$1/R$ curvature corrections as the source of the cosmological acceleration

Dan N. Vollick

Department of Physics and Astronomy and Department of Mathematics and Statistics, Okanagan University College, 3333 College Way, Kelowna, British Columbia, Canada V1V 1V7

(Received 1 July 2003; published 24 September 2003)

Corrections to Einstein's equations that become important at small curvatures are considered. The field equations are derived using a Palatini variation in which the connection and metric are varied independently. In contrast with the Einstein-Hilbert variation, which yields fourth order equations, the Palatini approach produces second order equations in the metric. The Lagrangian $L(R)=R-\alpha^2/R$ is examined and it is shown that it leads to equations whose solutions approach a de Sitter universe at late times. Thus, the inclusion of $1/R$ curvature terms in the gravitational action offers an alternative explanation for the cosmological acceleration.

DOI: 10.1103/PhysRevD.68.063510 PACS number(s): 95.35.+d

INTRODUCTION

One of the most interesting aspects of modern cosmology concerns the acceleration of the cosmological expansion. Recent supernovae [1–4] and cosmic microwave background radiation [5–10] observations indicate that the expansion of the universe is accelerating, contrary to previous expectations.

Most attempts to explain this acceleration involve the introduction of dark energy as a source of the Einstein field equations. The nature of the dark energy is unknown but it behaves like a fluid with a large negative pressure. One possible candidate for the dark energy is a very small cosmological constant. Another, possibly related, problem involves the existence of dark matter. Observations of spiral galaxies, elliptical galaxies and galactic clusters indicate that these objects contain a large amount of dark matter. The difference between dark energy and dark matter is that dark matter clusters with the visible matter and dark energy is more or less uniformly spread throughout the universe.

Of course, one possibility is that we do not understand gravity on these large scales. Since dark energy and dark matter are needed to explain phenomena in regions of low curvature, we can attempt to modify Einstein's theory by adding corrections that become important when the curvature is small (see [13–16] for other approaches that involve modifications of Einstein's theory). Recently two attempts [11,12] to explain the cosmic acceleration along these lines have been made. They involve adding a term proportional to $1/R$ to the Einstein-Hilbert action and varying the action with respect to the metric (see [17,18] for other papers that consider nonpolynomial terms in the action). This approach leads to complicated fourth order equations that can be simplified by performing a canonical transformation and introducing a fictitious scalar field. It was shown in these papers that the modified field equations can produce the observed cosmological acceleration without the need of dark energy.

In this paper, I also consider a correction to the action that is proportional to $1/R$, but I use the Palatini variational principle to derive the field equations. In this approach $g_{\mu\nu}$ and $\Gamma^{\alpha}_{\mu\nu}$ are taken as independent variables. In Einstein gravity the variation with respect to $\Gamma^{\alpha}_{\mu\nu}$ gives the usual relationship between $\Gamma^{\alpha}_{\mu\nu}$ and the metric, i.e. $\Gamma^{\alpha}_{\mu\nu}$ is the Christoffel symbol associated with the metric $g_{\mu\nu}$. The variation with respect to $g_{\mu\nu}$ gives $R_{\mu\nu}(\Gamma)-\frac{1}{2}g_{\mu\nu}R(\Gamma)=-\kappa T_{\mu\nu}$, where $\kappa=8\pi G$. Thus, the Palatini variation is equivalent to the Einstein-Hilbert variation in Einstein's theory. This is not the case however for other Lagrangians. In fact, it has been shown [19,20] that the Palatini variation gives the usual vacuum Einstein equations for generic Lagrangians of the form $L(R)$. This is to be contrasted with they purely metric variation that produces fourth order equations. If matter is included the Palatini variation still produces second order equations, but they are no longer identical to Einstein's equations. In this paper, I show that the Palatini variation of the Lagrangian $L(R)=R-\alpha^2/R$, where α is a constant, leads to field equations that give an accelerating universe at late times.

