Regularization of brane induced gravity

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We study the regularization of theories of ''brane induced'' gravity in codimension $N>1$. The brane can be interpreted as a thin dielectric with a large dielectric constant, embedded in a higher dimensional space. The kinetic term for the higher dimensional graviton is enhanced over the brane. A four dimensional gravitation is found on the brane at distances smaller than a critical distance $r<r_c$, and is due to the exchange of a massive resonant graviton. The crossover scale r_c is determined by the mass of the resonance. The suppression of the couplings of light Kaluza-Klein modes to brane matter results in a higher dimensional force law at large distances. We show that the resulting theory is free of ghosts or tachyons.

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I. INTRODUCTION

Brane world theories with large or infinite extra dimensions provide insight into a number of problems of high energy physics, cosmology, and the possible relations between them [1–5]. Those models may ultimately emerge as part of a fundamental higher dimensional theory such as string theory.[1]

The idea that our Universe may have infinite extra dimensions was rejected for a long time as being incompatible with the observed 4D nature of gravity. Since gravity probes all dimensions, the force between objects localized on a $(3+1)$-dimensional hypersurface would be higher dimensional. Nevertheless, it was pointed out in [1] that, in the presence of a brane in 5D space, the ordinary 5D action of gravity is modified by a 4D Einstein-Hilbert action on the brane. This term is compatible with all the symmetries of the theory and, therefore, can be generated by quantum corrections. This model, also known as the DGP (Dvali-Gabadadze-Porrati) model, is an example of a theory of ''brane induced gravity.'' There, gravity is 4D below a certain scale r_c, ranging from galactic to horizon size. One major problem of this scenario, related to the existence of $1/r_c^2$ singularities in off-shell trilinear interactions of longitudinal gravitons [7], was pointed out in [8] (see also [9]). There, it was shown that radiative corrections to the DGP model become uncontrollably large at a very low energy scale $(L_{Pl}r_c^2)^{-1/3}$. This scale is essentially a cutoff beyond which the DGP model needs a UV completion. A similar problem should arise also in codimensions greater than 1, in analogy with the UV behavior of massive gravity [10]. We will not address this problem in this paper, but we will concentrate on another, simpler one; namely, the proper definition and regularization of induced gravity in codimension $N>1$.

Setting aside the strong-coupling problem just mentioned here, brane induced gravity in five dimensions is compatible with all present measurement and observations [2] and allows the scale of 5D quantum gravity M_* to be as low as 10^{-3} eV [3]. Choosing r_c to be the horizon radius, M_* is instead 10^8 eV. In higher codimensions, M_* depends on the UV regularization scale, i.e., on the thickness of the brane. It is natural to choose this thickness to be $O(1/M_*)$; in this case M_* becomes $O(10^{-3}$ eV).

The investigation of brane induced gravity in a space with more than one infinite extra dimension is interesting because that framework may provide a solution to the cosmological constant problem [5,11].

Brane induced gravity in space with N transverse dimensions is described by the action

$$S = M_*^{2+N}\int d^4x d^N Y \sqrt{G}\mathcal{R}_{4+N} + M_P^2\int d^4x d^N Y \delta^N(Y)\sqrt{g}\mathcal{R}_4, \quad (1)$$

where Y_i, $i=1,\ldots,N$, are the coordinates of the N infinite extra dimensions, and the brane is taken to be infinitesimally thin and located at the origin. $\mathcal{R}_{4+N}$ is the $(4+N)$D curvature, G_{MN} is the $(4+N)$D metric, and $\mathcal{R}_4$ is the 4D curvature constructed from the induced metric on the brane $g_{\mu\nu} \equiv G_{MN}(y=0)\delta^M_\mu\delta^N_\nu$. The philosophy behind brane induced gravity is that there are two scales in the theory, a scale for bulk gravity M_*, and a much higher scale M_P, which characterizes the physics on the brane, i.e., the standard model and possible extensions like supersymmetry and grand unified theories. In order to obtain the correct value of Newton's constant, M_P is taken to be $\sim 10^{19}$ GeV.

Calculating the Green's function and Kaluza-Klein (KK) spectrum of Eq. (1) for the case $N=1$ is straightforward. The task is less clear for $N>1$ due to singularities that appear because of the zero thickness of the brane, and the resulting singularities in the Green's functions for $N>1$ [12].

In this paper we study a simple regularization of the action Eq. (1). We propose to consider an action for $(4+N)$-dimensional gravity, reminiscent of the action for a dielectric in electrodynamics:

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[1]A first attempt toward realizing them in string theory was proposed in [6].

$$\frac{S}{M_*^{2+N}} = \epsilon \int_{\rho<\Delta} d^{4+N}x \sqrt{G_{4+N}}\mathcal{R}_{4+N} + \int_{\rho>\Delta} d^{4+N}x \sqrt{G_{4+N}}\mathcal{R}_{4+N}, \quad (2)$$

with ϵ a large dimensionless quantity (to be defined more precisely in the text), ρ the radial coordinate in the N transverse dimensions, and Δ the width of the brane.

The brane in this theory is a thin ''gravitational'' dielectric. We will prove that, at distances smaller than a certain critical scale r_c and for $N>1$, the gravitational interaction is mediated by a massive resonant graviton, whereas at larger distances one rediscovers $(4+N)$-dimensional gravitation.

