

Critical collapse of a cylindrically symmetric scalar field in four-dimensional Einstein's theory of gravity

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Four-dimensional cylindrically symmetric spacetimes with homothetic self-similarity are studied in the context of Einstein's theory of gravity, and a class of exact solutions to the Einstein-massless scalar field equations is found. Their local and global properties are investigated and it is found that they represent gravitational collapse of a massless scalar field. In some cases the collapse forms black holes with cylindrical symmetry, while in the other cases it does not. The linear perturbations of these solutions are also studied and given in closed form. From the spectra of the unstable eigenmodes, it is found that there exists one solution that has precisely one unstable mode, which may represent a critical solution, sitting on a boundary that separates two different basins of attraction in the phase space.

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I. INTRODUCTION

The studies of nonlinearity of the Einstein field equations near the threshold of black hole formation reveal very rich phenomena [1], which are quite similar to critical phenomena in statistical mechanics and quantum field theory [2]. In particular, by numerically studying the gravitational collapse of a massless scalar field in $(3+1)$ -dimensional spherically symmetric spacetimes, Choptuik found that the mass of such formed black holes takes the form

$$M_{BH} = C(p)(p - p^*)^\gamma, \quad (1.1)$$

where $C(p)$ is a constant and depends on the initial data, and p parametrizes a family of initial data in such a way that when $p > p^*$ black holes are formed, and when $p < p^*$ no black holes are formed. It was shown that, in contrast with $C(p)$, the exponent γ is *universal* to all the families of initial data studied, and was numerically determined as $\gamma \sim 0.37$. The solution with $p = p^*$, usually called the critical solution, is found also universal. Moreover, for the massless scalar field it is periodic, too. Universality of the critical solution and the exponent γ as well as the power-law scaling of the black hole mass all have given rise to the name *critical phenomena in gravitational collapse*.

Choptuik's studies were soon generalized to other matter fields [3]. From all the work done so far, the following seems clear:

(a) There are two types of critical collapse, depending on whether or not the black hole mass takes the scaling form (1.1). When it takes the form, the corresponding collapse is called type II collapse, and when it does not it is called type I collapse. In the type II collapse, all the critical solutions found so far have either discrete self-similarity (DSS) or homothetic self-similarity (HSS), depending on the matter

fields. In the type I collapse, the critical solutions have neither DSS nor HSS. For certain matter fields, these two types of collapse can coexist.

(b) For the type II collapse, the corresponding exponent is universal only with respect to certain matter fields. Usually, different matter fields have different critical solutions and different exponents. But for a given matter field the critical solution and the exponent are universal.¹

(c) A critical solution for both the two types has *one and only one unstable mode*. This now is considered as one of the main criteria for a solution to be critical.

(d) The universality of the exponent is closely related to the number of unstable modes. In fact, the unstable mode, say, k_1 , of the critical solution is related to the exponent γ via the relation $\gamma = |k_1|^{-1}$, which can be obtained by using dimensional analysis [4].

From the above, one can see that to study critical collapse, one may first find some particular solutions by imposing certain symmetries, such as DSS or HSS. This can simplify the problem considerably. For example, in the spherically symmetric case, by imposing HSS symmetry the Einstein field equations will be reduced from partial differential equations to ordinary differential equations. Once the particular solutions are known, one can study their linear perturbations and find out the spectrum of the corresponding eigenmodes. If a solution has precisely one unstable mode, it may represent a critical solution, sitting on a boundary that separates two different basins of attraction in the phase space.

The studies of critical collapse have been mainly numerical so far, and analytical ones are still highly hindered by the complexity of the problem, even after imposing some symmetries. Lately, some progress has been achieved in the studies of critical collapse of a massless scalar field in an anti-de Sitter background in $(2+1)$ -dimensional spacetimes both

¹So far, the studies have been mainly restricted to spherically symmetric cases and their nonspherical linear perturbations [3]. Therefore, it is not really clear whether or not the critical solution and exponent are universal with respect to different symmetries of the spacetimes.

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numerically [5,6] and analytically [7–9]. This serves as the first analytical model in critical collapse.

In this paper, we shall present another analytical model that represents critical collapse of a massless scalar field in four-dimensional Einstein’s theory of gravity with cylindrical symmetry. Although spacetimes with cylindrical symmetry do not represent realistical models, the studies of them can provide deep insight into the nonlinearity of the Einstein field equations. In particular, they may throw some light on the possible roles that gravitational radiation and angular momentum may play in critical collapse. In fact, such studies have already been shown to be very useful in probing non-spherical gravitational collapse [10]. In addition, they may also provide a useful testbed for numerical relativity [11] and quantum gravity [12].

The rest of the paper is organized as follows. In Sec. II we first review the regularity conditions for a four-dimensional cylindrical spacetime, including the ones at the symmetry axis. Then we introduce the notion of homothetic self-similarity with cylindrical symmetry. In Sec. III, a class of exact solutions with such a symmetry to the Einstein-massless scalar field equations is presented. It is shown that they represent gravitational collapse of a scalar field, in which black holes can be formed. In Sec. IV, the linear perturbations of these solutions are studied and given in closed form. After properly imposing boundary conditions, the spectra of the unstable modes of the perturbations are determined. In particular, it is found that there exists a solution that has precisely one unstable mode, which may represent a critical solution, sitting on a boundary that separates two different basins of attraction in the phase space. In Sec. V, the main results are summarized and some concluding remarks are given. There are also two appendixes, A and B. In Appendix A, the Ricci tensor is given in terms of self-similar variables. The linear terms of perturbations of the Ricci tensor are also given there. In Appendix B, the expansions of the outgoing and ingoing radial null geodesics are calculated, from which trapped surfaces and apparent horizons are defined.

II. SPACETIMES WITH HOMOTHETIC SELF-SIMILARITY

The general metric for cylindrical spacetimes with two hypersurface orthogonal Killing vectors takes the form [13]

$$ds^2 = e^{-M(t,r)}(dt^2 - dr^2) - r^2 e^{-S(t,r)} \times (e^{V(t,r)} dw^2 + e^{-V(t,r)} d\theta^2), \tag{2.1}$$

where $x^\mu = \{t, r, w, \theta\}$ are the usual cylindrical coordinates, and the hypersurfaces $\theta = 0, 2\pi$ are identified. The two Killing vectors are given by $\xi_{(w)} = \partial_w$ and $\xi_{(\theta)} = \partial_\theta$. To have cylindrical symmetry, some physical and geometrical conditions needed to be imposed. In general this is not trivial. As a matter of fact, when the symmetry axis is singular, it is still an open question: which conditions should be imposed [14]. Since in this paper we are mainly interested in gravitational collapse, we would like to have the axis regular in the beginning of the collapse. By this way, we are sure that the

singularity to be formed later on the axis is indeed due to the collapse. Thus, following [15] we impose the following conditions:

(i) *There must exist a symmetry axis.* This can be written as

$$X \equiv |\xi_{(\theta)}^\mu \xi_{(\theta)}^\nu g_{\mu\nu}| \rightarrow 0, \tag{2.2}$$

as $r \rightarrow 0^+$, where we have chosen the radial coordinate r such that the axis is located at $r = 0$.

(ii) *The spacetime near the symmetry axis is locally flat.* This can be expressed as [13]

$$\frac{X_{,\alpha} X_{,\beta} g^{\alpha\beta}}{4X} \rightarrow -1, \tag{2.3}$$

as $r \rightarrow 0^+$, where $(\)_{,\alpha} \equiv \partial(\)/\partial x^\alpha$. Note that solutions failing to satisfy this condition it is sometimes acceptable, and are usually expected that the singularities located on the axis should be replaced by some kind of sources in more realistic models. A particular case of these is when the right-hand side of the above equation approaches a finite constant, and the singularity now can be related to a linelike source [16]. In this paper, since we are mainly interested in gravitational collapse, we shall not consider these possibilities and assume that the above condition holds strictly at the initial of the collapse.

