Simple sufficient conditions for the generalized covariant entropy bound

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The generalized covariant entropy bound is the conjecture that for any null hypersurface which is generated by geodesics with nonpositive expansion starting from a spacelike 2-surface B and ending in a spacelike 2-surface B', the matter entropy on that hypersurface will not exceed one quarter of the difference in areas, in Planck units, of the two spacelike 2-surfaces. We show that this bound can be derived from the following phenomenological assumptions: (i) matter entropy can be described in terms of an entropy current s_a; (ii) the gradient of the entropy current is bounded by the energy density, in the sense that $|k^a k^b \nabla_a s_b| \leq 2\pi T_{ab} k^a k^b / \hbar$ for any null vector k^a where T_{ab} is the stress energy tensor; and (iii) the entropy current s_a vanishes on the initial 2-surface B. We also show that the generalized Bekenstein bound—the conjecture that the entropy of a weakly gravitating isolated matter system will not exceed a constant times the product of its mass and its width—can be derived from our assumptions. Though we note that any local description of entropy has intrinsic limitations, we argue that our assumptions apply in a wide regime. We closely follow the framework of an earlier derivation, but our assumptions take a simpler form, making their validity more transparent in some examples.

I. INTRODUCTION AND SUMMARY

The covariant entropy bound [1] is a conjecture relating the area of two-dimensional spacelike surfaces to the entropy content of adjacent regions. Nonexpanding light rays emanating orthogonally from a spacelike 2-surface B of area A generate a null hypersurface L called a light sheet. (The light rays must be terminated before they begin to expand; this typically occurs when neighboring rays intersect.) The covariant bound is the conjecture that the matter entropy S_L on any such light sheet L satisfies[1]

$$S_L \leq \frac{A}{4 G_N \hbar},\qquad(1.1)$$

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[1]The conjecture requires that semiclassical corrections to general relativity are negligible in the construction of L. Strominger and Thompson [2] have recently proposed a modification to the covariant bound and conjectured that it accommodates one-loop quantum corrections to the geometry. Using suitable adaptations of the assumptions and of the proof given here, this modified bound can be derived in the context of the Callan-Giddings-Harvey-Strominger (CGHS) model [2].

where G_N is Newton's constant.[2]

When gravity is weak, this bound reduces to some earlier, more specialized proposals [3,4], which originally arose in attempts to allow the preservation of quantum-mechanical unitarity in the presence of black holes. It also generalizes and refines a cosmological bound conjectured by Fischler and Susskind [5]. No general derivation of the covariant bound (or of the earlier bounds) has been given. However, no realistic counterexamples have been constructed from known matter fields. Moreover, the bound has been verified explicitly in a variety of examples, including both weakly gravitating and gravitationally collapsing thermodynamic systems, as well as cosmological spacetimes. In addition, sufficient conditions for the bound have been identified which are readily seen to hold in a wide class of situations [6].

From a conceptual point of view, there is a tension between the scaling of maximal entropy with area asserted by the conjecture and the extensivity of quantum field theory. The holographic principle proposes that quantum gravity contains features that resolve this tension and that give rise to the covariant bound. Specifically, the holographic prin-

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[2]We use units with $c = k_B = 1$, but we retain Newton's constant G_N and Planck's constant $\hbar$. We use the $(-, +, +, +)$ metric signature. Also we work in four spacetime dimensions, although our analysis generalizes easily to D spacetime dimensions for all $D \geq 3$.

ciple asserts that in the fundamental theory, all physics on light sheets that have maximal area A can be described by roughly A binary degrees of freedom (in Planck units) [3,4,7]. A detailed review of the covariant entropy bound and related issues can be found in Ref. [8].

In Ref. [6] a stronger form of the covariant bound was suggested, which we call the generalized covariant entropy bound (GCEB). Consider a light sheet L some of whose generators are terminated prematurely (before they reach caustics). The end points of these prematurely terminating generators form a second spacelike 2-surface B' with nonvanishing area A'. The GCEB is the conjecture that the matter entropy S_L on such a light sheet satisfies

$$S_L \leqslant \frac{A - A'}{4 G_N \hbar}. \tag{1.2}$$

This bound reduces to the original covariant bound in the special case $A' = 0$. A key motivation for considering this generalization of the covariant entropy bound is that the GCEB, if true, would imply as a special case the generalized second law in classical regimes where the null energy condition is satisfied [6].

