Particle and light motion in a space-time of a five-dimensional rotating black hole

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We study the motion of particles and light in a space-time of a five-dimensional rotating black hole. We demonstrate that the Myers-Perry metric describing such a black hole, in addition to three Killing vectors, also possesses a Killing tensor. As a result, the Hamilton-Jacobi equations of motion allow a separation of variables. Using first integrals we present the equations of motion in the first-order form. We describe different types of motion of particles and light and study some interesting special cases. We prove that there are no stable circular orbits in equatorial planes in the background of this metric.

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I. INTRODUCTION

Brane world models with large extra dimensions have recently attracted a lot of interest [1]. An important generic feature of these models is that the fundamental quantum gravity scale may be very low (of the order of TeV) and the size of extra spatial dimensions may be much larger than the Planck length ($\sim 10^{-33}$ cm). In the models with "large extra dimensions" the minimal mass of a black hole can also be much smaller that the Planck mass (10^{16} GeV) and be of the order TeV. Such mini black holes could be produced in particle collisions in near future colliders and in cosmic ray experiments [2]. One can expect that the majority of black holes produced in such a way will be rotating [2,3]. Estimations show that such higher dimensional mini black holes can be described within the classical solutions of vacuum Einstein equations. These mini black holes can be attached to the brane or, if the brane tension is small, can leave the brane and travel in the bulk space [3]. Recently, the problem of higher dimensional black holes has attracted a lot of attention [4,5].

Higher dimensional black holes have also been studied in string theory. Motivated by ideas arising in various models in string theory a lot of work has been done studying supersymmetric higher dimensional black holes especially in five-dimensional space-time. For example, supersymmetric rotating black holes were studied in [6] and [7]. The solution analyzed there is not a vacuum solution of Einstein's equations and requires some special choice of parameters in order to accommodate the supersymmetry. In [8], rotating black holes were studied in the context of string theory. Although the solution described there is very general (a boosted vacuum solution of Einstein's equations) the authors mainly concentrate on scalar field gray body factors. In the previous paper [9], we studied some general geometrical properties of a five-dimensional rotating black hole and the propagation of a five-dimensional massless scalar field in the background of such a black hole.

In this paper, we extend our analysis to properties of the motion of particles and light in the space-time of a five-dimensional rotating black hole. We demonstrate the existence of a Killing tensor in such a space-time. We describe different types of motion of particles and light and study some interesting special cases. The five-dimensional metric is algebraically special and allows two families of principal null congruences. These congruences are geodesic but not shear-free. We also show that there are no stable circular orbits in equatorial planes in the background of this metric.

II. MYERS-PERRY METRIC AND ITS PROPERTIES

The metric for a five-dimensional rotating black hole[1] in the Boyer-Lindquist coordinates is [10,12]

$$ds^2=\frac{\rho^2}{4\Delta}\,dx^2+\rho^2\,d\theta^2-dt^2+(x+a^2)\sin^2\theta d\phi^2+(x+b^2)\cos^2\theta d\psi^2+\frac{r_0^2}{\rho^2}[dt+a\sin^2\theta\;d\phi+b\cos^2\theta d\psi]^2. \tag{2.1}$$

Here,

$$\rho^2=x+a^2\cos^2\theta+b^2\sin^2\theta, \tag{2.2}$$

$$\Delta=(x+a^2)(x+b^2)-r_0^2x. \tag{2.3}$$

The angles ϕ and ψ take values from the interval $[0,2\pi]$, while angle θ takes values from $[0,\pi/2]$. Note also that instead of the "radius" r we use the coordinate $x=r^2$. This will allow us to simplify calculations and make many of the expressions more compact.

The black hole horizon is located at $x=x_+$, where

$$x_\pm=\frac{1}{2}[r_0^2-a^2-b^2\pm\sqrt{(r_0^2-a^2-b^2)^2-4a^2b^2}]. \tag{2.4}$$

The angular velocities Ω_a and Ω_b and the surface gravity κ are

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[1]Note that the Myers-Perry solution is not the only black hole solution of the five-dimensional vacuum Einstein equations. For example, see [11].

$$\Omega_a=\frac{a}{x_+ + a^2}, \quad \Omega_b=\frac{b}{x_+ + b^2}, \quad \kappa=\left.\frac{\partial_x\Pi - r_0^2}{r_0^2\sqrt{x}}\right|_{x=x_+}. \tag{2.5}$$