We will also investigate smooth versions of this scenario. This is achieved by taking a varying ''dielectric constant.'' A scalar action of this kind was considered in [13], where it was pointed out that this kind of action can be realized physically in a dilaton gravity theory.

We thus consider the action of [13] generalized to arbitrary codimensions:

$$\frac{S}{M_*^{2+N}} = \int d^4x d^N Y \sqrt{G}[1+\epsilon\mathcal{F}(\rho/\Delta)]\mathcal{R}_{4+N}, \quad (3)$$

where $\mathcal{F}(\rho/\Delta)$ is the shape of the brane, which we take to depend only on the radial coordinate ρ of the N transverse dimensions. Again, Δ is the width of the brane and ϵ is a large, dimensionless parameter. The action Eq. (2) is obtained from Eq. (3) by choosing for the profile the step function $\mathcal{F}=\theta(\rho-\Delta)$.

The action Eq. (3) describes a smooth three-brane in a $(4+N)$-dimensional space. We will show that for sufficiently large values of ϵ, the effect of this brane ''dielectric'' constant is to convert the higher dimensional laws of gravity to four-dimensional laws at short distances.

On physical grounds one does not expect that the introduction of a ''medium'' (i.e., a brane) into the higher dimensional vacuum will result in the appearance of ghosts or tachyons and make the theory inconsistent. We will explicitly show this for the model (2). A brane with smoothly changing permeability $\epsilon\mathcal{F}$ can be thought of as a sequence of small regions of constant ϵ, and hence the proof should also hold for smooth $\mathcal{F}$.

The action Eq. (3) may provide a physical realization of Eq. (2). One can think of it as gravity coupled to a scalar that gets a nonconstant vacuum expectation value, in the form of a topological defect (domain wall, string, monopole, etc.).

This paper is organized as follows. In the first section we consider the scalar equivalent of the action Eq. (2). We analytically calculate the scalar field resonances and show that they lead to the crossover distance between four-dimensional and higher dimensional behavior for the force they mediate. We do that by studying the KK modes of the fields and prove that their coupling to the ''dielectric'' is determined by their wave function on the brane, for an appropriately chosen source. We will see that, generically, the coupling of heavy modes is strongly suppressed on the brane, and that the coupling of the modes is sharply peaked around a value m_g that is the mass of the effective four-dimensional scalar ''graviton.''

Next, we take a specific choice of a smooth brane and see that even there the scalar ''graviton'' is a massive resonance. Numerical results show that the behavior is the same as with the sharp-boundary dielectric. This shows that the regularization considered here is really shape independent.

In Sec. IV, we investigate the tensor structure of our regularized theory. We show that in the model (2) the four-dimensional *massive* graviton propagates with the correct degrees of freedom—i.e., the mass term generated in this theory has the Pauli-Fierz form. There are no ghost or tachyons propagating in our system, unlike in the regularization proposed in [14].

Section V summarizes the findings of this paper.

As we previously mentioned, it was pointed out in [8] and [9] that the DGP theory suffers from strong-coupling problems, as is typical for theories with massive gravitons [10]. Our regularization, of course, does not cure that problem.

II. A DIELECTRIC BRANE WITH A SHARP BOUNDARY

We start by looking at a model in which we imagine a thin ''dielectric'' with a sharp boundary and dielectric constant $\epsilon \gg 1$ centered around the origin of the transverse N-dimensional space. The results are easily generalized to smooth branes (smooth variation of the ''dielectric'' constant) by assuming that ϵ changes in small steps. We are interested in whether this system exhibits a four-dimensional behavior in a certain region of distance (or energy) scales.

Let us analyze this system first for the simplest case, a scalar field. Consider the following action ($M=0,\ldots,3+N$):

$$\frac{S}{M_*^{2+N}} = \epsilon \int_{\rho<\Delta} d^{4+N}x (\partial_M \Phi)^2 + \int_{\rho>\Delta} d^{4+N}x (\partial_M \Phi)^2, \quad (4)$$

where we define the dimensionless quantity ϵ as follows:

$$\epsilon \Delta^N = M_P^2/M_*^{2+N}, \quad (5)$$

a definition guaranteeing that in the limit $\Delta \to 0$, $\epsilon\Delta^N = \text{const}$, we approach the delta function limit. ρ is the radial coordinate in the extra N-dimensional space. To find a solution to this system, we solve the equation of motion in the two different regions, and impose the matching conditions at the boundary of the sphere. One obtains

$$\Box \Phi^I = 0, \quad \rho<\Delta, \quad (6)$$

$$\Box \Phi^o = 0, \quad \rho>\Delta, \quad (7)$$

and the boundary condition

$$\Phi^I = \Phi^o|_{\rho=\Delta},$$

$$\epsilon \partial_\rho \Phi^I = \partial_\rho \Phi^o|_{\rho=\Delta}. \quad (8)$$

