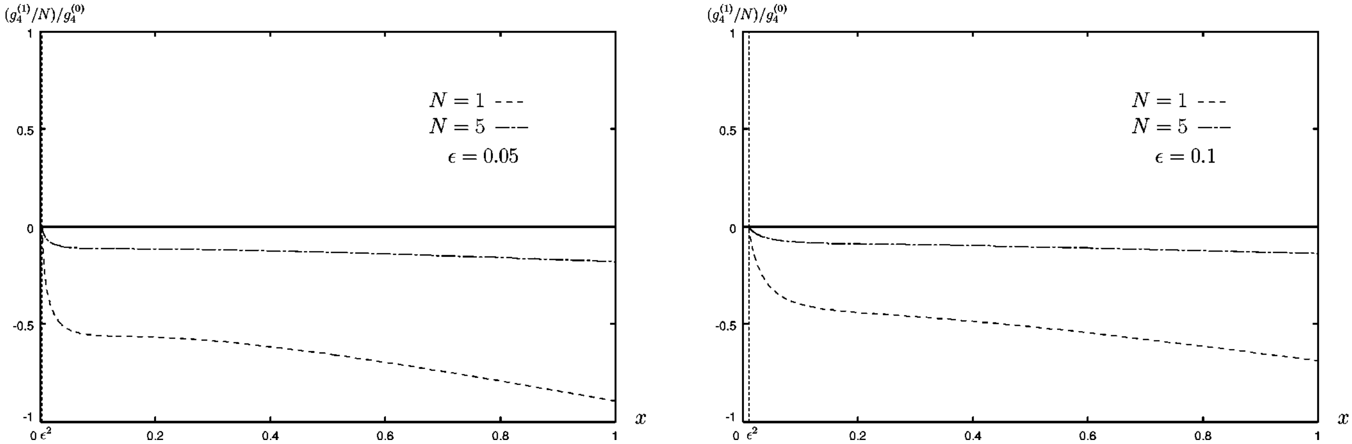

FIG. 3. The ratio $[g_4^{(1)}(x)/N]/g_4^{(0)}(x)$: the left graphs are for $\epsilon=0.05$ and the right for 0.1. The dotted line stands for $N=1$ and the dash-dotted line for $N=5$. It is recognized that the ratio remains less than unity even at $N=1$.

$$g_4^{(1)}(x)\equiv -4\int_0^{1/4} ds\, s\left(\frac{-\lambda_4 g_{4,x}^{(0)}+4\lambda_4(q_4+(x+s)q_{4,x})}{1-\lambda_4 g_4^{(0)}+4\lambda_4(x+s)q_4}+\frac{-\lambda_4 g_{4,x}^{(0)}+4\lambda_4 s q_{4,x}}{1-\lambda_4 g_4^{(0)}+4\lambda_4 s q_4}\right)$$

$$=-4(\sqrt{x}-\epsilon)\int_0^{1/4} ds\, s\left(\frac{-g_{4,x}^{(0)}+4(q_4+(x+s)q_{4,x})}{\epsilon g_4^{(0)}+4(\sqrt{x}-\epsilon)(x+s)q_4}+\frac{-g_{4,x}^{(0)}+4sq_{4,x}}{\epsilon g_4^{(0)}+4(\sqrt{x}-\epsilon)sq_4}\right), \tag{61}$$

$$g_3^{(1)}(x)\equiv -2\int_0^{1/4} ds\sqrt{s}\left(\frac{-\lambda_3 g_{3,x}^{(0)}+4\lambda_3(q_3+(x+s)q_{3,x})}{1-\lambda_3 g_3^{(0)}+4\lambda_3(x+s)q_3}+2\,\frac{-\lambda_3 g_{3,x}^{(0)}+4\lambda_3 s q_{3,x}}{1-\lambda_3 g_3^{(0)}+4\lambda_3 s q_3}\right)$$

$$=-2(\sqrt{x}-\epsilon)\int_0^{1/4} ds\sqrt{s}\left(\frac{-g_{3,x}^{(0)}+4(q_3+(x+s)q_{3,x})}{\epsilon g_3^{(0)}+4(\sqrt{x}-\epsilon)(x+s)q_3}+2\,\frac{-g_{3,x}^{(0)}+4sq_{3,x}}{\epsilon g_3^{(0)}+4(\sqrt{x}-\epsilon)sq_3}\right) \tag{62}$$

