

Bulk scalar stabilization of the radion without metric back reaction in the Randall-Sundrum model

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Generalizations of the Randall-Sundrum model containing a bulk scalar field Φ interacting with the curvature R through the general coupling $Rf(\Phi)$ are considered. We derive the general form of the effective 4D potential for the spin-zero fields and show that in the mass matrix the radion mixes with the Kaluza-Klein (KK) modes of the bulk scalar fluctuations. We demonstrate that it is possible to choose a nontrivial background form $\Phi_0(y)$ (where y is the extra dimension coordinate) for the bulk scalar field such that the exact Randall-Sundrum metric is preserved (i.e. such that there is no back reaction). We compute the mass matrix for the radion and the KK modes of the excitations of the bulk scalar relative to the background configuration $\Phi_0(y)$. We find that for any (consistent) $\Phi_0(y)$ the expected mass for the radionlike eigenstate is suppressed relative to the Planck scale by the standard warp factor needed to explain the hierarchy puzzle, implying that ~ 1 TeV is a natural order of magnitude for this mass. The general considerations are illustrated in the case of a model containing an $R\Phi^2$ interaction term.

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I. INTRODUCTION

Although the standard model (SM) of electroweak interactions describes successfully almost all existing experimental data, the model suffers from many theoretical drawbacks. One of these is the hierarchy problem: namely, the SM cannot consistently accommodate the weak energy scale $O(1 \text{ TeV})$ and a much higher scale such as the Planck mass scale $O(10^{18} \text{ GeV})$. Therefore, it is widely accepted that the SM is only an effective low-energy theory embedded in some more fundamental high-scale theory that presumably could contain gravitational interactions. Recently many models that incorporate gravity have been proposed in the context of higher dimensional space-time. These models have received tremendous attention since they might provide a solution to the hierarchy problem. One of the most attractive attempts has been formulated by Randall and Sundrum [1] who postulated a 5D universe with two 4D surfaces (“3-branes”) with the following action:

$$S = - \int d^4x \int_{-1/2}^{1/2} dy \left\{ \sqrt{|g|} \left(\frac{R}{2\kappa^2} + \Lambda \right) + \sum_{k=1,2} \sqrt{|g_k|} \Lambda_k \delta(y - y_k) \right\}, \quad (1)$$

where R is the Ricci scalar, $\kappa^2 = 8\pi G_N^{(5)}$ with $G_N^{(5)}$ the Newton constant in 5D and Λ , $\Lambda_1 \equiv \Lambda_{hid}$ and $\Lambda_2 \equiv \Lambda_{vis}$ are the cosmological constants in the bulk, on the hidden and visible branes, respectively. In the above, g_{ij} ($i, j = 0, 1, 2, 3, 4$) is the bulk metric and $(g_1)_{\mu\nu} \equiv (g_{hid})_{\mu\nu}(x) \equiv g_{\mu\nu}(x, y = y_1 = 0)$

and $(g_2)_{\mu\nu} \equiv (g_{vis})_{\mu\nu}(x) \equiv g_{\mu\nu}(x, y = y_2 = 1/2)$ ($\mu, \nu = 0, 1, 2, 3$) are the induced metrics on the branes.

It turns out that if the bulk and brane cosmological constants are related by

$$\Lambda = - \frac{6m_0^2}{\kappa^2}, \quad \Lambda_{hid} = - \Lambda_{vis} = \frac{6m_0}{\kappa^2} \quad (2)$$

and if periodic boundary conditions ($y \rightarrow y + 1$) with identification of (x, y) and $(x, -y)$ are imposed, then the following metric is a solution of the 5D Einstein equations:

$$ds^2 = e^{-2\sigma(y)} \eta_{\mu\nu} dx^\mu dx^\nu - b_0^2 dy^2, \quad (3)$$

where $\sigma(y) = m_0 b_0 \{ y [2\theta(y) - 1] - 2(y - 1/2)\theta(y - 1/2) \}$; b_0 is a constant parameter that is not determined by the equations of motion.

Within the Randall-Sundrum (RS) model all the SM particles as well as the nongravitational forces are assumed to be present on one of the 3-branes, the “visible brane.” Gravity lives on the visible brane, on the second brane (the “hidden brane”) and in the bulk. All mass scales in the 5D theory are of the order of the Planck mass. By placing the SM fields on the visible brane, the initial 5D electroweak mass scale $O(M_{Pl})$ is rescaled by an exponential suppression factor (the “warp factor”) $\gamma \equiv e^{-m_0 b_0 / 2}$, down to the weak scale $O(1 \text{ TeV})$ without any severe fine tuning. To achieve the necessary suppression, one needs $m_0 b_0 / 2 \sim 37$. This is a great improvement compared to the original problem of accommodating both the weak and the Planck scale within a single theory.

II. THE RADION EFFECTIVE POTENTIAL

A drawback of the RS model is the presence of a massless degree of freedom called the radion. There have been several

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attempts (see Refs. [2–4]) in the literature to generate the radion mass by introducing a bulk scalar field Φ that would induce an appropriate radion potential. Here we will derive the general form of the potential within a class of models containing the bulk scalar interacting with gravity in the following manner:

$$S = \int d^4x \int_{-1/2}^{1/2} dy \left\{ \sqrt{|g|} \left[-\frac{R}{2\kappa^2} - \Lambda - \alpha R f(\Phi) + \frac{1}{2} g^{ij} \Phi_{,i} \Phi_{,j} - V(\Phi) \right] - \sum_{k=1,2} \sqrt{|g_k|} [\Lambda_k + V_k(\Phi)] \times \delta(y - y_k) \right\}, \quad (4)$$

where we have introduced the bulk potential $V(\Phi)$ and the brane potentials $V_1(\Phi) \equiv V_{hid}(\Phi)$ and $V_2(\Phi) \equiv V_{vis}(\Phi)$. In addition to the standard scalar kinetic-energy term, we have allowed for a general coupling of the bulk scalar to gravity through the $\alpha R f(\Phi)$ interaction term. Since we would like to preserve the explanation of the hierarchy proposed by Randall and Sundrum, we will also require that the RS metric (3) remain an exact solution¹ of the Einstein equations even in the presence of the bulk scalar Φ . Therefore, it is useful to separate out in the action (4) both the bulk (Λ) and brane (Λ_{hid} , Λ_{vis}) cosmological constants that satisfy the RS conditions, Eq. (2).

In order to identify the radion, it is sufficient to consider scalar excitations of the metric around the background RS solutions. Hereafter, we will adopt the following parametrization (see Refs. [5,6]) of the metric fluctuations:

$$ds^2 = e^{-2\sigma(y) - 2b(x)e^{2\sigma(y)}} [\eta_{\mu\nu} + h_{\mu\nu}(x, y)] dx^\mu dx^\nu - b_0^2 [1 + 2b(x)e^{2\sigma(y)}]^2 dy^2, \quad (5)$$

where $h_{\mu\nu}(x, y)$ and $b(x)$ are related to the graviton² and the radion, respectively. Then from $-\sqrt{|g|}R/(2\kappa^2)$ in the action (4) one obtains [after expanding in powers of $b(x)$] the kinetic term for the radion,³

$$S = \frac{6}{\kappa^2 m_0} (e^{m_0 b_0} - 1) \int d^4x \frac{1}{2} (\partial_\mu b)(\partial^\mu b) + \dots \quad (6)$$

It is easy to verify that if interactions of the scalar field Φ are switched off, then there is no potential for $b(x)$ and consequently the radion would be massless.