Moreover, it has recently been shown [9] that the GCEB implies a generalized version of Bekenstein's bound [10,11] on the entropy S of any weakly gravitating isolated matter system of mass M traversed by a light sheet L of initially vanishing expansion:

$$S \leqslant \pi M x / \hbar. \tag{1.3}$$

The width x is defined as the longest distance (in the center of mass frame) traveled by any of the generators of L between entering and exiting the system. In the context of the original Bekenstein bound, one usually considers static, compact matter systems. For such systems, x is always smaller than the diameter d of the smallest sphere circumscribing the system. Moreover, if the GCEB holds for each light sheet traversing such a system along every spatial direction, x can be minimized by judicious orientation of the system relative to the light sheet. Therefore the bound (1.3) is stronger than the original bound $S \leqslant \pi M d / \hbar$ advocated by Bekenstein [10].

Since the GCEB also implies the original covariant bound (1.1), it is the strongest of the various conjectured bounds we have just reviewed. If entropy bounds are indeed related to a fundamental theory, this primacy would suggest that the GCEB bears one of that theory's more direct imprints. In any case, it is important to investigate whether or not nature obeys the GCEB.

The purpose of this paper is to establish the validity of the GCEB in a broad class of hydrodynamic regimes, that is, regimes in which matter admits a description in terms of continuum variables. [Note that we do not exclude solids.] Specifically, we consider only regimes in which matter entropy can be described in terms of a local entropy current s^a.[3] In such regimes, any appropriate fundamental definition of the total matter entropy S_L on a light sheet L should, to a good approximation, reduce to the integral

$$S_L = \int_L s^a \boldsymbol{\epsilon}_{abcd} \tag{1.4}$$

of the entropy current over L.[4] With entropy given by this formula, we will show that the GCEB can be derived from two postulated properties of the entropy current [Eqs. (3.5) and (3.6) below]. We will also argue that those assumptions should be valid in a large class of hydrodynamic regimes. Our approach is similar in spirit to that of Ref. [6]. However, the assumptions used here are somewhat simpler than the assumption that was used in Ref. [6] to derive the GCEB.

Before attempting to prove the GCEB in hydrodynamic regimes, we must confront a crucial limitation of the entropy current description. In Sec. II we note that apparent violations of the GCEB can be obtained by integrating any finite entropy current over sufficiently small distances. However, a hydrodynamic description of matter entropy becomes invalid at sufficiently small scales. We analyze a cosmological example due to Guedens [12] in which the onset of apparent violations of the GCEB is seen to coincide with the ultraviolet breakdown of the hydrodynamic description.

The analysis of Sec. II informs our choice of assumptions concerning the behavior of the entropy current s^a, which we present in Sec. III A. We argue that those assumptions should be valid in a large class of hydrodynamic regimes. Moreover, we note that the assumptions effectively impose an ultraviolet cutoff related to the local energy density. This effective cutoff allows our proof to evade the generic short-distance problems of the hydrodynamic approximation. In Sec. III B we derive the GCEB from our assumptions. Finally, in Sec. III C, we give a direct, purely nongravitational derivation of the generalized Bekenstein bound (1.3) from the same assumptions, without using the GCEB as an intermediate result.

We stress that this paper has nothing to say about the validity of the GCEB or of the generalized Bekenstein bound in regimes where our assumptions are not satisfied. Moreover, neither Ref. [8] (which demonstrated that the GCEB implies Bekenstein's bound) nor the present paper (which shows that both bounds can be derived from plausible assumptions in a hydrodynamic regime) bears on the much-debated question of whether any of the proposed entropy bounds follow from the generalized second law of thermodynamics [13–16].

Our result that the GCEB is valid in hydrodynamic regimes, together with the results of Ref. [6], eliminates a large

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[3]We will use the term "current" for a flow in the time direction. For typical matter, s^a will be a future-directed vector field. For example, in the local comoving frame of a fluid, the time component of s^a is the usual entropy density, and the spatial components vanish.

[4]Here the orientation chosen for L is that which gives a positive result for future-directed, timelike s^a.

set of possible counterexamples to the GCEB. However, if entropy bounds really have the significance ascribed to them by the holographic principle, one would expect the GCEB (or at least the original covariant bound) to apply more broadly. If it does, a complete proof may not be possible until the underlying quantum gravity theory is understood. In the meantime, further exploration of the bounds' domain of validity remains an important task which may produce clues about quantum gravity.