(iii) *No closed timelike curves (CTC’s).* In spacetimes with cylindrical symmetry, CTC’s can be easily introduced. To guarantee their absence, we impose the condition

$$\xi_{(\theta)}^\mu \xi_{(\theta)}^\nu g_{\mu\nu} < 0, \tag{2.4}$$

in the whole spacetime.

In addition to these conditions, it is usually also required that the spacetime be asymptotically flat in the radial direction. However, since we consider solutions with self-similarity, this condition cannot be satisfied by such solutions, unless we restrict the validity of them only up to a maximal radius, say, $r = r_0(t)$, and then join the solutions with others in the region $r > r_0(t)$, which are asymptotically flat in the radial direction. In this paper, we shall not consider such a possibility, and simply assume that the self-similar solutions are valid in the whole spacetime.

Spacetimes with *homothetic self-similarity* (or *self-similarity of the first kind*) are usually defined by the existence of a conform Killing vector ξ^μ that satisfies the equations [17],

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 2g_{\mu\nu}, \tag{2.5}$$

where a semicolon “;” denotes the covariant derivative. It can be shown that for the spacetimes given by Eq. (2.1) the conditions (2.5) imply that

$$M(t,r) = M(z), \quad S(t,r) = S(z), \quad V(t,r) = V(z), \tag{2.6}$$

where the self-similar variable z and the corresponding conform Killing vector ξ^μ are given by

$$\xi^\mu \partial_\mu = t \partial_t + r \partial_r, \quad z = \frac{r}{-t}. \quad (2.7)$$

It is interesting to note that under the coordinate transformations

$$t = a_1 \bar{t} + a_2 \bar{r}, \quad r = a_3 \bar{t} + a_4 \bar{r}, \quad (2.8)$$

the metric (2.1), the regular conditions (2.2)–(2.4), and the self-similar conditions (2.6) and (2.7) are all invariant, where a_i 's are real constants, subject to $a_1 a_2 - a_3 a_4 = 0$. Using this gauge freedom, we shall assume that

$$M(t, 0) = 0, \quad (2.9)$$

that is, the timelike coordinate t measures the proper time on the axis.

III. SELF-SIMILAR SOLUTIONS OF MASSLESS SCALAR FIELD

For a massless scalar field, the Einstein field equations read

$$R_{\mu\nu} = \kappa \phi_{,\mu} \phi_{,\nu}, \quad (3.1)$$

where $\kappa (\equiv 8\pi G/c^4)$ is the Einstein coupling constant. In this paper we shall choose units such that $\kappa = 1$. The scalar field satisfies the Klein-Gordon equation

$$g^{\alpha\beta} \phi_{;\alpha\beta} = 0. \quad (3.2)$$

However, this equation is not independent of the Einstein field equations (3.1) and can be obtained from the Bianchi identities $G_{\mu\alpha;\beta\gamma} g^{\alpha\beta} = 0$.

On the other hand, it can be shown that a massless scalar field $\phi(t, r)$ that is consistent with spacetimes with homothetic self-similarity must take the form

$$\phi(t, r) = 2q \ln(-t) + \varphi(z), \quad (3.3)$$

where q is an arbitrary constant, and $\varphi(z)$ is a function of z only, which will be determined by the Einstein field equations (3.1). Inserting Eqs. (A2) and (3.3) into Eq. (3.1) and considering the self-similar conditions (2.6), we find the following solutions:

$$\begin{aligned} M(z) &= 2q^2 \ln(1-z^2), & S(z) &= \ln(z), \\ V(z) &= -\ln(z), & \varphi(z) &= 0. \end{aligned} \quad (3.4)$$

When $q=0$ the corresponding spacetime is flat. Thus, in the following we shall assume that $q \neq 0$. Then, it can be shown that these solutions satisfy all the conditions (2.2)–(2.4) and (2.9), and the corresponding Ricci and Kretschmann scalars are given by

$$R = g^{\alpha\beta} \phi_{;\alpha} \phi_{;\beta} = 4q^2 \frac{(1-z^2)^{2q^2}}{t^2},$$

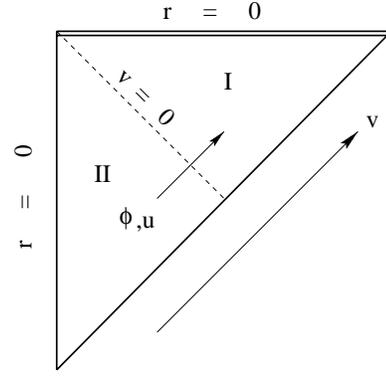


FIG. 1. The Penrose diagram for the solutions given by Eqs. (3.6) and (3.7) with $n \geq 2$ being an integer. The cylinders of constant t and r are all trapped in region I where $\Theta_t < 0$ and $\Theta_t \Theta_n > 0$, but not in region II, where $\Theta_t > 0$ and $\Theta_t \Theta_n < 0$. The dashed line $v=0$ represents the apparent horizon. When n is an odd integer, the spacetime is singular on the horizontal double line $r=0$, and when n is an even integer, the spacetime has no curvature singularity there.

$$I \equiv R^{\alpha\beta\lambda\sigma} R_{\alpha\beta\lambda\sigma} = 48q^4 \frac{(1-z^2)^{4q^2}}{t^4}. \quad (3.5)$$

From these expressions we can see that the spacetime is singular on the hypersurface $t=0$. On the other hand, although the metric is singular on $z=1$, the spacetime is not. Thus, to have a geodesically complete spacetime, we need to extend the metric beyond this surface. In order to do so, it is found convenient to study the two cases $0 < 2q^2 < 1$ and $2q^2 \geq 1$ separately.

A. $0 < 2q^2 < 1$

In this case, introducing two null coordinates u and v via the relations

$$\begin{aligned} t &= -[(-u)^n + (-v)^n] \equiv -f_+(u, v), \\ r &= (-u)^n - (-v)^n \equiv f_-(u, v), \end{aligned} \quad (3.6)$$

we find that in terms of u and v the metric and massless scalar field take the form

$$\begin{aligned} ds^2 &= n^2 4^{1/n} f_+^{2(n-1)/n} du dv - f_+^2 dw^2 - f_-^2 d\theta^2, \\ \phi &= 2q \ln[f_+(u, v)], \end{aligned} \quad (3.7)$$

where

$$n \equiv \frac{1}{1-2q^2} > 1. \quad (3.8)$$

From Eq. (3.6) we can see that the region $t \leq 0$, $r \geq 0$, $z < 1$ in the (t, r) plane is mapped into the region $u, v < 0$, $v \geq u$, which will be referred to as region II [cf. Fig. 1]. The half line $z=1$, $t \leq 0$ is mapped to $v=0$, $u \leq 0$. The region $u > 0$, $u \leq 0$, which will be referred to as region I, is an

extended region. Depending on the values of n , the nature of the extension is different. In particular, it is analytical only for the case where n is an integer. Otherwise, the extension is not analytical, and in some cases the metric and the scalar field even become not real in this extended region, as we can see from Eqs. (3.6) and (3.7). To have the extension unique, in the following we shall consider only analytical extensions, that is, the cases where n is an integer. Then, from Eqs. (3.7) and (B7) we find that

$$\begin{aligned}\phi_{,u} &= -2nq \frac{(-u)^{n-1}}{f_+}, & \phi_{,v} &= -2nq \frac{(-v)^{n-1}}{f_+}, \\ R &= \phi_{,\alpha} \phi^{,\alpha} = n^4 q^2 4^{1+1/n} \frac{(uv)^{n-1}}{f_+^{2/n}}, \\ I &= R^{\alpha\beta\lambda\sigma} R_{\alpha\beta\lambda\sigma} = 16^{2-1/n} \left(\frac{n-1}{n}\right)^2 \frac{(uv)^{2(n-1)}}{f_+^{4(2-1/n)}}, \\ \Theta_l &= \frac{4^{1-1/n} (-v)^{2n-1}}{nf_- f_+^{2(2n-1)/n}}, & \Theta_n &= -\frac{4^{1-1/n} (-u)^{2n-1}}{nf_- f_+^{2(2n-1)/n}}.\end{aligned}\quad (3.9)$$