The bulk scalar has been introduced here in order to generate a nontrivial potential for the radion. However, in general the presence of the scalar leads to a nontrivial interaction potential between the radion and the scalar in addition to the appearance of a radion potential. Therefore, the strategy that we will follow here will be to determine the background scalar configuration $\Phi(y)$ such that the RS background metric is preserved and then to expand the action (4) around it. First, one has to solve the Einstein equations together with the equation of motion for Φ . Let us start with the Einstein equations, keeping in mind that we would like to preserve the RS metric as a vacuum solution. We write

$$G_{ij} = \kappa^2 [T_{ij}^{(RS)} + (\delta T)_{ij}(\Phi)], \quad (7)$$

where G_{ij} is the Einstein tensor, $T_{ij}^{(RS)}$ denotes the RS contribution to the energy-momentum tensor and $(\delta T)_{ij}$ contains all new contributions emerging from interactions of the scalar Φ . It is useful to first calculate all the extra (compared to the pure RS model) contributions to the energy-momentum tensor. It is easy to show that

$$(\delta T)_{ij} = T_{ij}^{(\Phi)} + 2\alpha \{ D_{ij}[f(\Phi)] - G_{ij}f(\Phi) \} + \frac{1}{b_0} \sum_{k=1,2} V_k(\Phi) (g_k)_{\mu\nu} \delta_i^\mu \delta_j^\nu \delta(y - y_k), \quad (8)$$

where

$$T_{ij}^{(\Phi)} \equiv \nabla_i \Phi \nabla_j \Phi - g_{ij} \left[\frac{1}{2} g^{kl} \nabla_k \Phi \nabla_l \Phi - V(\Phi) \right] \quad (9)$$

$$D_{ij}[X] \equiv \nabla_i \nabla_j X - g_{ij} g^{kl} \nabla_k \nabla_l X. \quad (10)$$

For the RS background metric we obtain

$$G_{ij} = \begin{pmatrix} -3b_0^{-2} e^{-2\sigma} (2\sigma'^2 - \sigma'') \eta_{\mu\nu} & 0 \\ 0 & 6\sigma'^2 \end{pmatrix} \quad (11)$$

and

$$D_{ij}[f(\Phi)] = \begin{pmatrix} b_0^{-2} e^{-2\sigma} [-3\sigma' f(\Phi)' + f(\Phi)'] \eta_{\mu\nu} & 0 \\ 0 & 4\sigma' f(\Phi)' \end{pmatrix}, \quad (12)$$

¹One can, of course, consider slight modifications of the RS metric that would also solve the hierarchy problem in a similar manner. However, in this paper we would like to discuss a scenario with exactly the same metric as in the original RS model.

²In the following we will not discuss interactions with gravitons as they will not influence the potential for scalar degrees of freedom.

³Hereafter, the flat metric $\eta^{\mu\nu}$ will be assumed whenever repeated indices are summed.

where here, and in what follows, the prime denotes differentiation with respect to the 5th coordinate, y .

Since we demand that the RS metric be preserved even when the scalar is present (no back reaction from the scalar), we have to require that the extra contributions to the energy momentum tensor (calculated using the RS metric) vanish,

$$(\delta T)_{ij}(\Phi)=0. \quad (13)$$

Since we want to find a background solution for Φ that satisfies 4D Lorentz invariance, we will assume that the solution is only a function of the extra dimension coordinate, y . The (μ, ν) and $(5,5)$ components of Eq. (13) read, respectively,

$$\begin{aligned} &(\Phi')^2 + 12\alpha(2\sigma'^2 - \sigma'')f - 12\alpha\sigma'f' + 4\alpha f'' \\ &+ 2b_0^2 \left[V(\Phi) + \frac{1}{b_0} \sum_{k=1,2} V_k(\Phi) \delta(y-y_k) \right] = 0 \end{aligned} \quad (14)$$

$$(\Phi')^2 - 24\alpha\sigma'^2 f + 16\alpha\sigma'f' - 2b_0^2 V(\Phi) = 0. \quad (15)$$

Note that since $\Phi(y)$ should be a continuous function, the above equations imply $\Phi'(y)=0$ and $V(\Phi)=V_{vis}(\Phi)=V_{hid}(\Phi)=0$ for $\alpha=0$. Therefore, introduction of the extra coupling $\alpha Rf(\Phi)$ is essential in order to obtain a no-back reaction solution, $(\delta T)_{ij}(\Phi)=0$.

In addition, Φ must satisfy the following equation of motion:

$$\begin{aligned} &-\Phi'' + 4\sigma'\Phi' + 4\alpha(5\sigma'^2 - 2\sigma'') \frac{d f}{d \Phi} \\ &+ b_0^2 \left[\frac{dV}{d\Phi} + \frac{1}{b_0} \sum_{k=1,2} \frac{dV_k}{d\Phi} \delta(y-y_k) \right] = 0, \end{aligned} \quad (16)$$

where the RS metric was used.⁴ Once the vacuum solution (Φ_0) is determined, we expand the action, Eq. (4), adopting the parametrization of the scalar fluctuations of the metric given in Eq. (5) and the following definition for the Φ quantum fluctuation:

$$\Phi(x, y) \rightarrow \Phi_0(y) + \phi(x, y). \quad (17)$$

Then, in order to determine the effective 4D potential for the scalar degrees of freedom, we collect all nonderivative contributions to the d^4x integrand in the action of Eq. (4) containing $b(x)$ and $\phi(x, y)$.

⁴Both the Einstein equations (13) and the equation of motion (16) constrain the scalar Φ . However, it can be verified that the equations are not independent; a certain linear combination of derivatives with respect to y of the (μ, ν) and $(5,5)$ components of Eq. (13) is proportional to Eq. (16). Since we have not specified the potential $V(\Phi)$, we can find a consistent solution both for Φ and the potential. Details of the derivations for $f(\Phi)=3/32\Phi^2$ will be presented in Sec. III.

In other words, we expand the theory defined by the action (4) around the vacuum solutions for the metric (the RS solution) and the bulk scalar field Φ [the solution of Eqs. (13) and (16)] in terms of the scalar fluctuation of the metric, $b(x)$, and fluctuation of the scalar field, $\phi(x, y)$.

