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Dynamical system approach to cosmological models with a varying speed of light

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The methods of dynamical systems have been used to study homogeneous and isotropic cosmological models with a varying speed of light (VSL). We propose two methods for the reduction of the dynamics to the form of planar Hamiltonian dynamical systems for models with a time dependent equation of state. The solutions are analyzed on two-dimensional phase space in the variables $(x,\dot{x})$ where x is a function of a scale factor a. Then we show how the horizon problem may be solved on some evolutional paths. It is shown that the models with a negative curvature overcome the horizon and flatness problems. The presented method of reduction can be adapted to the analysis of the dynamics of the Universe with the general form of the equation of state $p=\gamma(a)\epsilon$. This is demonstrated using as an example the dynamics of VSL models filled with a noninteracting fluid. We demonstrate a new type of evolution near the initial singularity caused by a varying speed of light. Singularity-free oscillating universes are also admitted for a positive cosmological constant. We consider a quantum VSL Friedmann-Robertson-Walker closed model with radiation and show that the highest tunneling rate occurs for a constant velocity of light if $c(a)\propto a^n$ and $-1<n\leqslant 0$. It is also proved that the class of models considered is structurally unstable for the case of $n<0$.

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I. INTRODUCTION

Although the standard cosmological model is usually believed to be a correct picture of our Universe [1], it still has some difficulties, among which the flatness and horizon problems are most widely known. The existence of large-scale structure in the Universe extending to the limit of the deepest surveys is another mystery. Its very presence implies the appearance of some seeds for this structure in the early Universe. In the standard big-bang scenario they should be built in, which is a rather undesirable feature of the theory. Therefore, great attention has been paid to inflationary universe models, which, albeit invoking exotic (if not hypothetical) physics, were able to provide at least some hope for a consistent explanation of both the flatness and horizon problems as well as the origin of the seeds for the large-scale structure. The results of early universe physics lead us to expect the occurrence of phase transitions when the Universe was young, hot, and dense.

The varying speed of light (VSL) cosmology, seen as an alternative to the inflation theory, was proposed by Moffat [2,3], who conjectured that a spontaneous breaking of the local Lorentz invariance and diffeomorphism invariance associated with a first order phase transition can lead to variation of the speed of light in the early Universe. This idea was revived by Albrecht and Magueijo [4] and was given further consideration by Barrow [5,6]. Barrow showed that the conception of a VSL can lead to a solution of the flatness, horizon, and monopole problems if the speed of light falls at an appropriate rate. The dynamics of a VSL has been widely studied in theoretical as well as empirical contexts [7–16].

The main motivation for the study of VSL models is to seek explanations for some unusual properties of the Universe and to overcome some of the shortcomings of the inflation scenario [17]. In particular, there is empirical evidence of the fine structure constant varying with time in the context of the consistency of quasar absorption spectra [18]. Moreover, unlike inflation, the VSL theory provides a solution of the cosmological constant problem. However, it cannot solve the isotropy problem. It is also interesting to evaluate the power of this model in explaining the acceleration problem [19,20].

Of course, the VSL model as well as other models discussed in the literature have an *ad hoc* element (variable c) not yet firmly founded within any existing physical theories. This feature does not seem to be exotic enough to discard these models from discussion in the scientific community. Some brilliant arguments justifying this approach are given by Albrecht and Magueijo [4].

The varying speed of light models which provide decent fits to the real Universe are characterized by a speed of light or gravitational coupling which varies with time in the very early Universe but is nearly constant today. Because there are stringent bounds on how fast these constants can vary with time after the first few seconds, the models whose dynamics we study in the paper are relevant only in the very early Universe. This should be made very clear by using the phase space approach and its tools for classifying the qualitative types of solutions [21,22].

The present paper is a continuation of previous papers [23,24] on the dynamics of VSL cosmology. We introduce a simple framework which allows us to study the dynamics of VSL models in a general way, independent of any specific assumption about the equation of state, or the behavior of the

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scale factor $a(t)$ near the spatial singularity. We formulate a VSL Friedmann-Robertson-Walker cosmology as a two-dimensional dynamical system and we discuss its properties using phase portraits where the trajectories represent all solutions for all physically admissible initial conditions. The methods of dynamical systems allow us to indicate how the existence of certain desired physical effects depends on the choice of initial conditions and to analyze how these initial conditions determining the corresponding solutions are distributed in the phase space.

Our main goal was to perform a global analysis of the dynamics of VSL cosmological models. We avoid the assumption of power type evolution in the VSL models, which are represented by critical points (singular solutions) in the phase space. We analyze the dynamics of the models on the phase plane and discuss how different trajectories representing nonsingular solutions can solve cosmological puzzles. We conclude that models with negative curvature and positive cosmological constant are preferred (in the sense that they have the largest set of initial conditions leading to a solution of the flatness and horizon problems).

On the other hand, we present two arguments that distinguish the Friedmann-Robertson-Walker (FRW) models with constant velocity of light. The theoretical one is that the VSL FRW models are structurally unstable if $c(a)\propto a^n$ and $n<0$, contrary to classical FRW models. The quantum mechanical one is that if a closed universe was born from a quantum fluctuation via the quantum tunneling process then the most probable universe is that with $c=\mathrm{const}$. In this interval the potential function preserves its classical character and the universe tunnels from a zero size.

The dynamics of the cosmological models considered is reduced to the dynamics of a unit mass particle in a one-dimensional potential. Then different physical properties like the flatness, horizon, and cosmological constant problems can be formulated in terms of the diagram of the potential function of the system.

II. THE METHOD OF DYNAMICAL SYSTEM STABILITY

First of all, equations describing a cosmological model should be reduced to the form of a dynamical system

$$\dot{x}_i=\frac{dx_i}{dt}=f_i(x_1,\ldots,x_n),\quad i=1,\ldots,n,$$

in such a way that the solution with a static microspace (or other solutions of interest) is a critical point of the system $(x_1^*,\ldots,x_n^*)$, i.e., for every i, $f_i(x_1^*,\ldots,x_n^*)=0$ $(i=1,\ldots,n)$.

If a critical point is nondegenerate, i.e., at this point all real parts of the eigenvalues $\mathrm{Re}\,\lambda_i$ of the linearization matrix

$$A_j^i=\frac{\partial f_i}{\partial x_j}\bigg|_{x=x^*}$$

do not vanish, then there is a one-to-one continuous mapping of a neighborhood of this point which transforms trajectories of the original system into trajectories of the linearized system

$$\frac{dx_i}{dt}=\sum\frac{\partial f_i}{\partial x_j}(x^*)(x_j-x_j^*).$$

In this sense, the qualitative behavior of the original system is equivalent to the behavior of its linearized part. If $(\xi_i^1,\ldots,\xi_i^n)$ are eigenvectors of the linearization matrix A_j^i, the solution of the linearized system has, in general, the following form:

$$x_i(t)-x_i^*=\mathrm{Re}\sum_{k=1}^{n}C_k\xi_i^k e^{\lambda_k t},$$

where C_k are constants. A nondegenerate critical point is called an attracting point if, for all eigenvalues, $\mathrm{Re}\,\lambda_i<0$. In this case, all trajectories from the neighborhood of this point go to it when $t\rightarrow\infty$. A nondegenerate critical point is called a repulsing point if, for all eigenvalues, $\mathrm{Re}\,\lambda_i>0$. In this case, all trajectories from the neighborhood of this point go to it when $t\rightarrow-\infty$. A nondegenerate critical point is said to be an unstable saddle point if a dynamical system has, at $(x_1^*,\ldots,x_n^*)$, d eigenvalues with positive real parts and $n-d$ eigenvalues with negative real parts $(d=1,\ldots,n-1)$.

When investigating the stability of solutions with a static microspace the following theorem proves to be of special interest. If x^* is a nondegenerate critical point and if the dynamical system has, at x^*, d eigenvalues $\lambda_1,\ldots,\lambda_d$ with negative real parts, then there exists (locally) an invariant d-dimensional stable manifold W_{st}^d, on which all trajectories of the system go to x^* as $t\rightarrow\infty$. A manifold M is said to be an invariant manifold of a system if every trajectory passing through a nondegenerate point of M lies entirely in M (for $-\infty<t<+\infty$). For every such solution, there exists the asymptotic

$$\lim_{t\rightarrow\infty}t^{-1}\ln\left\{\sum_{j=1}^{n}[x_j(t)-x_j^*]^2\right\}^{1/2}=\alpha_i \tag{1}$$

for a certain i. Similarly, if at the critical point x^* the system has k eigenvalues with positive real parts then there exists an invariant k-dimensional unstable manifold W_{unst}^k on which all trajectories go away from the critical point [21].