From these expressions we can see that the spacetime is regular on the symmetry axis $v=u$ in region II, and $\phi_{,\alpha}$ is always timelike. On the hypersurface $v=0$, we have

$$\phi_{,v}(u,0) = 0, \quad (3.10)$$

and the only non-vanishing component of the energy-momentum tensor $T_{\mu\nu}$ is given by

$$T_{uu}(u,0) = \phi_{,u}^2(u,0) \neq 0, \quad (3.11)$$

which represents an energy flow, moving from region II into region I along the null hypersurfaces $u=\text{const}$. The expansion Θ_l of the null geodesics along the hypersurfaces $u=\text{const}$ is always positive in this region, while the expansion Θ_n of the null geodesics along the hypersurfaces $v=\text{const}$ is always negative. However, Θ_l becomes zero on the hypersurface $v=0$ and then negative in the extended region, I , where $v>0$, while Θ_n is negative even in this extended region. Thus, all the cylinders of constant t and r are trapped in the extended region, but not in region II. Then, the hypersurface $v=0$ defines an apparent horizon [18,19].

It should be noted that the above analysis is very important when we consider boundary conditions on the apparent horizon in the next section, as it shows clearly that it is the component $\phi_{,u}$ that represents the energy flow of the scalar waves that moves from region II into region I, while the component $\phi_{,v}$ represents the energy flow of the scalar field that moves in the opposite direction. Since now region I is a trapped region and no radiation is able to escape from this region. This can be seen clearly from Eq. (3.10).

The singularity behavior in region I depends on the values of n . In particular, when n is an odd integer, from Eq. (3.9) we can see that the spacetime becomes singular on the hypersurface $r=0$ or $u=-v$, which services as the up bound-

ary of the spacetime, and the corresponding Penrose diagram is that of Fig. 1. Thus, in this case region I can be considered as the interior of a black hole, and the corresponding solutions represent gravitational collapse of a massless scalar field in region II. The collapse always forms a black hole. Note that in this case $\phi_{,\alpha}$ is continuously timelike in the trapped region, I , as we can see from Eq. (3.9).

When n is an even integer, from Eq. (3.9) we can see that the Ricci and Kretschmann scalars are all finite at $r=0$ in region I, but both Θ_l and Θ_n become singular there. Thus, anything that moves along the null geodesics, defined by l^μ or n^μ , will be crashed to zero volume by the infinitely large contraction. Then, the hypersurface $r=0$ now represents a topological boundary of the spacetime, and the corresponding Penrose diagram is also given in Fig. 1, but now the spacetime is free of curvature singularities on the double horizontal line $r=0$.

It should be noted that apparent horizons and black holes are usually defined in asymptotically flat spacetimes [18]. To be distinguishable, Hayward called such apparent horizons as trapping horizons and defined black holes by the future outer trapping horizons [19]. For the sake of simplicity and without causing any confusions, in this paper we shall continuously use the notions of apparent horizons in the places of Hayward's trapping horizons, and define black holes in a little bit more general sense than that of Hayward in non-asymptotically flat spacetimes.

B. $2q^2 \geq 1$

In this case, introducing the two null coordinates u and v via the relations

$$t = u + v, \quad r = v - u, \quad (3.12)$$

we find that the metric and scalar field are given by

$$\begin{aligned}ds^2 &= 4^{1-2q^2} \left[\frac{(u+v)^2}{uv} \right]^{2q^2} dudv - (u+v)^2 dw^2 \\ &\quad - (u-v)^2 d\theta^2, \\ \phi &= 2q \ln[-(u+v)].\end{aligned}\quad (3.13)$$

To study the physics of the spacetime near the hypersurface $v=0$ or $z=1$ in some details, let us consider the radial null geodesics along the hypersurface $u=\text{const}$, say, $u=u_0$,

$$\ddot{v} - 2q^2 \frac{u_0 - v}{v(u_0 + v)} \dot{v}^2 = 0, \quad (3.14)$$

where an overdot denotes the ordinary differentiation with respect to the affine parameter λ along the null geodesics. Then, near the hypersurface $v=0$, Eq. (3.14) has the solution

$$v(\lambda) = \begin{cases} (b_1 \lambda + b_2)^{-1/(2q^2-1)}, & 2q^2 > 1 \\ e^{b_1 \lambda + b_2}, & 2q^2 = 1, \end{cases} \quad (3.15)$$

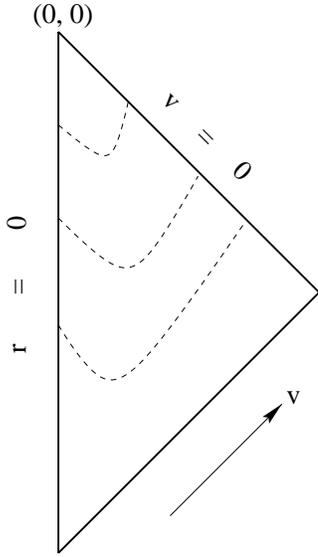


FIG. 2. The Penrose diagram for the solutions given by Eq. (3.13) for $2q^2 \geq 1$. The spacetime is geodesically maximal in the whole region $u, v < 0, v \geq u$, and singular only at $(u,v) = (0,0)$. The two surfaces of constant t and r are not trapped, because now we always have $\Theta_l > 0$ and $\Theta_n < 0$. The only exception is on the surface $v=0, u < 0$ where $\Theta_l(u,0) = 0$ and $\Theta_n(u,0) < 0$. The dashed lines represent the hypersurfaces $\phi(u,v) = \text{const}$, which are always spacelike.

where b_1 and b_2 are the integration constants. Thus, as $v \rightarrow 0$, we must have $\lambda \rightarrow \pm\infty$. That is, the “distance” between the point $(u_0,0)$ and any of the other points, say, $(u,v) = (u_0, v_0 < 0)$, along the null geodesics $u = u_0$ is infinite. Therefore, when $2q^2 \geq 1$ the hypersurface $v=0$ actually represents a natural boundary of the spacetime, and there is no need to extend the solutions beyond this surface, since now region II is already geodesically maximal. It should be noted that, although there is no spacetime singularity on the half line $v=0, u < 0$, the spacetime is singular at the point $(u,v) = (0,0)$, as can be seen from Eq. (3.9). The corresponding Penrose diagram is given in Fig. 2.