THE FIELD EQUATIONS

The field equations follow from the variation of the action,

$$S=\int\left[-\frac{1}{2\kappa}L(R)+L_M\right]\sqrt{g}d^4x \tag{1}$$

where

$$R^{\alpha}_{\mu\nu\beta}=\partial_\beta\Gamma^{\alpha}_{\mu\nu}-\partial_\nu\Gamma^{\alpha}_{\mu\beta}+\Gamma^{\lambda}_{\mu\nu}\Gamma^{\alpha}_{\beta\lambda}-\Gamma^{\lambda}_{\mu\beta}\Gamma^{\alpha}_{\nu\lambda}, \tag{2}$$

$R_{\mu\nu}=R^{\alpha}_{\mu\alpha\nu}$, $R=g^{\mu\nu}R_{\mu\nu}$, $\kappa=8\pi G$, and L_M is the matter Lagrangian. Here we consider a Palatini variation of the action, which treats $g_{\mu\nu}$ and $\Gamma^{\alpha}_{\mu\nu}$ as independent variables.

Varying the action with respect to $g_{\mu\nu}$ gives

$$L'(R)R_{\mu\nu}-\tfrac{1}{2}L(R)g_{\mu\nu}=-\kappa T_{\mu\nu} \tag{3}$$

where $T_{\mu\nu}$ is the energy-momentum tensor and is given by

$$T_{\mu\nu}=-\frac{2}{\sqrt{g}}\frac{\delta S_M}{\delta g^{\mu\nu}}. \tag{4}$$

Varying the action with respect to $\Gamma^{\alpha}_{\mu\nu}$ gives

$$\nabla_\alpha[L'\sqrt{g}g^{\mu\nu}]-\tfrac{1}{2}\nabla_\sigma[L'\sqrt{g}g^{\sigma\mu}]\delta^\nu_\alpha-\tfrac{1}{2}\nabla_\sigma[L'\sqrt{g}g^{\sigma\nu}]\delta^\mu_\alpha=0. \quad (5)$$

By contracting over α and μ it is easy to see that this is equivalent to

$$\nabla_\alpha[L'(R)\sqrt{g}g^{\mu\nu}]=0. \quad (6)$$

This equation can be solved for the connection using a similar approach to that used in general relativity. Alternatively, one can define a metric $h_{\mu\nu}=L'g_{\mu\nu}$ and it is easy to see that Eq. (6) implies that the connection is the Christoffel symbol with respect to the metric $h_{\mu\nu}$. A conformal transformation back to the metric $g_{\mu\nu}$ gives (see [21] for the details on conformal transformations)

$$\Gamma^\alpha_{\mu\nu}=\left\{\begin{matrix}\alpha\\ \mu\nu\end{matrix}\right\}+\frac{1}{2L'}[2\delta^\alpha_{(\mu}\partial_{\nu)}L'-g_{\mu\nu}g^{\alpha\beta}\partial_\beta L'] \quad (7)$$

where the first term is the Christoffel symbol with respect to the metric $g_{\mu\nu}$. At first sight it might appear that this does not really define the connection since L' contains derivatives of the connection. However, if we contract the field equation (3) we get

$$RL'(R)-2L(R)=-\kappa T. \quad (8)$$

If this equation can be solved for $R=R(T)$, as we will assume here, then the terms in Eq. (7) involving $L'(R)$ can be expressed as derivatives of T. Since T contains only the metric and not its derivatives, the connection will involve only first derivatives of the metric and the field equations will then be second order in the metric $g_{\mu\nu}$.

The Ricci tensor and Ricci scalar are given by

$$R_{\mu\nu}=R_{\mu\nu}(g)-\tfrac{3}{2}(L')^{-2}\nabla_\mu L'\nabla_\nu L'+(L')^{-1}\nabla_\mu\nabla_\nu L'+\tfrac{1}{2}(L')^{-1}g_{\mu\nu}\Box L' \quad (9)$$

and

$$R=R(g)+3(L')^{-1}\Box L'-\tfrac{3}{2}(L')^{-2}\nabla_\mu L'\nabla^\mu L' \quad (10)$$

where $R_{\mu\nu}(g)$ is the usual expression for $R_{\mu\nu}$ in terms of $g_{\mu\nu}$ and $R=g^{\mu\nu}R_{\mu\nu}$.

