Cardy-Verlinde formula and Achúcarro-Ortiz black hole

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In this paper it is shown that the entropy of the black hole horizon in Achúcarro-Ortiz spacetime, which is the most general two-dimensional black hole derived from the three-dimensional rotating Bañados-Teitelboim-Zanelli black hole, can be described by the Cardy-Verlinde formula. The latter is supposed to be an entropy formula of conformal field theory in any dimension.

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INTRODUCTION

Holography is believed to be one of the fundamental principles of the true quantum theory of gravity [1,2]. An explicitly calculable example of holography is the much-studied anti–de Sitter (AdS)/conformal field theory (CFT) correspondence.

Lately, the duality of quantum gravity has been proposed, in a de Sitter (dS) space, to a certain Euclidean CFT, living on a spacelike boundary of the dS space [3] (see also earlier works [4–7]). This duality is defined in analogy to the $\mathrm{AdS}_d/\mathrm{CFT}_{d-1}$ correspondence. Following this idea, several works on dS space have been carried out [6–29].

The Cardy-Verlinde formula proposed by Verlinde [30] relates the entropy of a certain CFT with its total energy and its Casimir energy in arbitrary dimensions. Using the $\mathrm{AdS}_d/\mathrm{CFT}_{d-1}$ and $\mathrm{dS}_d/\mathrm{CFT}_{d-1}$ correspondences, this formula has been shown to hold exactly for the cases of Schwarzschild-dS, Reissner-Nordström-dS, Kerr-dS, dS Kerr Newman-dS, and topological dS black holes. In this paper, by using the Cardy-Verlinde formula, we have reobtained the entropy of the Achúcarro-Ortiz black hole which is a two-dimensional black hole derived from the three-dimensional rotating Bañados-Teitelboim-Zanelli (BTZ) black hole.

In 1992, Bañados, Teitelboim, and Zanelli [31,32] showed that $(2+1)$-dimensional gravity has a black hole solution. This black hole is described by two (gravitational) parameters: the mass M and the angular momentum (spin) J. It is locally AdS and thus it differs from Schwarzschild and Kerr solutions since it is asymptotically anti–de Sitter instead of flat spacetime. Additionally, it has no curvature singularity at the origin. AdS black holes are members of this two-parametric family of BTZ black holes and they are very interesting in the framework of string theory and black hole physics [33,34].

For systems that admit 2D CFTs as duals, the Cardy formula [35] can be applied directly. This formula gives the entropy of a CFT in terms of the central charge c and the eigenvalue of the Virasoro operator l_0. However, it should be pointed out that this evaluation is possible as soon as one has explicitly shown (e.g., using the $\mathrm{AdS}_d/\mathrm{CFT}_{d-1}$ correspondence) that the system under consideration is in correspondence with a 2D CFT [36,37].

Even in the most favorable case presented above, the use of the Cardy formula for the computation of the entropy of the gravitational system is far from trivial, since the central charge c and the eigenvalue l_0 of the Virasoro operator have to be expressed in terms of the gravitational parameters (M,J) [38].

The 2D limit of the Cardy-Verlinde proposal is interesting for various reasons. The main motivation to study 2D black holes and 2D gravity is to use them as the useful laboratory tools for more complicated 4D cousins [39–41]. At the same time, quite much is known about (basically quantum) Cardy-Verlinde formula in four dimensions. For instance, the quantum origin for 4D entropy in relation with Cardy-Verlinde formula is explicitly discussed in a number of works [42–44]. Using the $\mathrm{AdS}_d/\mathrm{CFT}_{d-1}$ correspondence, it is known that there are 2D gravitational systems that admit 2D CFTs as duals [36,45]. As mentioned before, one can make direct use of the original Cardy formula [35] to compute the entropy [36,45,46]. A comparison of this result with the corresponding one derived from a 2D generalization of the Cardy-Verlinde formula could be very useful, in particular, for understanding of the puzzling features of the $\mathrm{AdS}_d/\mathrm{CFT}_{d-1}$ correspondence in two dimensions [47].

Also of great interest in extending the Cardy-Verlinde formula to $d=2$ is the clarification of the meaning of the holographic principle for 2D spacetimes. The boundaries of spacelike regions of 2D spacetimes are points, therefore the notion of holographic bound is far from trivial. Furthermore, a generalization of the work of Verlinde to two spacetime dimensions presents several difficulties, essentially for dimensional reasons. Since the black hole horizons are isolated points, one cannot establish an area law [48]. Additionally, the spatial coordinate is not a "radial" coordinate due to a

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scale symmetry and hence one cannot impose a natural normalization on it [49].