For the metric (2.1),

$$\sqrt{-g}=\frac{1}{2}\sin\theta\cos\theta\rho^2. \tag{2.6}$$

We shall also need the following expressions for the contravariant components of the metric:

$$g^{tt}=\frac{1}{\rho^2}\left[(a^2-b^2)\sin^2\theta-\frac{(x+a^2)[\Delta+r_0^2(x+b^2)]}{\Delta}\right],$$

$$g^{t\phi}=\frac{ar_0^2(x+b^2)}{\rho^2\Delta}, \quad g^{t\psi}=\frac{br_0^2(x+a^2)}{\rho^2\Delta},$$

$$g^{\phi\phi}=\frac{1}{\rho^2}\left[\frac{1}{\sin^2\theta}-\frac{(a^2-b^2)(x+b^2)+b^2r_0^2}{\Delta}\right],$$

$$g^{\psi\psi}=\frac{1}{\rho^2}\left[\frac{1}{\cos^2\theta}+\frac{(a^2-b^2)(x+a^2)-a^2r_0^2}{\Delta}\right],$$

$$g^{\phi\psi}=-\frac{abr_0^2}{\rho^2\Delta}, \quad g^{xx}=4\frac{\Delta}{\rho^2}, \quad g^{\theta\theta}=\frac{1}{\rho^2}. \tag{2.7}$$

The metric (2.1) is invariant under the following transformation:

$$a\leftrightarrow b, \quad \theta\leftrightarrow\left(\frac{\pi}{2}-\theta\right) \qquad \phi\leftrightarrow\psi. \tag{2.8}$$

It possesses three Killing vectors ∂_t, ∂_ϕ, and ∂_ψ. For $a=b$ the metric has two additional Killing vectors [9]:

$$\cos\partial_{\bar\theta}-\cot\bar\theta\sin\bar\phi\,\partial_{\bar\phi}+\frac{\sin\bar\phi}{\sin\bar\theta}\partial_{\bar\psi} \tag{2.9}$$

and

$$-\sin\bar\phi\,\partial_{\bar\theta}-\cot\bar\theta\cos\bar\phi\partial_{\bar\phi}+\frac{\cos\bar\phi}{\sin\bar\theta}\partial_{\bar\psi}, \tag{2.10}$$

where $\bar\phi=\psi-\phi$, $\bar\psi=\psi+\phi$, and $\bar\theta=2\theta$.

In the next section we shall demonstrate that in the general case the metric (2.1) also has the Killing tensor $K^{\mu\nu}$ satisfying the equation

$$K_{(\mu\nu;\sigma)}=0. \tag{2.11}$$

III. EQUATIONS OF MOTION FOR PARTICLES AND LIGHT: FIRST INTEGRALS

The equations of motion of a test particle of mass m in a curved space-time given by a metric $g_{\mu\nu}$ are

$$\frac{D^2x^\mu}{D\tau^2}=0, \tag{3.1}$$

where $D/D\tau$ denotes the covariant derivative with respect to proper time τ. In this section we study these equations in the Myers-Perry metric (2.1) by using the Hamilton-Jacobi method. This consideration is similar to the approach developed by Carter for the four-dimensional Kerr metric [13] (see also [14]).

Equations (3.1) can be derived from the Lagrangian

$$L=\frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu, \tag{3.2}$$

where an overdot denotes the partial derivative with respect to an affine parameter λ. For consistency, we chose

$$\tau=m\lambda, \tag{3.3}$$

which is equivalent to

$$g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu=-m^2. \tag{3.4}$$

The conjugate momenta following from Eq. (3.2) are

$$p_\mu=g_{\mu\nu}\dot{x}^\nu. \tag{3.5}$$

Thus, the Hamiltonian is

$$H=\frac{1}{2}g^{\mu\nu}p_\mu p_\nu. \tag{3.6}$$

The momenta calculated for the metric (2.1) are

$$p_t=\left(-1+\frac{r_0^2}{\rho^2}\right)\dot{t}+\frac{r_0^2}{\rho^2}\dot\phi+\frac{r_0^2b\cos^2\theta}{\rho^2}\dot\psi,$$

$$p_\phi=\frac{r_0^2a\sin^2\theta}{\rho^2}\dot{t}+\left(x+a^2+\frac{r_0^2a^2\sin^2\theta}{\rho^2}\right)\sin^2\theta\dot\phi+\frac{r_0^2ab\sin^2\theta\cos^2\theta}{\rho^2}\dot\psi,$$

$$p_\psi=\frac{r_0^2b\cos^2\theta}{\rho^2}\dot{t}+\frac{r_0^2ab\sin^2\theta\cos^2\theta}{\rho^2}\dot\phi+\left(x+b^2+\frac{r_0^2b^2\cos^2\theta}{\rho^2}\right)\cos^2\theta\dot\psi,$$

$$p_x=\frac{\rho^2}{4\Delta}\dot{x},$$

$$p_\theta=\rho^2\dot\theta. \tag{3.7}$$

From the symmetries of the metric (2.1) it follows that at least three constants of motion should exist. They correspond to conservation of energy E and angular momenta in two independent planes defined by the angles ϕ and ψ. Thus,

$$p_t = -E, \quad p_\phi = \Phi, \quad \text{and} \quad p_\psi = \Psi. \tag{3.8}$$

We also have the constant of motion corresponding to conservation of rest mass which is given by Eq. (3.3).