If we decompose the field into Kaluza-Klein modes, $\Phi(x,Y)=\Sigma\sigma^m(x)\phi_m(Y)$, $\Box_4\sigma^m=m^2\sigma^m$, the equations of motion and the boundary condition on the sphere reduce to equations for each one of the modes:

$$\epsilon\partial_\rho\phi_m^I=\partial_\rho\phi_m^o|_{\rho=\Delta},$$

$$\Box_N\phi_m=-m^2\phi_m. \tag{9}$$

We must make some important remarks at this point. We are interested in the coupling of the KK modes to the matter on the brane, which is given by the convolution of the wave function and the matter profile $\Psi(y)$:

$$\tilde{\Phi}_m(0)=\int_{\rho<\Delta}d^{4+N}Y\Psi(Y)\Phi_m(Y). \tag{10}$$

Now we can assume that the profile of the matter fields is spherically symmetric in the extra dimensions, $\Psi(Y)=\Psi(\rho)$. We can take it that the standard model matter is given by the zero modes of some fields with a characteristic scale $M_{SM}\gg M_*$. These zero modes have spherically symmetric profiles in the N transverse dimensions. The excited states with nonzero angular momentum l have masses that are naturally of order M_{SM}, so that they decouple from low energy physics. That is why we can assume that only KK modes with $l=0$ couple to the ''spherically'' symmetric brane matter. When calculating the tree level exchange, we are taking into account only a single tower of radial KK excitations.

Since we can assume that the matter profiles are sharply peaked around the origin, the coupling of the KK modes is approximately given by $\phi_m(0)$.

We look at the case of codimension 1 first. The DGP model is perfectly regular for $N=1$, so we expect to discover the same results as obtained in [1]. The general solution to the 5D equations of motion is (using plane wave normalization)

$$\phi_m^I=A_m\cos(m\rho), \tag{11}$$

$$\phi_m^o=\cos(m\rho+\varphi_m). \tag{12}$$

The odd modes do not couple significantly to matter located on the brane, as explained in the previous paragraph. A straightforward calculation yields the following equation for φ_m:

$$\epsilon\tan(m\Delta)=\tan(m\Delta+\varphi_m). \tag{13}$$

On the other hand, A_m is given by

$$A_m\cos(m\Delta)=\cos(m\Delta+\varphi_m). \tag{14}$$

Since we are interested in four-dimensional distances larger than the width of the brane, modes heavier than $1/\Delta$ decouple. Later we will assume that $\Delta\sim 1/M_*$. Since M_* is the cutoff of the effective field theory for the bulk gravity, we take $m\Delta\ll 1$. Then we can approximate $\tan(m\Delta)\approx m\Delta$, so we find for A_m

$$A_m\approx\frac{1}{\sqrt{1+(\epsilon m\Delta)^2}}. \tag{15}$$

Using the definition of ϵ (5), we see that $\epsilon=r_c/\Delta$, where $r_c=M_P^2/M_*^3$. As explained above, the coupling of the Kaluza-Klein mode of mass m to the brane matter is given by

$$|\phi_m(\rho=0)|^2\approx\frac{1}{1+(mr_c)^2}. \tag{16}$$

We see that we get a resonance, and that for $m\gg r_c^{-1}$ the coupling is strongly suppressed. The result is independent of the shape of the brane [13] and it is exactly what is found in the DGP model [1], as we anticipated. In Ref. [1] it was explained how this resonance leads to a $1/r$ potential for $r<r_c$ and to a $1/r^2$ potential for $r>r_c$.

Now consider for instance the case of codimension $N=3$. The theory with a delta function type brane is known to have singular behavior in this case. Let us apply our regularization prescription to this case. We expect a regularization (Δ) dependent result in this case. The solution in the two regions is

$$\phi_m^I=A_m\frac{\sin(m\rho)}{\rho},$$

$$\phi_m^o=\frac{\sin(m\rho+\varphi_m)}{\rho}. \tag{17}$$

The ϕ_m^I solution is determined by asking for regularity at $\rho=0$. We keep only the $l=0$ terms in the expansion in spherical harmonics as explained before. The boundary conditions give the following equations:

$$A_m\sin(m\Delta)=\sin(m\Delta+\varphi_m), \tag{18}$$

$$\epsilon[m\Delta\cot(m\Delta)-1]=m\Delta\cot(m\Delta+\varphi_m)-1. \tag{19}$$

Expanding the cotangent, we get

$$\cot(m\Delta+\varphi_m)\approx\frac{1}{m\Delta}-\frac{\epsilon m\Delta}{3}. \tag{20}$$

We obtain for the amplitude

$$A_m\approx\frac{1}{\sqrt{1-2\epsilon(m\Delta)^2/3+(\epsilon^2/9)(m\Delta)^4}}. \tag{21}$$

The amplitude squared is a sharply peaked function around its maximum, obtained at $\epsilon(m\Delta)^2\approx 3$. This gives, using our definition of ϵ, $m_g^2\approx M_*^5\Delta/M_P^2$. The coupling of the modes, given by the wave function squared, is now $\phi_m(0)^2\approx m^2A_m^2$. We must emphasize that the fact that A_m is singular at its maximum is due to our approximation. Indeed, the denominator receives corrections at $O[(m\Delta)^2]$, which gives $\phi_m(m_g)^2\approx 1/\Delta^2$: a finite and ϵ independent result. For KK modes with mass larger than m_g, the coupling to the brane is