First, let us calculate the Ricci scalar for the metric (5) and collect all the terms containing derivatives with respect to the extra component y ,

$$R = \frac{20}{b_0^2} \sigma'^2 - \frac{8}{b_0^2} \frac{\sigma''}{1 + 2be^{2\sigma}} + \dots \quad (18)$$

where ellipses contain only (x, y) -derivatives of the graviton and x -derivatives of the radion. Since we are going to calculate the potential, derivatives of $b(x)$ will be dropped hereafter. As has already been mentioned, we will not consider fluctuations of the $\eta_{\mu\nu}$ part of the metric. Therefore, we will also neglect all terms containing $h_{\mu\nu}(x, y)$.⁵

Using the contributions to the Ricci scalar displayed in Eq. (18), one gets the following form of the effective 4D potential from the action (4):

$$\begin{aligned} V_{\text{eff}}(b, \phi) = & \int_{-1/2}^{1/2} dy e^{-4\sigma - 4be^{2\sigma}} \left\{ \frac{1}{2} \left[\frac{1}{b_0(1 + 2be^{2\sigma})} \right. \right. \\ & \times (\Phi_0' + \phi')^2 + 2b_0(1 + 2be^{2\sigma})V(\Phi_0 + \phi) \left. \right] \\ & + \frac{2}{\kappa^2 b_0} [5(1 + 2be^{2\sigma})\sigma'^2 - 2\sigma''] \\ & \times [1 + 2\kappa^2 \alpha f(\Phi_0 + \phi)] + \Lambda b_0(1 + 2be^{2\sigma}) \\ & \left. + \sum_{k=1,2} [\Lambda_k + V_k(\Phi_0 + \phi)] \delta(y-y_k) \right\} \quad (19) \end{aligned}$$

where $\Phi_0 = \Phi_0(y)$ denotes the vacuum solution (that preserves the RS metric) for the scalar Φ . Note that Φ_0 is determined as a solution of the equations of motion for the RS background metric. As a result, it does not contain any dependence on the radion field $b(x)$. It is easy to verify that contributions from the pure RS model to $V_{\text{eff}}(b, \phi)$ vanish when the relations (2) are satisfied: a nontrivial potential requires an extension of the minimal RS model.

Next, it is easy to show from Eq. (8) that if Φ is independent of x then the following identity holds:

⁵Adopting the traceless gauge, $h_{\mu}^{\mu}=0$, one can eliminate possible mixing between b or ϕ and $h_{\mu\nu}$. Consequently, graviton interactions cannot influence scalar masses. Thus, $h_{\mu\nu}$ will be suppressed in the following.

$$\begin{aligned}
(\delta T)_\mu^\mu(\Phi) &= \frac{2}{b_0} \left[\frac{1}{b_0} (\Phi')^2 + 2b_0 V(\Phi) \right] \\
&+ \frac{24\alpha}{b_0^2} (2\sigma'^2 - \sigma'') f(\Phi) + \frac{8\alpha}{b_0^2} [-3\sigma' f(\Phi)' \\
&+ f(\Phi)''] + \frac{4}{b_0} \sum_{k=1,2} V_k(\Phi) \delta(y-y_k). \quad (20)
\end{aligned}$$

Multiplying the above equation by $\exp\{-4\sigma\}(b_0/4)$, integrating the $(-3\sigma'f+f'')$ term by parts and using the RS relations, Eq. (2), one obtains the following simple relation between $(\delta T)_\mu^\mu(\Phi_0)$ and the minimum of the effective potential of Eq. (19) at its $[b(x)=0, \phi(x,y)=0]$ minimum:

$$V_{\text{eff}}(0,0) = \frac{b_0}{4} \int_{-1/2}^{1/2} dy e^{-4\sigma} (\delta T)_\mu^\mu(\Phi_0). \quad (21)$$

Note that in Eq. (21) we employ the trace from Eq. (20) calculated for the background solution Φ_0 . Since the no-back-reaction requirement, Eq. (13), implies $(\delta T)_\mu^\mu(\Phi_0) = 0$, the relation (21) shows that the effective potential must vanish at the minimum

$$V_{\text{eff}}(0,0) = 0. \quad (22)$$

It is straightforward to verify that linear terms in $b(x)$ and $\phi(x,y)$ disappear by virtue of Eq. (13) and Eq. (16), respectively.

In order to determine scalar masses one has to expand the action (4) up to terms quadratic in b and ϕ . First, let us define the KK modes of the scalar fluctuations,

$$\phi(x,y) = \sum_{n=0}^{\infty} \varphi_n(x) \frac{\chi_n(y)}{b_0^{1/2}}, \quad (23)$$

with orthonormal functions $\chi_n(y)$,

$$\int_{-1/2}^{1/2} dy e^{-2\sigma(y)} \chi_n(y) \chi_m(y) = \delta_{nm}. \quad (24)$$

The resulting mass terms are the following:

$$\frac{1}{2} \begin{pmatrix} r & \varphi_m \end{pmatrix} \begin{pmatrix} M_{rr}^2 & M_{rm}^2 \\ M_{rm}^2 & M_{mn}^2 \end{pmatrix} \begin{pmatrix} r \\ \varphi_n \end{pmatrix}, \quad (25)$$

where r is the canonically normalized radion [see Eq. (6)],

$$r(x) = \left(\frac{6}{\kappa^2 m_0} \right)^{1/2} e^{m_0 b_0/2} b(x). \quad (26)$$

Inputting the equation of motion (16), the elements of the mass matrix read

$$M_{rr}^2 = \frac{2}{3} \kappa^2 \frac{m_0}{b_0} e^{-m_0 b_0} \int_{-1/2}^{1/2} (\Phi_0')^2 dy \quad (27)$$

$$\begin{aligned}
M_{rm}^2 &= \left(\frac{2}{3} \kappa^2 \frac{m_0}{b_0} e^{-m_0 b_0} \right)^{1/2} \frac{1}{b_0} \int_{-1/2}^{1/2} e^{-2\sigma(y)} \chi_n(y) \\
&\times \left[\Phi_0'' + 2\sigma' \Phi_0' + 20\alpha \sigma'^2 \frac{df}{d\Phi}(\Phi_0) + b_0^2 \frac{dV}{d\Phi}(\Phi_0) \right] \quad (28)
\end{aligned}$$

$$\begin{aligned}
M_{nm}^2 &= \frac{1}{b_0^2} \int_{-1/2}^{1/2} e^{-4\sigma(y)} \left\{ \chi_n'(y) \chi_m'(y) + \chi_n(y) \chi_m''(y) \right. \\
&\times \left[4\alpha (5\sigma'^2 - 2\sigma'') \frac{d^2 f}{d\Phi^2} + b_0^2 \left(\frac{d^2 V}{d\Phi^2} + \frac{1}{b_0} \right. \right. \\
&\left. \left. \times \sum_{k=1,2} \frac{d^2 V_k}{d\Phi^2} \delta(y-y_k) \right) \right] \left. \right\}. \quad (29)
\end{aligned}$$

Note that $M_{rr}^2 > 0$ is automatic.

Before we can estimate the size of the elements of the mass matrix, we must discuss first the constraint that is imposed on the model by the requirement of maintaining the standard strength of classical 4D gravity. Adopting the metric defined by Eq. (5), one can calculate the coefficient of the 4D Ricci scalar obtained for $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. In order to reproduce the standard result, the coefficient should be $M_{Pl}^2/2$. The resulting constraint is

$$M_{Pl}^2 = \frac{1 - \gamma^2}{\kappa^2 m_0} + 2\alpha b_0 \int_{-1/2}^{1/2} dy e^{-2\sigma} f[\Phi_0(y)], \quad (30)$$

where $M_{Pl} \sim 2 \times 10^{18}$ GeV is the reduced Planck mass and $\gamma = \exp(-m_0 b_0/2)$. In order to solve the hierarchy problem, one needs $m_0 b_0/2 \sim 37$. Therefore, terms of order γ^2 can be safely neglected in Eq. (30). It is clear that the most natural scenario⁶ emerges when all the mass parameters of the 5D theory are of the order of M_{Pl} . In this case, the elements of the scalar mass matrix defined by Eq. (25) are of the following order of magnitude:

$$\sim M_{Pl}^2 \begin{pmatrix} a\gamma^2 & b^{1/2}\gamma \\ b^{1/2}\gamma & 1 \end{pmatrix}, \quad (31)$$

where a and b are calculable coefficients of the order of 1. It is clear that for $m_0 b_0/2 \sim 37$ the lowest scalar mass is of order

$$\sim \frac{\gamma M_{Pl} (a-b)^{1/2}}{\sqrt{2}} \sim \frac{246 \text{ GeV}}{\sqrt{2}}. \quad (32)$$

⁶Of course, an appropriate cancellation between contributions coming from parameters that differ even by many orders of magnitude is in principle also possible. However, since we would like to preserve the solution of the hierarchy problem proposed by Randall and Sundrum, we should assume that all the mass parameters in the fundamental 5D theory are of the same order. Then, the only necessary fine tuning is to keep $m_0 b_0/2 \sim 37$.