On the other hand, from Eq. (B7) we find that

$$\begin{aligned}\Theta_l &= -4^{1-2q^2} \left[\frac{(u+v)^2}{uv} \right]^{2q^2} \frac{v}{u^2-v^2}, \\ \Theta_n &= 4^{1-2q^2} \left[\frac{(u+v)^2}{uv} \right]^{2q^2} \frac{u}{u^2-v^2}.\end{aligned}\quad (3.16)$$

Thus, in the whole spacetime now we always have $\Theta_l > 0$ and $\Theta_n < 0$, except on the half hypersurface $v=0, u < 0$ where we have $\Theta_l = 0, \Theta_n < 0$. That is, all the two surfaces of constant t and r are not trapped for $u, v < 0$, and become marginally trapped only on the half surface $v=0, u < 0$. In addition, we have

$$R = \phi_{,\alpha} \phi^{,\alpha} = 4^{1+2q^2} q^2 \frac{(uv)^{2q^2}}{(u+v)^{2(1+2q^2)}}, \quad (3.17)$$

which is always positive for $u, v < 0$, and zero only when $v=0$. That is, the scalar field is always timelike, except on the hypersurface $v=0$ where it becomes null. Thus, in this case the corresponding solution can be considered as representing gravitational collapse of a massless scalar field. Although now no black holes are formed, a pointlike spacetime singularity is indeed developed at the point $(u,v) = (0,0)$, as we can see from Eq. (3.5). It is interesting to note that this singularity is not naked, and an observer can see it only when he/she arrives at that point.

IV. LINEAR PERTURBATIONS OF THE SELF-SIMILAR SOLUTIONS

To see if the above solutions represent critical collapse, we need to do their linear perturbations, because by definition a critical solution has one and only one unstable mode. To study such perturbations, it is found convenient to use the self-similar variables τ and z defined by Eq. (A1) but still work in the (t,r) coordinates. Then, the linear perturbations can be written as

$$F(\tau, z) = F_0(z) + \epsilon F_1(z) e^{k\tau}, \quad (4.1)$$

where $F \equiv \{M, S, V, \phi\}$, and ϵ is a very small real constant. Quantities with subscripts “1” denote perturbations, and those with “0” denote the background self-similar solutions given by Eq. (3.4). It is understood that there may be many perturbation modes for different values (possibly complex) of the constant k . Then, the general perturbations will be the sum of these individual ones. Modes with $\text{Re}(k) > 0$ grow as $\tau \rightarrow \infty$ and are referred to as unstable modes, and the ones with $\text{Re}(k) < 0$ decay and are referred to as stable modes.

It should be noted that in writing Eq. (4.1), we have already used some of the gauge freedom to write the perturbations such that they preserve the form of the metric (2.1). However, this does not completely fix the gauge. We shall return to this point later when we consider the gauge modes.

To the first order of ϵ , the Ricci tensor is given by Eqs. (A4)–(A10). Applying them to the background solutions given by Eq. (3.4), and using the Einstein field equations (3.1), we find that there are only four independent equations, which can be cast in the form

$$\begin{aligned}kM_1(z) &= 2z^2 S_1'' + 2zV_1' + 2z \left[(1+k) - 4q^2 \frac{z^2}{1-z^2} \right] S_1' + kV_1 \\ &\quad + k \left(1 - 4q^2 \frac{z^2}{1-z^2} \right) S_1 + 4qz\phi_1',\end{aligned}\quad (4.2)$$

and

$$\begin{aligned}z(1-z^2)V_1'' + [1 - (1+2k)z^2]V_1' - k^2zV_1 \\ = kzS_1 - (1-z^2)S_1',\end{aligned}\quad (4.3)$$

$$\begin{aligned}z(1-z^2)\phi_1'' + [1 - (1+2k)z^2]\phi_1' - k^2z\phi_1 \\ = 2qz(zS_1' + kS_1),\end{aligned}\quad (4.4)$$

$$z(1-z^2)S_1'' + 2(1-kz^2)S_1' + k(1-k)zS_1 = 0, \quad (4.5)$$

where a prime denotes the ordinary differentiation with respect to the indicated argument. It can be shown that Eq. (4.5) has the general solution,

$$S_1(z) = \frac{1}{z} [c_1(1+z)^{2-k} + c_2(1-z)^{2-k}], \quad (4.6)$$

where c_1 and c_2 are two integration constants. Substituting the above solution into Eqs. (4.3) and (4.4) we find that these two equations can be written in the form

$$y(1-y) \frac{d^2 Z_i}{dy^2} + [e - (a+b+1)y] \frac{dZ_i}{dy} - abZ_i = \frac{1}{4y^{1/2}} f_i(y), \quad (4.7)$$

where $y \equiv z^2$, $\{Z_i\} = \{V_1, \varphi_1\}$, $a = b = k/2$, $e = 1$, and

$$\begin{aligned} f_1(y) &\equiv kzS_1 - (1-z^2)S_1' \\ &= \frac{1}{y} \{c_1[1 - (2-k)y^{1/2} + y](1+y^{1/2})^{2-k} \\ &\quad + c_2[1 + (2-k)y^{1/2} + y](1-y^{1/2})^{2-k}\}, \\ f_2(z) &\equiv 2qz(zS_1' + kS_1) \\ &= -2q\{c_1[(1-k) - y^{1/2}](1+y^{1/2})^{1-k} \\ &\quad + c_2[(1-k) + y^{1/2}](1-y^{1/2})^{1-k}\}. \end{aligned} \quad (4.8)$$

Equation (4.7) is the inhomogeneous hypergeometric equation [20], and the general solution of the associated homogeneous equation is a linear combination of the two independent solutions, $F_1^{(i)}(z)$, where

$$\begin{aligned} F_1^{(1)}(z) &= F\left(\frac{1}{2}k, \frac{1}{2}k; 1; z^2\right), \\ F_1^{(2)}(z) &= F\left(\frac{1}{2}k, \frac{1}{2}k; k; 1-z^2\right), \end{aligned} \quad (4.9)$$

with $F(a, b; c; z)$ denoting the hypergeometric function. From the above two independent solutions, we can construct particular solutions of the inhomogeneous equation (4.7), and then find that the general solutions for $V_1(z)$ and $\varphi_1(z)$ can be written as

$$\begin{aligned} V_1(z) &= [a_1^{(2)} + A_1^{(2)}(z)]F_1^{(1)}(z) + [a_1^{(1)} - A_1^{(1)}(z)]F_1^{(2)}(z), \\ \varphi_1(z) &= [a_2^{(2)} + A_2^{(2)}(z)]F_1^{(1)}(z) + [a_2^{(1)} - A_2^{(1)}(z)]F_1^{(2)}(z), \end{aligned} \quad (4.10)$$

where $a_j^{(i)}$'s are integration constants, and

$$A_j^{(i)}(z) \equiv \int^z \frac{f_j(z)F_1^{(i)}(z)dz}{z(1-z^2)\Delta(z)},$$

$$\Delta(z) \equiv F_1^{(2)}(z) \frac{d}{dz} [F_1^{(1)}(z)] - F_1^{(1)}(z) \frac{d}{dz} [F_1^{(2)}(z)]. \quad (4.11)$$

To have physically acceptable perturbations, we need to impose boundary conditions. In general relativity, this is a very subtle problem and there are no fixed rules to follow. In this paper we shall choose the axis $r=0$ and the hypersurface $z=1$ as the places where we impose the boundary conditions.

Since the axis for the background solutions is regular, and the conditions (2.2), (2.3) and (2.9) are satisfied by them, we would expect that the linear perturbations also satisfy these conditions. In particular, it can be shown that the condition (2.9) requires $M_1(0)=0$, and the ones (2.2) and (2.3) require, respectively, $G_1(z) \rightarrow 0$ and $zG_1'(z) \rightarrow 0$, as $z \rightarrow 0$, where $G_1(z) \equiv S_1(z) + V_1(z)$. On the other hand, the condition that $R = \phi_{,\alpha}\phi^{,\alpha}$ is regular on the axis further requires $z\varphi_1' + k\varphi_1 \rightarrow 0$.² In summary, on the axis we shall impose the following conditions:

$$M_1(z) \rightarrow 0,$$

$$G_1(z) \rightarrow 0,$$

$$zG_1'(z) \rightarrow 0,$$

$$z\varphi_1'(z) + k\varphi_1(z) \rightarrow 0, \quad (4.12)$$

as $z \rightarrow 0$.