II. INTRINSIC LIMITATIONS OF A LOCAL DESCRIPTION OF ENTROPY

On sufficiently large scales a description of entropy can often be given in terms of a local entropy current s^a. In such situations, the actual entropy contained in any spatial region $\mathcal{R}$ is, to a good approximation, given by the integral of the entropy current over $\mathcal{R}$, as long as $\mathcal{R}$ is much larger than some microphysical length scale Λ which is determined by the physical system under consideration. For example, for a bath of thermal radiation, Λ is of order the wavelength of a typical quantum of radiation. Thus the entropy current s^a makes sense only when integrated over sufficiently large regions. It is important not to take this local description of entropy too literally. Entropy is fundamentally nonlocal, and there is no physical justification for computing an entropy for regions smaller than Λ by integrating the entropy current s^a.

Indeed, if the entropy-current approximation for entropy could be extrapolated to arbitrarily short distances, the GCEB could clearly be violated. Consider an initial 2-surface B on which the expansion θ vanishes everywhere. Construct a very short light sheet L, for which the final 2-surface B' is allowed to approach the initial 2-surface B arbitrarily closely, so that the affine parameter interval $\Delta\lambda$ along the null generators goes to zero. Then the change $A - A'$ in area is quadratic in $\Delta\lambda$, since the area of cross sections of L has a local maximum at B. But in the entropy-current approximation, the total entropy S_L is given by the integral (1.4) and scales linearly with $\Delta\lambda$ for small $\Delta\lambda$:

$$S_L \propto \Delta\lambda. \tag{2.1}$$

Therefore the GCEB will be violated for sufficiently small $\Delta\lambda$. It follows in particular that the GCEB cannot be derived from any assumptions that permit the integration of a nonvanishing entropy current over arbitrarily short light sheets with initially vanishing expansion.

An example of this type was first pointed out by Raf Guedens [12]. Its original purpose was to show that the GCEB cannot be derived from the second set of assumptions (1.10) and (1.11) of Ref. [6], as the example explicitly satisfies those assumptions. It is instructive to review the Guedens example before attempting to formulate more suitable assumptions.

Consider a closed, radiation-dominated Friedmann-Robertson-Walker cosmological model, for which the metric is

$$ds^2 = a(\eta)^2[-d\eta^2 + d\chi^2 + \sin^2\chi(d\vartheta^2 + \sin^2\vartheta \, d\varphi^2)], \tag{2.2}$$

with scale factor

$$a(\eta) = a_{\mathrm{m}}\sin\eta. \tag{2.3}$$

Here a_{m} is the radius of the spatial 3-sphere at the moment $\eta = \pi/2$ of maximum expansion. Next, we choose a light sheet L and compute its area decrease, $A - A'$, and entropy, S_L. Let L begin on the 2-sphere B of maximum radius in the spacetime, given by $\chi = \eta = \pi/2$, and let it end on the nearby 2-sphere B' given by $\chi = \eta = \pi/2 + \Delta\chi$.

We shall work to leading order in $\Delta\chi$, which means that we can use $\Delta\chi$ as an affine parameter. The area decrease is quadratic:

$$A - A' = 8\pi a_{\mathrm{m}}^2\Delta\chi^2[1 + O(\Delta\chi)]. \tag{2.4}$$

In the fluid approximation, the total entropy S_L is just the product of the entropy density[5] s and the volume $4\pi a_{\mathrm{m}}^3\Delta\chi$ of the projection of L onto the $\eta = \pi/2$ hypersurface. For a radiation-dominated universe, standard thermodynamics implies

$$s = \frac{4\rho}{3T}, \tag{2.5}$$

where $\rho = 3/(8\pi G_{\mathrm{N}}a^2)$ is the energy density and T the temperature. This yields

$$S_L = \frac{2a_{\mathrm{m}}\Delta\chi}{G_{\mathrm{N}}T}[1 + O(\Delta\chi)]. \tag{2.6}$$

Comparison with the area change (2.4) shows that the bound (1.2) is apparently violated when

$$a_{\mathrm{m}}\Delta\chi \lesssim \frac{\hbar}{\pi T}. \tag{2.7}$$

Equation (2.7) says that the proper length $a_{\mathrm{m}}\Delta\chi$ in the cosmological rest frame of the light sheet is shorter than the thermal wavelength $\hbar/T$. Therefore the light sheet is shorter than even a single wavelength of the radiation filling the Universe. In this regime it is clear that our computation of the entropy (2.6) is invalid.