From the above theorem it follows that, for a saddle point, there are two invariant manifolds W_{st}^d and W_{unst}^{n-d} containing this point and filled with trajectories (separatrices) going to and away from the critical point. All other trajectories (not contained in W_{st}^d or in W_{unst}^{n-d}) do not meet the critical point in question.

For the complete construction of phase portraits in a plane it is necessary to know how the trajectories of a dynamical system behave at infinity. Let us take as an example the two-dimensional system

$$\dot{x}=P(x,y), \tag{2}$$

$$\dot{y} = Q(x,y). \tag{3}$$

In the case of polynomial right-hand sides one usually introduces projective coordinates, e.g., $z=1/x$, $u=y/x$ or $v=1/y$, $w=x/y$. Two maps (z,u) and (v,w) are equivalent if and only if $u\neq 0$ and $v\neq 0$. Infinitely distant points of the (x,y) plane correspond to a circle S^1 which can be covered by two lines $z=0$, $-\infty<u<\infty$, and $w=0$, $-\infty<v<\infty$. The original system in the projective coordinates (z,u) and after the time reparametrization $\tau\to\tau_1 : d\tau_1 = x d\tau$ assumes the form

$$\dot{z} = zP^*(z,u), \tag{4}$$

$$\dot{u} = Q^*(z,u) - uP^*(z,u), \tag{5}$$

where

$$P^*(z,u) = z^2 P(1/z, u/z),$$

$$Q^*(z,u) = z^2 Q(1/z, u/z),$$

and the overdot denotes differentiation with respect to the new time τ_1.

In a similar way, in the projective coordinates (v,w) and in the new time $\tau_2 : d\tau_2 = y d\tau$, we obtain

$$\dot{v} = -vQ^*(v,w), \tag{6}$$

$$\dot{w} = P^*(v,w) - wQ^*(v,w), \tag{7}$$

where

$$P^*(v,w) = v^2 P(1/v, w/v),$$

$$Q^*(v,w) = v^2 Q(1/v, w/v),$$

and the overdot denotes differentiation with respect to time τ_2.

The idea of structural stability was introduced by Andronov and Pontryagin [25]. A dynamical system S is said to be structurally stable if there exist dynamical systems in the space of all dynamical systems that are close, in the metric sense, to S or are topologically equivalent to S. Instead of finding and analyzing an individual solution of a model, the space of all possible solutions is investigated. A given physical property is believed to be "realistic" if it can be attributed to large subsets of models within a space of all possible solutions or if it possesses a certain stability, i.e., if it is also shared by a slightly perturbed model. There is a well established opinion among specialists that realistic models should be structurally stable. What does structural stability mean in physics? The problem is in principle open in higher than the two-dimensional case where according to Smale there are large subsets of structurally unstable systems in the space of all dynamical systems [26]. For two-dimensional dynamical systems, as in the considered case, Peixoto's theorem says that structurally stable dynamical systems on compact manifolds form open and dense subsets in the space of all dynamical systems on the plane. Therefore, it is reasonable to require the model of a real two-dimensional problem to be structurally stable.

In our further considerations we will investigate the dynamics of dynamical systems in a finite domain of phase space as well as at infinity. At this point we would like to recommend the presentation of the actual state of the art in the field of application of dynamical systems to general relativity [27].

III. BASIC EQUATIONS OF THE THEORY

Albrecht and Magueijo [4] and Barrow [5] set up a useful framework to discuss the VSL models, assuming that the time variable c should not introduce changes in the curvature terms of the gravitational field equations and that the Einstein equations must hold. Because varying c breaks Lorentz invariance, the VSL cosmology requires a specific reference frame (including a specific choice of a time coordinate) in which changes in the field equations are minimal and one postulates that it coincides with the cosmological comoving frame.

In the case of the VSL version of the FRW models (with $\Lambda=0$) the scale factor obeys the following dynamical equations:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G(t)\rho}{3} - \frac{Kc^2(t)}{a^2(t)}, \tag{8}$$

$$\frac{\ddot{a}(t)}{a} = -\frac{4\pi G(t)}{3}\left(\rho + \frac{3p}{c^2(t)}\right). \tag{9}$$

Equation (9) is called the Raychaudhuri equation, and from the above system one can build a generalized conservation equation

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2(t)}\right) = -\rho\frac{\dot{G}}{G} + \frac{3Kc^2}{8\pi G a^2}\frac{\dot{c}}{c}, \tag{10}$$

in which the time dependence of fundamental constants was explicitly taken into account. Alternatively, one can think of the Raychaudhuri equation together with the generalized conservation equation as a fundamental system for which Eq. (8) is a first integral.

The fundamental difficulty concerning the system (8)–(10) is that it is a nonautonomous system with unknown functions $G(t)$ and $c(t)$. In order to be specific in further analysis, we adopt Barrow's power-law ansatz

$$G(t) = G_0 a(t)^q, \quad c(t) = c_0 a(t)^n. \tag{11}$$

Moreover, we assume the hydrodynamical energy-momentum tensor with the equation of state for the noninteracting multifluid

$$p=\frac{\sum_{i=1}^{l}\gamma_i\rho_i c^2}{\sum_{i=1}^{l}\rho_i c^2}\epsilon=\gamma(a)\rho c^2, \tag{12}$$

where $\rho_i=\rho_{i0}a^{-3(\gamma_i+1)}$ and the energy density $\epsilon=\rho c^2$.

In the special case of a matter and radiation mixture, the factor $\gamma(a)$ depending on the scale factor a takes the form

$$\gamma(a)=\frac{1}{3}\frac{1}{\alpha a+1}, \quad p=0+\frac{1}{3}\epsilon_{\rm r}, \quad \epsilon=\epsilon_{\rm m}+\epsilon_{\rm r} \tag{13}$$

where $\alpha=\rho_{\rm m0}/\rho_{\rm r0}$.

If we substitute $l=1$ into Eq. (12) then we obtain models filled with single matter and with the equation of state $p=\gamma\rho c^2$. Generally, for noninteracting fluids with pressure $p=\Sigma_i\gamma_i\rho_i c^2$, the equation of state assumes the form $p=\gamma(a)\rho c^2$ where the factor γ can be parametrized by the scale factor. This fact is crucial for the reduction procedure.

The power-law ansatz (11) turns the field equations back into an autonomous system. Now we can think about extensions of the baseline equations. One can include the cosmological constant Λ in a straightforward way by introducing the pressure p_Λ and energy density ρ_Λ:

$$p_\Lambda=-\rho_\Lambda c^2(t), \tag{14}$$

$$\rho_\Lambda=\frac{\Lambda c^2(t)}{8\pi G(t)}. \tag{15}$$

The system (8)–(10) with a cosmological constant can be cast into the form

$$\left(\frac{\dot a}{a}\right)^2=\frac{8\pi G(t)\rho}{3}-\frac{Kc^2(t)}{a^2(t)}+\frac{\Lambda c^2(t)}{3}, \tag{16}$$

$$\frac{\ddot a}{a}=-\frac{4\pi G(t)}{3}\left(\rho+\frac{3p}{c^2(t)}\right)+\frac{\Lambda c^2(t)}{3}, \tag{17}$$

$$\dot\rho+3\frac{\dot a}{a}\left(\rho+\frac{p}{c^2}c^2(t)\right)=-\rho\frac{\dot G}{G}+\frac{3Kc^2}{8\pi G a^2}\frac{\dot a}{a}. \tag{18}$$

Equation (18) is easy to solve only for the case of $c=\text{const}$. Therefore, in our consideration of the general formulation of the dynamics we cannot use an explicit form of a solution of Eq. (18). To avoid this difficulty we consider a special procedure of reduction.