with

$$q_{D,x}\equiv\frac{\partial q_D}{\partial x}. \tag{63}$$

In Eqs. (61) and (62), we have replaced λ_D by the tree value $(\sqrt{x}-\epsilon)/[\sqrt{x}g_D^{(0)}(x)]$ in the first terms. In Figs. 1 and 2 we show the shape of $g_D^{(0)}(x)$ as well as $g_D^{(1)}(x)$ for $0\leqslant x\equiv(m+\varepsilon)^2/\Lambda^2\leqslant 1$. We choose two cases for the infrared cutoff $\epsilon\equiv\varepsilon/\Lambda$, 0.05 and 0.1.

The gap equation, up to the one-loop order of the auxiliary fields, therefore, is found to be

$$\frac{1}{\lambda_D}=\frac{\sqrt{x}}{\sqrt{x}-\epsilon}\left(g_D^{(0)}(x)+\frac{1}{N}g_D^{(1)}(x)\right), \tag{64}$$

whose right hand side diverges at $\sqrt{x}=\epsilon$, implying that $\lambda_D=0$. As was discussed before, this is physically reasonable since we have the current quark mass ε even in a free $\lambda=0$ theory. In view of Eq. (64), an important role of the current quark mass ε under the loop expansion is recognized: when $\varepsilon=0$ ($\epsilon=0$) the right hand side reads

$$\left.\left(g_D^{(0)}(x)+\frac{1}{N}g_D^{(1)}(x)\right)\right|_{\epsilon=0},$$

whose second term becomes infinite. [Note that in Eqs. (61) and (62) $q_{D,x}$ is singular at $x=0$ under $\epsilon=0$.] Thus the second term surpasses the first, causing a breakdown of the loop expansion. On the contrary, if $\epsilon\neq 0$, the second term is much smaller in the dangerous region $\sqrt{x}\sim\epsilon$ owing to the factor $\sqrt{x}-\epsilon$ in front of the integrals in Eqs. (61) and (62).

In Figs. 3 and 4, we plot the ratios $(g_D^{(1)}/N)/g_D^{(0)}$ in $D=4$ and 3, respectively. It is recognized that even at $N=1$ (dotted line) the loop expansion is legitimate, since the ratio remains less than unity. It is also shown that the smaller ϵ becomes the greater the ratio goes. The critical values that cause the ratio to exceed unity at $N=1$ are then found such that $\epsilon=0.0326$ at $x=1$ in $D=4$ and $\epsilon=0.040$ at $x=0.0052$ in $D=3$. The values of ϵ can be set smaller when N is larger.

Finally we plot the right hand side of the gap equation (64) for $D=4$ and 3 in Figs. 5 and 6. The horizontal line again stands for the value of $x=(m+\varepsilon)^2/\Lambda^2$ and the vertical line for $1/\lambda_D$. The solid line is for $\sqrt{x}g_D^{(0)}(x)/(\sqrt{x}-\epsilon)$, namely, the tree or $N=\infty$ case. The dotted and the dash-dotted lines include $O(1/N)$ contributions with $N=1$ and 5,