In order to solve the system of equations of motion completely, we need one more constant of motion. This can be obtained using the Hamilton-Jacobi method of solving the equations of motion.

From the Hamiltonian (3.6) we have Hamilton-Jacobi equations in the form

$$-\frac{\partial S}{\partial \lambda} = H = \frac{1}{2} g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} \tag{3.9}$$

where S is the Hamilton-Jacobi action. Since it was proven in [9] that the equation of motion can be separated, the action must take the form

$$S = \frac{1}{2} m^2 \lambda - Et + \Phi \phi + \Psi \psi + S_\theta + S_x, \tag{3.10}$$

where S_θ and S_x are functions of θ and x, respectively. From Eqs. (3.9) and (3.10) we can conclude that

$$\left(\frac{\partial S_\theta}{\partial \theta}\right)^2 + (m^2 - E^2)(a^2 \cos^2\theta + b^2 \sin^2\theta) + \frac{1}{\sin^2\theta}\Phi^2 + \frac{1}{\cos^2\theta}\Psi^2 = K \tag{3.11}$$

and

$$4\Delta\left(\frac{\partial S_x}{\partial x}\right)^2 + (m^2 - E^2)x - \frac{r_0^2 (x+a^2)(x+b^2)}{\Delta} \mathcal{E}^2 - (a^2 - b^2) \times \left(\frac{\Phi^2}{(x+a^2)} - \frac{\Psi^2}{(x+b^2)}\right) = -K, \tag{3.12}$$

where

$$\mathcal{E} = E + \frac{a\Phi}{x+a^2} + \frac{b\Psi}{x+b^2}. \tag{3.13}$$

Here K is a new constant of the motion. We used the freedom $K \to K + \text{const}$ (in this case $\text{const} = a^2E^2$) to obtain the equations in a form invariant under the transformation (2.8). Using the relations $p_\theta = S_{,\theta}$ and $p_x = S_{,x}$ we can replace $(S_\theta)_{,\theta}$ and $(S_x)_{,x}$ by p_θ and p_x, respectively.

The conserved quantity K is of the second order in the momenta p_μ and it is related to the Killing tensor $K_{\mu\nu}$ as follows:

$$K^{\mu\nu} p_\mu p_\nu = K. \tag{3.14}$$

By comparing Eq. (3.11) with Eq. (3.14), one obtains

$$K^{\mu\nu} = -(a^2 \cos^2\theta + b^2 \sin^2\theta)(g^{\mu\nu} + \delta_t^\mu \delta_t^\nu) + \frac{1}{\sin^2\theta}\delta_\phi^\mu \delta_\phi^\nu + \frac{1}{\cos^2\theta}\delta_\psi^\mu \delta_\psi^\nu + \delta_\theta^\mu \delta_\theta^\nu. \tag{3.15}$$

A similar result for the four-dimensional Kerr metric was obtained by Carter in 1968 [13]. We used the GRTENSOR program to check directly that $K^{\mu\nu}$ does obey Eq. (2.11).

Equations (3.11) and (3.12) can be written in a compact form:

$$\frac{\partial S_\theta}{\partial \theta} = \sigma_\theta \sqrt{\Theta}, \quad \frac{\partial S_x}{\partial x} = \sigma_x \sqrt{X}. \tag{3.16}$$

Here the functions Θ and X are given by

$$\Theta = (E^2 - m^2)(a^2 \cos^2\theta + b^2 \sin^2\theta) - \frac{1}{\sin^2\theta}\Phi^2 - \frac{1}{\cos^2\theta}\Psi^2 + K, \tag{3.17}$$

$$X = \frac{\mathcal{X}}{4\Delta^2}, \tag{3.18}$$

$$\mathcal{X} = \Delta\left[x(E^2 - m^2) + (a^2 - b^2)\left(\frac{\Phi^2}{(x+a^2)} - \frac{\Psi^2}{(x+b^2)}\right) - K\right] + r_0^2 (x+a^2)(x+b^2)\mathcal{E}^2. \tag{3.19}$$

The sign functions $\sigma_\theta = \pm$ and $\sigma_x = \pm$ in the right hand sides of the two equations (3.16) are independent of one another. In each of the equations the change of sign occurs at a turning point where the expression in the right hand side vanishes.