IV. REDUCTION TO A PLANAR HAMILTONIAN SYSTEM: THE GENERAL SOLUTION OF A DYNAMICAL PROBLEM

To construct a dynamical system we assume the form of the equation of state $p=\gamma(a)\rho c^2$ and calculate the density ρ using both equation (15) and (16). For simplicity we focus our attention on the case of $\Sigma=0$ corresponding to the VSL model with matter in the multifluid form. Then we obtain from Eq. (16)

$$\frac{8\pi G\rho}{3}=\frac{\dot a^2}{a^2}+\frac{Kc^2(t)}{a^2}-\frac{\Lambda c^2}{3} \tag{19}$$

and from Eq. (17)

$$-\frac{8\pi G\rho}{3}=\frac{1}{1+3\gamma(a)}\left(\frac{2\ddot a}{a}-\frac{2\Lambda c^2}{3}\right). \tag{20}$$

By adding the sides of the above equations we obtain a second order nonlinear equation with respect to the variable a,

$$\ddot a+\psi(a)\dot a^2+\kappa(a)=0, \tag{21}$$

where

$$\psi(a)=\frac{1+3\gamma(a)}{2a},$$

$$\kappa(a)=\left[\frac{K}{2a}[1+3\gamma(a)]-\frac{\Lambda}{2}a[1+\gamma(a)]\right]c^2(a)$$

and only the term $\kappa(a)$ depends on the cosmological constant.

Equation (21) can be rewritten as an autonomous dynamical system

$$\dot a=p, \tag{22}$$

$$\dot p=-\psi(a)p^2-\kappa(a). \tag{23}$$

To apply dynamical system theory, it is useful to reduce the system (22),(23) to a form with polynomial right-hand sides:

$$a'=\frac{da}{d\eta}=pa, \tag{24}$$

$$p'=\frac{dp}{d\eta}=-\frac{1}{2}[3\gamma(a)+1]p^2-\phi(a), \tag{25}$$

where $\phi(a)=a\kappa(a)$ and $t\to\eta$:$dt/a=d\eta$. The solution of Eqs. (24),(25) represents a phase curve in the phase space $(a,p)=(a,\dot a)$.

The solution of Eq. (21) may be given after the substitution $\dot a=p(a)$ or equivalently by taking the quotient of Eqs. (22) and (23). Then we obtain

$$p\frac{dp}{da}+\psi(a)p^2+\kappa(a)=0. \tag{26}$$

Equation (26) takes the form of the Bernoulli equation and after the standard substitution $u(a)=p^2$, $u'=du/da$ we obtain the nonautonomous system

$$u' + 2\psi(a)u + 2\kappa(a) = 0. \tag{27}$$

Finally, the solution of Eq. (21), passing through the point $(a_0, p_0(a_0))$, can be given in the following form:

$$u(a) = \exp\left(-2\int_{a_0}^{a}\psi(a)da\right)\left[p_0^2(a_0) - \int_{a_0}^{a}2\kappa(a)\exp\left(2\int_{a_0}^{a}\psi(a)da\right)da\right],$$

i.e.,

$$p^2(a) = \exp\left(-\int_{a_0}^{a}\frac{3\gamma(a)+1}{a}da\right)\left\{p_0^2(a_0) - 2\int_{a_0}^{a}\left[\frac{K[1+3\gamma(a)]}{2a} - \Lambda a[\gamma(a)+1]\right]\exp\left(\int_{a_0}^{a}\frac{3\gamma(a)+1}{a}da\right)da\right\}.$$

To consider the case of a mixture of matter and radiation, we substitute the special form of $\gamma(a)$ from the formula (13) and then we obtain

$$p^2(a) = \frac{a_0^2(\alpha a+1)}{a^2(\alpha a_0+1)}\left\{p_0^2(a_0) - 2\int_{a_0}^{a}\frac{a^2}{\alpha a+1}da\left[\frac{K[1+3\gamma(a)]}{2a} - \Lambda a\left(1+\frac{1}{3(\alpha a+1)}\right)\right]c^2(a)\right\} \tag{28}$$

and the general solution of Eq. (27) has the form

$$u(a) = \exp\left(-2\int_{a_0}^{a}\psi(a)da\right)\times\left[C - 2\int_{a_0}^{a}\kappa(a)\exp\left(2\int_{a_0}^{a}\psi(a)da\right)da\right]. \tag{29}$$

This allows us to formulate an expression that can be treated as a first integral of the system (24),(25). It is characteristic for dynamical systems of general relativity and cosmology that a first integral can be used in constructing a Hamiltonian function. A first integral can be represented as algebraic curves in the phase space. These algebraic curves are given by

$$p^2(a)\exp\left(2\int_{a_0}^{a}\psi(a)da\right) + 2\int_{a_0}^{a}\kappa(a)\exp\left(2\int_{a_0}^{a}\psi(a)da\right)da = C. \tag{30}$$

Thus in the case considered we obtain

$$\dot{a}^2\frac{a^2}{\alpha a+1} + 2\int_{a_0}^{a}\kappa(a)\frac{a^2}{\alpha a+1}da = C. \tag{31}$$

Now if we introduce a new variable x such that

$$\frac{1}{\sqrt{2}}dx = \frac{a\,da}{\sqrt{\alpha a+1}}$$

Eq. (31) can be written as

$$\frac{\dot{x}^2}{2} + V(x) = V(a_0) = \text{const}, \tag{32}$$

where

$$V(x) = 2\int^{x}\frac{\kappa[a(x)]a(x)}{\alpha a(x)+1}dx$$

plays the role of the potential with $a(x):(1/\sqrt{2})x = (2/3\alpha^2)\sqrt{\alpha a+1}(\alpha a-2)$.

This procedure works successfully for any function $\psi(a)$. It is sufficient to replace the expression $a^2/(\alpha a+1)$ in the considered case by $\exp[2\int^a\psi(a)da] \equiv \phi^2$.

In the special case of $\alpha = 0$ we obtain that $p = \epsilon/3$ and we can use the standard formalism considered in [28].

Now we can see from formula (28) that algebraic curves on which lie trajectories of the system take a complicated form. Therefore it is useful to visualize them in the phase space. Using the form of the first integral (32), we can also classify all possible solutions by considering a limiting curve $\dot{x} = 0$ and derive the relation $\Lambda(a)$ as was presented in the classical case [28].

There are two kinds of different methods of reducing Eq. (21) to the form of a Newtonian equation of motion in a one-dimensional configuration space. First, after introducing the new rescaled variable $a \to x$, we obtain the dynamics in the form $\ddot{x} = -\partial V/\partial x$. Second, after introducing the new time variable, say $\tau(t)$, defined in such a way that the term $\psi(a)\dot{a}^2$ can be dropped in this parametrization, we obtain the dynamics in the form $x'' = d^2a/d\tau^2 = -\partial V/\partial a$.

Let us note that the function $V(a(x))$ plays the role of the potential for a particle whose position is given by x and motion described by

$$\ddot{x} = -\frac{dV(a(x))}{dx}.$$

Szydłowski *et al.* [28] showed that the choice of new variables $x = x(a)$ allows one to reduce the dynamics of classical FRW models with matter or radiation to a one-dimensional Newtonian equation of motion. It is also interesting that there is a similar possibility of reducing the system (21) to the Hamiltonian form for any equation of state $\gamma = \gamma(a)$, for example, for any mixture of noninteracting fluids. To perform this let us consider a general nonlinear reparametrization of the variable a such that

$$x = a^{D[\gamma(a)]} = a^{D(a)}. \tag{33}$$

It can be shown that Eq. (21) can be reduced to the form of a Newtonian equation of motion, $\ddot{x}=-dV(x)/dx$, if the multiplicative coefficient appearing in $\dot{a}^2$ vanishes. This condition gives us the following:

$$D_{aa}(\ln a)+\frac{2D_a}{a}-\frac{D}{a^2}\left(D_a\ln a+\frac{D}{a}\right)^2=\psi(a)\left(D_a\ln a+\frac{D}{a}\right), \tag{34}$$

where $D_a=\partial D/\partial a$. In the special case of $\gamma(a)=\text{const}$ ($D_a=0$), the classical results can be recovered [28]; we have $D=2$ for pure radiation, $D=3/2$ for dust, and $D=\frac{3}{2}(1+\gamma)$ for a perfect fluid with $p=\gamma\rho$. If $\gamma=\gamma(a)$ is any function of a then $D(a)$ is a solution of Eq. (34). After the simple substitution

$$D_a(\ln a)+\frac{D(a)}{a}=z(a), \tag{35}$$

we obtain that $z(a)$ is a solution of the equation

$$z+\frac{1}{z}\frac{dz}{da}=\psi(a). \tag{36}$$

In this new variable

$$\ddot{x}=-\kappa[a(x)]z[a(x)]x=-\frac{\partial V}{\partial x},$$

and the dynamics is reduced to the case of a nonlinear oscillator with "springlike" tension $k(x)=\kappa(a)z(a)$.