Thus the onset of apparent violations of the GCEB coincides in the Guedens example with the breakdown of the local description of entropy, namely when the condition (2.7) holds. In the regime (2.7), most of the occupied modes cannot be localized within L but will spill over beyond the boundaries of the light sheet. Thus we cannot conclude that the GCEB is violated in any physically meaningful sense.

A number of questions arise concerning the applicability and precise formulation of the GCEB at short distances, such as the regime (2.7) in the Guedens example. Is there a meaningful definition of entropy for such short light sheets? It is conceivable that the entropy S_L computed from a sufficiently

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[5]We use s to denote an entropy density per unit affine parameter and per unit proper transverse area [see Eq. (3.2)]. Here it was convenient to choose the normalization of the affine parameter $\Delta\chi$ such that s coincides with the usual entropy per unit proper volume (in the local rest frame of the cosmological fluid). Note that this choice differs from the normalization convention we will use in our proof in Sec. III below.

general and fundamental prescription will satisfy the GCEB at all scales. Alternatively, it is possible that the GCEB will only apply to the statistical entropy of complete, isolated systems. If this is true, one would need to supplement the statement of the GCEB with a suitable criterion defining "isolation." Resolution of these issues would sharpen the conjecture and may contribute to a deeper understanding of its meaning.

III. DERIVATION OF ENTROPY BOUNDS

In this section we will derive the GCEB and the generalized Bekenstein bound in the regime defined by Eq. (1.4), subject to two assumptions. We start by introducing some notations to describe the integral (1.4) over the light sheet L. The affine parameter, λ, along any null generator of L is taken to run from 0 at B to 1 at B'.[6] The null vector $k^a \equiv (d/d\lambda)^a$ is normal (and tangent) to L. We introduce a coordinate system $x = (x^1, x^2)$ on the initial 2-surface B, and we label the geodesic generators of L with these coordinates, thereby defining a coordinate system (x^1, x^2, λ) on L. We denote by $h_{AB}(x, \lambda)$ the two-dimensional induced metric on the cross-sections $\lambda = $ const of L, and define $h = \det h_{AB}$. With these notations the integral (1.4) can be written as [6]

$$S_L = \int_B d^2x \int_0^1 d\lambda \sqrt{h(x, \lambda)} s(x, \lambda). \tag{3.1}$$

Here s is the entropy density on L, or more precisely the entropy per unit affine parameter and per unit cross-sectional area. It is given by

$$s = \pm k^a s_a, \tag{3.2}$$

where the minus sign applies for future directed k^a and the plus sign for past directed k^a. The determinant factor in Eq. (3.1) can be written as [6]

$$\sqrt{h(x, \lambda)} = \mathcal{A}(x, \lambda) \sqrt{h(x, 0)}, \tag{3.3}$$

where $\mathcal{A}$ is an area decrease factor associated with a given generator given by

$$\mathcal{A}(\lambda) \equiv \exp \left[\int_0^\lambda d\bar{\lambda} \, \theta(\bar{\lambda}) \right]. \tag{3.4}$$

Here $\theta = \nabla_a k^a$ is the expansion of the generators of L which by assumption is nonpositive everywhere on L. A prime will be used to denote the operator $k^a \nabla_a = d/d\lambda$.

A. Assumptions

We assume that the entropy density s on a light sheet L satisfies two conditions.

(1) The *gradient assumption* that

$$s' \leq 2\pi T_{ab} k^a k^b / \hbar \tag{3.5}$$

at every point on L, where T_{ab} is the stress tensor.

(2) The *isolation assumption*[7] that the entropy density should vanish on the initial 2-surface:

$$s_{|B} = 0. \tag{3.6}$$

We now discuss the motivation for these assumptions. The first assumption essentially states that the entropy gradient must be smaller than the energy density in natural units. This holds for free Bose and Fermi gases in local thermal equilibrium [6]. More generally, the assumption is plausible if (i) both the entropy density and the stress tensor are smeared (in all spatial directions) over a distance Λ set by the largest wavelength of any of the modes that contribute significantly to the entropy, and if (ii) the effective number of scalar fields in the Lagrangian, N, is not very large. In this case, one can apply an estimate given in Ref. [6] which we refine here.