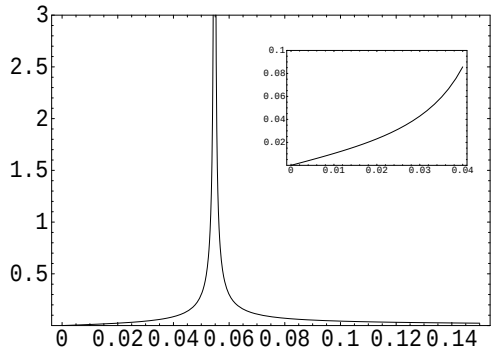

FIG. 1. Wave functions of KK modes on the brane for $\epsilon = 1000$. The scalar ''graviton'' is a very sharp massive resonance. Inset: light modes are suppressed as $\sim m$, giving rise to a $1/r^4$ potential at large distances.

heavily suppressed. In this case we find that m_g is Δ dependent, as we anticipated before.

For Δ, one might take the natural value $\Delta = M_*^{-1}$, in which case the crossover distance and the graviton width become

$$r_c \equiv m_g^{-1} \sim \frac{M_P}{M_*^2}, \quad \Gamma_{m_g} \sim \frac{M_*^3}{M_P^2}. \tag{22}$$

This is exactly what is found for the regularized delta function type brane theory regularized with higher derivative operators [12,14]. In Fig. 1 we show the value of the wave function $\phi_m(0)$ that is roughly the coupling of KK modes to the brane matter.

In codimension $N=3$, the effective 4D ''graviton'' is a very narrow resonance. The crossover from 4D to 7D behavior does not happen because of graviton ''decay.'' At large distances, the 4D ''graviton'' simply becomes too heavy to contribute to the tree-level exchange.

At large distances $r \gg r_c$ we evaluate the potential from the coupling of light KK modes to brane matter. Their coupling is suppressed linearly $\phi_m(0) \sim m$ and the potential is

$$V_{New}(r) \approx \int_0^\infty \frac{m^2}{M_*^5} \frac{\exp(-mr)}{r} dm \sim \frac{1}{M_*^5} \frac{1}{r^4}, \quad r \gg r_c. \tag{23}$$

This is just the 7D potential that appears at distances larger than the crossover distance.

At short distances, the Newtonian potential between two masses, due to exchange of KK modes, ϕ_m, is dominated by the exchange of the lowest mass resonance. So we can approximate our integral by the area of the resonance peak. This gives

$$V_{New}(r) \approx \int_{m_g-\Gamma_{m_g}}^{m_g+\Gamma_{m_g}} \frac{|\phi_m(0)|^2}{M_*^5} \frac{\exp(-mr)}{r} dm \sim \frac{1}{M_P^2} \frac{1}{r}, \quad r \ll r_c. \tag{24}$$

We see that the weak four-dimensional gravity is due to the exchange of a massive resonant ''graviton.''

III. A SMOOTH DIELECTRIC BRANE

In this section we study the 4D effective field theory of a scalar system in which the kinetic term of a bulk field is not homogeneous through the extra dimensions. We shall obtain results qualitatively similar to the ones obtained in the previous section. As we remarked earlier, a smooth ''dielectric'' constant may be achieved in dilaton gravity theories. The reader not interested in the details of this regularization can skip this section and proceed to Sec. IV, where the sharp-boundary approximation is applied to true spin-2 gravity.

The action that describes a scalar in a smoothly varying ''dielectric'' medium is

$$\mathcal{L} = M_*^{2+N} f(Y_i) \partial_A \Phi \partial^A \Phi. \tag{25}$$

We take the localizing profile $f(Y_i)$ to be a spherically symmetric function of radial variable in N extra dimensions. For the system Eq. (3) that we are studying the localizing profile f is

$$f(\rho/\Delta) = 1 + \epsilon \mathcal{F}(\rho/\Delta). \tag{26}$$

As before, we are interested in the four-dimensional spectrum of Eq. (25), and we decompose the field Φ into its Kaluza-Klein modes,

$$\Phi(x_\mu, Y_i) = \sum_m \Phi_m(Y_i) \sigma_m(x_\mu), \quad \Box_4 \sigma_m = m^2 \sigma_m. \tag{27}$$

The differential equation for wave functions $\Phi_m(Y_i)$ is

$$f(\rho) \nabla_N^2 \Phi_m + \nabla_N f(\rho) \nabla_N \Phi_m + f(\rho) m^2 \Phi_m = 0. \tag{28}$$

Let us look at the Schrödinger equation for wave functions Φ_m for a general profile $f(\rho)$ in the case $N=2$. The Laplacian and gradient in $N=2$ have both radial and angular parts, so the solutions can be separated as $\Phi_m(\rho,\theta) = \Sigma \Phi_n(\rho) \Theta_m(\theta)$, where θ is the polar angle in the transverse plane. The scalar ''graviton'' couples to matter located on the stringlike brane $\epsilon \mathcal{F}(\rho)$. We assume, as explained in the previous section, that only the radial KK modes (no angular dependence) of our scalar field couple to matter density localized on the soliton $\Psi(\rho) T(x)$. If we substitute

$$\Phi_m(\rho) \equiv \frac{1}{\sqrt{\rho f(\rho)}} \phi_m(\rho), \tag{29}$$

we obtain a Schrödinger equation for modes ϕ_m,

$$-\phi_m'' + \left[\frac{1}{2} \frac{f''}{f} + \frac{1}{2} \frac{f'}{f} \left(\frac{1}{\rho} - \frac{f'}{2f} \right)^2 - \frac{1}{4\rho^2} \right] \phi_m = m^2 \phi_m. \tag{30}$$