The information about the equation of state is hidden in the function $\psi(a)$ and after finding the solution $z(a)$ for a specific form of the equation of state it should be easy to find $D(a)$ from Eq. (35). It can be easily shown that the corresponding equation determining $z(a)$ is the Bernoulli equation, for which the solution is

$$z(a)=\frac{\phi(a)}{\int^a\phi(a)da}=\frac{d}{da}\ln\left(\int^a\phi(a)da\right),$$

where $\phi(a)\equiv\exp[\int\psi(a)da]$. If we put $z(a)$ into Eq. (35) we can find that

$$D(a)=\frac{\ln\int^a\phi(a)da}{\ln a}=\log_a\int^a\phi(a)da.$$

For the case of a mixture of radiation and dust it has the simple form

$$z(a)=\begin{cases}\dfrac{3\alpha^2}{2}\left|\dfrac{a}{(\alpha a+1)(\alpha a-2)}\right| & \text{for } \alpha\neq 0,\\[2mm] \dfrac{3}{2a} & \text{for } \alpha=\infty,\\[2mm] \dfrac{2}{a} & \text{for } \alpha=0,\end{cases}$$

where $\phi(a)=a/\sqrt{\alpha a+1}$ and $\int^a\phi(a)da=(2/3\alpha^2)|\sqrt{\alpha a+1}(\alpha a-2)|$.

Therefore we obtain for the case of noninteracting matter and radiation

$$D(a)=\begin{cases}\log_a\left(\dfrac{2\sqrt{\alpha a+1}|\alpha a-2|}{3\alpha^2}\right) & \text{for } \alpha\neq 0,\\[2mm] 2 & \text{for } \alpha=0,\\[2mm] \dfrac{3}{2} & \text{for } \alpha=\infty.\end{cases}$$

It can be proved that in a general situation we have the following relation:

$$\int^a\phi(a)da=x(a)=\int^a\sqrt{a}\exp\left(\frac{3}{2}\int^a\frac{\gamma(a')}{a'}da'\right)da \tag{37}$$

and

$$x(a)=\begin{cases}a^{\log_a(2\sqrt{\alpha a+1}|\alpha a-2|/3\alpha^2)} & \text{for } \alpha\neq 0,\\ a^2 & \text{for } \alpha=0,\\ a^{3/2} & \text{for } \alpha=\infty,\end{cases}$$

i.e., for any fluid (or its mixture) that satisfies the equation of state for so-called "quintessence" matter $p=\gamma(a)\rho c^2(a)$ we can always find the corresponding $D(a)$.

Due to Eq. (37) the equation of motion can be rewritten in the simplest form

$$\ddot{x}=-\kappa[a(x)]\phi[a(x)],\quad V(x)=\int^x\kappa[a(x)]\phi[a(x)]dx. \tag{38}$$

Let us note that there is also the possibility of generalizing such a result to the case of nonvanishing shear in $B(I)$ or $B(V)$ models when $\sigma\propto x^{3/2}$.

In the second approach it is useful to reparametrize the time variable t in such a way that

$$t\to\tau: dt=\phi[a(\tau)]d\tau, \tag{39}$$

where $\phi(a)$ is a yet-to-be-determined function which should be chosen in such a way that the term $\psi(a)\dot{a}^2$ is absent in Eq. (21). We can do that provided that ϕ satisfies the condition

$$\phi\equiv\exp\int^a\psi(a)da. \tag{40}$$

Then Eq. (21) assumes a form similar to the equation for the motion of a nonrelativistic particle in the external field with the potential $V(a)$, namely,

$$\frac{d^2a}{d\tau^2}\equiv a''=-\frac{\partial V}{\partial a} \tag{41}$$

and

$$V(a)=\int^{a}\kappa(a)\phi^{2}(a)da, \tag{42}$$

where

$$\psi(a)=\frac{1+3\gamma(a)}{2a},$$

$$\kappa(a)=\left[\frac{K}{2a}[1+3\gamma(a)]-\frac{\Lambda}{2}a[1+\gamma(a)]\right]c^{2}(a).$$

Therefore for such a system the Hamiltonian takes the form

$$\mathcal{H}(a,p)=\frac{p^{2}}{2}+V(a)=E=\text{const}, \tag{43}$$

where the correspondence with the vacuum case is reached after putting $E=0$ and $\gamma(a)=0$.

The advantage of this procedure of reduction is its simplicity. The new time variable τ is a monotonic function of Newtonian time t and motion is represented in the form of a one-dimensional Hamiltonian system with the potential

$$V(a)=\int^{a}\kappa(a)\phi^{2}(a)da=c_{0}^{2}\int^{a}\left[\frac{K}{2a}[1+3\gamma(a)]-\frac{\Lambda}{2}a[1+\gamma(a)]\right]a^{2n}\phi^{2}(a)da. \tag{44}$$

By comparison of the potentials (32) and (44), we can observe that both procedures give rise to the same form of the potential function as a function of a. Whereas the second approach seems to be simpler, the first one has the advantage that it allows one to discuss the dynamics in the origin time. In both cases we do not explicitly integrate the continuity equation (18) which gives us the relation $\rho(a)$. The effect of matter content is included in $\rho(a)$ and the energy constant E.

For the special case of matter content in the form of noninteracting matter and radiation, we obtain

$$V(a)=\frac{Kc_{0}^{2}}{2}\int^{a}\frac{(\alpha a+2)a^{2n+1}}{(\alpha a+1)^{2}}da+\frac{\Lambda c_{0}^{2}}{2}\int^{a}\frac{(3\alpha a+4)a^{2n+3}}{(\alpha a+1)^{2}}da, \tag{45}$$

where integrals in the above form of the potential can be given explicitly.

In the special case of $\Lambda=n=0$ we have

$$V(a)=\frac{Ka^{4}}{2(\alpha a+1)}. \tag{46}$$

Therefore the dynamics is given by the Hamiltonian equations

$$a'=\frac{\partial\mathcal{H}}{\partial p}, \tag{47}$$

$$p'=-\frac{\partial\mathcal{H}}{\partial a}, \tag{48}$$

which constitute the two-dimensional dynamical system. The entire evolution is represented by an evolutional path on the plane (a,p). The domain of acceleration $\ddot{a}(t)>0$ is determined by the condition

$$a''-(a')^{2}\psi(a)>0. \tag{49}$$

The above condition can be rewritten as

$$-\frac{\partial V}{\partial a}-(a')^{2}\psi(a)>0. \tag{50}$$

Let us note that Eq. (50) can be formulated equivalently as the following condition in some domain of the configuration space $\{a:a\geqslant 0\}$:

$$-\frac{\partial V}{\partial a}-2[E-V(a)]\psi(a)>0 \tag{51}$$

or

$$\frac{1}{\psi(a)}\frac{\partial V}{\partial a}-2V(a)<-2E, \tag{52}$$

where $\partial V/\partial a=-\kappa(a)\phi^{2}(a)$ and $\psi(a)=[1+3\gamma(a)]/2a$.

V. EVOLUTION OF THE VSL DYNAMICAL SYSTEM IN PHASE DIAGRAMS

A. Background

In the further qualitative analysis of the dynamical system (22),(23) we consider the matter as a mixture of radiation and dust. Then the system (22),(23) takes the form of an autonomous system with rational right-hand sides:

$$\dot{a}=p, \tag{53}$$

$$\dot{p}=-\frac{\alpha a+2}{2a(\alpha a+1)}p^{2}-\left(\frac{K(\alpha a+2)}{2a(\alpha a+1)}-\frac{3\alpha a+4}{6(\alpha a+1)}\Lambda a\right)c^{2}(a), \tag{54}$$

where $c^{2}(a)=c_{0}^{2}a^{2n}$, $n\leqslant 0$, $\alpha=\epsilon_{\mathrm{m}0}/\epsilon_{\mathrm{r}0}$ [for regularization of the system at $a=0$ it is useful as in Eqs. (24),(25) to introduce the time $\eta: dt/a=d\eta$]. Of course the above system possesses the first integral in the form (28).