On the other hand, when $0 < 2q^2 < 1$ the hypersurface $z=1$ is an apparent horizon (cf. Fig. 1), and we required that the background solutions be analytical across it with respect to the null coordinate v . Otherwise, it was found that the extension was not unique. Clearly, this condition should hold also for the perturbations. In addition, since the hypersurface $z=1$ represents an apparent horizon and region I is a trapped region, so nothing should be able to escape from it. In particular, for the scalar field this implies $\phi_{,v}(u, 0) = 0$ [cf. Eq. (3.11)]. On the other hand, as shown in [21], the gravitational wave component that moves out of region I is represented by Ψ_0 , which is a function of $V_{,vv}$, $V_{,v}$, $M_{,v}$ and $S_{,v}$. Then, we can see that the condition that no gravitational waves come out from region I requires $V_{,vv}$, $V_{,v}$, $S_{,v}$, $M_{,v}$, $\phi_{,v} \rightarrow 0$ as $v \rightarrow 0$. Changing to the self-similar variable z , it can be shown that these conditions are equivalent to

²It should be noted that this condition is not independent of the ones (2.2) and (2.3). In fact, using the Einstein field equations (3.1) we can deduce it from Eqs. (2.2) and (2.3). However, without loss of generality, in this paper we shall impose it independently.

$$\begin{aligned}
 F_1(z) &\sim \text{analytical with respect to } v, \\
 (1-z)^{(n-1)/n} \frac{dF_1(z)}{dz} &\rightarrow 0, \\
 (1-z)^{(n-2)/n} \frac{dV_1(z)}{dz} &\rightarrow 0, \quad (0 < 2q^2 < 1), \quad (4.13)
 \end{aligned}$$

as $z \rightarrow 1$, where $F_1(z) \equiv \{M_1, S_1, V_1, \varphi_1\}$.

When $2q^2 \geq 1$ the hypersurface $z=1$ is a future null infinity of the spacetime and marginally trapped (cf. Fig. 2). Since it is the future null infinity, we would expect that the perturbations be finite there. On the other hand, the hypersurface $z=1$ is also marginally trapped, so there we should have only outgoing scalar field and gravitational waves. Note that the amplitude of gravitational wave components is always proportional to $e^{M_0} \sim (1-z)^{2q^2}$ [21]. Thus, now we must require $(1-z)^{2q^2}(V_{1,v}, V_{1,v}, S_{1,v}, M_{1,v}) \rightarrow 0$, as $z \rightarrow 1$. On the other hand, using the Einstein field equations we can show that these conditions also imply $(1-z)^{2q^2}\varphi_{1,v} \rightarrow 0$. In terms of the self-similar variable z , and noticing that now t and r are given by Eq. (3.12), we find that these conditions can be written as

$$\begin{aligned}
 F_1(z) &\rightarrow \text{finite}, \\
 (1-z)^{2q^2} \frac{dF_1(z)}{dz} &\rightarrow 0, \\
 (1-z)^{2q^2} \frac{d^2V_1(z)}{dz^2} &\rightarrow 0 \quad (2q^2 \geq 1), \quad (4.14)
 \end{aligned}$$

as $z \rightarrow 1$.

Once we have the boundary conditions, let us first consider the gauge modes. We note that the metric (2.1) is invariant under the coordinate transformations

$$\begin{aligned}
 t &= a(\bar{t} + \bar{r}) + b(\bar{t} - \bar{r}), \\
 r &= a(\bar{t} + \bar{r}) - b(\bar{t} - \bar{r}), \quad (4.15)
 \end{aligned}$$

where $a(\bar{t} + \bar{r})$ and $b(\bar{t} - \bar{r})$ are arbitrary functions of their indicated arguments, subject to $a'b' \neq 0$. Thus, let us consider the gauge transformations

$$\begin{aligned}
 t &\rightarrow t + \epsilon[A(t+r) + B(t-r)], \\
 r &\rightarrow r + \epsilon[A(t+r) - B(t-r)], \quad (4.16)
 \end{aligned}$$

where $A(t+r)$ and $B(t-r)$ are other arbitrary functions. Then, we find that under the above coordinate transformations the resultant perturbations are given by

$$F(\tau, z) = F_0(z) + \delta F(\tau, z), \quad (4.17)$$

with

$$\begin{aligned}
 \delta M(\tau, z) &= 2(A' + B') - \frac{4q^2 z}{t(1-z^2)} [(A-B) + (A+B)z], \\
 \delta S(\tau, z) &= \frac{1}{r} [(A-B) + (A+B)z], \\
 \delta V(\tau, z) &= -\frac{1}{r} [(A-B) + (A+B)z], \\
 \delta \varphi(\tau, z) &= \frac{4q}{t} (A+B). \quad (4.18)
 \end{aligned}$$

In order to have the above expressions in the form of Eq. (4.1), we must choose

$$\begin{aligned}
 A(t+r) &= -c_2 t_0 \left(\frac{-t}{t_0}\right)^{1-k} (1-z)^{1-k}, \\
 B(t-r) &= c_1 t_0 \left(\frac{-t}{t_0}\right)^{1-k} (1+z)^{1-k}, \quad (4.19)
 \end{aligned}$$

for which Eq. (4.18) can be written as

$$\delta F(\tau, z) = F_1(z) e^{k\tau}, \quad (4.20)$$

with

$$\begin{aligned}
 M_1(z) &= 2\{c_1[(1-k) + 2q^2 z](1+z)^{-k} - c_2[(1-k) - 2q^2 z] \\
 &\quad \times (1-z)^{-k}\},
 \end{aligned}$$

$$S_1(z) = \frac{1}{z} [c_1(1+z)^{2-k} + c_2(1-z)^{2-k}],$$

$$V_1(z) = \frac{1-z^2}{z} [c_1(1+z)^{-k} + c_2(1-z)^{-k}],$$

$$\varphi_1(z) = -2q[c_1(1+z)^{1-k} - c_2(1-z)^{1-k}], \quad (4.21)$$

where c_1 and c_2 are arbitrary constants. It can be shown that the above perturbations do not satisfy the boundary conditions (4.12)–(4.14), or in other words, the boundary conditions imposed in this paper limit all the gauge modes.

Now let us consider the above boundary conditions for the particular background given by Eq. (3.4). We first consider the conditions at the axis given by Eq. (4.12). Let us first note that

$$F_1^{(1)}(z) = 1 + \frac{1}{4}k^2 z^2 + O(z^4),$$

$$F_1^{(2)}(z) = A_2(k) \{ [A_1(k) - 2 \ln(z)] F_1^{(1)}(z) + \frac{1}{2}k(k-2)z^2 \}, \quad (4.22)$$

as $z \rightarrow 0$, where

$$A_1(k) = \sum_{n=0}^{\infty} \frac{2(2-k)}{(n+1)(2n+k)}, \quad A_2 \equiv \frac{\Gamma(k)}{\Gamma^2\left(\frac{1}{2}k\right)}, \quad (4.23)$$

and $\Gamma(k)$ denotes the Gamma function. Inserting Eqs. (4.22) into Eqs. (4.11) and (4.10), after tedious calculations we find that

$$z\varphi_1'(z) + k\varphi_1(z) \rightarrow -2ka_2^{(1)}A_2(k)\ln(z) + A_0(k), \quad (4.24)$$

as $z \rightarrow 0$, where $A_0(k)$ is a finite constant. Thus, the last condition of Eq. (4.12) requires

$$a_2^{(1)} = 0. \quad (4.25)$$

Similarly, one can show that

$$\begin{aligned} G_1(z) &\equiv S_1(z) + V_1(z) \\ &= \frac{2(c_1 + c_2)}{z} - 2a_1^{(1)}A_2(k)\ln(z) + (2-k)(c_1 - c_2) \\ &\quad + [a_1^{(1)}A_1(k)A_2(k) + a_1^{(2)}] + O(z), \end{aligned} \quad (4.26)$$

from which we can see that the second condition of Eq. (4.12) requires

$$c_1 = -c_2 = c, \quad a_1^{(1)} = 0, \quad a_1^{(2)} = 2(k-2)c. \quad (4.27)$$

Once Eq. (4.27) holds, it can be shown that

$$\begin{aligned} zG_1'(z) &\rightarrow O(z^2), \\ M_1(z) &\rightarrow O(z^2), \end{aligned} \quad (4.28)$$

as $z \rightarrow 0$. That is, the first and third conditions of Eq. (4.12) do not impose further restrictions on the free parameters.