We can write the Hamilton-Jacobi action in terms of these functions:

$$S = \frac{1}{2} m^2 \lambda - Et + \Phi\phi + \Psi\psi + \sigma_\theta \int^\theta \sqrt{\Theta}\, d\theta + \sigma_x \int^x \sqrt{X}\, dx. \tag{3.20}$$

By differentiating with respect to K, m, E, Φ, and Ψ, we obtain the solution of the Hamiltonian-Jacobi equations as

$$\int^\theta \frac{d\theta}{\sqrt{\Theta}} = \int^x \frac{dx}{4\Delta\sqrt{X}}, \tag{3.21}$$

$$\lambda = \int^\theta \frac{1}{\sqrt{\Theta}}(a^2 \cos^2\theta + b^2 \sin^2\theta)\, d\theta + \int^x \frac{1}{\sqrt{X}} \frac{x}{4\Delta} dx, \tag{3.22}$$

$$t=\int^{\theta}\frac{1}{\sqrt{\Theta}}E(a^2\cos^2\theta+b^2\sin^2\theta)d\theta+\int^{x}\frac{1}{\sqrt{X}}\left[\frac{r_0^2(x+a^2)(x+b^2)}{4\Delta^2}\mathcal{E}+\frac{xE}{4\Delta}\right]dx, \quad (3.23)$$

$$\phi=\int^{\theta}\frac{1}{\sqrt{\Theta}}\frac{\Phi}{\sin^2\theta}d\theta-\int^{x}\frac{1}{\sqrt{X}}\left[\frac{ar_0^2(x+b^2)}{4\Delta^2}\mathcal{E}+\frac{(a^2-b^2)\Phi}{4\Delta(x+a^2)}\right]dx, \quad (3.24)$$

$$\psi=\int^{\theta}\frac{1}{\sqrt{\Theta}}\frac{\Psi}{\cos^2\theta}d\theta-\int^{x}\frac{1}{\sqrt{X}}\left[\frac{br_0^2(x+a^2)}{4\Delta^2}\mathcal{E}-\frac{(a^2-b^2)\Psi}{4\Delta(x+b^2)}\right]dx. \quad (3.25)$$

Often it is more convenient to rewrite these equations in the form of the first-order differential equations

$$\rho^2\dot{\theta}=\sigma_\theta\sqrt{\Theta}, \quad (3.26)$$

$$\rho^2\dot{x}=\sigma_x 2\sqrt{\mathcal{X}}, \quad (3.27)$$

$$\rho^2\dot{t}=E\rho^2+\frac{r_0^2(x+a^2)(x+b^2)}{\Delta}\mathcal{E}, \quad (3.28)$$

$$\rho^2\dot{\phi}=\frac{\Phi}{\sin^2\theta}-\frac{ar_0^2(x+b^2)}{\Delta}\mathcal{E}-\frac{(a^2-b^2)\Phi}{(x+a^2)}, \quad (3.29)$$

$$\rho^2\dot{\psi}=\frac{\Psi}{\cos^2\theta}-\frac{br_0^2(x+a^2)}{\Delta}\mathcal{E}+\frac{(a^2-b^2)\Psi}{(x+b^2)}. \quad (3.30)$$

These equations can be obtained from Eqs. (3.21) by direct differentiation with respect to the affine parameter λ.

IV. GENERAL TYPES OF MOTION AND SPECIAL CASES

A. Radial motion

The geodesic world line of a particle in the Myers-Perry metric is completely determined by the first integrals of motion E, Φ, Ψ, and K. We discuss the motion in the black hole exterior, so we assume that $\Delta>0$. Consider $\mathcal{X}$ given by Eq. (3.19) as a function of x for fixed values of the other parameters. At large distances the leading term of X contains the factor E^2-m^2. For $E^2<m^2$ the function X becomes negative at large x. Hence only orbits with $E^2>m^2$ can extend to infinity. These orbits are called *unbounded*. For $E^2<m^2$ the orbit is always *bounded*, that is, the particle cannot reach infinity.

To study qualitative characteristics of the motion of test particles in the metric (2.1), it is convenient to use the *effective potential*. Let us write $\mathcal{X}$ as

$$\mathcal{X}=\alpha E^2-2\beta E+\gamma, \quad (4.1)$$

where

$$\alpha=\Delta x+r_0^2(x+a^2)(x+b^2), \quad (4.2)$$

$$\beta=-r_0^2(x+a^2)(x+b^2)\left(\frac{a\Phi}{x+a^2}+\frac{b\Psi}{x+b^2}\right), \quad (4.3)$$

$$\gamma=\Delta\left[-m^2x+(a^2-b^2)\left(\frac{\Phi^2}{x+a^2}-\frac{\Psi^2}{x+b^2}\right)-K\right]+r_0^2(x+a^2)\times(x+b^2)\left[\frac{a\Phi}{x+a^2}+\frac{b\Psi}{x+b^2}\right]^2. \quad (4.4)$$

The radial turning points $\mathcal{X}=0$ are defined by the condition $E=V_\pm(x)$, where

$$V_\pm=\frac{\beta\pm\sqrt{\beta^2-\alpha\gamma}}{\alpha}. \quad (4.5)$$

The quantities $V_\pm$ are called the *effective potentials*. They are functions of x and the integrals of motion Φ, Ψ, and K.