In the finite domain, the system (53),(54) has at most one critical point, which corresponds to an extremum of the function $V(a):(dV/da)|_{a=a_{0}}=0$, $p_{0}(a_{0})=0$. The stability of this point is determined from the convexity of a diagram of the potential function $V(a)$. There are two limit cases corresponding to the equation of state: of dust $(\alpha\rightarrow\infty)$ and of pure radiation ($\alpha=0$). From the system (53),(54) in the latter case we obtain the VSL system with pure radiation

$$\dot{a}=p, \tag{55}$$

$$\dot{p}=-\frac{1}{a}p^2-\left(\frac{K}{a}-\frac{2}{3}\Lambda a\right)c^2(a). \tag{56}$$

In the above system instead of a in the spirit of the first approach we introduce a new variable $x\equiv a^2$ and we obtain the system in a simpler form:

$$\dot{x}=y, \tag{57}$$

$$\dot{y}=-2\left(K-\frac{2}{3}\Lambda x\right)c^2[a(x)], \tag{58}$$

where $c^2[a(x)]=c_0^2a^{2n}=c_0^2x^n$. The phase portraits of the system (57),(58) for $n=-2$ are presented on Fig. 1.

In the projective coordinates (z,u) the above system takes the form

$$\dot{z}=-zu, \tag{59}$$

$$\dot{u}=-2\left(Kz-\frac{2}{3}\Lambda\right)c_0^2z^{-n}-u^2. \tag{60}$$

This form of the system is useful in the analysis of the behavior of trajectories at infinity, because $z=0$ corresponds a circle at infinity, $x=\infty$, which bounds the phase plane. The phase portraits of the system (59),(60) for $n=-2$ are presented on Fig. 2.

In this case the first integral (28) takes the form

$$\frac{\dot{x}^2}{2}+2\int\left(K-\frac{2}{3}\Lambda x\right)c^2(x)dx=\bar{C}=\text{const}. \tag{61}$$

Let us note that in the special cases of $n=-1$ and $n=-2$ the potential takes the particular form $V(x)=2[K\ln x-(2/3)\Lambda x]$, $V(x)=2[-K/x-(2/3)\Lambda\ln x]$.

It is clear that the first integral (61) is in fact the integral of energy, because the system (57),(58) is a Hamiltonian dynamical system with the Hamiltonian

$$\mathcal{H}(p,x)=\frac{p^2}{2}+2\left(\frac{Kx^{n+1}}{n+1}-\frac{2}{3}\Lambda\frac{x^{n+2}}{n+2}\right)\equiv\bar{C}=\text{const}>0.$$

Now the integral of energy can be used in the classification of all possible evolutions modulo their quantitative (i.e., in accuracy to differential type) properties.

In the qualitative classification, for any case of $\gamma(a)$, the first integral (28) may be useful as in the method of classification previously used in [28]. It assumes the form

$$\frac{p^2}{2}+V(a(x))=V(a_0) \tag{62}$$

where

$$V(a)=\int^{a(x)}\frac{a^2}{\alpha a+1}\left[\frac{K[1+3\gamma(a)]}{2a}-\Lambda a\left(1+\frac{1}{3(\alpha a+1)}\right)\right]c^2(a)da,$$

$V(a_0)=\text{const}>0$, $x=(2\sqrt{2}/\alpha^2)t^2(t-1)$, $t=\sqrt{\alpha a+1}$, and $x=a^2$ for $\alpha=0$.

From Eq. (62), after imposing the condition $p=0$, we can calculate Λ from the expression of the function $\Lambda(a)$ which constitutes a boundary of the domain of configuration space admissible for motion:

$$p^2\geqslant 0\Leftrightarrow V(a)-V(a_0)<0$$

and

$$\Lambda(a)\geqslant\frac{\int^a[a(\alpha a+2)/2(\alpha a+1)^2]Kc^2(a)da-V(a_0)}{\int^a[(3\alpha a+4)a^2/3(\alpha a+1)^2]c^2(a)da}. \tag{63}$$

By consideration of the boundary $\partial D:\{(a,\Lambda):\Lambda(a)=0\}$ and construction of the levels $\Lambda=\text{const}$ we obtain a qualitative classification of all possible trajectories in the space (Λ,a) or (Λ,x).

B. Interpretation of the acceleration of scale factors and the absence of a particle horizon

A great advantage of the phase-space dynamical description is the ability to discuss the distribution of models with given properties. In other words, one can imagine an ensemble of models starting from different initial conditions and ask how a given property is distributed in the ensemble. Now we formulate sufficient conditions for solving the flatness and horizon problems in terms of phase-space relations. Let us recall that the flatness problem is solved whenever the scale factor's acceleration is positive,

$$\ddot{a}(t)>0.$$

This condition is satisfied in the subspace $\mathcal{D}_{\text{flat}}$ of the phase space:

$$\mathcal{D}_{\text{accel}}=\left\{(a,\dot{a}):-\frac{\alpha a+2}{2a(\alpha a+1)}p^2-\left(\frac{K(\alpha a+2)}{2a(\alpha a+1)}-\frac{3\alpha a+4}{6(\alpha a+1)}\Lambda a\right)c^2(a)>0\right\}. \tag{64}$$

This means that trajectories representing the histories of VSL universes undergo an accelerated expansion while staying in the region $\mathcal{D}_{\text{flat}}$. One can restate relation (64) using the Hamiltonian constraint $p^2=2[V(a_0)-V(a)]$. It is easy to see that the corresponding condition expressed purely in terms of configuration space reads