We neglect factors of order unity. Because of the lack of features at distances smaller than Λ, the entropy gradient obeys $\nabla s \lesssim s/\Lambda$. Now consider a sphere of radius Λ. Since Λ is the largest wavelength, particles can be considered localized on this scale, and we can define n to be the number of particles inside the sphere. Since we assume that modes with wavelength Λ contribute significantly to the entropy, we can obtain the order of magnitude of the entropy by considering only such modes. Then the number of states will be $(N+n-1)!/[(N-1)!n!]$ [or $(N+n)!/(N!n!)$ if states with fewer than n particles are also allowed]. The total entropy $\mathcal{S}$ in the sphere is the logarithm of this number and hence obeys $\mathcal{S} \lesssim n \ln N$ (saturation occurs for small n). Unless the number of fields is very large, $\ln N$ will be of order unity. The entropy density therefore satisfies $s \lesssim n/\Lambda^3$. On the other hand, the energy density is bounded from below: $\rho \gtrsim n\hbar/\Lambda^4$. It follows that $\nabla s \lesssim \rho/\hbar$.

The gradient assumption is clearly closely related to Bekenstein's bound. One might even interpret it as a kind of local formulation of Bekenstein's bound, or at least as a first step in that direction. The same may be said of the quasi-local assumption (1.9) of Ref. [6], which was shown in that reference to independently imply the GCEB.

Consider now the second assumption (3.6). It is clearly satisfied if the initial 2-surface B is in a vacuum region outside a compact matter system. If the initial 2-surface lies instead inside the matter system so that the initial entropy current $s_{a|B}$ is nonvanishing, then the assumption is violated. However, we can imagine a slightly different matter system in which the entropy current in a thin sliver near the initial 2-surface B has been modified to achieve $s_{a|B} = 0$ without violating the gradient assumption (3.5). This will only require changing the entropy current within one or two wave-

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[6]We neglect any generators of L whose affine parameter length is infinite; this is justified in Ref. [6].

[7]Strominger and Thompson have shown that this assumption can be replaced with the requirement that $s \leq -\theta/4$ on the initial 2-surface, which can be interpreted as demanding that the GCEB is satisfied in an infinitesimal neighborhood of B [2].

lengths of the initial surface, and thus the change in the total integrated entropy will be negligible (so long as the extent of L is significantly larger than one wavelength). But the fluid approximation is in any case valid only for regions much larger than a typical wavelength. Hence, our second assumption poses no significant additional restriction on the range of applicability of our proof.

In the previous section we pointed out that the hydrodynamic description of entropy breaks down at short distances, where it would generically lead to violations of the GCEB. These difficulties will not plague our derivations, as our two assumptions together exclude the arbitrarily short light sheets with finite entropy density that led to the problematic scaling (2.1). Namely, Eq. (3.6) ensures that the entropy density vanishes on the initial surface, and Eq. (3.5) prevents it from turning on too rapidly.

In this sense the second assumption (3.6) addresses the uncertainty as to the formulation of the GCEB at short distances. It can be interpreted to do so via either of the two approaches to this formulation outlined at the end of Sec. II. Most straightforwardly, Eq. (3.6) may be regarded as expressing the requirement that the matter on the light sheet L be isolated, in the sense that modes carrying significant amounts of entropy should be fully contained on L and should not spill over beyond the initial boundary of L.

The second interpretation is to regard Eq. (3.6) as an imprint of a more general prescription for defining the entropy S_L through L. By insisting on setting $s=0$ on the initial boundary B, and allowing s to increase only at a rate limited by Eq. (3.5), the contributions of "spill-over" modes are effectively removed from the vicinity of the boundary. Note that under such a prescription, s could not be obtained as the contraction $k^a s_a$ of k^a with an absolute entropy current s^a that is the same for all light sheets. Rather, such a prescription would entail light-sheet dependent entropy currents. However, in view of the nonlocal nature of entropy, some kind of subadditive, light-sheet dependent prescription may indeed be appropriate. Light-sheet dependent currents have previously been considered, with a similar motivation, in the assumption (1.9) of Ref. [6]. Note that if s does arise from a global, absolute entropy current vector field s^a, our second condition (3.6) will be satisfied for every light sheet in the spacetime if and only if s^a satisfies $|k^a k^b \nabla_a s_b| \leq 2\pi T_{ab} k^a k^b / \hbar$ for all null vector fields k^a (the condition cited in the abstract). We stress, however, that our proof applies both to light-sheet dependent and absolute entropy currents; as long as s satisfies the required conditions on some light sheet L, we prove that the GCEB will hold on L.