Now let us turn to consider the boundary conditions at $z = 1$. It is found convenient to consider the cases $0 < 2q^2 < 1$ and $2q^2 \geq 1$ separately.

A. $0 < 2q^2 < 1$

In this case we have $1 - z \sim (-v)^n$, as $v \rightarrow 0$. Then, we find that

$$S_1(z) \sim c[2^{2-k} - (-v)^{(2-k)n}], \quad (4.29)$$

as $v \rightarrow 0$. Thus, in order to have $S_1(z)$ analytical, the constant k has to take the values

$$k = 2 - \frac{m}{n} \quad (m \geq 1), \quad (4.30)$$

where m is a positive integer ($m = 1, 2, 3, \dots$). When $k < 0$ the corresponding modes are stable, which we are not so interested in. Thus, in the following we shall consider only the case where $k > 0$, which together with Eq. (4.30) implies $2n > m \geq 1$. To study the boundary conditions at $z = 1$ further, let us consider the cases $k \neq 1$ and $k = 1$ separately.

Case A. $k \neq 1$: In this case it can be shown that

$$\begin{aligned} F_1^{(1)}(z) &= B_1(k)F_{11}^{(1)}(z) + B_2(k)F_{12}^{(1)}(z)x^{1-k}, \\ F_1^{(2)}(z) &= F_{11}^{(1)}(z) \quad (k \neq 1), \end{aligned} \quad (4.31)$$

where

$$\begin{aligned} F_{11}^{(1)}(z) &\equiv 1 + \frac{1}{4}kx + D_1(k)x^2 + O(x^3), \\ F_{12}^{(1)}(z) &\equiv 1 + \frac{1}{4}(2-k)x + D_2(k)x^2 + D_3(k)x^3 \\ &\quad + O(x^4), \end{aligned} \quad (4.32)$$

with $x \equiv 1 - z^2$ and

$$\begin{aligned} B_1(k) &= \frac{\Gamma(1-k)}{\Gamma^2\left(1 - \frac{1}{2}k\right)}, \quad B_2(k) = \frac{\Gamma(k-1)}{\Gamma^2\left(\frac{1}{2}k\right)}, \\ D_1(k) &= \frac{k(2+k)^2}{32(1+k)}, \quad D_2(k) = \frac{(2-k)(4-k)^2}{32(3-k)}, \\ D_3(k) &= \frac{(2-k)(4-k)(6-k)^2}{384(3-k)}. \end{aligned} \quad (4.33)$$

Inserting the above expressions into Eqs. (4.2), (4.10) and (4.11) and after tedious calculations we find that

$$\begin{aligned} kM_1(z) &= \frac{1}{2}k\{c(k-2)(k-1-2q^2)2^k - 2B_2(k)[(k-1)a_1^{(2)} \\ &\quad + 2kqa_2^{(2)}]\}x^{1-k} - 2\{c(k-2)(k-1-2q^2)2^k \\ &\quad + 2(1-k)B_2(k)[a_1^{(2)} + 2qa_2^{(2)}]\}x^{-k} + A_3(c, k, q) \\ &\quad + O(x^k) + O(x^{2-k}), \end{aligned} \quad (4.34)$$

$$\varphi_1(z) = a_2^{(2)}[B_2(k)x^{1-k} + B_1(k)] + O(x), \quad (4.35)$$

$$V_1(z) = a_1^{(2)}B_2(k)x^{1-k} + O(x^{2-k}) \quad (k \neq 1), \quad (4.36)$$

as $z \rightarrow 1$, where $A_3(c, k, q)$ is a finite constant, and $a_1^{(2)}$ is given by Eq. (4.27). From the last two equations we can see that the analytical conditions of $V_1(z)$ and $\varphi_1(z)$ require $k < 1$, which together with Eq. (4.30) implies

$$2n > m > n. \quad (4.37)$$

The condition that $M_1(z)$ is analytical across the hypersurface $z = 1$ requires

$$a_2^{(2)} = \frac{c(k-2)}{2} \left(\frac{(k-1-2q^2)2^k}{2(k-1)B_2(k)} - 2 \right). \quad (4.38)$$

On the other hand, it can be also shown that

$$\begin{aligned} x^{1-1/n} \frac{dM_1(x)}{dx} &\sim \frac{c(k-2)}{2} [(k-1-2q^2)2^k + 4(1-k) \\ &\quad \times B_2(k)] x^{(m-n-1)/n} + O(x^{(n-1)/n}), \end{aligned}$$

$$\begin{aligned}
 x^{1-1/n} \frac{dV_1(x)}{dx} &\sim (1-k)a_1^{(2)}B_2(k)x^{(m-n-2)/n} \\
 &\quad + O(x^{1-2/n}) + O(x^{(m-2)/n}), \\
 x^{1-1/n} \frac{d\varphi_1(x)}{dx} &\sim (1-k)a_2^{(2)}B_2(k)x^{(m-n-1)/n} \\
 &\quad + O(x^{(n-1)/n}), \\
 x^{1-1/n} \frac{dS_1(x)}{dx} &\sim \frac{ck}{2^k}x^{(n-1)/n} + O(x^{(m-1)/n}) \quad (k \neq 1),
 \end{aligned} \tag{4.39}$$

as $x \rightarrow 0$. Thus, the last two conditions of Eq. (4.13) further require

$$2n > m > n + 2 \quad (k \neq 1). \tag{4.40}$$

Case B. $k=1$: In this case it can be shown that

$$\begin{aligned}
 F_1^{(1)}(z) &= F_{11}^{(1)}(x) - F_{12}^{(1)}(x)\ln(x), \\
 F_1^{(2)}(z) &= \pi F_{12}^{(1)}(x) \quad (k=1),
 \end{aligned} \tag{4.41}$$

but now with

$$\begin{aligned}
 F_{11}^{(1)}(x) &\equiv \frac{2}{\pi} \left[C_0 + \frac{1}{4}(C_0-1)x + \frac{3}{128}(6C_0-7)x^2 + O(x^3) \right], \\
 F_{12}^{(1)}(x) &\equiv \frac{1}{\pi} \left[1 + \frac{1}{4}x + \frac{9}{64}x^2 + O(x^3) \right], \\
 C_0 &\equiv \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)}.
 \end{aligned} \tag{4.42}$$

Then, it can be shown that

$$V_1(z) \rightarrow \frac{2c}{\pi} \ln(x) + O(x \ln(x)) \quad (k=1), \tag{4.43}$$

as $x \rightarrow 0$. Thus, the analytical condition of $V_1(z)$ across the hypersurface $x=0$ ($z=1$) requires $c=0$. It can be further shown that

$$\varphi_1(z) \rightarrow a_2^{(2)} \ln(x) + O(x \ln(x)) \quad (k=1), \tag{4.44}$$

as $x \rightarrow 0$. Thus, the analytical condition of $\varphi_1(z)$ across the hypersurface $x=0$ requires $a_2^{(2)}=0$. Considering Eqs. (4.25) and (4.27) we find that the boundary conditions in the present case limit all the perturbations, that is,

$$M_1(z) = S_1(z) = V_1(z) = \varphi_1(z) = 0 \quad (k=1). \tag{4.45}$$

Therefore, from Eq. (4.40) we can see that for any given n , the solution has

$$N = n - 3 \quad (0 < 2q^2 < 1) \tag{4.46}$$

unstable modes. In particular, the solution with $n=2$ or $n=3$ has no unstable mode, and consequently is stable with

respect to the linear perturbations. The solution with $n=4$ has only one unstable mode, which may represent a critical solution, sitting on a boundary that separates two attractive basins in the phase space. All the solutions with $n > 4$ have more than one unstable modes and are not stable with respect to the linear perturbations.