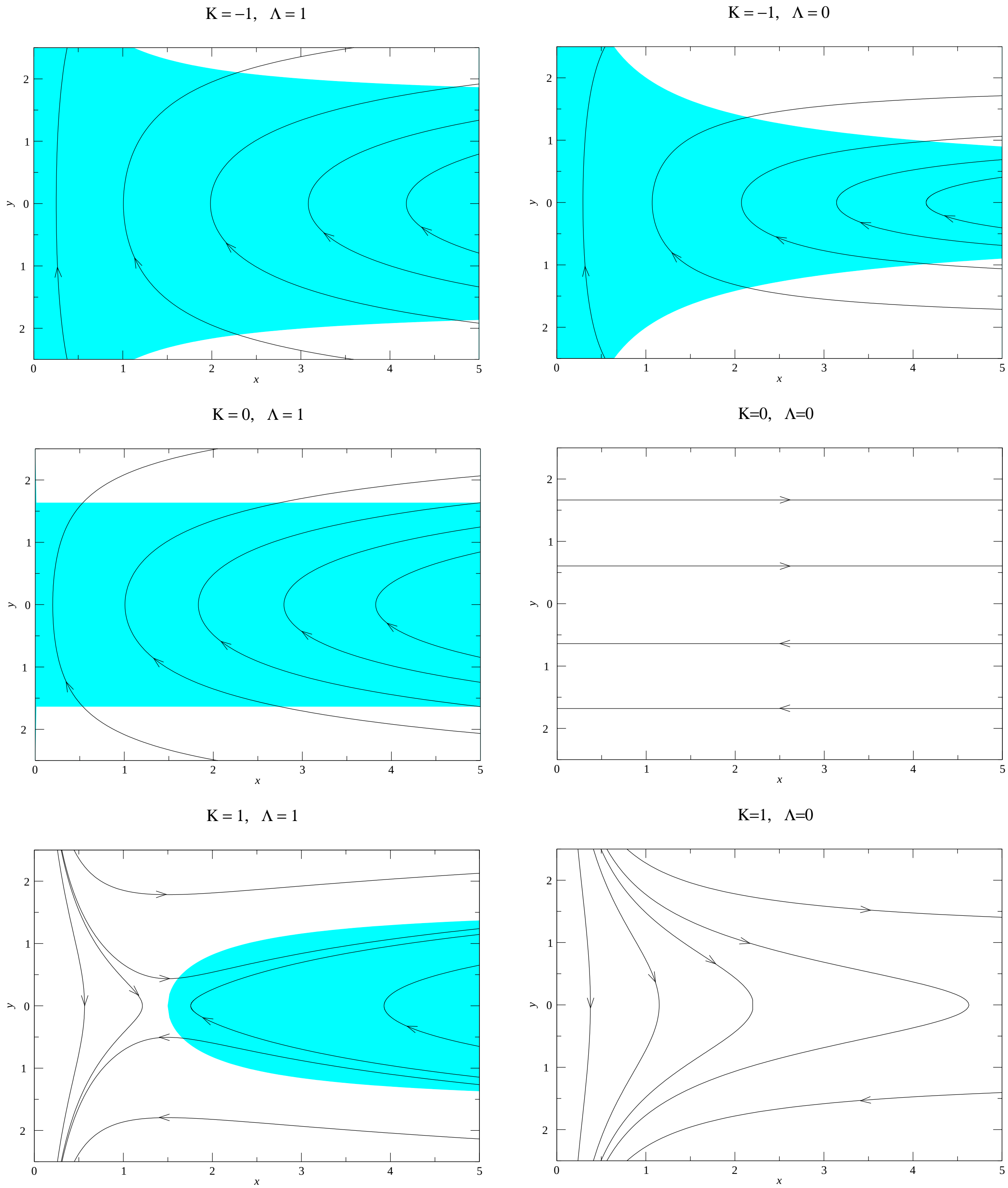

FIG. 1. The phase portrait for the system (57),(58) for $n=-2$. For the cases $(K=-1,\Lambda=1)$, $(K=-1,\Lambda=0)$, $(K=0,\Lambda=1)$ the qualitative structure of the phase space is the same. In the finite domains of phase space there are no critical points. The typical trajectories represent the solution starting from the singularity-free stage $x=\infty$ at $t=0$, then reaching the stage $x=x_0, y=0$, and going to infinity. In these three cases the trajectories pass through the acceleration region (the shaded region). The largest acceleration region is for $(K=-1,\Lambda=1)$. These models accelerate for a finite interval of time, and this acceleration happens to models without a cosmological constant. The closed models for $(K=1,\Lambda=0)$ are the typical oscillating models. There is no acceleration for $(K=1,\Lambda=0)$, $(K=0,\Lambda=0)$. In the case of $(K=1,\Lambda=1)$ a saddle appears and we have two additional types of evolution: the Lemaître model with a quasistatic stage of evolution and the aforementioned open and flat models. The acceleration region is in the middle of the quasistatic phase of evolution of the Lemaître model.

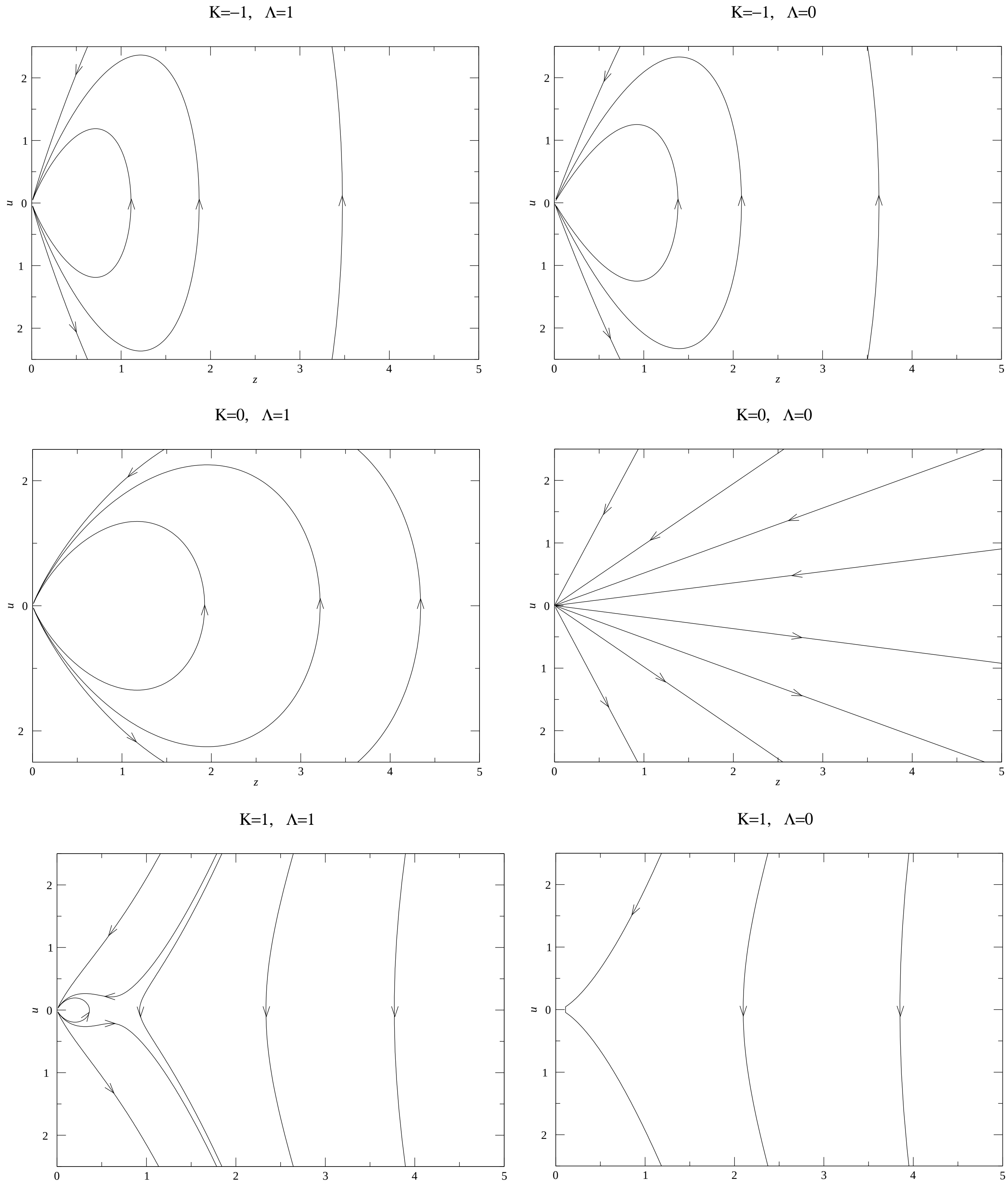

FIG. 2. The phase portrait for the system (59),(60) for $n=-2$ in the projective coordinates $z=1/x, u=y/x$. All critical points at infinity correspond to $z_0=0$. There are two types of critical points $(z_0,u_0)=(0,0)$ and $(z_0,u_0)=(2\Lambda/3K,0)$. For all points $\mathrm{tr}\,A=0$ (where A is a linearization matrix of the system considered), i.e., at least one of the eigenvalues is zero. If $(\Lambda>0,\ K=1)$ we have a saddle point at $(z_0,0)$. The presence of degenerate points at infinity (0,0) indicates that the system is structurally unstable, in contrast to models with a positive cosmological constant and constant velocity of light where $(0,\sqrt{4\Lambda/3})$ represents a stable node (the de Sitter stage). For $n>0$ there is no critical point at (0,0) and the models are structurally stable.

$$\mathcal{D}_{\mathrm{accel}}=\left\{a:-\frac{\alpha a+2}{a(\alpha a+1)}[V(a_0)-V(a)]-\left(\frac{K(\alpha a+2)}{2a(\alpha a+1)}-\frac{3\alpha a+4}{6(\alpha a+1)}\Lambda a\right)c^2(a)>0\right\}. \tag{65}$$

Analogous criteria for acceleration if the dynamics is covered by Eqs. (47),(48) are given by Eq. (50) in phase space and Eq. (52) in the configuration space.

Another interesting question concerns the horizon problem. It is easy to prove the following criterion for avoiding the horizon problem.

Theorem 1. The FRW cosmological model does not have an event horizon near the singularity if $\dot{a}(t)c^{-1}(t)$ tends to a constant while $a(t)$ tends to zero.