The modes are orthogonal, and the normalization integrals are

$$\int f(\rho)\Phi_n\Phi_m\rho d\rho=\alpha_m\delta_{m,n},$$

$$\frac{\int f(\rho)\partial_\rho\Phi_n\partial_\rho\Phi_m\rho d\rho}{\int f(\rho)\Phi_n\Phi_m\rho d\rho}=m^2. \tag{31}$$

The effective 4D action is

$$\mathcal{L}=\frac{1}{2}M_*^{2+N}\sum_m(\partial_\mu\sigma_m\partial^\mu\sigma_m-m^2\sigma_m^2)$$

$$+\sum_m\frac{\tilde{\Phi}_m(0)}{M_*^2\sqrt{\alpha_m}}\sigma_m T(x). \tag{32}$$

Here $\tilde{\Phi}_m(0)$ is the convolution of the wave function $\Phi_m(\rho)$ and the matter profile $\Psi(\rho)$:

$$\tilde{\Phi}_m(0)=\int\rho d\rho\Psi(\rho)\Phi_m(\rho). \tag{33}$$

As in the previous section, this represents the coupling of KK graviton modes of mass m to brane-localized matter, and we can approximate their couplings to be $\tilde{\Phi}_m(0)\approx\Phi_m(0)$.

Let us illustrate the main properties of the smooth brane model in $N=2$ transverse dimensions. Consider the following profile:

$$\mathcal{F}=\frac{\exp(-\rho^2/\Delta^2)}{\rho/\Delta}. \tag{34}$$

This profile multiplied by $1/\Delta^2$ is a regularization of a delta function in $N=2$ dimensions. It is instructive to see what the spectrum is when one neglects the constant 1 in Eq. (26), and then treats the problem in some approximation. This is equivalent to taking the limit $\epsilon\to\infty$. Taking $f=1+\epsilon\mathcal{F}$ one finds in Eq. (30) the Schrödinger potential

$$V(\rho)=(1/\Delta)^2[(\rho/\Delta)^2-1]. \tag{35}$$

This is the potential of a simple harmonic oscillator with the spectrum

$$m_n=\sqrt{2n}/\Delta,\quad n=0,2,4,\ldots. \tag{36}$$

Here n is an even integer, since the wave function and derivative at the origin must be continuous, and we restricted ourselves to modes with no angular dependence. As a first approximation, we can say that the spectrum of the model in Eq. (3) consists of a tower of metastable modes with the masses Eq. (36). A metastable zero mode (with mass exactly zero in the $\epsilon\to\infty$ limit) will be responsible for the $1/r$ potential.[2] The Schrödinger potential resulting from the profile (26) reads

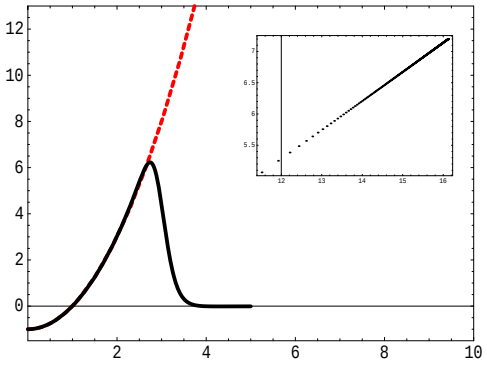

FIG. 2. Potential Eq. (37) $N=2$ (solid line) together with potential Eq. (35) (dashed line). ρ is in units of Δ, the potentials are in units of $1/\Delta^2$, while $\epsilon=10000$. The inset shows the log-log plot of the integral in Eq. (38) vs ϵ.

$$V(\rho)=(1/\Delta)^2\frac{\exp(2\rho^2)-4\epsilon[2\rho^3\exp(\rho^2)+\epsilon(-1+\rho^2)]}{4[\epsilon+\rho\exp(\rho^2)]^2}, \tag{37}$$

where ρ is given in units of the brane thickness Δ. This potential is shown in Fig. 2. The tunneling rate represents the decay width of the resonance and can be calculated in the WKB approximation as

$$\Gamma_0\sim\frac{1}{\Delta}\exp\left(-2\int_{y_1}^{y_2}\sqrt{V}dy\right), \tag{38}$$

where y_1, y_2 are the classical turning points. The width of the potential well is $\sim\Delta$. For large values of ϵ, we are well within the limit of validity of the WKB approximation, namely, $V'/(2V)^{3/2}\ll 1$. The integral in Eq. (38) can be evaluated numerically. The dependence of the integral on ϵ is $\Gamma_0^{N=2}\sim 1/\Delta\epsilon^{0.931}$ (shown in a log-log plot in the inset in Fig. 2).