If $T=0$ then the solutions to Eq. (8) will be constants and this implies that $R_{\mu\nu}=R_{\mu\nu}(g)$, and $R=R(g)$. Thus, in a vacuum the field equations will reduce to the Einstein field equations with a cosmological constant for a generic $L(R)$ (see [19,20] for the vacuum case). The field equations (3) can be written in the Einstein form

$$R_{\mu\nu}(g)-\tfrac{1}{2}R(g)g_{\mu\nu}=-\kappa S_{\mu\nu} \quad (11)$$

with a modified source $S_{\mu\nu}$ given by

$$S_{\mu\nu}=(L')^{-1}T_{\mu\nu}-\frac{1}{\kappa}\left[\frac{3}{2}(L')^{-2}\nabla_\mu L'\nabla_\nu L'-(L')^{-1}\nabla_\mu\nabla_\nu L'-\frac{1}{2}g_{\mu\nu}(L')^{-1}\Box L'+\frac{1}{2}\left(R-\frac{L}{L'}\right)g_{\mu\nu}\right]. \quad (12)$$

Now consider the Lagrangian

$$L(R)=R-\frac{\alpha^2}{3R} \quad (13)$$

where α is a positive constant with the same dimensions as R and the factor of 3 is introduced to simplify future equations. The field equations for this Lagrangian are

$$\left[1+\frac{\alpha^2}{3R^2}\right]R_{\mu\nu}-\frac{1}{2}\left[R-\frac{\alpha^2}{3R}\right]g_{\mu\nu}=-\kappa T_{\mu\nu}. \quad (14)$$

Contracting the indices gives

$$R^2-\kappa TR-\alpha^2=0 \quad (15)$$

and the solution to this algebraic equation is

$$R=\tfrac{1}{2}[\kappa T\pm\sqrt{\kappa^2T^2+4\alpha^2}]. \quad (16)$$

For large $|T|$ we expect the above to reduce to $R=\kappa T$, which follows from the Einstein field equations. Thus, if $T>0$ we need to select the positive sign and if $T<0$ we need to select the negative sign. In a universe filled with an ideal fluid, $T=-(\rho-3P)$, so that $T<0$ if the dominant energy condition holds (i.e., $\rho>0$ and $\rho\geqslant 3|P|$). Thus we have

$$R=\tfrac{1}{2}[\kappa T-\sqrt{\kappa^2T^2+4\alpha^2}]. \quad (17)$$

The vacuum solution is

$$R=-\alpha \quad (18)$$

so that at late times, as $T\to 0$, the universe will approach a de Sitter spacetime and the expansion of the universe will accelerate.

From Eqs. (14) and (16), we see that the field equations reduce to the Einstein equations if $|\kappa T|\gg\alpha$. Thus, in a dust filled universe the evolution will be governed by the Einstein field equations at early times. Eventually the corrections to the equations of motion will become important and the universe will make a transition to a de Sitter universe at late times. To match the observations of the cosmological acceleration we must take $\alpha\sim 10^{-67}\ (\mathrm{eV})^2\sim 10^{-53}\ \mathrm{m}^{-2}$. Note that in a vacuum we get the Einstein field equations plus a small cosmological constant, so that this theory will pass all the solar system tests that general relativity has passed. It is also interesting to note that in a radiation dominated universe, $T=0$ so that the dynamics is not governed by the Einstein's equations even at large curvature. As we will see below [see Eq. (24)] the equations of motion are the Einstein field equations with a cosmological constant and a modified Newton's constant.

Now consider the evolution of a universe with metric,

$$ds^2=-dt^2+a(t)^2[dx^2+dy^2+dz^2] \tag{19}$$

at late times, when $\alpha \gg \kappa T$. In this regime

$$R_{\mu\nu} \simeq R_{\mu\nu}(g)+\frac{1}{L'}\nabla_\mu \nabla_\nu L'+\frac{1}{2L'}g_{\mu\nu}\Box L', \tag{20}$$

$$R \simeq \tfrac{1}{2}\kappa T-\alpha, \tag{21}$$

$$L \simeq -\frac{2}{3}\alpha\left[1-\frac{\kappa T}{\alpha}\right] \tag{22}$$

$$L' \simeq \frac{4}{3}\left[1+\frac{\kappa T}{4\alpha}\right], \tag{23}$$

and the field equations are

$$R_{\mu\nu} \simeq -\frac{1}{4}\alpha g_{\mu\nu}-\kappa\left[\frac{3}{4}T_{\mu\nu}-\frac{5}{16}Tg_{\mu\nu}\right]. \tag{24}$$

The matter in the present universe can be approximated by dust with $T=\rho_0/a^3$, where ρ_0 is a constant. The nonvanishing components of the Ricci tensor are

$$R_{tt}=\frac{3\ddot{a}}{a}+\frac{9\kappa\rho_0}{8\alpha a^3}\left[\frac{\ddot{a}}{a}-3\left(\frac{\dot{a}}{a}\right)^2\right] \tag{25}$$

and

$$R_{ij}=-\left[a\ddot{a}+2\dot{a}^2+\frac{3\kappa\rho_0}{8\alpha a}\left(\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}\right)\right]\delta_{ij}. \tag{26}$$