Proof. When all events whose coordinates at past time are located beyond some distance d_H can never communicate with the observer at the coordinate $r=0$ in the Robertson-Walker metric, we can define the distance d_H as the past event horizon distance. It is given by

$$d_H(t)=a(t)\int_0^{t_0}\frac{dt'c(t')}{a(t')}=a(t)I.$$

Whenever I diverges as $t\to 0$ there is no past event horizon in the space time geometry. On the other hand, when I converges the space time exhibits a past horizon

$$\int_t^{t_0}\frac{dtc(t)}{a(t)}=\int_t^{t_0}\frac{a^n da}{a da/dt}=\int_t^{t_0}\frac{1}{c^{-1}\dot{a}}\frac{da}{a}.$$

Let $c^{-1}(t)\dot{a}<A$; then

$$I\geqslant\frac{1}{A}\int_0^{a_0}\frac{da}{a}=\frac{1}{A}(\ln a_0+\infty).$$

On the other hand, when $\dot{a}<\bar{A}$ is bounded then

$$I=\int_t^{t_0}\frac{dtc(t)}{a(t)}=\int_t^{t_0}\frac{a^n da}{a da/dt}=\int_t^{t_0}a^{n-1}\frac{da}{\dot{a}}$$

and $I\geqslant(1/\bar{A})\int_0^{a_0}a^{n-1}da$ or

$$\bar{A}\int_t^{t_0}\frac{dtc(t)}{\dot{a}(t)}\geqslant\frac{a^n}{n}.$$

Therefore I diverges as $a\to 0$ if $n<0$ and $\dot{a}<\bar{A}$. ■

The above criterion can be reformulated in the language of phase space in the form

$$a\to 0 \quad\text{and}\quad c^{-1}(t)\dot{a}(t)\to\text{const},$$

$$\text{i.e.,}\quad a^{-2n}\left(\frac{da}{dt}\right)^2\to(\text{const})^2.$$

For example, for the radiation case $a=\sqrt{x}$ and then the past horizon is eliminated only if $\dot{x}^2x^{-2n-1}\to\text{const}$ as $x\to 0$. After substituting the first integral (61) we obtain that there is an n such that the horizon disappears if and only if $n<-1$. Let us note that the above proof is based on the Hamiltonian constraint and is independent of any specific assumption about an equation of state or $a(t)$ near the singularity. If we assume power-law behavior of $a(t)$ then Barrow's result can be simply achieved [17], namely, $a(t)\propto t^{2/3(\gamma-1)}$ if $2\geqslant 2n+3(\gamma+1)$.

C. The potential function for a mixture of dust and radiation

The boundary of the domain in the configuration space (x space or a space) admissible for motion is determined by the expression

$$V(a(x))=Kc_0^2\int^{a(x)}\frac{a^{1+2n}(\alpha a+2)}{(\alpha a+1)^2}da-\Lambda c_0^2\int^{a(x)}\frac{(3\alpha a+4)a^{3+2n}}{3(\alpha a+1)^2}da=V(a_0)>0.$$

The equation $V(a(x))-V(a_0(x_0))=0$ can be represented in the space (Λ,a) or more correctly in the space (Λ,x) as a boundary curve for the classification. Instead of the inverse function $a(x)$ (it is difficult to give it in a simple form), we take the function $x(a)$ as

$$V(a_0)-Kc_0^2\int^a\frac{a^{1+2a}}{2(\alpha a+1)}da-Kc_0^2\int^a\frac{a^{1+2n}}{2(\alpha a+1)^2}da+\Lambda c_0^2\left[\int^a\frac{a^{3+2n}}{3(\alpha a+1)}da+\frac{1}{3}\int^a\frac{a^{3+2n}}{(\alpha a+1)^2}da\right]=0,$$

where the physical domain is $V(a_0)-V(a)>0$, i.e.,

$$\Lambda\geqslant\frac{-V(a_0)+\frac{1}{2}Kc_0^2\int^a[a^{1+2n}/(\alpha a+1)]da+\frac{1}{2}Kc_0^2\int^a[a^{1+2n}/(\alpha a+1)^2]da}{c_0^2\left[\int^a[a^{3+2n}/3(\alpha a+1)]da+\frac{1}{3}\int^a[a^{3+2n}/(\alpha a+1)^2]da\right]},$$

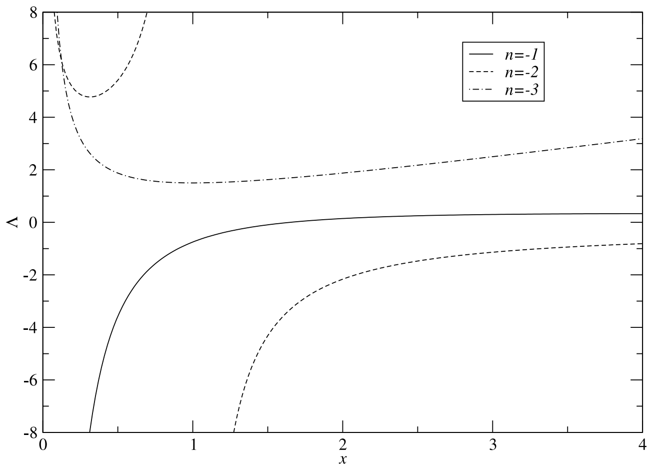

FIG. 3. The relationship between Λ and x for $K=1$. The qualitative evolution of the models is represented by levels of $\Lambda=$const. The domain under the characteristic curves $\Lambda(a)$ is classically forbidden (apart from the case of $n=-2$ and $\Lambda<0$). For $n=-1$ and $\Lambda<0$ all models start from a singularity and oscillate. For $n=-1$ and $\Lambda>0$ all models start from the singularity and expand. For $n=-2$ and $\Lambda<0$ all models start from a finite scale factor and expand. For $n=-2$ and $\Lambda>0$ there are oscillating universes which start from a finite scale factor. For $n=-3$ all models are oscillating and have no singularity.

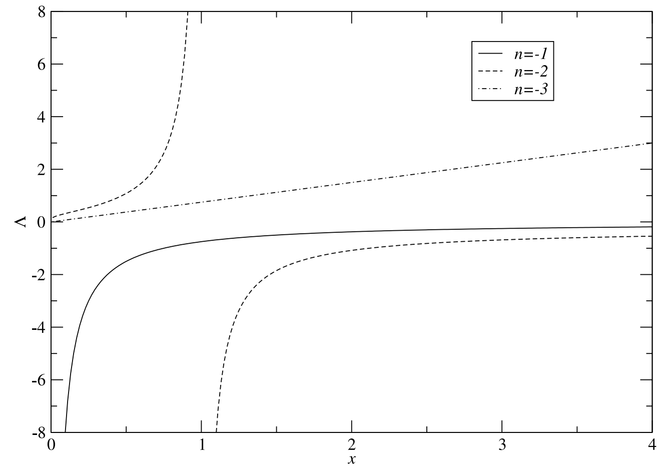

FIG. 4. The relationship between Λ and x for $K=0$. The qualitative evolution of the models is represented by levels of $\Lambda=$const. The domain under the characteristic curves $\Lambda(a)$ is classically forbidden (apart from the case of $n=-2$ and $\Lambda<0$). For $n=-1$ and $\Lambda<0$ all models oscillate. For $n=-1$ and $\Lambda>0$ the models evolve from a singularity to infinity. For $n=-2$ and $\Lambda<0$ the models expand. For $n=-2$ and $\Lambda>0$ the models also oscillate starting from a singularity. For $n=-3$ and $\Lambda<0$ there is no solution. For $n=-3$ and $\Lambda>0$ all models oscillate.

where

$$\int \frac{da}{a^m(\alpha a+1)}=\sum_{k=1}^{m-1}\frac{(-1)^k\alpha^{k-1}}{(m-k)!a^{m-k}}+(-1)^m\alpha^{m-1}\ln\frac{\alpha a+1}{a},$$

$$\int \frac{da}{a^l(\alpha a+1)^2}=-\frac{1}{\alpha}\left[\frac{1}{a^l(\alpha a+1)}+l\left(\sum_{k=1}^{l}\frac{(-1)^k\alpha^{k-1}}{(l+1-k)!a^{l+1-k}}+(-1)^{l+1}\alpha^l\ln\frac{\alpha a+1}{a}\right)\right].$$

On Figs. 3–5, for simplicity of presentation without loss of generality, it is assumed that $c_0=1$, $V(a_0)=1$, K, α are parameters, and n is chosen as -1, -2, -3.

VI. TUNNELING IN *n* DECAYING COSMOLOGIES

In the classical VSL cosmology a particle trajectory is determined through a knowledge of both a position x and a canonically conjugate momentum p_x. In the quantum VSL cosmology the notion of trajectories loses its classical meaning due to the uncertainty relation (x and p_x are replaced by noncommuting operators). The Wheeler-DeWitt equation

$$\left[\frac{\partial^2}{\partial x^2}-V(x)\right]\psi(a)=0 \tag{66}$$

is identical to the one-dimensional time-independent Schrödinger equation for a one-half unit particle of energy E subject to the potential $V(x)$ [$\psi(a)$ is known as the wave function of the Universe].