The limiting values of the effective potentials $V_\pm$ at infinity and at the horizon are

$$V_\pm(x=\infty)=\pm m, \quad V_\pm(x_+)=\Phi\Omega_a+\Psi\Omega_b. \quad (4.6)$$

The motion of a particle with energy E is possible only in the regions where either $E\geqslant V_+$ or $E\leqslant V_-$. The expression (4.5) remains invariant under the transformations $E\to -E$, $\Phi\to -\Phi$, $\Psi\to -\Psi$ relating these regions. In the absence of rotation, $a=b=0$, the second region $E\leqslant V_-$ is excluded.

B. Motion in the θ direction

Consider now motion in the θ direction. Since $\Theta\geqslant 0$, bounded motion with $E^2<m^2$ is possible only if $K\geqslant 0$. For bounded motion and $a\neq 0$, $b\neq 0$, $K>0$. For $K>0$ there exist both bounded and unbounded trajectories. A particle can reach the subspace $\theta=0$ only if $\Phi=0$ and the subspace $\theta=\pi/2$ only if $\Psi=0$. For $\Psi=0$ the orbit is in the $\theta=\pi/2$ plane if $K=\Phi^2-(E^2-m^2)b^2$. Similarly, for $\Phi=0$ the orbit is in the $\theta=0$ plane if $K=\Psi^2-(E^2-m^2)a^2$.

A special class of motion is the case when particles are moving quasiradially along trajectories on which the value of the angle θ remains constant, $\theta=\theta_0$. The relation between the integrals of motion corresponding to this type of motion can be found by solving simultaneously the equations

$$\Theta(\theta_0)=\left.\frac{d\Theta}{d\theta}\right|_{\theta_0}=0. \quad (4.7)$$

These equations are of the form

$$(E^2-m^2)(a^2\cos^2\theta_0+b^2\sin^2\theta_0)-\frac{\Phi^2}{\sin^2\theta_0}-\frac{\Psi^2}{\cos^2\theta_0}+K=0, \quad (4.8)$$

$$(E^2-m^2)(b^2-a^2)+\frac{\Phi^2}{\sin^4\theta_0}-\frac{\Psi^2}{\cos^4\theta_0}=0. \tag{4.9}$$

In the second equation we excluded the cases $\theta_0=0$ and $\theta_0=\pi/2$.

C. Circular orbits

A characteristic property of the four-dimensional gravitational field is the existence of bounded orbits located in the exterior of the black hole. For such orbits there are two turning points, corresponding to the minimum and maximum of the radial coordinate for the particle trajectory. This means that the line $E=$const crosses the effective potential $V(r)$ curve at two points, r_1 and r_2. Between these points $E>V(r)$. The function $V(r)$ has a minimum at $r_1<r_{min}<r_2$. For the special case when $E=V(r_{min})$ the orbit is circular. Thus the existence of stable circular orbits is a necessary condition for the existence of bounded orbits located in the exterior of the black hole.

It is well known that there are no stable circular orbits in the Newtonian gravity in a space-time with more than three spatial dimensions. This is also true in general relativity. That is, the Schwarzschild metric

$$ds^2=-Fdt^2+\frac{dr^2}{F}+r^2d\Omega_{k+2}^2,\quad F=1-\left(\frac{r_0}{r}\right)^{k+1}, \tag{4.10}$$

where $d\Omega_{k+2}^2$ is the metric on a unit $(k+2)$-dimensional sphere, does not allow stable circular orbits. The effective potential for the radial motion of a particle with mass m in this field is

$$V_S^2(r)=-F\left(1+\frac{L^2}{m^2r^2}\right), \tag{4.11}$$

where L is the total angular momentum. It is easy to see that this potential does not have any minima in the interval (r_0,∞).