Here, the difference from the expected result $\Gamma\sim 1/\epsilon$ is due to the WKB approximation. Assuming that the width of the zero mode is $\sim 1/\epsilon$, and by knowing that the zero mode is responsible for the 4D potential on the brane, we can calculate the value of the ''Newton'' constant G. The square of the KK graviton coupling to matter (32) multiplied by the width of the zero mode resonance gives the cumulative effect of the tree-level exchange of continuum KK modes in the resonance. The effective 4D Newton constant is thus $G_{N=2}\sim 1/M_P^2$ for two choices of parameters: (1) $\Delta\sim 1/M_*$ and $\epsilon\sim(M_P/M_*)^2$; (2) $\Delta\sim 1/M_P$ and $\epsilon\sim(M_P/M_*)^4$.

Four-dimensional interactions between masses on the brane are mediated by the exchange of the zero mode reso-

[2]In fact, in the case $N=1$ [13] this is an exact picture. The metastable mode of mass zero has a width $\sim 1/\epsilon$, which results in the crossover distance $r_c=M_P^2/M_*^3$ between four- and five-dimensional gravity.

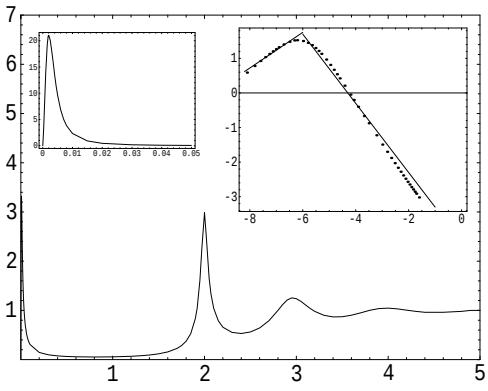

FIG. 3. Dependence of the coupling of a KK mode on its mass for $\epsilon=5000$. Peaks are at the location of resonant states of a harmonic oscillator with $m=0,2,\sqrt{8},\ldots$. In the inset, a peak is shown at $m_g\sim 1/\Delta\sqrt{\epsilon}$, for $\epsilon=50000$ on both normal (left) and log-log (right) scale. The ''graviton'' resonance is asymmetrical. To the left of the peak, it grows as $\sim\sqrt{m}$, giving rise to a 6D potential at large distances, while to the right it decays as $\sim 1/m$. The solid lines have coefficients 0.5 and -1, respectively.

nance, of width $\sim 1/\Delta\epsilon$. The WKB approximation gives the value of Newton's constant; however, it misses most of the features of brane induced gravity in $N=2$ transverse dimensions. In particular, it does not predict the transition between the four-dimensional and the higher dimensional regime and it does not give the distance at which this transition happens (crossover distance). In the rest of this section, we will study the spectrum and the couplings of KK modes in brane induced gravity. We will see that the graviton resonance has a finite mass, and that its width is not $\sim 1/\epsilon$. However, the correct value of the Newton constant is obtained by exchange of a massive graviton resonance.

The Schrödinger equation can be solved numerically, and its solution can be used to find the convolution of the KK wave functions with the wave function of localized matter. We will investigate the couplings of different modes in order to determine how the transition between four-dimensional gravity and higher dimensional gravity occurs and what is the crossover distance.

The suppression of the KK graviton couplings to brane matter is shown in Fig. 3. We notice the peak positioned close to zero mass that is responsible for 4D gravity, as well as higher resonant modes at the positions of harmonic oscillator levels, with masses $m=2/\Delta,\sqrt{8}/\Delta$, etc. Since we are interested in the large distance behavior, we show a magnified picture of a graviton resonance on a log-log scale. The behavior is as follows: for the mode of mass zero, the coupling is zero. Then the coupling rises as $\sim\sqrt{m}$, with the peak positioned at $m_g\sim 1/\Delta\sqrt{\epsilon}$. To the right of the peak, the coupling dies off as $\sim 1/m$.

We see that the graviton is a massive asymmetric resonance. The mass and the width of the resonance are

$$m_g\sim\frac{1}{\Delta\sqrt{\epsilon}},\quad \Gamma_{m_g}\sim\frac{1}{\Delta\sqrt{\epsilon}}. \tag{39}$$

Starting from zero mass, the resonance grows as

$$\phi_m(0)\sim\sqrt{\epsilon m}. \tag{40}$$

Now we can choose two possible sets of values for ϵ and Δ in order to reproduce the correct 4D Newton constant $\Delta\sim 1/M_*,\epsilon\sim(M_P/M_*)^2$; or $\Delta\sim 1/M_P,\epsilon\sim(M_P/M_*)^4$. Both choices give, for the crossover distance (inverse graviton mass) and the resonance width,

$$r_c^{N=2}\equiv m_g^{-1}\sim\frac{M_P}{M_*^2},\quad \Gamma_{m_g}\sim\frac{M_*^2}{M_P}. \tag{41}$$