There are two essential conclusions. First, we see that the lowest scalar mass receives the standard suppression from the warp factor $\gamma = \exp(-m_0 b_0/2)$ that is necessary for the solution of the hierarchy problem. As a result, the mass expected in the presence of a bulk scalar $Rf(\Phi)$ interaction is of the order of the electroweak scale. Second, it is clear that in order to find precise values for scalar masses for any particular choice of the interaction function $f(\Phi)$, one has to take into account the mixing between the radion and the Kaluza-Klein modes of the bulk scalar fluctuation. To fully explore the phenomenology of the theory, it would be necessary to calculate all the entries of the mass matrix; however, this is beyond the scope of this paper. The most crucial aspect of such a calculation would be the demonstration that all eigenvalues of the mass-squared matrix of Eq. (25) are positive (for the stability of our solution).⁷

Finally, we close this section by reiterating the fact that if there is no $Rf(\Phi)$ interaction, i.e. if $\alpha=0$, then necessarily $\Phi'_0(y)=0$ and $V(\Phi_0)=V_{vis}(\Phi_0)=V_{hid}(\Phi_0)=0$, which in turn would lead to a vanishing mass matrix.

III. THE $R\Phi^2$ INTERACTION

In this section, we will illustrate the general discussion from Sec. II, choosing a specific form of the interaction between the bulk scalar and gravity,

$$f(\Phi) = \frac{3}{32} \Phi^2. \quad (33)$$

The function $f(\Phi)$ has been normalized such that $\alpha = -1$ corresponds to a 5D conformally invariant interaction. This coupling was discussed in various contexts by many authors in the past, see e.g. Ref. [7].

In this case, the conditions for $(\delta T)_{ij}(\Phi) = 0$, Eq. (13), as written out in Eqs. (14) and (15), read

$$\begin{aligned} & -3\sigma'(\Phi^2)' + (\Phi^2)'' + \frac{8}{3\alpha}(\Phi')^2 + 3(2\sigma'^2 - \sigma'')\Phi^2 \\ & + \frac{16b_0^2}{3\alpha} \left[V(\Phi) + \frac{1}{b_0} \sum_{k=1,2} V_k(\Phi) \delta(y-y_k) \right] = 0; \end{aligned} \quad (34)$$

$$\sigma'(\Phi^2)' + \frac{2}{3\alpha}(\Phi')^2 - \frac{4b_0^2}{3\alpha}V(\Phi) - \frac{3\sigma'^2}{2}\Phi^2 = 0. \quad (35)$$

Eliminating $V(\Phi)$ from Eqs. (34), (35), one obtains the following equation for Φ :

⁷We believe this can be accomplished by choosing $d^2V_k/d\Phi^2$ on the $y=0$ and/or $y=1/2$ branes (appearing in the M_{nm}^2 entries) appropriately. (As will be apparent from subsequent discussion, there are no constraints on these terms.) In particular, these terms can be taken large enough that the mass-squared matrix reduces to an approximately block diagonal form in which the M_{rn}^2 elements are sufficiently small that stability reduces to the requirement that the M_{nm}^2 submatrix has positive eigenvalues.

$$\begin{aligned} & \sigma'(\Phi^2)' + (\Phi^2)'' + \frac{16}{3\alpha}(\Phi')^2 - 3\sigma''\Phi^2 + \frac{16b_0}{3\alpha} \\ & \times \sum_{k=1,2} V_k(\Phi) \delta(y-y_k) = 0. \end{aligned} \quad (36)$$

Away from the branes, we find the solution

$$\Phi_0(y) = d[1 - ce^{-\sigma(y)}]^{1/\beta}, \quad (37)$$

where c, d are integration constants and $\beta \equiv 2 + 8/(3\alpha)$ is required for consistency. If $1/\beta$ is not an integer, we must also demand that $1 - ce^{-\sigma(y)} > 0$ in order that $\Phi_0(y)$ be well-defined. Recalling that $\sigma'' = 2m_0b_0[\delta(y) - \delta(y-1/2)]$, and noting that $(\Phi^2)''$ will contain a term proportional to σ'' , the conditions that the coefficients of the δ functions in Eq. (36) vanish reduce to

$$m_0d^2g(c) + \beta(\beta-2)V_{hid}(0) = 0 \quad (38)$$

$$m_0d^2g(c\gamma) - \beta(\beta-2)V_{vis}\left(\frac{1}{2}\right) = 0, \quad (39)$$

where $\gamma \equiv \exp(-m_0b_0/2)$, $g(x) \equiv (1-x)^{(2/\beta)-1}[x(2+3\beta) - 3\beta]$, $V_{hid}(0) \equiv V_{hid}[\Phi_0(0)]$ and $V_{vis}(\frac{1}{2}) \equiv V_{vis}[\Phi_0(\frac{1}{2})]$, and we have introduced the notation $V_{vis} = V_2$ and $V_{hid} = V_1$.

Insertion of the solution $\Phi_0(y)$ into, for example, Eq. (35) fixes the form of the bulk potential,

$$\begin{aligned} V(\Phi) = & \frac{3}{\beta-2}m_0^2\Phi^2 \left\{ \frac{4}{3\beta} \left[\left(\frac{\Phi}{d} \right)^{-\beta} - 1 \right] \right. \\ & \left. + \frac{\beta-2}{6\beta^2} \left[\left(\frac{\Phi}{d} \right)^{-\beta} - 1 \right]^2 - 1 \right\}. \end{aligned} \quad (40)$$

In addition, Φ must satisfy its equation of motion as obtained from Eq. (16) for the form Eq. (33),

$$\begin{aligned} & 4\sigma'\Phi' - \Phi'' + \frac{3\alpha}{4}(5\sigma'^2 - 2\sigma'')\Phi + b_0^2 \left[\frac{dV}{d\Phi} + \frac{1}{b_0} \right. \\ & \left. \times \sum_{k=1,2} \frac{dV_k(\Phi)}{d\Phi} \delta(y-y_k) \right] = 0. \end{aligned} \quad (41)$$

It is easily verified that the bulk form for $\Phi_0(y)$, Eq. (36), also satisfies the equation of motion in the bulk, Eq. (41). However, cancellation of the $\delta(y)$ and $\delta(y-1/2)$ brane delta function pieces yields matching conditions that are different from Eqs. (38), (39). For consistency of Eq. (41) we require

$$2m_0dh(c) + \beta(\beta-2)V'_{hid}(0) = 0 \quad (42)$$

$$2m_0dh(c\gamma) - \beta(\beta-2)V'_{vis}\left(\frac{1}{2}\right) = 0, \quad (43)$$

where $h(x) \equiv (1-x)^{(1/\beta)-1}[x(2+3\beta) - 4\beta]$, $V'_{hid}(0) \equiv dV_{hid}/d\Phi|_{\Phi=\Phi_0(0)}$ and $V'_{vis}(\frac{1}{2}) \equiv dV_{vis}/d\Phi|_{\Phi=\Phi_0(1/2)}$.