With a change of variables $y=r_0^2/r^2$ (y goes from 0 to 1) we can write

$$V_S^2(y)=(1-y^{(1+k)/2})\left(1+\frac{L^2}{r_0^2m^2}y\right). \tag{4.12}$$

The second derivative of the function $V_S^2(y)$ with respect to y is

$$\frac{\partial^2(V_S^2(y))}{\partial y^2}=-\frac{k^2-1}{4}y^{(k-3)/2}-\frac{(k+3)(k+1)L^2}{4r_0^2m^2}y^{(k-1)/2}. \tag{4.13}$$

This function is zero only for $y\leqslant 0$, which means that V_S^2 cannot have more than one extremum for positive y. Note also that $V_S^2(y)$ takes the value 1 at $y=0$ and the value 0 at $y=1$. Since $V_S^2(y)$ is non-negative, we conclude that either it is a monotonically decreasing function or it has one maximum in this interval without possibility of having a minimum. The new variable y is a monotonic function of r and the original function $V_S^2(r)$ [and therefore $V_S(r)$] does not have a minimum either. Thus, the potential $V_S(r)$ does not allow any motion that is confined in a finite interval of the radial coordinate.

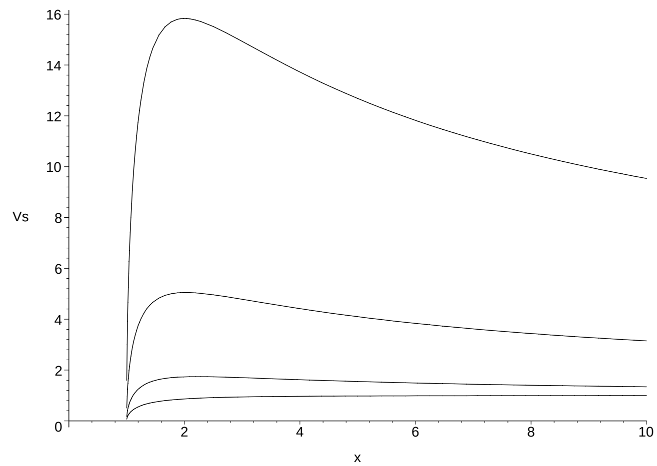

FIG. 1. The effective potential V_S for radial motion in the five-dimensional Schwarzschild space-time for several values of the angular momentum L. We set $m=r_0=1$ and the four different curves correspond to $L^2=1$, 10, 100, and 1000 from the bottom up.

The other way to see that this potential does not support any stable circular orbits is to note that the second derivative (4.13) is always negative for $y>0$ and arbitrary k. A similar conclusion can be derived if we consider an effective potential for angular motion [15].

Figure 1 shows a plot of the potential V_S for the five-dimensional Schwarzschild space-time for several values of the angular momentum L.

We shall prove now that stable circular orbits do not exist in the Myers-Perry metric, at least in the case when these orbits are in the "equatorial" planes, $\theta=0$ and $\theta=\pi/2$. We give the proof for the case $\theta=\pi/2$, and the other case is similar.

Circular orbits $x=x_0=$const are defined by the equations

$$\mathcal{X}=0,\quad \frac{\partial\mathcal{X}}{\partial x}=0, \tag{4.14}$$

where $\mathcal{X}$ is given by Eq. (3.18). For orbits in the plane $\theta=\pi/2$ one has $\Psi=0$ and $K=\Phi^2-(E^2-m^2)b^2$ (see the previous subsection). Substituting these relations into Eq. (4.14), we obtain a system of quadratic equations[2] for the variables E and Φ. The solutions are

$$E_\sigma=\sigma E\quad \Phi_\sigma=\sigma\Phi,\quad \sigma=\pm, \tag{4.15}$$

[2]This method of analyzing circular orbits is similar to the method used by Bardeen, Press, and Teukolsky [16] for a four-dimensional Kerr metric.

$$\frac{E}{m}=\frac{y}{\sqrt{y^2-y_0^2}}, \tag{4.16}$$

$$\frac{\Phi}{m}=\frac{(y+a^2-b^2-r_0^2)r_0}{\sqrt{y^2-y_0^2}}, \tag{4.17}$$

where

$$y=x\mp r_0 a+b^2-r_0^2, \quad y_0=r_0\sqrt{(r_0\pm a)^2-b^2}. \tag{4.18}$$

Since the constants E and Φ are to be real, the expressions under the square root in the denominators of Eqs. (4.16) and (4.17) are to be positive. This happens when

$$y^2>y_0^2. \tag{4.19}$$

But for these values of y we have $E^2>m^2$ and the potential must allow unbounded motion. This means that if the minimum of the effective potential exists, there must also be at least one maximum. But the equation for the second derivative

$$\frac{\partial^2\mathcal{X}}{\partial x^2}=2[(E^2-m^2)(3x+a^2+2b^2)-\Phi^2+r0^2m^2]=0 \tag{4.20}$$

has only one solution for x, and this situation is impossible. Thus, we conclude that there are no stable circular orbits around the rotating five-dimensional Myers-Perry black hole, at least in the ''equatorial'' planes.