In the case of $N=2$, the graviton mass is roughly the same as the resonance width. On the brane, at distances $r\gg r_c$, we can evaluate the ''Newtonian'' potential V_{New} from the coupling given in Eq. (32), and the shape of the peak's left ridge Eq. (40). The exchange of modes with mass $m>1/r$ contributes weakly, so one can integrate from zero to infinity. The coupling of graviton KK modes is proportional to $\sim\sqrt{m}$, so the potential is given by

$$V(r)\approx\int_0^\infty\frac{m}{M_*^4}\frac{\exp(-mr)}{r}dm=\frac{1}{M_*^4}\frac{1}{r^3},\quad r\gg M_P/M_*^2. \tag{42}$$

This is just the 6D Newton's potential that appears at distances larger than the crossover distance. At short distances, the exchange of a single massive-graviton resonance gives rise to the 4D Newton potential:

$$V(r)\approx\frac{1}{M_P^2}\frac{1}{r},\quad r\ll M_P/M_*^2. \tag{43}$$

Here we approximated the area under the square of the curve determining the coupling with twice the area on the left side of the peak.

IV. THE SPIN-2 CASE

After having studied the scalar theory, in detail we can proceed with a true gravitational theory. Although the scalar gravity example captures the main features of brane induced gravity, it is crucial to show that our regularization gives a consistent (4D) spin-2 theory. In particular, we have to show that no tachyons or ghosts propagate in our framework. In the effective 4D theory, the higher dimensional graviton is represented by four-dimensional spin-2 states and a set of scalars. We should check that the masses given to those fields are not negative, and that the (massive) graviton propagates with the right number of degrees of freedom, which is assured if the mass term is of the Pauli-Fierz form.

The action to be considered is

$$\frac{S}{M_*^{2+N}} = \epsilon \int_{\rho<\Delta} d^{4+N}x \sqrt{G_{4+N}} \mathcal{R}_{4+N} + \int_{\rho>\Delta} d^{4+N}x \sqrt{G_{4+N}} \mathcal{R}_{4+N}. \quad (44)$$

We can do an expansion in KK modes for the gravitational field, analogous to the scalar case:

$$G_{MN} = \sum_m G^m_{MN}(x) \phi_m(Y). \quad (45)$$

The boundary conditions now are the well-known Israel matching conditions [15], which give an equation for the extrinsic curvature of the sphere which divides the $(4+N)$-dimensional space into two parts:

$$\epsilon K^I_{MN}(x,Y) = K^o_{MN}(x,Y)|_{\rho=\Delta}. \quad (46)$$

These equations become particularly simple when we use Gaussian normal coordinates, since then the extrinsic curvature is given by $K_{MN} = \frac{1}{2}\partial G_{MN}/\partial\rho$. One sees that the boundary conditions for the KK modes are completely analogous to the scalar case equations treated in the previous section. This reduces the spin-2 case essentially to the spin-0 case. In the following equations we split the $(4+N)$-dimensional indices $M,N,\ldots$ into 4D indices $\mu = 0,\ldots,3$, and transverse indices $i = 4,\ldots,3+N$. γ_{ab} are the fluctuations of the metric around the flat background and $\gamma = \gamma_4 + \gamma_N$ is the trace of the fluctuation. The Einstein tensor is denoted by $\mathcal{G}_{MN}$:

$$\mathcal{G}_{i\mu} = \frac{1}{2}\partial^\lambda \partial_i \gamma_{\mu\lambda} + \frac{1}{2}\partial^j \partial_i \gamma_{\mu j} + \frac{1}{2}\partial^\lambda \partial_\mu \gamma_{i\lambda} + \frac{1}{2}\partial^j \partial_\mu \gamma_{ij} - \frac{1}{2}\Box_4 \gamma_{i\mu} - \frac{1}{2}\Box_N \gamma_{i\mu} - \frac{1}{2}\partial_\mu \partial_i \gamma, \quad (47)$$

$$\mathcal{G}_{\mu\nu} = \partial^\lambda \partial_{(\mu}\gamma_{\nu)\lambda} - \frac{1}{2}\Box_4 \gamma_{\mu\nu} - \frac{1}{2}\partial_\mu \partial_\nu (\gamma_4 + \gamma_N) + \partial^i \partial_{(\mu}\gamma_{\nu)i} - \frac{1}{2}\Box_N \gamma_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}[\partial^\lambda \partial^\rho \gamma_{\lambda\rho} - (\Box_4 + \Box_N)\gamma_4] - \frac{1}{2}\eta_{\mu\nu}[\partial^i \partial^j \gamma_{ij} + 2\partial^i \partial^\mu \gamma_{i\mu} - (\Box_4 + \Box_N)\gamma_N]. \quad (48)$$

We now perform a shift on $\gamma_{\mu\nu}$:

$$\gamma_{\mu\nu} \to \gamma_{\mu\nu} + \alpha \eta_{\mu\nu}\gamma_N. \quad (49)$$

After some algebra we get

$$\mathcal{G}_{\mu\nu} = \partial^\lambda \partial_{(\mu}\gamma_{\nu)\lambda} - \frac{1}{2}\Box_4 \gamma_{\mu\nu} - \frac{1}{2}\partial_\mu \partial_\nu \gamma_4 + \partial^i \partial_{(\mu}\gamma_{\nu)i} - \frac{1}{2}\Box_N \gamma_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}[\partial^\lambda \partial^\rho \gamma_{\lambda\rho} - (\Box_4 + \Box_N)\gamma_4] - \frac{1}{2}\eta_{\mu\nu}[\partial^i \partial^j \gamma_{ij} - (\Box_4 + \Box_N)\gamma_N] + \frac{1}{2}\alpha \eta_{\mu\nu}\Box_N \gamma_N. \quad (50)$$