B. $2q^2 \geq 1$

In this case the boundary conditions at $z=1$ are those given by Eq. (4.14). From Eq. (4.6) we can see that to have $S_1(z)$ finite as $z \rightarrow 1$, we must assume that $\text{Re}(z) < 2$. To study the boundary conditions further, let us first consider the case $k \neq 1$.

When $k \neq 1$, it can be shown that Eqs. (4.31)–(4.36) also hold in the present case. Thus, the conditions that $V_1(z)$ and $\varphi_1(z)$ are finite as $z \rightarrow 1$ require that

$$\text{Re}(k) < 1, \tag{4.47}$$

while the condition that $M_1(z)$ is finite further requires that the constant $a_2^{(2)}$ has to take the values given by Eq. (4.38). On the other hand, from Eqs. (4.34)–(4.36) we can see that the last two conditions of Eq. (4.14) require

$$\text{Re}(k) < 2q^2 - 1. \tag{4.48}$$

When $k=1$, it can be shown that Eqs. (4.41)–(4.44) hold in this case, too, and following the analysis given there we find that the boundary conditions at $z=1$ limit all the perturbations for $k=1$.

Therefore, for any given q with $2q^2 > 1$, Eqs. (4.47) and (4.48) show that there always exists a continuous spectrum of k such that

$$0 < \text{Re}(k) < \text{Re}(k_{min}) \quad (2q^2 > 1), \tag{4.49}$$

where

$$\text{Re}(k_{min}) = \begin{cases} 1, & q^2 > 1 \\ 2q^2 - 1, & \frac{1}{2} < q^2 < 1. \end{cases} \tag{4.50}$$

That is, in the case $2q^2 > 1$, there are infinite numbers of unstable modes. However, when $2q^2 = 1$, from Eq. (4.48) we find that

$$\text{Re}(k) < 0 \quad (2q^2 = 1). \tag{4.51}$$

Thus, the solution with $2q^2 = 1$ is stable against the linear perturbations.

V. SUMMARY AND CONCLUDING REMARKS

In this paper we have first introduced the notion of homothetic self-similarity to four-dimensional spacetimes with cylindrical symmetry, and then presented a class of exact solutions to the Einstein-massless scalar field equations, which is parametrized by a constant q . It has been shown that for $0 < 2q^2 < 1$, the corresponding spacetimes have black hole structures but with cylindrical symmetry. These black holes

are formed from the gravitational collapse of a massless scalar field. When $2q^2 \geq 1$ the corresponding solutions also represent gravitational collapse of the scalar field but no black holes are formed. Instead, a pointlike singularity is developed, which is not naked and can be seen by an observer only when he/she arrives at the singularity.

Then, the linear perturbations of all these solutions have been given analytically in closed form in terms of hypergeometric functions. After properly imposing boundary conditions at the axis and on the horizons, it has been shown that the solutions with $n=2,3$ and the one with $2q^2=1$ are stable, where n is an integer and given by $n \equiv 1/(1-2q^2)$. For any given $n \geq 4$, the corresponding solution has $N=n-3$ unstable modes. In particular, the one with $n=4$ has precisely one unstable mode, which may represent a critical solution sitting on a boundary that separates two attractive basins in the phase space. The solution for any given q with $2q^2 > 0$ has a continuous spectrum of unstable eigenmodes, given by Eqs. (4.49) and (4.50).

It should be noted that in this paper we have shown that black holes can be formed from gravitational collapse of the massless scalar field, and the ones with $n=2,3$ are stable against the linear perturbations. However, these spacetimes are not asymptotically flat in the radial direction, and thus may not be considered as representing counterexamples to the hoop conjecture [22]. To have an asymptotically flat spacetime in the radial direction, we may restrict the distri-

bution of the scalar field only to a finite region, say, $r \leq r_0(t)$, and then join it with an asymptotically flat region.

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APPENDIX A: THE RICCI TENSOR AND ITS LINEAR PERTURBATIONS IN TERMS OF SELF-SIMILAR VARIABLES

The general metric for cylindrical spacetimes with two hypersurface orthogonal Killing vectors takes the form of Eq. (2.1). Introducing the self-similar dimensionless variables τ and z via the relations

$$\tau = -\ln\left(-\frac{t}{t_0}\right), \quad z = \frac{r}{(-t)}, \quad (\text{A1})$$

where t_0 is a dimensional constant, we find that the non-vanishing components of the Ricci tensor are given by

$$\begin{aligned} R_{tt} &= \frac{e^{2\tau}}{2zt_0^2} \{2z^3 S_{,zz} - (1-z^2)(zM_{,zz} + 2M_{,z}) - z^3(S_{,z}^2 + V_{,z}^2) + z(1+z^2)M_{,z}S_{,z} + 4z^2S_{,z} + z[2S_{,\tau\tau} + M_{,\tau\tau} - (S_{,\tau}^2 + V_{,\tau}^2)] \\ &\quad + (2+M_{,\tau})S_{,\tau} + M_{,\tau}\} + z^2[4S_{,\tau z} + 2M_{,\tau z} - 2(S_{,\tau}S_{,z} + V_{,\tau}V_{,z}) + (M_{,\tau}S_{,z} + M_{,z}S_{,\tau})], \\ R_{tr} &= \frac{e^{2\tau}}{2zt_0^2} \{2z^2 S_{,zz} - z^2(S_{,z}^2 + V_{,z}^2) - 2z(1-zS_{,z})M_{,z} + 4zS_{,z} + z[2S_{,\tau z} - (S_{,\tau}S_{,z} + V_{,\tau}V_{,z}) + (M_{,\tau}S_{,z} + M_{,z}S_{,\tau})] \\ &\quad + 2(S_{,\tau} - M_{,\tau})\}, \\ R_{rr} &= \frac{e^{2\tau}}{2zt_0^2} \{2zS_{,zz} + z(1-z^2)M_{,zz} - z(S_{,z}^2 + V_{,z}^2) + z(1+z^2)M_{,z}S_{,z} - 2(1+z^2)M_{,z} + 4S_{,z} - z[M_{,\tau\tau} + (1-S_{,\tau})M_{,\tau}] \\ &\quad - z^2[2M_{,\tau z} - (M_{,\tau}S_{,z} + M_{,z}S_{,\tau})]\}, \\ R_{22} &= \frac{1}{2}e^{M+V-S} \{z^2(1-z^2)[(S_{,zz} - V_{,zz}) - S_{,z}(S_{,z} - V_{,z})] + 2z(1-z^2)(S_{,z} - V_{,z}) - 2(1-zS_{,z}) - z^3[2(S_{,\tau z} - V_{,\tau z}) \\ &\quad - 2S_{,\tau}S_{,z} + (S_{,\tau}V_{,z} + S_{,z}V_{,\tau})] - z^2[(S_{,\tau\tau} - V_{,\tau\tau}) + (1-S_{,\tau})(S_{,\tau} - V_{,\tau})]\}, \\ R_{33} &= \frac{1}{2}e^{M-V-S} \{z^2(1-z^2)[(S_{,zz} + V_{,zz}) - S_{,z}(S_{,z} + V_{,z})] + 2z(1-z^2)(S_{,z} + V_{,z}) - 2(1-zS_{,z}) - z^3[2(S_{,\tau z} + V_{,\tau z}) \\ &\quad - 2S_{,\tau}S_{,z} - (S_{,\tau}V_{,z} + S_{,z}V_{,\tau})] - z^2[(S_{,\tau\tau} + V_{,\tau\tau}) + (1-S_{,\tau})(S_{,\tau} + V_{,\tau})]\}. \end{aligned} \quad (\text{A2})$$