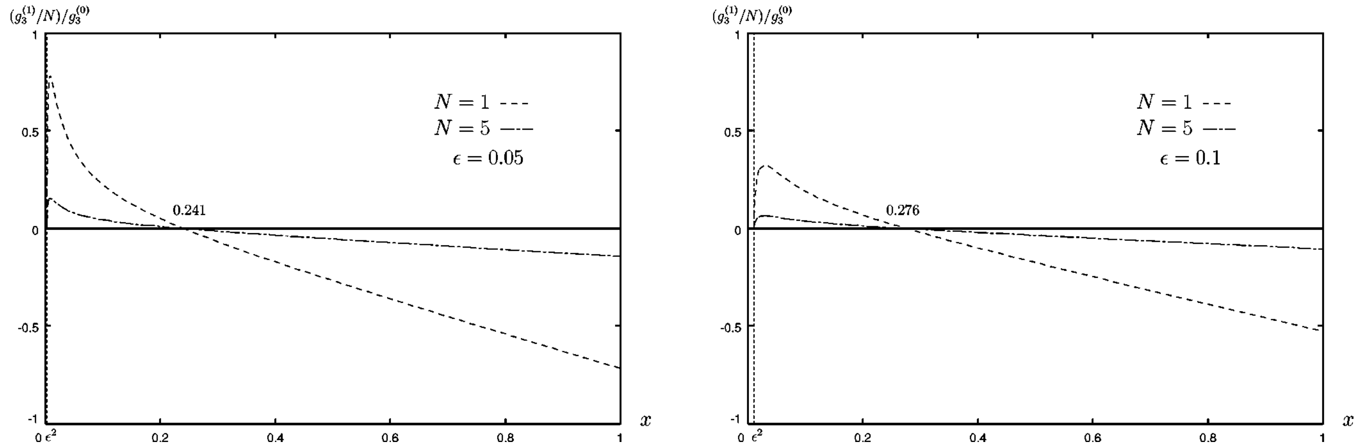

FIG. 4. The ratio $[g_3^{(1)}(x)/N]/g_3^{(0)}(x)$: the left graphs are for $\epsilon=0.05$ and the right for 0.1. The dotted line stands for $N=1$ and the dash-dotted line for $N=5$. It is recognized that the ratio remains less than unity even at $N=1$.

respectively. We have set $\epsilon=0.05$ (left graphs) and $\epsilon=0.1$ (right graphs). In $D=4$ magnified figures for $0\leqslant x<0.15$ are shown together with the whole plots as insets.

It is seen that for a fixed four-Fermi coupling λ_D, that is, with respect to a (supposed) horizontal line, the mass x is a monotonically increasing function of N in $D=4$, but in $D=3$ the dependence is not so simple because of the zeros in $g_3^{(1)}(x)$ (see Figs. 1 and 2); x is monotonically decreasing (increasing) in the small (large) mass or four-Fermi coupling region. Physically speaking, due to quantum effects, χSB is restored in $D=4$ at any coupling: Meanwhile, in $D=3$ it is restored (enhanced) in the strong (weak) coupling region.

IV. DISCUSSION

In this paper we have examined the higher order (= quantum) effect of auxiliary fields on the gap equation in the NJL model. Contrary to the observation by Kleinert and Bossche [11], we find that auxiliary fields still play a significant role for nonvanishing current quark mass ε. "Pions can still survive" in the NJL model. We cannot set the intrinsic fermion mass at zero but it is at the order of $\Lambda/100$ when $N=1$, in order to overcome the infrared divergences and to ensure the loop expansion. In $D=4$ the dynamical mass x is a monotonically increasing function of N at any fixed four-Fermi coupling constant. But in $D=3$ x is monotonically decreasing (increasing) in the small (large) mass or coupling region. In other words, the dynamical mass shrinks by means of quantum effects in the strong coupling regime for $D=3$ as well as for $D=4$. In contrast, it swells in the weak coupling regime in $D=3$. We have already encountered a similar situation in $D=3$ in Ref. [5]; the dynamical mass is a complicated function of the magnitude of the background magnetic fields (MBMFs) under the influence of quantum gluons. In the small mass or coupling region, it is a monotonically increasing function of the MBMFs, but a decreasing function in the larger mass or coupling region.

In this way, we recognize that the auxiliary field method for the NJL model can survive with an infrared cutoff. The power of the auxiliary field method is shown in Ref. [10], using zero- and one-dimensional examples. The case has been made, however, only for bosonic models, so that an