One may conjecture that the absence of bounded stable orbits in the black hole exterior is a generic property of higher dimensional black holes.

D. Principal null congruences

We consider now null rays moving along $\theta_0=$const surfaces. One can use the conditions (4.8) and (4.9) (with $m=0$) to determine two of the constants of motion. After this a general solution is specified by E, K, θ_0, and t_0, ϕ_0, ψ_0. The last 3 constants are required as the initial data for three Killing variables t, ϕ, and ψ. Let us note that the parameter E is not important. It can be ''gauged away'' be rescaling of the affine parameter $\lambda\rightarrow E\lambda$. Thus, we have a five-parameter family of null geodesics. In order to have a congruence of null geodesics one needs to fix one more parameter. The following special choice is very convenient:

$$\Phi=-Ea\sin^2\theta_0, \quad \Psi=-Eb\cos^2\theta_0. \tag{4.21}$$

For this choice, Eq. (4.9) is automatically satisfied, while Eq. (4.8) gives

$$K=E^2(a^2-b^2)(\sin^2\theta_0-\cos^2\theta_0). \tag{4.22}$$

The null geodesics from this congruence are uniquely specified by the parameters θ_0, t_0, ϕ_0, and ψ_0. The null vectors tangent to the geodesics from this congruence are

$$L_\pm^\mu\partial_\mu=\frac{(x+a^2)(x+b^2)}{\Delta}\left[\partial_t-\frac{a}{x+a^2}\partial_\phi-\frac{b}{x+b^2}\partial_\psi\right]\pm 2\sqrt{x}\partial_x. \tag{4.23}$$

The two different congruences differ by the choice of sign in Eq. (3.27). By analogy with similar congruences in the four-dimensional Kerr geometry, we call the congruences generated by $L_\pm^\mu$ *principal null congruences*.

By using the GRTENSOR program one can check that both of the null principal congruences obey the condition

$$L_\pm[{}_\alpha C_\beta]\gamma\delta\epsilon L_\pm^\gamma L_\pm^\delta=0, \tag{4.24}$$

where $C_{\beta\gamma\delta\epsilon}$ is the Weyl tensor. In the four- dimensional case, a similar condition means that the space-time is algebraically special and belongs to the Petrov class D (or more degenerate).[3]

One can check that the shear $\sigma_\pm$ defined by Eq. (A6) for $L_\pm^\gamma$ is

$$\sigma_\pm=\sqrt{\frac{2}{3}}\frac{(a^2\cos^2\theta+b^2\sin^2\theta)}{\rho^2\sqrt{x}}, \tag{4.25}$$

and it does not vanish.[4] To perform the calculation of σ we used the GRTENSOR program. Direct calculations based on relation (A5) of the Appendix result in very long expressions, so that the standard GRTENSOR simplification procedure does not work. We found that calculations based on the second form of the expression for σ in Eq. (A6) are much shorter and allow one to arrive at the result (4.25) much faster.

The principal null geodesic congruences (4.23) were used in [10] to establish relations between Boyer-Lindquist and Kerr-Schild coordinates for the Myers-Perry metric.

Similarly to the four-dimensional Kerr metric both of the null principal vectors $L_\pm^\mu$ are eigen vectors of the tensor $\xi_{(t)\mu;\nu}$, where $\xi_{(t)}^\mu\partial_\mu=\partial_t$ is the Killing vector, which is time like at infinity. That is the following relations are valid:

$$\xi_{(t)\mu;\nu}L_\pm^\nu=\kappa_\pm L_{\pm\mu}, \tag{4.26}$$

where

$$\kappa_\pm=\pm\sqrt{\frac{x\quad r_0^2}{\rho^4}}. \tag{4.27}$$

[3] Petrov classification for five-dimensional metrics of Euclidean signature was given in [17]. The classification of higher dimensional space-times and its relation to the existence of principal null geodesics is an interesting problem. In our paper we use the term ''algebraically special'' only in the sense that the space-time allows principal null geodesics.

[4] In the four-dimensional case the following theorem proved by Goldberg and Sachs [18] is valid: If a vacuum gravitational field is algebraically special then principal null congruences must be shear-free. The result (4.25) indicates that this theorem might not be valid in the five-dimensional case.

V. CONCLUSIONS

We discussed the motion of particles and light in the space-time of a five-dimensional rotating black hole. There is an intriguing similarity of this problem with the case of the four-dimensional Kerr metric. In both cases the Hamilton-Jacobi equations for particles and light allow separation of variables. This property follows from the existence of a Killing tensor in addition to the Killing vectors.