B. Derivation of the generalized covariant bound

In Ref. [6] it was shown that, to prove the GCEB using a local entropy current, it is sufficient to focus on each individual null generator of the light sheet L, one at a time. Specifically, we need only show that

$$\int_0^1 d\lambda\, s(\lambda) \mathcal{A}(\lambda) \leq \frac{1}{4 G_N \hbar} [1 - \mathcal{A}(1)] \qquad (3.7)$$

for each null generator of L of finite affine parameter length, where $\mathcal{A}(\lambda)$ is the area-decrease factor (3.4).

The Raychaudhuri equation along the geodesic can be written in the form

$$8\pi G_N T_{ab} k^a k^b + \hat{\sigma}_{ab} \hat{\sigma}^{ab} = \frac{-2 G''(\lambda)}{G(\lambda)}, \qquad (3.8)$$

where $G(\lambda) \equiv \sqrt{\mathcal{A}(\lambda)}$ and $\hat{\sigma}_{ab}$ is the shear tensor, and primes denote derivatives with respect to λ. Thus the gradient assumption (3.5) implies that[8]

$$s'(\lambda) \leq \frac{-1}{2 G_N \hbar} \frac{G''(\lambda)}{G(\lambda)}. \qquad (3.9)$$

Using our second assumption, $s(0)=0$, we integrate this expression to obtain a bound on the scalar entropy density:

$$s(\lambda) = \int_0^\lambda d\bar{\lambda}\, s'(\bar{\lambda})$$

$$\leq \frac{-1}{2 G_N \hbar} \int_0^\lambda d\bar{\lambda} \frac{G''(\bar{\lambda})}{G(\bar{\lambda})}. \qquad (3.10)$$

Integration by parts now gives

$$s(\lambda) \leq \frac{1}{2 G_N \hbar} \left[\frac{G'(0)}{G(0)} - \frac{G'(\lambda)}{G(\lambda)} - \int_0^\lambda d\bar{\lambda} \frac{G'(\bar{\lambda})^2}{G(\bar{\lambda})^2} \right]. \qquad (3.11)$$

The first term in the square brackets in Eq. (3.11) is nonpositive by the nonexpansion condition $\theta \leq 0$, since $G'(0) = \theta(0)/2$. The last term in the square brackets is manifestly nonpositive. Hence both terms can be discarded, and we obtain

$$s(\lambda) \leq \frac{-1}{2 G_N \hbar} \frac{G'(\lambda)}{G(\lambda)}. \qquad (3.12)$$

Inserting this into the left-hand side of Eq. (3.7) and using $\mathcal{A} = G^2$ we obtain

$$\int_0^1 d\lambda\, s(\lambda) \mathcal{A}(\lambda) \leq \frac{-1}{2 G_N \hbar} \int_0^1 d\lambda\, G(\lambda) G'(\lambda)$$

$$= \frac{1}{4 G_N \hbar} [G(0)^2 - G(1)^2]. \qquad (3.13)$$

Finally using $G(0)=1$ and $G(1)^2 = \mathcal{A}(1)$ yields Eq. (3.7). This completes the proof.

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[8]Note that the corresponding Eq. (2.17) of Ref. [6] contains a typographical error; the role of the barred and unbarred quantities should be interchanged.

C. Derivation of the generalized Bekenstein bound

We have derived the GCEB for light sheets on which our assumptions hold, and in Ref. [8] the GCEB was shown to imply the generalized Bekenstein bound Eq. (1.3) for weakly gravitating matter systems. Curiously, in this sequence of implications neither our assumptions nor the Bekenstein bound contain Newton's constant, whereas the GCEB does. It seems strange that we should have to go via a bound involving gravity to derive the Bekenstein bound from our assumptions, using the Raychaudhuri equation once in Eq. (3.8) to introduce G_N, and then a second time in Ref. [9] to eliminate it. Here we point out that it is indeed possible to obtain the generalized Bekenstein bound directly from our assumptions, without using general relativity.