At late times the universe will almost be in a de Sitter phase and we can take

$$a(t)=e^{Ht}+b(t) \tag{27}$$

where $|b(t)| \ll e^{Ht}$ and $\alpha=12H^2$. To lowest order in b the field equations are

$$\ddot{b}-H^2 b=-\frac{\kappa\rho_0}{12}e^{-2Ht} \tag{28}$$

and

$$\ddot{b}+4H\dot{b}-5H^2 b=\frac{\kappa\rho_0}{4}e^{-2Ht}. \tag{29}$$

Subtracting these two equations gives the first order equation

$$\dot{b}-Hb=\frac{\kappa\rho_0}{12H}e^{-2Ht}, \tag{30}$$

and the particular solution to this equation is

$$b(t)=-\frac{\kappa\rho_0}{36H^2}e^{-2Ht}. \tag{31}$$

Thus, at late times the universe approaches a de Sitter spacetime exponentially fast. This behavior is analogous to the cosmic no hair theorem for fourth order gravity discussed by Kluske and Schmidt [22].

CONCLUSION

Using a Palatini variation the field equations for a nonlinear gravitational Lagrangian coupled to matter were found. The vacuum field equations are the Einstein equations with a cosmological constant. Thus, at late times as $T_{\mu\nu}\rightarrow 0$ our universe will approach a de Sitter spacetime. The inclusion of matter gives field equations that differ from Einstein's equations. Using these equations it was shown that the approach to de Sitter space is exponentially fast when the $1/R$ term dominates. Thus, the inclusion of nonpolynomial curvature terms in the gravitational action offers an alternative explanation for the cosmological acceleration.

[1] J.L. Tonry *et al.*, astro-ph/0305008.
[2] S. Perlmutter *et al.*, Astrophys. J. **517**, 565 (1999).
[3] A.G. Riess *et al.*, Astron. J. **116**, 1009 (1998).
[4] S. Perlmutter *et al.*, Bull. Am. Astron. Soc. **29**, 1351 (1997).
[5] C.L. Bennett *et al.*, astro-ph/0302207.
[6] C.B. Netterfield *et al.*, Astrophys. J. **571**, 604 (2002).
[7] N.W. Halverson *et al.*, Astrophys. J. **568**, 38 (2002).
[8] A.H. Jaffe *et al.*, Phys. Rev. Lett. **86**, 3475 (2001).
[9] A.E. Lange *et al.*, Phys. Rev. D **63**, 042001 (2001).
[10] A. Melchiorri *et al.*, Astrophys. J. Lett. **536**, L63 (2000).
[11] S.M. Carroll, V. Duvvuri, M. Trodden, and M. Turner, astro-ph/0306438.
[12] S. Capozziello, S. Carloni, and A. Troisi, astro-ph/0303041.
[13] C. Deffayet, G. Dvali, and G. Gabadadze, Phys. Rev. D **65**, 044023 (2002).
[14] G. Dvali, G. Gabadadze, and M. Shifman, Phys. Rev. D **67**, 044020 (2003); hep-th/0208096.
[15] G. Dvali, A. Gruzinov, and M. Zaldarriaga, Phys. Rev. D **68**, 024012 (2003).
[16] G. Dvali and M. Turner, astro-ph/0301510.
[17] H. Kleinert and H.-J. Schmidt, Gen. Relativ. Gravit. **34**, 1295 (2002).
[18] R. Brandenberger, V. Mukhanov, and A. Sornborger, Phys. Rev. D **48**, 1629 (1993).
[19] M. Ferraris, M. Francaviglia, and I. Volovich, Nuovo Cimento Soc. Ital. Fis., B **108B**, 1313 (1993).
[20] M. Ferraris, M. Francaviglia, and I. Volovich, Class. Quantum Grav. **11**, 1505 (1994).
[21] S.W. Hawking and G.F.R. Ellis, *The Large Scale Structure of Spacetime* (Cambridge University Press, Cambridge, 1973), p. 42.
[22] S. Kluske and H.-J. Schmidt, Astron. Nachr. **317**, 337 (1996).