Now we make the gauge choice

$$\partial^i \gamma_{\nu i} = 0, \quad \partial^i \partial^j \gamma_{ij} - \alpha \Box_N \gamma_N = 0. \quad (51)$$

For $\alpha = -\frac{1}{2}$ we get the following equation of motion for the four-dimensional part of the metric:

$$\mathcal{G}^4_{\mu\nu} - \frac{1}{2}\Box_N \gamma_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}\Box_N \gamma_4 = T_{\mu\nu}. \quad (52)$$

We assume that the matter distribution is confined to the brane, so only the 4D part of the energy-momentum tensor is nonzero. If we contract this equation with ∂^μ, we get, by conservation of the matter energy-momentum tensor, and thanks to the Bianchi identities,

$$\Box_N(\partial^\mu \gamma_{\mu\nu} - \partial_\nu \gamma_4) = 0. \quad (53)$$

This equation gives us a constraint on $\gamma_{\mu\nu}$. Indeed, remembering that $\Box_N$ applied to the fields gives their (nonzero) masses, we can conclude that

$$\partial^\mu \gamma_{\mu\nu} - \partial_\nu \gamma_4 = 0. \quad (54)$$

On the other hand, if we take the trace of Eq. (52), we get

$$\frac{3}{2}\Box_N \gamma_4 = T. \quad (55)$$

This means that γ_4 is determined ''algebraically'' by T; $\gamma_4 = (2/3\Box_N)T$. By ''algebraically,'' we mean that the equation for T is local in four dimensions. Next, after a short calculation, the equations of motion for $\gamma_{\mu i}$ become

$$\frac{1}{2}\partial^\lambda F^i_{\lambda\mu} + \frac{1}{2}\Box_N \gamma_{\mu i} = 0, \quad (56)$$

where $F^i_{\mu\nu}$ is the field strength of the four-dimensional vectors $\gamma_{\mu i}$. This equation describes N massive vector fields in four dimensions.

We are left with the equations of motion of the N^2 scalars γ_{ij}. One obtains

$$\Box_4 \gamma_{ij} + \Box_N \gamma_{ij} + \partial_i \partial_j \gamma + \eta_{ij}\left(\frac{1}{2}\Box_4 \gamma_N + \frac{1}{2}\Box_N \gamma_N - \Box_N \gamma_4\right) = 0. \quad (57)$$

Observe that the divergence of $\mathcal{G}_{ij}$ is identically zero by virtue of the gauge choice $\partial^j \gamma_{ij} + \frac{1}{2}\partial_i \gamma_N = 0$.[3]

Using Eq. (55) we finally obtain

$$\Box_4\left(\gamma_{ij} + \frac{1}{2}\eta_{ij}\gamma_N\right) + \Box_N\left(\gamma_{ij} + \frac{1}{2}\eta_{ij}\gamma_N\right) = (\eta_{ij}\Box_N - \partial_i\partial_j)\frac{2}{3\Box_N}T. \tag{58}$$

Looking at Eqs. (52) and (58), we see that all the mass terms have the correct form, with $m^2 = -\Box_N$. In particular, the spin-2 mass terms have the Pauli-Fierz form. This is of course what we expect in a generic Kaluza-Klein reduction [16]. We can thus be sure that in our regularization no ghost propagates.

V. CONCLUSIONS

In this paper we have proposed a well-defined regularization scheme for ''brane induced'' gravity in codimension $N > 1$. At its simplest, the scheme replaces an infinitely thin brane with a ''dielectric'' sphere of radius Δ in the transverse space. This regularization allows for a simple analytic computation of the spectrum of massive resonances, and it accounts simply for the tensor structure of gravity. Unlike other regularizations already existing in the literature, ours introduces neither ghosts nor tachyons. In this respect we must notice an important point. Our definition of the 4D graviton involves a shift [see Eq. (49)], which is the linearized version of a conformal rescaling. By doing this shift, we couple the (scalar) internal components of the metric fluctuation to the trace of the matter stress-energy tensor, even though it has only 4D components. In other words, extra scalar, dilatonlike degrees of freedom couple to matter. This is not a disaster *per se*, because, even in their absence, linearized massive gravity propagates an extra spin-zero degree of freedom [17]. Their presence only worsens the vDVZ discontinuity. In the regularization of Ref. [14], instead, when the regularization parameter is removed, one recovers *massless* 4D gravity. This is achieved at the price of introducing light ghosts into the system.

We notice also that in codimension $N = 1$ no extra scalars propagate, as shown by Eq. (58).

Finally, our ''sharp'' regularization can easily be extended to cover the case of vector fields in $4 + N$ dimensions. The smooth version of our regularization may also be realized in a physical setting by coupling $(4 + N)$-dimensional gravity to a scalar field which admits a soliton solution (kink, etc.).

[3]As it should be, using global energy-momentum conservation and remembering that the energy-momentum tensor has only 4D components.

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