The derivation of the generalized Bekenstein bound involves two light sheets $L_\pm$ which nearly coincide with each other but are oppositely directed [9]. Each has initially vanishing expansion, and each fully contains the matter system. Specifically, one requires that the matter system occupies a world volume W of compact spatial support in approximately Minkowski space and that no world-line in W fails to intersect $L_\pm$. As indicated in the Introduction, the width of the system, x, is defined in terms of the light sheets $L_\pm$. Thus the light sheets $L_\pm$ are an integral part of the generalized Bekenstein bound. [This was not the case for the original Bekenstein bound, which was formulated mainly for static systems and employs a different, more lenient definition of width.]

The GCEB had to be assumed to hold for both L_+ and L_- at the outset of the derivation in Ref. [9]. In order to give a direct, nongravitational derivation of the generalized Bekenstein bound in the hydrodynamic regime (1.4), we should therefore assume that our conditions (3.6) and (3.6) hold for L_+ and L_-. In Ref. [9] the assumption of weak gravity allowed the construction of nearly coinciding light sheets $L_\pm$ with small relative area decrease. In the limit $G \to 0$, the same construction results in exactly coinciding light sheets, $L_+ = L_- = L$ with everywhere vanishing expansion, in flat Minkowski space. For a direct nongravitational derivation of the generalized Bekenstein bound from our assumptions, we will set G_N to zero from the start and work with the simpler object L.

Let L be a congruence of future-directed parallel null geodesics ($\theta = 0$) orthogonal to a compact 2-surface B. Let each ray extend a finite spatial distance (in a fixed Lorentz frame), so that the congruence ends on a second 2-surface B'. All cross sections of the congruence including B and B' have the same area A, since spacetime is flat. Thus we can regard L as a light sheet L_+ originating at B, with an affine parameter λ_+ that varies from 0 at B to 1 at B'. Alternatively, we can regard L as a light sheet L_- originating at B', with an affine parameter λ_- that varies from 0 at B' to 1 at B'. The two affine parameters are related by

$$\lambda_- = 1 - \lambda_+. \tag{3.14}$$

We require our two assumptions to hold in either case; in particular, the isolation condition, $s = 0$, is assumed to hold on both terminal 2-surfaces.

The entropy density s at any point on L can thus be computed in two ways, by integrating s' starting from either end:

$$s(\lambda_+) = \int_0^{\lambda_+} d\bar{\lambda}_+ \frac{ds}{d\bar{\lambda}_+} = \int_0^{1-\lambda_+} d\bar{\lambda}_- \frac{ds}{d\bar{\lambda}_-}. \tag{3.15}$$

By adding both expressions, applying the gradient assumption to each integrand, and using $\lambda_- = 1 - \lambda_+$, we find

$$2s(\lambda_+) \leq 2\pi \int_0^1 d\bar{\lambda}_+ T_{ab} k^a k^b / \hbar, \tag{3.16}$$

where $k^a = (d/d\lambda_+)^a$. Since the gradient assumption applies with respect to both directions, and $ds/d\lambda_+ = -ds/d\lambda_-$, the null energy condition must hold and hence the integrand is positive definite. By construction, the direction of k^a is the same for all light-rays, and we can only increase the right-hand side by replacing k^b with k^b_{max}, the null vector with the largest time component in a fixed Lorentz frame, maximizing over all the generators of L. Hence

$$s(\lambda_+) \leq \pi \int_0^1 d\lambda_+ T_{ab} k^a k^b_{max} / \hbar. \tag{3.17}$$

Further integration over the light sheet L then yields

$$S \leq \pi P_b k^b_{max} / \hbar, \tag{3.18}$$

where $P_b = \int_B \int d\lambda_+ T_{ab} k^a$ is the total four-momentum of the matter present on L [8]. This inequality takes its simplest form in the rest frame of the matter system, for which the spatial components of P_b vanish. Then P_0 is given by the system's rest mass, M, and k^0_{max} is a proper length corresponding to the width of the system, x. (Recall that x is defined as the greatest spatial distance traversed by any of the generators of L, in the rest frame of W.) Thus we obtain the generalized Bekenstein bound (1.3).

ACKNOWLEDGMENTS

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