Equations (38) and (42) can be solved for the parameter c in terms of $V_{hid}(0)$ and $V'_{hid}(0)$. Two solutions are possible for c , specified by

$$c_i = f_i(\beta, -R_{hid}) \quad i = 1, 2, \quad (44)$$

where

$$R_{hid} \equiv \frac{V'_{hid}(0)}{4m_0 V_{hid}(0)}. \quad (45)$$

The functions f_i denote the two possible solutions of the quadratic equations for c ,

$$f_i(\beta, R) \equiv \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}, \quad (46)$$

where $i = 1, 2$ corresponds to the $+$ and $-$ signs in front of the square root, respectively. The quantities A, B, C in the above are given by

$$\begin{aligned} A &= -(2 + 3\beta) \left[1 + \frac{2 + 3\beta}{R\beta(\beta - 2)} \right] \\ B &= (2 + 3\beta) \left[1 + \frac{8}{R(\beta - 2)} \right] + 3\beta \\ C &= -\beta \left[3 + \frac{16}{R(\beta - 2)} \right]. \end{aligned}$$

Positivity of $B^2 - 4AC$ leads to the requirement $1 - (2 + 3\beta)/R > 0$.

Once c is determined, we can compute $c\gamma$ in terms of $V'_{vis}(\frac{1}{2})$ and $V_{vis}(\frac{1}{2})$ using Eqs. (39) and (43). One finds

$$c_i \gamma_i = f_j(\beta, R_{vis}) \quad \text{for } i = 1, 2, \quad (47)$$

where

$$R_{vis} \equiv \frac{V'_{vis}(\frac{1}{2})}{4m_0 V_{vis}(\frac{1}{2})} \quad (48)$$

and the appropriate branch $j = 1$ or $j = 2$ is determined by the need to obtain a very small value for the warp factor γ_i , i.e. $f_j(\beta, R_{vis}) \sim 0$. The latter is most straightforwardly achieved by requiring $C_{vis} \approx 0$ and choosing the $j = 1, +$ ($j = 2, -$) solution for $f_j(\beta, R_{vis})$ [see Eq. (46)] for $B_{vis} > 0$ ($B_{vis} < 0$), respectively. From Eq. (47), the requirement that $C_{vis} \approx 0$ is equivalent to

$$\beta \approx -\frac{2}{3} \left(\frac{8}{R_{vis}} - 3 \right) \quad \text{or} \quad R_{vis} \approx \frac{8}{3 \left(1 - \frac{1}{2}\beta \right)}. \quad (49)$$

For R_{vis} as above,⁸ the positivity of $\sqrt{B_{vis}^2 - 4A_{vis}C_{vis}}$ is automatic so long as B_{vis} is not extremely tiny. For R_{vis} as given in Eq. (49), one finds $B_{vis} \approx 3\beta/2 - 1$, so that we must use the $j = 1, +$ solution for $\beta > 2/3$ and the $j = 2, -$ solution for $\beta < 2/3$. For $\beta = 2/3$, the choice becomes ambiguous.

For convenience, in what follows we denote by $A_{vis}, B_{vis}, C_{vis}$ the values of A, B, C for $R = R_{vis}$ and by $A_{hid}, B_{hid}, C_{hid}$ the values of A, B, C for $R = -R_{hid}$.

From Eqs. (44) and (47), the warp factor γ_i (and hence the distance between the branes) for a given solution c_i is given by

$$\gamma_i \equiv e^{-m_0 b_{0i}/2} = \frac{f_j(\beta, R_{vis})}{f_i(\beta, -R_{hid})} \quad \text{for } i = 1, 2, \quad (50)$$

with $j = 1$ for $\beta > 2/3$ and $j = 2$ for $\beta < 2/3$, as discussed earlier. In practice, we will require that $\gamma_i = \gamma \equiv e^{-37}$ for either choice of i . Further, one can use (for example) Eq. (42) to determine d_i ,

$$d_i = -\frac{\beta(\beta - 2)V'_{hid}(0)}{2m_0 h[f_i(\beta, -R_{hid})]}. \quad (51)$$

Once d_i, c_i and γ_i have been fixed as specified above, Eqs. (38) and (39) imply a consistency constraint on the model parameters,

$$\frac{V_{hid}(0)}{V_{vis}(\frac{1}{2})} = -\frac{g[f_i(\beta, -R_{hid})]}{g[f_j(\beta, R_{vis})]} \approx \frac{g[f_i(\beta, -R_{hid})]}{3\beta}, \quad (52)$$

where $\gamma_i \sim 0$ has been used to obtain the last approximate form. Using $\gamma_i \sim 0$, Eqs. (39) and (43) also simplify to

$$V_{vis}(\frac{1}{2}) \sim -3\frac{m_0 d_i^2}{\beta - 2}, \quad V'_{vis}(\frac{1}{2}) \sim -8\frac{m_0 d_i}{\beta - 2}, \quad (53)$$

with d_i as given in Eq. (51) for solution branch i . Note that, for a given value of β and choice of solution branch i , fixing $V_{vis}(\frac{1}{2})$ corresponds to fixing the normalization d_i of Φ_0 in terms of m_0 and that fixing also $V'_{vis}(\frac{1}{2})$ then fixes both m_0 and d_i .

Finally, it is important to note that the definition of R_{hid} , Eq. (45), yields the following constraint on the relative signs of R_{hid} and $V_{hid}(0)$:

$$V'_{hid}(0) = 4m_0 R_{hid} V_{hid}(0) > 0. \quad (54)$$

Using Eqs. (53) and (52), the condition Eq. (54) can be converted to a requirement expressed entirely in terms of R_{hid} and β for a given solution branch i ,

⁸For $R_{vis} = 8/[3(1 - \frac{1}{2}\beta)] \equiv r(\beta)$ exactly, $\gamma = 0$. As R_{vis} changes (for a fixed R_{hid}) from a value slightly larger than this to a value slightly smaller, γ will change sign. Whether a positive or negative shift relative to $r(\beta)$ is required to give $\gamma_i > 0$ is determined by the sign of c_i , which can vary.