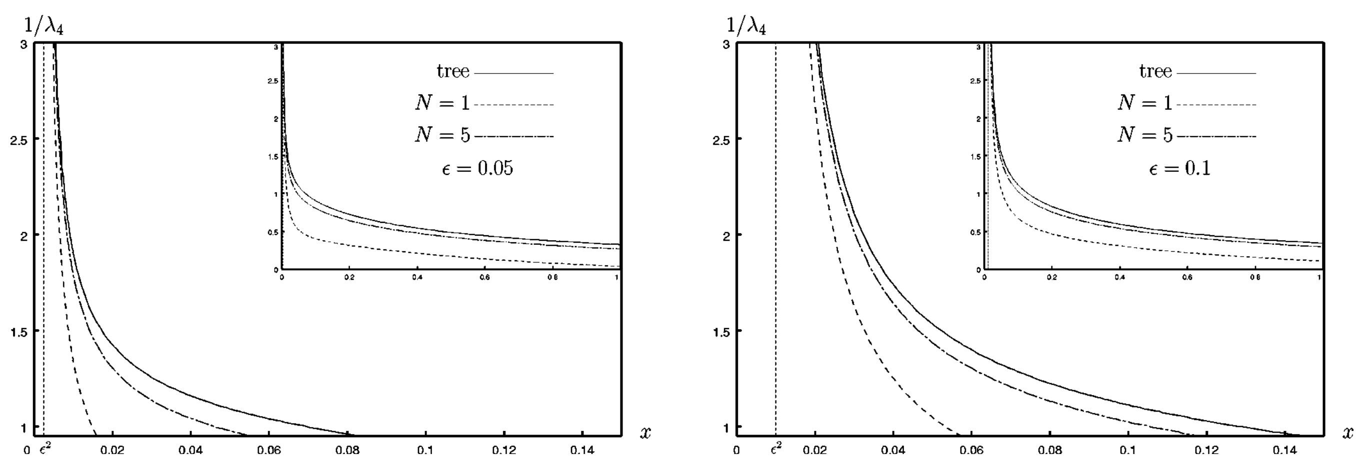

FIG. 5. The right hand side of the gap equation in $D=4$: the solid line designates the tree order or $N=\infty$, while the dotted and the dash-dotted ones include the one-loop effects with $N=1$ and $N=5$, respectively. ϵ is set at 0.05 (left) and 0.1 (right). Graphs with $0\leqslant x<0.15$ are shown; the insets are the whole shapes $0\leqslant x\leqslant 1$. x is recognized as a monotonically increasing function of N with respect to a (supposed) horizontal line, namely, a fixed four-Fermi coupling.

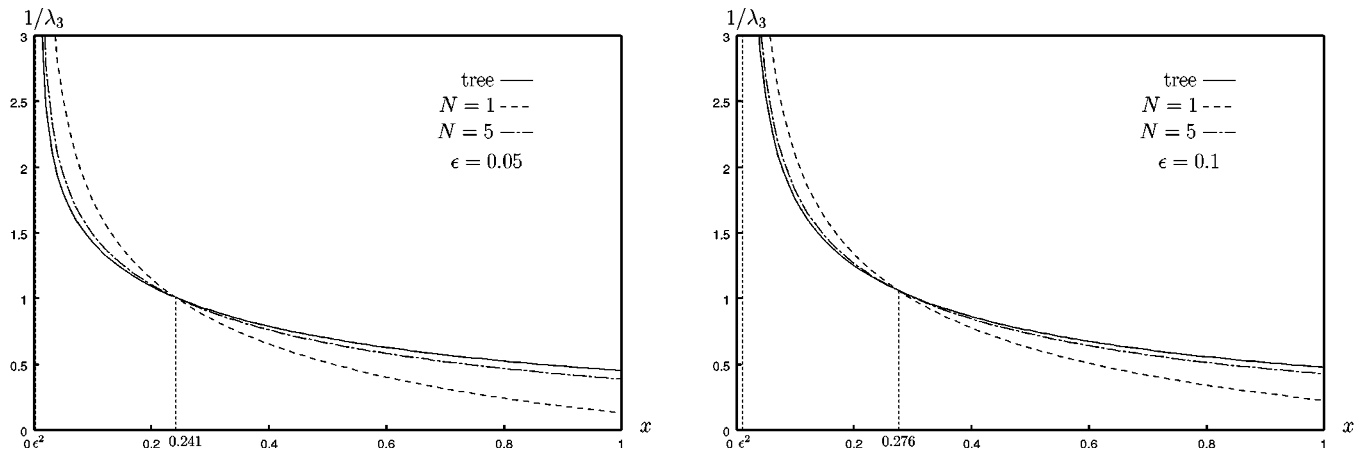

FIG. 6. The right hand side of the gap equation in $D=3$: the solid line designates the tree order or $N=\infty$, while the dotted and the dash-dotted lines include the one-loop effects with $N=1$ and $N=5$, respectively. ϵ is set at 0.05 (left) and 0.1 (right). For $x \leqslant 0.241$ (0.276) x is a monotonically decreasing function of N for a fixed coupling, but on the contrary at $x > 0.241$ (0.276) it is an increasing function.

analysis for fermionic models is necessary. The zero-dimensional fermionic model, the Grassmann integration model, is studied to fulfill our expectation that inclusion of higher-loop effects of auxiliary fields will make the result much better [17]. The one-dimensional, quantum mechanical case is now under study.

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