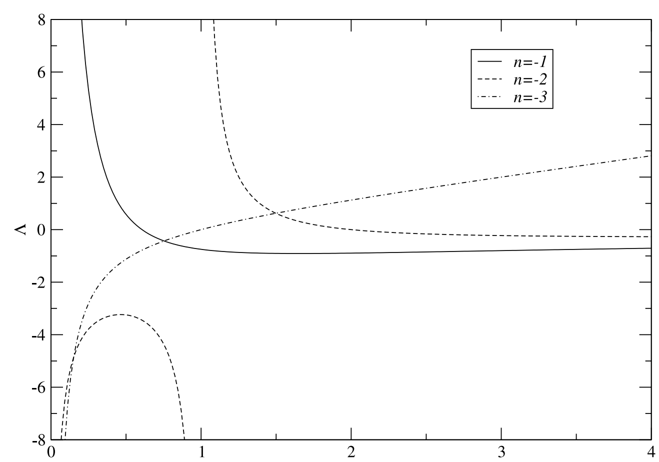

FIG. 5. The relationship between Λ and x for $K=-1$. The qualitative evolution of the models is represented by levels of $\Lambda=$const. The domain under the characteristic curves $\Lambda(a)$ is classically forbidden (apart from the case of $n=-2$ and $\Lambda<0$). For $n=-1$ and $\Lambda<0$ there is no solution. For $n=-1$ and $\Lambda>0$ the models start from finite size and expand. For $n=-2$ and $\Lambda<0$ all models oscillate without a singularity. For $n=-2$ and $\Lambda>0$ the models start from a finite size of the scale factor to $x=\infty$. For $n=-3$ all models oscillate with a singularity.

The "particle-universe" can quantum mechanically tunnel through the potential barrier. Let us consider the case when the region beneath the barrier $0<a<a_0$ is classically forbidden whereas the region $a\geqslant a_0$ is classically allowed.

We can adopt a simple method of calculating the amplitude for the quantum creation of the VSL FRW universe from zero size to $a=a_0=\sqrt{3(n+2)/2(n+1)\Lambda}$:

$$|\langle \text{VSLFRW}(a_0)|\text{nothing}\rangle|^2 = P \cong \exp\left[-\frac{2}{\hbar}\int_0^{x_0}\sqrt{2(E-V)}dx\right]. \tag{67}$$

The above formula (called Gamov's formula) gives us the tunneling probability because the quantized VSL FRW universe is mathematically equivalent to a one-dimensional particle of unit mass.

As an example, let us go back to the previously considered case of the compact vacuum VSL model. The Hamiltonian for this case can be obtained if we put $E=0$ into $\mathcal{H}(p,x)$.

The region of the barrier $0<x<x_0$ is classically forbidden for a zero energy particle. Therefore one can find the probability that a particle at $x=0$ can tunnel to $x=x_0:V(x_0)=0;\ x_0=3(n+2)/2(n+1)\Lambda$.

After rescaling the variable $x\mapsto x/x_0=\kappa$ we obtain for Eq. (67)

$$P\cong \exp\left[-\frac{4}{\hbar}\left(\frac{3(n+2)K}{2(n+1)\Lambda}\right)^{(n+3)/2}\right]\int_0^1 \kappa^{n/2}\sqrt{\kappa(1-\kappa)}d\kappa, \tag{68}$$

where we assume $K=1,\ \Lambda\geqslant 0$, and $-1<n<0$; the potential $V(x)=x^{n+1}(1+x)$ has two extrema and two zeros. The relation (68) can be rewritten in the form

$$P\cong \exp\left[-\frac{2}{\hbar}F(n)\right] \tag{69}$$

where

$$F(n)=\left[\frac{3(n+2)}{2(n+1)\Lambda}\right]^{(n+3)/2}\frac{\sqrt{\pi}}{2}\frac{\Gamma((n+3)/2)}{\Gamma(3+n/2)}. \tag{70}$$

It is most probable that the closed and vacuum VSL FRW model with $-1<n<0$ is created when we have the maximum permissible energy density or the least size a_0. It occurs that the creation of a universe with constant c $(n=0)$ is most probable when classical spacetime emerges via the quantum tunneling process, whereas $c(a)$ is a decreasing function during the evolution of the Universe.

VII. CONCLUSIONS

Let us assume that one takes the idea of the varying speed of light seriously as a physical effect that might have happened in the very early Universe and today is confined to a very narrow range admissible by the inaccuracy of existing bounds on the variability of c. One of the problems arising then is to see how this modification of physics would change the evolution of standard Friedmann-Robertson-Walker cosmological models. So far only specific qualitative results are known concerning the solution of the flatness and horizon problems in VSL models. In the present work we attempted to extend this qualitative discussion in the sense that by constructing phase-space portraits of VSL cosmological models we were able to obtain a global view of their dynamics. In order to achieve this we used a power-law ansatz for the function $c(t)$ and investigated the classical Einstein equations with c allowed to be a function of time.

Two procedures of reduction of the dynamics are proposed. In the first case we reduced the dynamics of VSL models to a two-dimensional Hamiltonian dynamical system with a quadratic kinetic energy form and a potential function depending on a generalized scale factor. In the second one we reparametrized the time variable but the scale factor remained a state variable. In both cases the shape of the potential and the existence of the energy integral were used to classify possible evolutions of VSL models. These possibilities comprise models evolving from a singularity to infinity, oscillatory behavior between initial and final singularities, Einstein–de Sitter type models evolving from a singularity to the static world, Lemaître-Eddington type models evolving from the static Einstein solution to infinity, models expanding to infinity from a finite size, and finally models starting and ending with finite scale factors.

We have dealt with the full global dynamics of VSL models. From the theoretical point of view the size of the class of models without horizon or with solved cosmological puzzles is important. We call this class of models generic if their inset in the open phase is open or has nonzero measure. This point of view is justified by the fact that if the solution of a cosmological puzzle is an attribute of a trajectory with given initial conditions, it should also be an attribute of another trajectory which starts with neighboring initial conditions.

We have shown that the assumed time dependence of the speed of light leads to a uniform evolution pattern of VSL models on the phase space. The criteria for solving the flatness and horizon problems were formulated in terms of the phase space. It is an advantage of the phase-space approach that one can trace the patterns of evolution for all possible initial conditions. We have depicted, on respective phase portraits, the regions where the flatness problem is solved. The models where the region of initial conditions leading to flatness and horizon problem avoidance is large play a distinguished role. From this perspective open $(K=-1)$ models with positive cosmological constant $\Lambda>0$ are preferred in the class of VSL FRW models filled with radiation.

The formalism presented in this paper can easily be extended to the case where the matter content of the model is a mixture of different types of matter and to the case of models with shear (e.g., Bianchi type I or V models).

This formalism can also be treated as a starting point for the application of quantum cosmology to the description of early stages of evolution of the Universe [29,30]. The tunneling rate with an exact prefactor can be calculated to the first order in $\hbar$ for the closed VSL FRW model with a decaying variable velocity of light term $c(a)$. The tunneling probability P can be calculated in the WKB approximation given in the $V \gg E$ limit by Eq. (67). We consider closed vacuum VSL FRW models for which the potential is qualitatively classical. This implies that $-1<n$. In the interval $-1<n\leqslant 0$ the probability of tunneling increases as $F(n)$ monotonically decreases with increasing n. It is shown that the highest tunneling rate occurs for $n=0$; it corresponds to the standard FRW model.

In our work we showed the effectiveness of dynamical system methods in the investigation of VSL FRW models, namely, in the class of open models with a cosmological constant the acceleration has ''transitional'' character, i.e., there is a finite time when the trajectories are in the acceleration region, and the measure of this region normalized to the area of the phase plane is finite, even in the case of $\Lambda=0$.

We can argue that the VSL models considered are structurally unstable (Fig. 2) because of the presence of degenerate critical points at infinity for $n<0$. From the theoretical point of view such a situation seems to be unsatisfactory because in the space of all dynamical systems on the plane they form a set of zero measure (the Peixoto theorem).

The advantage of representing the dynamics in terms of the Hamiltonian is to allow discussion of how trajectories with interesting properties are distributed on the phase plane.

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