$$\frac{R_{hid}g[f_i(\beta, -R_{hid})]}{\beta(\beta-2)} < 0. \quad (55)$$

The conformal limit of $\alpha = -1$ ($\beta = -2/3$) requires special treatment,⁹ since for this choice $A_{hid} = 0$. In this case, $g(x) = 2(1-x)^{-4}$ and $h(x) = \frac{8}{3}(1-x)^{-5/2}$. Equations (38) and (42) then yield

$$m_0 d^2 (1-c)^{-4} = -\frac{8}{9} V_{hid}(0),$$

$$m_0 d (1-c)^{-5/2} = -\frac{1}{3} V'_{hid}(0), \quad (56)$$

respectively, from which we conclude that $V_{hid}(0) < 0$ and $V'_{hid}(0) < 0$ are required, which also implies that [see Eqs. (45) and (54)] $R_{hid} < 0$. By combining Eq. (56) and Eq. (45) we find

$$c = 1 + \frac{2}{R_{hid}}. \quad (57)$$

Note that $R_{hid} < -2$ is required for $0 < c < 1$, but that c is negative for $-2 < R_{hid} < 0$. There is nothing obvious to forbid this latter choice since $(1 - ce^{-\sigma(y)})$ will automatically be positive for all y if $c < 0$. In an analogous spirit, utilizing Eqs. (39) and (43), one can show that

$$c\gamma = 1 - \frac{2}{R_{vis}}. \quad (58)$$

Combining Eqs. (57) and (58), one obtains the following result for the warp factor:

$$\gamma = \frac{R_{hid} R_{vis} - 2}{R_{vis} R_{hid} + 2}. \quad (59)$$

In order to have a phenomenologically acceptable small value for the warp factor γ , either $R_{vis} \sim 2$ or $R_{hid} \sim 0$ is required. The remaining constraint [the analog of Eq. (52)] in this case reads

$$\frac{V_{hid}(0)}{V_{vis}\left(\frac{1}{2}\right)} = -\left(\frac{R_{hid}}{R_{vis}}\right)^4. \quad (60)$$

⁹In addition to the general solutions discussed in the main text for this case, there exists a special background solution, $\Phi_0(y) \propto e^{3/2\sigma(y)}$, for which there are no matching conditions since the solution satisfies all the necessary equations everywhere, including the boundaries. For this particular special conformally symmetric case, substituting the form of Φ_0 , as given above, into Eq. (35) leads to a vanishing bulk potential, $V(\Phi_0) = 0$. Similarly, Eq. (36) gives $V_{1,2}(\Phi) = 0$. Because all the potentials are zero, one finds $M_{rr}^2 = 0$. We are only interested in cases for which a nonzero mass is generated for the radion.

Combining this equation with the earlier-noted constraint that $V_{hid}(0) < 0$ results in the requirement that $V_{vis}\left(\frac{1}{2}\right) > 0$, implying that $R_{vis} > 0$. Combined with Eq. (59) and the requirement that $\gamma > 0$, the only allowed choices are

$$(0 < R_{vis} < 2 \text{ and } -2 < R_{hid} < 0)$$

$$\text{or } (R_{vis} > 2 \text{ and } R_{hid} < -2). \quad (61)$$

[Note that Eq. (49) does not apply for the conformal choice of α .] For $R_{vis} \sim 2$, as generally needed for small γ , one finds

$$V_{vis}\left(\frac{1}{2}\right) = \frac{9}{8} m_0 d^2, \quad V_{hid}(0) = -\frac{9}{144} m_0 d^2 R_{hid}^4. \quad (62)$$

In Fig. 1, we display $\Phi_0(y)/d$ as a function of y for three cases. In all cases, we have chosen input parameters so that¹⁰ $m_0 b_0/2 \approx 37$ as required for the warp factor $\gamma = e^{-m_0 b_0/2} \sim 1 \text{ TeV}/M_{Pl}$. In the first case, we have taken $\beta \sim +2/3$, equivalent to $R_{vis} = 4$ from Eq. (49), and $R_{hid} = +1$. For this choice, $c \sim 0.7835$. In the second (third) cases, we make the conformal choice of $\beta = -2/3$, take $R_{vis} \sim 2$ (for small γ) and employ $R_{hid} = -4$ ($R_{hid} = -1$) for which $c = 1/2$ ($c = -1$). In the $\beta = 2/3$ case (which is representative of cases with $\beta > 0$), we see that $\Phi_0(y)$ is repulsed from the hidden brane located at $y = 0$. The second (third) case is representative of a $\beta = -2/3$ case for which $\Phi_0(y)$ is strongly peaked on (strongly repulsed from) the hidden brane.

A useful cross-check is to adopt the explicit form of the solution (37) to verify that indeed $V(0,0)$ vanishes as predicted by Eq. (22). In order to calculate the radion potential at the minimum we use Eq. (35) to eliminate $V(\Phi)$ in the general formula (19). The result is the simplified expression

$$V_{eff}(0,0) = \frac{10m_0^2 b_0}{\beta-2} \int_{-1/2}^{1/2} dy e^{-4\sigma} (\Phi_0(y))^2$$

$$+ \frac{1}{b_0} \int_{-1/2}^{1/2} dy e^{-4\sigma} (\Phi_0'(y))^2$$

$$+ \frac{8m_0}{\beta-2} \left\{ [\Phi_0(0)]^2 - \left[\Phi_0\left(\frac{1}{2}\right) \right]^2 e^{-2m_0 b_0} \right\}$$

$$+ V_{hid}(0) + V_{vis}\left(\frac{1}{2}\right) e^{-2m_0 b_0}. \quad (63)$$

Then, inserting the solution Eq. (37) into Eq. (63), one obtains

¹⁰We reemphasize that $m_0 b_0/2$ is calculable in terms of the input parameters using Eq. (50) or Eq. (59).

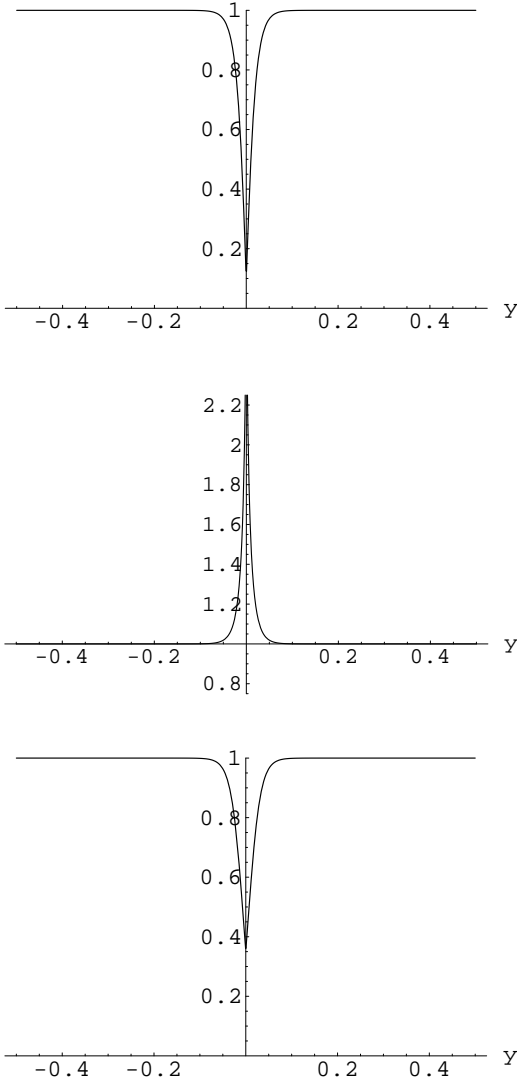


FIG. 1. The normalized scalar background function $\Phi_0(y)/d = [1 - ce^{-\sigma(y)}]^{1/\beta}$ for $m_0 b_0/2 \sim 37$ in three cases: (top) $R_{hid} = 1$ and $\beta \sim +2/3$ (equivalent to $R_{vis} = 4$); (middle) $\beta = -2/3$ and $R_{hid} = -4$ (implying $c = +0.5$); (bottom) $\beta = -2/3$ and $R_{hid} = -1$ (implying $c = -1$). In the latter two (conformal) cases, $m_0 b_0/2 = 37$ requires $R_{vis} \sim 2$.