I. ACHÚCARRO-ORTIZ BLACK HOLE

The black hole solutions of Bañados, Teitelboim, and Zanelli [31,32] in $(2+1)$ spacetime dimensions are derived from a three-dimensional theory of gravity

$$S=\int dx^3\sqrt{-g}({}^{(3)}R+2\Lambda) \tag{1}$$

with a negative cosmological constant $(\Lambda=1/l^2>0)$.

The corresponding line element is

$$ds^2=-\left(-M+\frac{r^2}{l^2}+\frac{J^2}{4r^2}\right)dt^2+\frac{dr^2}{\left(-M+\frac{r^2}{l^2}+\frac{J^2}{4r^2}\right)}+r^2\left(d\theta-\frac{J}{2r^2}dt\right)^2. \tag{2}$$

There are many ways to reduce the three-dimensional BTZ black hole solutions to the two-dimensional charged and uncharged dilatonic black holes [50,51]. The Kaluza-Klein reduction of the $(2+1)$-dimensional metric (2) yields a two-dimensional line element

$$ds^2=-g(r)dt^2+g(r)^{-1}dr^2, \tag{3}$$

where

$$g(r)=\left(-M+\frac{r^2}{l^2}+\frac{J^2}{4r^2}\right), \tag{4}$$

with M the Arnowitt-Deser-Misner mass, J the angular momentum (spin) of the BTZ black hole, and $-\infty<t<+\infty$, $0\leqslant r<+\infty$, $0\leqslant\theta<2\pi$.

The outer and inner horizons, i.e., r_+ (henceforth simply black hole horizon) and r_-, respectively, concerning the positive mass black hole spectrum with spin $(J\neq 0)$ of the line element (2) are given as

$$r_\pm^2=\frac{l^2}{2}\left(M\pm\sqrt{M^2-\frac{J^2}{l^2}}\right) \tag{5}$$

and therefore, in terms of the inner and outer horizons, the black hole mass and the angular momentum are given, respectively, by

$$M=\frac{r_+^2}{l^2}+\frac{J^2}{4r_+^2} \tag{6}$$

and

$$J=\frac{2\,r_+r_-}{l} \tag{7}$$

with the corresponding angular velocity to be

$$\Omega=\frac{J}{2r^2}. \tag{8}$$

The Hawking temperature T_H of the black hole horizon is [52]

$$T_H=\frac{1}{2\pi r_+}\sqrt{\left(\frac{r_+^2}{l^2}+\frac{J^2}{4r_+^2}\right)^2-\frac{J^2}{l^2}}=\frac{1}{2\pi r_+}\left(\frac{r_+^2}{l^2}-\frac{J^2}{4r_+^2}\right). \tag{9}$$

The area $\mathcal{A}_H$ of the black hole horizon is

$$\mathcal{A}_H=2\pi l\left(\frac{M+\sqrt{M^2-\frac{J^2}{l^2}}}{2}\right)^{1/2} \tag{10}$$

$$=2\pi r_+, \tag{11}$$

and thus the entropy of the two-dimensional Achúcarro-Ortiz black hole, if we employ the well-known Bekenstein-Hawking area formula (S_{BH}) for the entropy [53–55], is given by

$$S_{bh}=\frac{1}{4\hbar G}\mathcal{A}_H=S_{BH}. \tag{12}$$

Using the BTZ units where $8\hbar G=1$, the entropy of the two-dimensional Achúcarro-Ortiz black hole takes the form

$$S_{bh}=4\pi r_+. \tag{13}$$

II. CARDY-VERLINDE FORMULA

In a recent paper, Verlinde [30] propounded a generalization of the Cardy formula, which holds for the $(1+1)$-dimensional CFT, to $(n+1)$-dimensional spacetime described by the metric

$$ds^2=-dt^2+R^2d\Omega_n, \tag{14}$$

where R is the radius of a n-dimensional sphere.

The generalized Cardy formula (hereafter named Cardy-Verlinde formula) is given by

$$S_{CFT}=\frac{2\pi R}{\sqrt{ab}}\sqrt{E_C(2E-E_C)}, \tag{15}$$

where E is the total energy and E_C is the Casimir energy. The definition of the Casimir energy is derived by the violation of the Euler relation as

$$E_C\equiv n(E+pV-TS-\Phi Q), \tag{16}$$

where the pressure of the CFT is defined as $p=E/nV$. The total energy may be written as the sum of two terms,

$$E(S,V)=E_E(S,V)+\tfrac{1}{2}E_C(S,V), \tag{17}$$

where E_E is the purely extensive part of the total energy E. The Casimir energy E_C as well as the purely extensive part of energy E_E expressed in terms of the radius R and the entropy S are written as

$$E_C=\frac{b}{2\pi R}S^{1-1/n} \tag{18}$$

$$E_E=\frac{a}{4\pi R}S^{1+1/n}. \tag{19}$$

After the work of Witten on $\mathrm{AdS}_d/\mathrm{CFT}_{d-1}$ correspondence [56], Savonije and Verlinde proved that the Cardy-Verlinde formula (15) can be derived using the thermodynamics of AdS-Schwarzschild black holes in arbitrary dimension [57].