We described different types of motion of particles and light in the background of a five-dimensional rotating black hole, including some interesting special cases (like radial motion and motion with constant θ). In many aspects the qualitative properties of different types of motion are similar in four and five dimensions. Both four- and five-dimensional metrics are algebraically special and allow two families of principal null congruences. In both cases these congruences are geodesic. The principal null rays are also eigenvectors of the tensor $\xi_{\mu;\nu}$. However, there are some differences. While in four dimensions the principal null congruences are shear-free, in five dimensions this is not the case. Also, in four dimensions there exist stable circular orbits around the rotating black hole, while for the five-dimensional Myers-Perry black hole they are absent, at least in the "equatorial" planes.

It would be interesting to generalize these results to rotating black holes in arbitrary numbers of dimensions.

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APPENDIX: OPTICAL SCALARS FOR NULL CONGRUENCES IN A HIGHER DIMENSIONAL SPACE-TIME

Consider a congruence of null geodesics in n-dimensional space-time and let l^μ be a tangent vector, then

$$l^\mu l_\mu = 0, \qquad l^\mu{}_{;\nu} l^\nu = 0, \qquad l^\mu{}_{;\nu} l_\mu = 0. \tag{A1}$$

Denote by Σ the $(n-1)$-dimensional null plane which is spanned by vectors z^μ obeying the condition $z^\mu l_\mu = 0$. Let us choose another null vector n^ν, normalized by the condition $n^\mu l_\mu = -1$, and denote by S the $(n-2)$-dimensional subspace of Σ formed by vectors that are orthogonal to both null vectors. Denote by $h_{\mu\nu}$ a projector onto S:

$$h_{\mu\nu} = g_{\mu\nu} + 2 l_{(\mu} n_{\nu)}. \tag{A2}$$

One has

$$h_{\mu\nu} l^\nu = h_{\mu\nu} n^\nu = 0, \qquad h_\mu{}^\alpha h_{\alpha\nu} = h_{\mu\nu}. \tag{A3}$$

As usual, let us decompose $l_{\mu;\nu}$ as follows:

$$l_{\mu;\nu} = l_{[\mu;\nu]} + \sigma_{\mu\nu} + \frac{1}{n-2} \vartheta h_{\mu\nu}, \tag{A4}$$

where

$$\vartheta = l^\alpha{}_{;\alpha} = h^{\alpha\beta} l_{\alpha;\beta}, \qquad \sigma_{\mu\nu} = l_{(\mu;\nu)} - \frac{1}{n-2} \vartheta h_{\mu\nu}. \tag{A5}$$

The parameters of twist, $\hat{\rho}$, and shear, σ, are defined as follows:

$$\hat{\rho}^2 = l_{[\mu;\nu]} l^{[\mu;\nu]},$$

$$\sigma^2 = \sigma_{\mu\nu}\sigma^{\mu\nu} = h^{\alpha\mu} h^{\beta\nu}\, l_{(\alpha;\beta)} l_{(\mu;\nu)} - \frac{1}{n-2} \vartheta^2. \tag{A6}$$

It should be emphasized that the choice of the null vector n^μ Contains the ambiguity $n^\mu \to n^\mu + z^\mu$ where z^μ is any vector from Σ. Under this transformation $h_{\mu\nu} \to h_{\mu\nu} + 2 l_{(\mu}\, z_{\nu)}$, while the expansion ϑ, twist $\hat{\rho}$, and shear σ remain invariant.

In a space-time with two principal null geodesic congruences $L^\mu_\pm$, the natural choice of the projector $h_{\mu\nu}$ is

$$h_{\mu\nu} = g_{\mu\nu} - 2\alpha L_{+(\mu} L_{-\nu)}, \qquad \alpha = (L_{+\mu} L_-{}^\mu)^{-1}. \tag{A7}$$

For the five-dimensional metric (2.1) and the principal null vectors (4.23) one has

$$\alpha = \frac{-\Delta}{2x\rho^2}, \tag{A8}$$

$$h^{\mu\nu} = \frac{1}{x\rho^2} \times \begin{bmatrix} x(a^2 \sin^2\theta + b^2 \cos^2\theta) + a^2 b^2 & 0 & 0 & -a(x+b^2) & -b(x+a^2) \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 0 \\ -a(x+b^2) & 0 & 0 & \dfrac{x + b^2 \sin^2\theta}{\sin^2\theta} & ab \\ -b(x+a^2) & 0 & 0 & ab & \dfrac{x + a^2\cos^2\theta}{\cos^2\theta} \end{bmatrix}. \tag{A9}$$

[The matrix $h^{\mu\nu}$ is written in the basis $(t, x, \theta, \phi, \psi)$.]

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