$$\begin{aligned}
& \beta(\beta-2)V_{eff}(0,0) \\
&= [m_0 d^2(1-c)^{(2/\beta)-1} [c(2+3\beta)-3\beta] \\
&\quad + \beta(\beta-2)V_{hid}(0)] - \gamma^4 [m_0 d^2(1-c\gamma)^{(2/\beta)-1} \\
&\quad \times [c\gamma(2+3\beta)-3\beta] - \beta(\beta-2)V_{vis}(1/2)] \\
&= [m_0 d^2 g(c) + \beta(\beta-2)V_{hid}(0)] \\
&\quad - \gamma^4 [m_0 d^2 g(c\gamma) - \beta(\beta-2)V_{vis}(1/2)]. \quad (64)
\end{aligned}$$

The two bracketed expressions on the right hand side of the above equation vanish by virtue of the fact that they correspond precisely to the expressions appearing in the matching conditions that emerge from the requirement of $(\delta T)_{\mu\nu} = 0$, Eqs. (38), (39).

Finally, if we insert the solution Eq. (37) into the formula for the M_{rr}^2 mass-squared matrix entry, Eq. (27), one obtains [after employing Eqs. (38) and (39)] the following result for M_{rr}^2 :

$$\begin{aligned}
M_{rr}^2 = & \frac{2}{3} \kappa^2 m_0 \gamma^2 \left[-V_{hid}(0) \frac{c(2-\beta)+\beta}{c(2+3\beta)-3\beta} \right. \\
& \left. - V_{vis} \left(\frac{1}{2} \right) \frac{c\gamma(2-\beta)+\beta}{c\gamma(2+3\beta)-3\beta} \right], \quad (65)
\end{aligned}$$

where c and $c\gamma$ are given by Eqs. (44) and (47) or, in the conformal case, Eqs. (57) and (58). Note the presence of the warp factor γ that reduces the radion mass from the typical 5D scale [a natural choice for which would be $O(M_{Pl})$ if $V_{hid}(0)$ and $V_{vis}(\frac{1}{2})$ are $O(M_{Pl}^4)$] down to the electroweak scale. Of course, Eq. (27) guarantees that $M_{rr}^2 \geq 0$ for any given form of $f(\Phi)$ and any background solution $\Phi_0(y)$ that is *real* and fully consistent. The conditions for reality and consistency are (a) the constraint Eq. (52) or, in the conformal case, Eq. (60) is satisfied; (b) the positivity condition Eq. (54) is satisfied; and (c) $1-c > 0$ when $1/\beta$ is not an integer. However, substantial variation of M_{rr}^2 is possible. In particular, $M_{rr}^2 = 0$ at special points when considered as a function of R_{hid} at fixed β . A final form for M_{rr}^2 can be derived in the $\gamma \ll 1$ limit by using Eqs. (52) and (53) in Eq. (65):

$$\begin{aligned}
M_{rr}^2 = & \frac{2}{9} \kappa^2 m_0 \gamma^2 V_{vis} \left(\frac{1}{2} \right) \left[1 - \frac{g(c)}{\beta} \frac{c(2-\beta)+\beta}{c(2+3\beta)-3\beta} \right] \\
= & \frac{2}{3} (\kappa m_0 d \gamma)^2 \frac{1}{\beta-2} \left[(1-c)^{(2/\beta)-1} \left(1-c+c\frac{2}{\beta} \right) - 1 \right]. \quad (66)
\end{aligned}$$

This shows the fundamental importance of the scale of $V_{vis}(\frac{1}{2})$ in determining M_{rr}^2 . Of course, one should continue to keep in mind the relation Eq. (53) between $V_{vis}(\frac{1}{2})$ and $m_0 d^2$ as well as the relation Eq. (52) between $V_{vis}(\frac{1}{2})$ and $V_{hid}(0)$.

To illustrate how M_{rr}^2 depends upon the parameters and how other constraints enter, consider first the conformal case with $\beta = -2/3$. For $R_{vis} \sim 2$ (so that $\gamma \ll 1$) and using Eqs. (57) and (60), one can rewrite Eq. (66) in terms of $V_{vis}(\frac{1}{2})$ (which, as noted earlier, fixes the value of d in terms of m_0). The result is

$$M_{rr}^2 = \frac{2}{9} \kappa^2 m_0 \gamma^2 V_{vis} \left(\frac{1}{2} \right) K(R_{hid}) = \frac{1}{4} (\gamma \kappa m_0 d)^2 K(R_{hid}), \quad (67)$$

where

$$K(R_{hid}) \equiv \frac{3}{16} R_{hid}^4 + \frac{1}{2} R_{hid}^3 + 1 > 0. \quad (68)$$

In Fig. 2, we plot $K(R_{hid})$ as a function of R_{hid} . We observe that $M_{rr}^2 > 0$ everywhere except at $R_{hid} = -2$ [a point where $c = 0$ is required by Eq. (57) and $\Phi_0(y)$ becomes trivial].

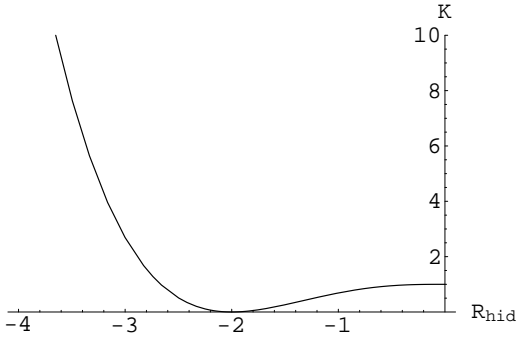


FIG. 2. The function $K(R_{hid})$ is plotted as a function of R_{hid} in the (allowed) $R_{hid} < 0$ region.

Our two earlier $\beta = -2/3$ plots of the wave function thus correspond to choices for which $M_{rr}^2 > 0$.