On the other hand, it can be shown that the Klein-Gordon equation $\square\phi=0$ for the massless scalar field takes the form

$$z(1-z^2)\phi_{,zz} - 2z^2\phi_{,\tau z} - z\phi_{,\tau\tau} + [z(zS_{,z} - 1) + zS_{,\tau}]\phi_{,\tau} + [(1-z^2)(2-zS_{,z}) + z^2S_{,\tau}]\phi_{,z} = 0. \quad (\text{A3})$$

Now let us consider the linear perturbations of Eq. (4.1). To first order in ϵ , it can be shown that the non-vanishing components of the Ricci tensor are given by

$$R_{tt}^{(1)} = \frac{e^{(2+k)\tau}}{2zt_0^2} \{2z^3 S_1'' - z(1-z^2)M_1'' - 2z^3 V_0' V_1' - [2 - 2(1+k)z^2 - z(1+z^2)S_0']M_1' + z[4(1+k)z + (1+z^2)M_0' - 2z^2 S_0']S_1' - 2kz^2 V_0' V_1 + kz(1+k+zS_0')M_1 + kz[2(1+k) + z(M_0' - 2S_0')]S_1\}, \quad (\text{A4})$$

$$R_{tr}^{(1)} = \frac{e^{(2+k)\tau}}{2zt_0^2} \{2z^2 S_1'' - 2z^2 V_0' V_1' + 2z[(2+k) + z(M_0' - S_0')]S_1' - 2z(1-zS_0')M_1' - kzV_0' V_1 - k(2-zS_0')M_1 + k[2 + z(M_0' - S_0')]S_1\}, \quad (\text{A5})$$

$$R_{rr}^{(1)} = \frac{e^{(2+k)\tau}}{2zt_0^2} \{2zS_1'' + z(1-z^2)M_1'' - 2zV_0' V_1' + [4 + z(1+z^2)M_0' - 2zS_0']S_1' - [(1+z^2)(2-zS_0') + 2kz^2]M_1' - kz(1+k-zS_0')M_1 + kz^2 M_0' S_1\}, \quad (\text{A6})$$

$$R_{22}^{(1)} = e^{k\tau} \{ (M_1 + V_1 - S_1)R_{22}^{(0)} + \frac{1}{2}ze^{M_0 + V_0 - S_0}\bar{R}_{22}^{(1)} \}, \quad (\text{A7})$$

$$R_{33}^{(1)} = e^{k\tau} \{ (M_1 - V_1 - S_1)R_{33}^{(0)} + \frac{1}{2}ze^{M_0 - V_0 - S_0}\bar{R}_{33}^{(1)} \}, \quad (\text{A8})$$

where $R_{22}^{(0)}$ and $R_{33}^{(0)}$ are the corresponding components of the Ricci tensor for the background solution, and

$$\bar{R}_{22}^{(1)} = z(1-z^2)(S_1'' - V_1'') + [2kz^2 + (1-z^2)(zS_0' - 2)]V_1' + [4 - 2(1+k)z^2 - z(1-z^2)(2S_0' - V_0')]S_1' + kz(1+k-zS_0')V_1 - kz[(1+k) - z(2S_0' - V_0')]S_1, \quad (\text{A9})$$

$$\bar{R}_{33}^{(1)} = z(1-z^2)(S_1'' + V_1'') - [2kz^2 + (1-z^2)(zS_0' - 2)]V_1' + [4 - 2(1+k)z^2 - z(1-z^2)(2S_0' + V_0')]S_1' - kz(1+k-zS_0')V_1 - kz[(1+k) - z(2S_0' + V_0')]S_1. \quad (\text{A10})$$

APPENDIX B: APPARENT HORIZONS IN SPACETIMES WITH CYLINDRICAL SYMMETRY

In [23], the ingoing and outgoing radial null geodesics were studied in double null coordinates, and the corresponding expansions of them were calculated. In this paper, we shall define apparent horizons in spacetimes with cylindrical symmetry, using those quantities. Before doing so, we would first like to note the difference between the double null coordinates used in this paper and the ones used in [23]. As a matter of fact, the roles of u and v are exchanged in this paper.

To write the metric (2.1) in the double null coordinates, let us first introduce the two null coordinates u and v via the relations

$$t = \alpha(u) + \beta(v), \quad r = \alpha(u) - \beta(v), \quad (\text{B1})$$

where $\alpha(u)$ and $\beta(v)$ are two arbitrary functions of their indicated arguments, subject to

$$\alpha'(u)\beta'(v) \neq 0, \quad (\text{B2})$$

where a prime denotes the ordinary differentiation. Then, in terms of u and v , the metric (2.1) takes the form

$$ds^2 = 2e^{2\sigma(u,v)} du dv - r^2 e^{-S(t,r)} (e^{V(t,r)} d\omega^2 + e^{-V(t,r)} d\theta^2), \quad (\text{B3})$$

where

$$\sigma(u,v) = \frac{1}{2} [\ln(2\alpha'\beta') - M]. \quad (\text{B4})$$

Introducing two null vectors l^λ and n^λ by

$$l_\lambda \equiv \frac{\partial u}{\partial x^\lambda} = \delta_\lambda^u, \quad n_\lambda \equiv \frac{\partial v}{\partial x^\lambda} = \delta_\lambda^v, \quad (\text{B5})$$

we find that

$$l_{\mu;\nu} l^\nu = 0 = n_{\mu;\nu} n^\nu, \quad (\text{B6})$$

which means that each of them defines an affinely parametrized null geodesic congruence. In particular, l^μ defines the one moving along the null hypersurfaces $u = \text{const}$, while n^μ defines the one moving along the null hypersurfaces $v = \text{const}$. Then, the expansions of these null geodesics are defined as

$$\Theta_l \equiv g^{\alpha\beta} l_{\alpha;\beta} = e^{-2\sigma} \frac{\mathcal{R}_{,v}}{\mathcal{R}},$$

$$\Theta_n \equiv g^{\alpha\beta} n_{\alpha;\beta} = e^{-2\sigma} \frac{\mathcal{R}_{,u}}{\mathcal{R}}, \quad (\text{B7})$$

where

$$\mathcal{R} \equiv \|\partial_w \cdot \partial_w\| \cdot \|\partial_\theta \cdot \partial_\theta\| = r^2 e^{-S}. \quad (\text{B8})$$

We call the cylinders of constant t and r trapped if $\Theta_l \Theta_n > 0$, marginally trapped if $\Theta_l \Theta_n = 0$, and untrapped if $\Theta_l \Theta_n < 0$. An apparent horizon is defined as a hypersurface foliated by marginally trapped surfaces [18,19].

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