III. ENTROPY OF ACHÚCARRO-ORTIZ BLACK HOLE IN CARDY-VERLINDE FORMULA

We would like to derive the entropy of the two-dimensional Achúcarro-Ortiz black hole (13) from the Cardy-Verlinde formula (15). First, we evaluate the Casimir energy E_C using Eq. (16). It is easily seen from Eqs. (9) and (13) that

$$T_H S_{bh}=2\left(\frac{r_+^2}{l^2}-\frac{J^2}{4r_+^2}\right), \tag{20}$$

while from Eqs. (7) and (8) we have

$$\Omega_+ J=\frac{J^2}{2r_+^2}. \tag{21}$$

Since the two-dimensional Achúcarro-Ortiz black hole is asymptotically anti–de Sitter, the total energy is $E=M$ and thus the Casimir energy, substituting Eqs. (6), (20), and (21) in Eq. (16), is given as

$$E_C=\frac{J^2}{2r_+^2}, \tag{22}$$

where in our analysis the charge Q is the angular momentum J of the two-dimensional Achúcarro-Ortiz black hole, the corresponding electric potential Φ is the angular velocity Ω, and $n=1$. Making use of expression (18), Casimir energy E_C can also be written as

$$E_C=\frac{b}{2\pi R}. \tag{23}$$

Additionally, it is obvious that the quantity $2E-E_C$ is given, by substituting Eqs. (20) and (21) in Eq. (16), as

$$2E-E_C=2\frac{r_+^2}{l^2}. \tag{24}$$

The purely extensive part of the total energy E_E by substituting Eq. (24) in Eq. (17), is given as

$$E_E=\frac{r_+^2}{l^2}, \tag{25}$$

while by substituting Eq. (13) in Eq. (19), it takes the form

$$E_E=\frac{4\pi a}{R}r_+^2. \tag{26}$$

At this point it is useful to evaluate the radius R. By equating the right hand sides of Eqs. (22) and (23), the radius is written as

$$R=\frac{br_+^2}{\pi J^2}, \tag{27}$$

while by equating the right hand sides of Eqs. (25) and (26) it can also be written as

$$R=4\pi a l^2. \tag{28}$$

Therefore, the radius expressed in terms of the arbitrary positive coefficients a and b is

$$R=2r_+\left(\frac{l}{J}\right)\sqrt{ab}. \tag{29}$$

Finally, we substitute expressions (22), (24), and (29) which were derived in the context of thermodynamics of the two-dimensional Achúcarro-Ortiz black hole in the Cardy-Verlinde formula (15), which in turn was derived in the context of CFT:

$$S_{CFT}=\frac{2\pi}{\sqrt{ab}}2r_+\left(\frac{l}{J}\sqrt{ab}\right)\sqrt{\frac{J^2}{2r_+^2}\frac{2r_+^2}{l^2}} \tag{30}$$

and we get

$$S_{CFT}=S_{bh}. \tag{31}$$

It has been proven that the entropy of the two-dimensional Achúcarro-Ortiz black hole can be expressed in the form of Cardy-Verlinde formula.

CONCLUSIONS

Among the family of $\mathrm{AdS}_d/\mathrm{CFT}_{d-1}$ dualities, the pure gravity case $\mathrm{AdS}_3/\mathrm{CFT}_2$ is the best understood. In contrast, although some progress has been made in understanding AdS/CFT correspondence in two spacetime dimensions

[36,47], AdS_2/CFT_1 remains quite enigmatic. The aim of this paper is to further investigate the AdS_2/CFT_1 correspondence in terms of Cardy-Verlinde entropy formula. Naively, one might expect that holographic dualities in a two-dimensional bulk context would be the simplest cases of all. This may certainly be true on a calculational level; however, one finds such two-dimensional dualities to be plagued with conceptually ambiguous features [58]. One of the remarkable outcomes of the AdS/CFT and dS/CFT correspondences has been the generalization of Cardy's formula (Cardy-Verlinde formula) for arbitrary dimensionality as well as for a variety of AdS and dS backgrounds. In this paper, we have shown that the entropy of the black hole horizon of Achúcarro-Ortiz spacetime can also be rewritten in the form of Cardy-Verlinde formula.

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