For fixed $\beta \neq -2/3$, and $\gamma \ll 1$, M_{rr}^2 is approximately a function of R_{hid} only, where R_{hid} is to be restricted to those values such that a given solution c_1 or c_2 satisfies the other consistency constraints. We explore the behavior of M_{rr}^2 as follows. First, recall that $V_{vis}(\frac{1}{2})$, $V_{hid}(0)$, R_{vis} , R_{hid} and β are the input model parameters. At fixed β , in order to obtain $\gamma \ll 1$, as desired on phenomenological grounds, we adjust R_{vis} according to Eq. (49). Then, for a chosen R_{hid} we calculate c_i using Eq. (44). In fact, there are the two solution branches, $c_{1,2}(R_{hid}) = f_{1,2}(\beta, -R_{hid})$. Positivity of $B_{hid}^2 - 4A_{hid}C_{hid}$, as determined by the sign of $1 - (2 + 3\beta)/(-R_{hid})$, requires $R_{hid} < \min[-(2 + 3\beta), 0]$ or $R_{hid} > \max[0, -(2 + 3\beta)]$ (the limit cross over taking place at the conformal point of $\beta = -2/3$). Once c_1 (c_2) is chosen, if $1/\beta$ is not an integer we check to see if $1 - c_1 > 0$ ($1 - c_2 > 0$), as required for a well-defined $\Phi_0(y) = d(1 - ce^{-2\sigma(y)})^{1/\beta}$. Finally, for any values of β and R_{hid} , and given a choice of branch i , we check the positivity requirement of Eq. (54). To compute M_{rr}^2 , we adopt Eq. (66), which takes into account the consistency constraint (52) such that $V_{hid}(0)$ is expressed in terms of $V_{vis}(\frac{1}{2})$. Equation (66) makes it clear that, for a given value of β , M_{rr}^2 is proportional to $V_{vis}(\frac{1}{2})$ [which fixes $m_0 d_i^2$ through Eq. (53)] and depends nontrivially on R_{hid} , the other parameters being held fixed. Remarkably, one finds $(M_{rr}^2)_i > 0$ (as expected) so long as (a) R_{hid} is such that $1 + (2 + 3\beta)/R_{hid} > 0$ (so that $c_{1,2}$ are real); (b) $1 - c_i > 0$ when $1/\beta$ is not an integer; and (c) the positivity condition, Eq. (54), which we abbreviate as $R_{hid}V_{hid} > 0$, is satisfied. There are many different cases. Here we simply describe a couple of illustrative possibilities.

Consider first two choices of β such that $2 + 3\beta > 0$.

(i) For $\beta = 2/3$, any value of $R_{hid} > 0$ gives $1 - c_{1,2} > 0$ and $(R_{hid}V_{hid})_{1,2} > 0$ and substantial values for the corresponding $(M_{rr}^2)_1$ and $(M_{rr}^2)_2$. For $R_{hid} < -(2 + 3\beta) = -4$, any value of R_{hid} gives $1 - c_1 > 0$ and substantial $(M_{rr}^2)_1$. In all these cases, M_{rr}^2 is a relatively smooth function of R_{hid} . However, in the $R_{hid} \leq -4$ region, $1 - c_2$ and $(R_{hid}V_{hid})_2$ are both only positive for $-4.5 \leq R_{hid} \leq -4$ and $(M_{rr}^2)_2$ varies rapidly, as illustrated in Fig. 3. This case illustrates the

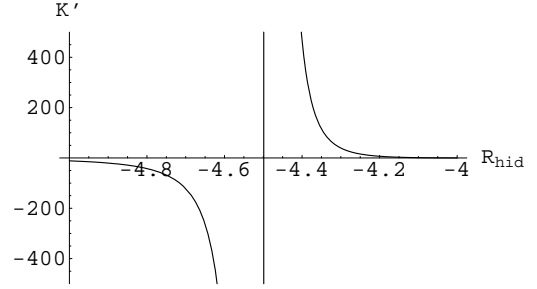


FIG. 3. For $\beta = 2/3$, we plot $[K']_2 \equiv [M_{rr}^2 / \{\frac{2}{3}(\kappa m_0 d \gamma)^2\}]_2$, see Eq. (66), as a function of R_{hid} in the $R_{hid} \leq -(2 + 3\beta) = -4$ region. Both $1 - c_2$ and $(R_{hid}V_{hid})_2$ display the same sign changes and singular behavior as $[M_{rr}^2]_2$. Only the $[K']_2 > 0$ region corresponds to a solution consistent with all constraints.

extreme sensitivity that M_{rr}^2 can have to R_{hid} and shows that very large and very small values of M_{rr}^2 are quite possible.

(ii) For $\beta = 1/2$, since $1/\beta$ is an integer, the sign of $1 - c$ is unconstrained. One finds $(R_{hid}V_{hid})_{1,2} > 0$ for all $R_{hid} > 0$ and $(M_{rr}^2)_{1,2} > 0$ and behaves smoothly. For all $R_{hid} < -7/2$, $(R_{hid}V_{hid})_1 < 0$ and this solution branch fails the positivity constraint. In the small region $-4.65 \leq R_{hid} < -7/2$, $(R_{hid}V_{hid})_2 > 0$ and $(M_{rr}^2)_2 > 0$, blowing up at $R_{hid} \sim -4.65$.

We also briefly describe the following interesting $2 + 3\beta < 0$ case: For $\beta = -1$, again the $1 - c > 0$ constraint is not necessary. For all $R_{hid} \geq -(2 + 3\beta) = 1$, both $(R_{hid}V_{hid})_1 < 0$ and $(R_{hid}V_{hid})_2 < 0$ so that the c_1 and c_2 solutions both fail the positivity constraint. In contrast, for all $R_{hid} < 0$ one finds $(R_{hid}V_{hid})_1 > 0$ and, of course, $(M_{rr}^2)_1 > 0$. However, $(R_{hid}V_{hid})_2 > 0$ only for $-0.32 \leq R_{hid} < 0$ and $R_{hid} \leq -1.78$ —in these two ranges $(M_{rr}^2)_2 > 0$ aside from a zero at the top end, $R_{hid} = -1.78$, of the 2nd range.

Overall it is clear that there is a large range of possible models that satisfy all the constraints necessary for the exact Randall-Sundrum metric with positive radion mass-squared. (Some particular choices are somewhat more fine-tuned than others.) We have not understood how to choose between the various models; possibly, the conformal models with $\beta = -2/3$ should be regarded as the more attractive.

IV. CONCLUSIONS

We have considered a class of generalizations of the Randall-Sundrum model containing a bulk scalar field Φ . We demonstrated that no-back reaction from the scalar on the Randall-Sundrum metric solution *requires* the existence of an extra interaction between gravity and the scalar. Here, we considered the coupling $Rf(\Phi)$. A general form of the potential for the fluctuation of the compactification volume (the radion) and the Kaluza-Klein modes of the excitation of the bulk scalar was derived and the mass matrix was determined. In order to obtain the values of scalar masses, one has to take into account the mixing between the radion and the Kaluza-Klein modes of the fluctuation of the bulk scalar. Here, we demonstrated that a positive value for the M_{rr}^2 mass-squared matrix entry associated with the radion field r

can be generated using a choice of the background bulk field, $\Phi_0(y)$, that preserves the RS metric (no back reaction). We found that M_{rr} receives the same suppression from the warp factor that is necessary to explain the hierarchy puzzle. Thus, ~ 1 TeV is a natural order of magnitude for the mass of the radionlike eigenstate. Finally, we illustrated the general scenario for the case of $f(\Phi) \propto \Phi^2$, for which the scalar background solution that preserves the Randall-Sundrum metric was explicitly found.

Since the mass-squared matrix for the radion and KK bulk scalar excitations is nondiagonal, it is clear that the introduction of Higgs-radion mixing on the visible brane through a term in the Lagrangian of the form $\xi \sqrt{|g_{vis}|} R(g_{vis}) \hat{H}^\dagger \hat{H}$, as considered for example in [8], would result in a complicated situation where the Higgs field, the radion and the KK excitations of the bulk scalar would all mix. A phenomenological study of the magnitudes of the various mixings, as a function of the available parameters, would be required to understand the extent to which the phenomenology of the physical Higgs

boson eigenstate can be modified. Such a study is beyond the scope of this paper.

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